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Some fixed point results via R-functions

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Abstract

We establish existence and uniqueness of fixed points for a new class of mappings, by using R-functions and lower semi-continuous functions in the setting of metric spaces. As consequences of this results, we obtain several known fixed point results, in metric and partial metric spaces. An example is given to support the new theory. A homotopy result for operators on a set endowed with a metric is given as application.

MSC: 47H10; 54H25

Keywords: R-function; R-λ-contraction; fixed point; metric space; partial metric space

1 Introduction

Metric fixed point theory is a fundamental topic, which gives basic methods and notions for establish practical problems in mathematics and the other sciences. As an example, we consider the existence of solutions of mathematical problems reducible to equivalent fixed point problems. Thus, we recall that Banach contraction principle [1] is at the foundation of this theory. However, the potentiality of fixed point approaches attracted many scientists and hence there is a wide literature available for interested reader; see for instance [2–7]. We give some details on the notions and ideas used in this study.

First, the notion of partial metric space was introduced in 1994 by Matthews [8] as a part of the study of denotational semantics of data for networks. Clearly, this setting is a generalization of the classical concept of metric space. Also, some authors discussed the existence of several connections between partial metrics and topological aspects of domain theory; see for instance [9–12].

Second, the notion of $\mathcal{Z}$-contraction was introduced in 2014 by Khojasteh *et al.* [13]. This concept is a new type of nonlinear contraction defined by using a specific function, called simulation function. Consequently, they proved the existence and uniqueness of fixed points for $\mathcal{Z}$-contraction mappings (see [13], Theorem 2.8). The notion of R-contraction was introduced in 2015 by Roldán López de Hierro and Shahzad [14]. Also this notion is a new type of nonlinear contraction defined by using a specific function called R-function. Naturally, they proved the existence and uniqueness of fixed points for R-contraction mappings (see [14], Theorem 27). We point out that the advantage of these methods is in providing a unifying point of view for several fixed point problems; see recent results in [15–17].

Finally, Samet *et al.* [18], and Vetro and Vetro [19] discussed fixed point results, by using semi-continuous functions in metric spaces, that generalize and improve many existing fixed point theorems in the literature. As an application of presented results, the authors gave some theorems in the setting of partial metric spaces. In this paper, we use the ideas in [18, 19] and the notion of R-function to establishing the existence and uniqueness of fixed points that belong to the zero set of a certain function. As consequences of this study, we deduce several related fixed point results, in metric and partial metric spaces. Also, an example is given to support the new theory. As application, a homotopy result for operators on a set endowed with a metric is given.

2 Preliminaries

We will start with a brief recollection of basic notions and results in partial metric spaces that can be found in [8, 11, 20, 21].

A partial metric on a non-empty set Z is a function $p : Z \times Z \to [0, +\infty[$ such that, for all $u, v, w \in Z$, we have

(p_1) $u = v \Leftrightarrow p(u, u) = p(u, v) = p(v, v)$;
(p_2) $p(u, u) \le p(u, v)$;
(p_3) $p(u, v) = p(v, u)$;
(p_4) $p(u, v) \le p(u, w) + p(w, v) - p(w, w)$.

A partial metric space is a pair (Z, p), where Z is a non-empty set and p is a partial metric on Z.

Every partial metric $p : Z \times Z \to [0, +\infty[$ generates a T_0 topology τ_p on Z, which has as a base the family of open p-balls $\{U_p(u, \rho) : u \in Z, \rho > 0\}$, where $U_p(u, \rho) = \{v \in Z : p(u, v) < p(u, u) + \rho\}$ for all $u \in Z$ and $\rho > 0$.

Let (Z, p) be a partial metric space and $\{u_j\} \subset Z$. Then

(i) $\{u_j\}$ converges to a point $u \in Z$ if and only if $p(u, u) = \lim_{j\to+\infty} p(u, u_j)$;
(ii) $\{u_j\}$ is called a Cauchy sequence if there exists $\lim_{i,j\to+\infty} p(u_i, u_j)$ (and it is finite);
(iii) (Z, p) is said to be complete if every Cauchy sequence $\{u_j\}$ in Z converges, with respect to τ_p, to a point $u \in Z$ such that $p(u, u) = \lim_{i,j\to+\infty} p(u_i, u_j)$.

It is elementary to verify that the function $d^p : Z \times Z \to [0, +\infty[$ defined by

$$d^p(u, v) = 2p(u, v) - p(u, u) - p(v, v) \tag{1}$$

is a metric on Z whenever p is a partial metric on Z. Moreover, $\lim_{j\to+\infty} d^p(u_j, u) = 0$ if and only if

$$p(u, u) = \lim_{j\to+\infty} p(u_j, u) = \lim_{i,j\to+\infty} p(u_i, u_j). \tag{2}$$

The following lemma shows that the function $\lambda : Z \to [0, +\infty[$ defined by $\lambda(u) = p(u, u)$ for all $u \in Z$ is continuous in (Z, d^p).

Lemma 2.1 *Let (Z, p) be a partial metric space and let $\lambda : Z \to [0, +\infty[$ be defined by $\lambda(u) = p(u, u)$ for all $u \in Z$. Then the function λ is continuous in the metric space (Z, d^p).*

Proof Let $u \in Z$ and $\{u_j\}$ be a sequence which converges to u in the metric space (Z, d^p). By (2), we get

$$p(u,u) = \lim_{j\to+\infty} p(u_j, u_j)$$

and hence λ is continuous in (Z, d^p). □

The following lemma correlates the Cauchy sequences of the spaces (Z,p) and (Z,d^p).

Lemma 2.2 ([8, 20]) *Let (Z,p) be a partial metric space. Then*

(1) *$\{u_j\}$ is a Cauchy sequence in (Z,p) if and only if it is a Cauchy sequence in the metric space (Z,d^p);*
(2) *a partial metric space (Z,p) is complete if and only if the metric space (Z,d^p) is complete.*

3 New fixed point theorems in complete metric spaces

In this section, we consider the family $\mathcal{R}$ of R-functions introduced by Roldán López de Hierro and Shahzad in [14]. Precisely, a function $\eta : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$ is called R-function if the following conditions hold:

(η_1) for each sequence $\{t_n\} \subset]0,+\infty[$ such that $\eta(t_{n+1}, t_n) > 0$ for all $n \in \mathbb{N}$, we have $\lim_{n\to+\infty} t_n = 0$;
(η_2) for every two sequences $\{t_n\}, \{s_n\} \subset]0,+\infty[$ such that $\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n = L \geq 0$, then $L = 0$ whenever $L < t_n$ and $\eta(t_n, s_n) > 0$ for all $n \in \mathbb{N}$.

Now, we use R-functions to define a new class of contractions. Let (Z,d) be a metric space. Denote by Λ the family of lower semi-continuous functions $\lambda : Z \to [0,+\infty[$. Let $h : Z \to Z$ be a self-mapping and $\lambda \in \Lambda$. In the sequel, we will use the following notation

$$D(u,v;\lambda) := d(u,v) + \lambda(u) + \lambda(v) \quad \text{for all } u,v \in Z.$$

Now, we define the new family of contractions.

Definition 3.1 Let (Z,d) be a metric space and let $h : Z \to Z$ be a mapping. The mapping h is a R-λ-contraction if there exist an R-function $\eta : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$ and a function $\lambda \in \Lambda$ such that

$$\eta\big(D(hu,hv;\lambda), D(u,v;\lambda)\big) > 0 \tag{3}$$

for all $u,v \in Z$ with $D(u,v;\lambda) > 0$.

In the following theorem, we establish a result of existence and uniqueness of a fixed point for R-λ-contractions that belong to $\{x \in Z : \lambda(x) = 0\}$.

Theorem 3.1 *Let (Z,d) be a complete metric space and let $h : Z \to Z$ be a R-λ-contraction. Assume that, at least, one of the following conditions holds:*

(1) *h is continuous;*

(2) *for every two sequences* $\{t_i\}, \{s_i\} \subset]0, +\infty[$ *such that* $\lim_{i\to+\infty} s_i = 0$ *and* $\eta(t_i, s_i) > 0$ *for all* $i \in \mathbb{N}$, *then* $\lim_{i\to+\infty} t_i = 0$;

(3) $\eta(t, s) \leq s - t$ *for all* $t, s \in]0, +\infty[$.

Then h *has a unique fixed point* $x \in Z$ *such that* $\lambda(x) = 0$ *and, for any choice of the starting point* $z_0 \in Z$, *the sequence* $\{z_n\}$ *defined by* $z_n = hz_{n-1}$ *for each* $n \in \mathbb{N}$ *converges to the point* x.

Proof We fix arbitrarily a point z_0 of Z and we consider the Picard sequence $\{z_i\}$ of h starting at z_0, that is, the sequence defined by $z_i = hz_{i-1}$ for all $i \in \mathbb{N}$. If for some $j \in \mathbb{N} \cup \{0\}$ we have $z_{j+1} = z_j$, then z_j is evidently a fixed point of h. Also, we claim that $\lambda(z_j) = 0$. First, from $z_j = z_{j+1}$, we deduce that $z_i = z_j$ for all $i \in \mathbb{N} \cup \{0\}$ with $i \geq j$. Assume $\lambda(z_j) > 0$ and let $t_i := D(z_{j+i}, z_{j+i+1}; \lambda)$, that is, a positive real number for all $i \in \mathbb{N}$. Since h by hypothesis is an R-λ-contraction, we get

$$\eta(t_{i+1}, t_i) = \eta\big(D(hz_{j+i}, hz_{j+i+1}; \lambda), D(z_{j+i}, z_{j+i+1}; \lambda)\big) > 0 \quad \text{for all } i \in \mathbb{N}.$$

By property (η_1) of the function η it follows that $\lambda(z_j) = \lambda(z_{j+i}) \to 0$ as $i \to +\infty$ and so $\lambda(z_j) = 0$ and hence the conclusion follows if $z_{j+1} = z_j$ for some $j \in \mathbb{N} \cup \{0\}$. Therefore, we can suppose that $z_{i-1} \neq z_i$ for all $i \in \mathbb{N}$.

We shall divide the proof in three parts. First, we show that

$$\lim_{i\to+\infty} d(z_{i-1}, z_i) = 0 \quad \text{and} \quad \lim_{i\to+\infty} \lambda(z_i) = 0. \tag{4}$$

From $z_{i-1} \neq z_i$ for all $i \in \mathbb{N}$, we deduce that

$$t_{i-1} = D(z_{i-1}, z_i; \lambda) > 0 \quad \text{for all } i \in \mathbb{N}.$$

Thus the sequence $\{t_i\} \subset]0, +\infty[$. Since h is an R-λ-contraction, from (3) with $u = z_i$ and $v = z_{i+1}$, we get

$$\begin{aligned}\eta(t_{i+1}, t_i) &= \eta\big(D(z_{i+1}, z_{i+2}; \lambda), D(z_i, z_{i+1}; \lambda)\big)\\ &= \eta\big(D(hz_i, hz_{i+1}; \lambda), D(z_i, z_{i+1}; \lambda)\big) > 0\end{aligned}$$

for all $i \in \mathbb{N} \cup \{0\}$. The property (η_1) of the function η allows one to state that $t_i \to 0$ as $i \to +\infty$. Consequently, $d(z_{i-1}, z_i) \to 0$ and $\lambda(z_i) \to 0$, that is, (4) holds.

The second part is to show that the sequence $\{z_i\}$ is Cauchy. Let us assume that $\{z_i\}$ is not a Cauchy sequence. Then there exist $\sigma > 0$ and two subsequences $\{z_{j(k)}\}$ and $\{z_{i(k)}\}$ of $\{z_i\}$ with $k \leq j(k) < i(k)$ and

$$d(z_{j(k)}, z_{i(k)-1}) \leq \sigma < d(z_{j(k)}, z_{i(k)})$$

for all $k \in \mathbb{N}$. The above restrictions and $\lim_{i\to+\infty} d(z_{i-1}, z_i) = 0$ imply

$$\lim_{k\to+\infty} d(z_{j(k)}, z_{i(k)}) = \lim_{k\to+\infty} d(z_{j(k)-1}, z_{i(k)-1}) = \sigma.$$

Since $\lambda(z_i) \to 0$ as $i \to +\infty$, we get

$$\sigma = \lim_{k\to+\infty} D(z_{j(k)}, z_{i(k)}; \lambda) = \lim_{k\to+\infty} D(z_{j(k)-1}, z_{i(k)-1}; \lambda).$$

The previous equality allows us to assume $D(z_{j(k)-1}, z_{i(k)-1}; \lambda) > 0$ for each $k \in \mathbb{N}$. Now, we consider the sequences $\{t_k\}, \{s_k\}$ given by

$$t_k := D(z_{j(k)}, z_{i(k)}; \lambda) \quad \text{and} \quad s_k := D(z_{j(k)-1}, z_{i(k)-1}; \lambda) \quad \text{for all } k \in \mathbb{N}.$$

From (3) with $u = z_{j(k)-1}$ and $v = z_{i(k)-1}$, we obtain

$$\begin{aligned} \eta(t_k, s_k) &= \eta\bigl(D(z_{j(k)}, z_{i(k)}; \lambda), D(z_{j(k)-1}, z_{i(k)-1}; \lambda)\bigr) \\ &= \eta\bigl(D(hz_{j(k)-1}, hz_{i(k)-1}; \lambda), D(z_{j(k)-1}, z_{i(k)-1}; \lambda)\bigr) > 0 \end{aligned} \tag{5}$$

for all $k \in \mathbb{N}$. Let $L = \sigma$; from $L = \sigma < d(z_{j(k)}, z_{i(k)}) \leq D(z_{j(k)}, z_{i(k)}; \lambda) = t_k$ and (5), by property (η_2) of the function η, we obtain $\sigma = L = 0$, which is a contradiction. Hence $\{z_i\}$ is a Cauchy sequence. As (Z, d) is by hypothesis a complete metric space, there exists $x \in Z$ such that $z_i \to x$ as $i \to +\infty$. The hypothesis that λ is lower semi-continuous implies that

$$0 \leq \lambda(x) \leq \liminf_{i \to +\infty} \lambda(z_i) = 0,$$

that is, $\lambda(x) = 0$. The third part is to prove that x is a fixed point of h. We consider the following three steps.

First step. h is a continuous mapping, that is, condition (1) holds. From $z_{i+1} = hz_i \to hx$, we get $x = hx$.

Second step. Hypothesis (2) holds. If there exists a subsequence $\{z_{i(k)}\}$ of $\{z_i\}$ such that $hz_{i(k)} = hx$ for all $k \in \mathbb{N}$, then x is a fixed point of h. If this does not happen, then we can assume that $z_i \neq x$ and $hz_i \neq hx$ for all $i \in \mathbb{N}$. Now, consider the sequences

$$t_i := D(hz_i, hx; \lambda) \quad \text{and} \quad s_i := D(z_i, x; \lambda)$$

for all $i \in \mathbb{N}$. Such a choice ensures that $\{t_i\}, \{s_i\} \subset]0, +\infty[$. Clearly, by (4) and $\lambda(x) = 0$, $s_i \to 0$ and since h is a R-ϕ-contraction, we have also

$$\eta(t_i, s_i) = \eta\bigl(D(hz_i, hx; \lambda), D(z_i, x; \lambda)\bigr) > 0 \quad \text{for all } i \in \mathbb{N}.$$

Then, by condition (2), we get $t_i \to 0$. This allows one to state that

$$d(z_{i+1}, hx) = d(hz_i, hx) \to 0$$

and hence $x = hx$.

Third step. Hypothesis (3) holds, that is, $\eta(t, s) \leq s - t$ for all $t, s \in]0, +\infty[$. Since (3) ensures that condition (2) holds, we conclude that x is a fixed point of h.

Finally, let us to verify that x is a unique fixed point of h. Proceeding by contradiction, we suppose that there exists $z \neq x$ such that $z = hz$. Let $t_i := D(z, x; \lambda) > 0$ for all $i \in \mathbb{N}$. Therefore

$$\begin{aligned} \eta(t_{i+1}, t_i) &= \eta\bigl(D(z, x; \lambda), D(z, x; \lambda)\bigr) \\ &= \eta\bigl(D(hz, hx; \lambda), D(z, x; \lambda)\bigr) > 0, \end{aligned}$$

for all $i \in \mathbb{N}$. Then by the property (η_1) of the function η, we obtain $t_i \to 0$, which contradicts the fact that $d(z,x) \neq 0$. Therefore $z = x$ and so h has a unique fixed point. □

Now, we present some particular results of fixed point in metric spaces, by choosing an appropriate R-function. The first corollary is a generalization of Geraghty's fixed point theorem [22] and it is obtained by taking in Theorem 3.1 as R-function $\eta(t,s) = \psi(s)s - t$ for all $t,s \in [0,+\infty[$, where ψ is endowed with a suitable property.

Corollary 3.1 *Let (Z,d) be a complete metric space and $h : Z \to Z$ be a mapping. Suppose that there exists a function $\lambda \in \Lambda$ such that*

$$D(hu,hv;\lambda) \leq \psi\big(D(u,v;\lambda)\big)D(u,v;\lambda) \quad \textit{for all } u,v \in Z \textit{ with } D(u,v;\lambda) > 0,$$

where $\psi : [0,+\infty[\to [0,1[$ is a function such that $\lim_{i\to+\infty} \psi(t_i) = 1$ implies $\lim_{i\to+\infty} t_i = 0$, for all $\{t_i\} \subset [0,+\infty[$. Then h has a unique fixed point $x \in Z$ such that $\lambda(x) = 0$ and, for any choice of the initial point $z_0 \in Z$, the sequence $\{z_i\}$ defined by $z_i = hz_{i-1}$ for each $i \in \mathbb{N}$ converges to the point x.

Remark 3.1 From Corollary 3.1, we obtain Geraghty fixed point theorem [22], if the function $\lambda \in \Lambda$ is defined by $\lambda(u) = 0$ for all $u \in Z$. Clearly, the Geraghty result is a generalization of Banach's contraction principle.

In the following corollary we give a result inspired by well-known results in [4, 23, 24]. It is obtained by taking in Theorem 3.1 as R-function $\eta(t,s) = \psi(s)s - t$ for all $t,s \in [0,+\infty[$, where ψ is endowed with a suitable property.

Corollary 3.2 *Let (Z,d) be a complete metric space and $h : Z \to Z$ be a mapping. Suppose that there exists a function $\lambda \in \Lambda$ such that*

$$D(hu,hv;\lambda) \leq \psi\big(D(u,v;\lambda)\big)D(u,v;\lambda) \quad \textit{for all } u,v \in Z \textit{ with } D(u,v;\lambda) > 0,$$

where $\psi : [0,+\infty[\to [0,1[$ is a function such that $\limsup_{t\to r^+} \psi(t) < 1$, for all $r > 0$. Then h has a unique fixed point $x \in Z$ such that $\lambda(x) = 0$ and, for any choice of the initial point $z_0 \in Z$, the sequence $\{z_i\}$ defined by $z_i = hz_{i-1}$ for each $i \in \mathbb{N}$ converges to the point x.

If in Theorem 3.1 we consider as R-function $\eta(t,s) = s - \psi(t)$ for all $t,s \in [0,+\infty[$, where ψ is a right continuous function, then we deduce the following corollary.

Corollary 3.3 *Let (Z,d) be a complete metric space and $h : Z \to Z$ be a mapping. Suppose that there exists a function $\lambda \in \Lambda$ such that*

$$\psi\big(D(hu,hv;\lambda)\big) \leq D(u,v;\lambda) \quad \textit{for all } u,v \in Z \textit{ with } D(u,v;\lambda) > 0,$$

where $\psi : [0,+\infty[\to [0,1[$ is a right continuous function such that $\psi(t) > t$, for all $t > 0$. Then h has a unique fixed point $x \in Z$ such that $\lambda(x) = 0$ and, for any choice of the initial point $z_0 \in Z$, the sequence $\{z_i\}$ defined by $z_i = hz_{i-1}$ for each $i \in \mathbb{N}$ converges to the point x.

From the previous corollary, we deduce the following result of integral type.

Corollary 3.4 *Let (Z,d) be a complete metric space and $h : Z \to Z$ be a mapping. Suppose that there exists a function $\lambda \in \Lambda$ such that*

$$\int_0^{D(hu,hv;\lambda)} \xi(\tau)\,d\tau \leq D(u,v;\lambda) \quad \text{for all } u,v \in Z \text{ with } D(u,v;\lambda) > 0, \tag{6}$$

where $\xi : [0,+\infty[\to [0,+\infty[$ is a function such that $\int_0^t \xi(\tau)\,d\tau$ exists and $\int_0^t \xi(\tau)\,d\tau > t$, for every $t > 0$. Then h has a unique fixed point $x \in Z$ such that $\lambda(x) = 0$ and, for any choice of the initial point $z_0 \in Z$, the sequence $\{z_i\}$ defined by $z_i = hz_{i-1}$ for each $i \in \mathbb{N}$ converges to the point x.

Example 3.1 Let $Z = [0, \frac{15}{8}] \cup \{2\}$ endowed with the usual metric $d(u,v) = |u-v|$ for all $u,v \in Z$. Obviously, (Z,d) is a complete metric space. Consider the function $h : Z \to Z$ defined by

$$hu = \begin{cases} \frac{u}{2} & \text{if } u \in [0, \frac{15}{8}], \\ \frac{3}{2} & \text{if } u = 2. \end{cases}$$

Clearly, h satisfies condition (6) with respect to the function $\xi : [0,+\infty[\to [0,+\infty[$ given by

$$\xi(t) = 1 + \frac{1}{(t+1)^2} \quad \text{for all } t \in [0,+\infty[$$

and the lower semi-continuous function $\lambda : Z \to [0,+\infty[$ defined by $\lambda(u) = u$ for all $u \in Z$. Indeed, if $u \leq v$ and $u,v \in [0, \frac{15}{8}]$, then

$$\int_0^{D(hu,hv;\lambda)} \xi(\tau)\,d\tau = \frac{v+2}{v+1} v \leq 2v = D(u,v;\lambda).$$

If $u \in [0, \frac{15}{8}]$ and $v = 2$, or $u = v = 2$, then

$$\int_0^{D(hu,h2;\lambda)} \xi(\tau)\,d\tau = \frac{3+2}{3+1} 3 \leq 4 = D(u,2;\lambda).$$

Since all the conditions of Corollary 3.4 are satisfied, the mapping T has a unique fixed point $x = 0$ in Z. Clearly, $\lambda(x) = 0$.

From $d(h0,h2) = 3/2$ and $d(0,2) = 2$, we deduce that

$$\int_0^{d(h0,h2)} \xi(\tau)\,d\tau = \frac{21}{10} \geq 2 = d(0,2).$$

Thus h is not a R-contraction with respect to the R-function $\eta : [0,+\infty[\times [0,+\infty[\to \mathbb{R}$ defined by

$$\eta(t,s) = s - \int_0^t \xi(\tau)\,d\tau, \quad \text{for all } t,s \in [0,+\infty[.$$

It follows that Theorem 27 of [14] cannot be used to deduce that h has a fixed point with respect to this R-function. The previous consideration also shows that the role of the function λ is decisive in enlarging the class of self-mappings satisfying condition (6) and hence condition (3).

4 Fixed points in partial metric spaces

In this section, from our Theorem 3.1, we deduce easily various fixed point theorems on partial metric spaces including the Matthews fixed point theorem.

Theorem 4.1 *Let (Z,p) be a complete partial metric space and let $h : Z \to Z$ be a mapping. Suppose that there exists a R-function η such that*

$$\eta\bigl(p(hu,hv),p(u,v)\bigr) > 0 \quad \text{for all } u,v \in Z, u \neq v. \tag{7}$$

Assume that, at least, one of the following conditions holds:

(j) *h is continuous with respect to metric d^p*;

(jj) *for every two sequences $\{t_i\},\{s_i\} \subset]0,+\infty[$ such that $\lim_{i\to+\infty} s_i = 0$ and $\eta(t_i,s_i) > 0$ for all $i \in \mathbb{N}$, then $\lim_{i\to+\infty} t_i = 0$*;

(jjj) *$\eta(t,s) \leq s - t$ for all $t,s \in]0,+\infty[$.*

Then h has a unique fixed point $x \in Z$ such that $p(x,x) = 0$.

Proof From (1), we say that

$$p(u,v) = \frac{d^p(u,v) + p(u,u) + p(v,v)}{2} \quad \text{for all } u,v \in Z. \tag{8}$$

The hypothesis that (Z,p) is complete, by Lemma 2.2, ensures that the metric space $(Z,2^{-1}d^p)$ is complete. Also, by Lemma 2.1, the function $\lambda : Z \to [0,+\infty[$ defined by $\lambda(u) = 2^{-1}p(u,u)$ is continuous and so lower semi-continuous in $(Z,2^{-1}d^p)$. Now, from (7) and (8), we see that for the mapping h the condition

$$\eta\bigl(2^{-1}d^p(hu,hv) + \lambda(hu) + \lambda(hv), 2^{-1}d^p(u,v) + \lambda(u) + \lambda(v)\bigr) > 0$$

holds for all $u,v \in Z$, $u \neq v$. Consequently, for the mapping h all the conditions of Theorem 3.1 hold with respect to the metric space $(Z,2^{-1}d^p)$. To conclude that the mapping h has a unique fixed point $x \in Z$ such that $p(x,x) = 2\,\lambda(x) = 0$. □

From Theorem 4.1 if consider as R-function $\eta(t,s) = ks - t$ for all $t,s \in [0,+\infty[$ with $k \in [0,1[$, we obtain the Matthews fixed point theorem.

Corollary 4.1 *Let (Z,p) be a complete partial metric space and let $h : Z \to Z$ be a mapping. Suppose that there exists $k \in [0,1[$ such that*

$$p(hu,hv) \leq k\,p(u,v) \quad \text{for all } u,v \in Z,\ u \neq v.$$

Then h has a unique fixed point $x \in Z$ such that $p(x,x) = 0$.

From Theorem 4.1 if consider as R-function $\eta(t,s) = \psi(s)s - t$ for all $t,s \in [0,+\infty[$, then we obtain a result of Geraghty type in partial metric spaces.

Corollary 4.2 *Let (Z,p) be a complete partial metric space and let $h : Z \to Z$ be a mapping. Suppose that there exists a function $\psi : [0,+\infty[\to [0,1[$ such that $\lim_{j\to+\infty} \psi(t_j) = 1$ implies $\lim_{j\to+\infty} t_j = 0$, for all $\{t_j\} \subset [0,+\infty[$. If*

$$p(hu,hv) \le \psi\bigl(p(u,v)\bigr)p(u,v) \quad \text{for all } u,v \in Z,$$

then h has a unique fixed point such that $p(x,x) = 0$ and, for any choice of the initial point $z_0 \in Z$, the sequence $\{z_j\}$ defined by $z_j = hz_{j-1}$ for each $j \in \mathbb{N}$ converges to the point x.

Other known results of fixed point in the setting of partial metric spaces we can get considering suitable simulation functions.

5 An application to homotopy

In this section, we give an application of our results to homotopy theory. Denote by Γ the family of nondecreasing upper semi-continuous functions $\rho : [0,+\infty[\to [0,+\infty[$ such that $\rho(s) < s$ for all $s > 0$ and with the following property:

$$\lim_{i,j\to+\infty} \bigl[s_{i,j} - \rho(s_{i,j})\bigr] = 0 \quad \text{implies} \quad \lim_{i,j\to+\infty} s_{i,j} = 0 \tag{9}$$

for every sequence $\{s_{i,j}\} \subset [0,+\infty[$.

Theorem 5.1 *Let (Z,d) be a complete metric space, let U an open subset of Z and V a closed subset of Z with $U \subset V$. Assume that the operator $Q : V \times [0,1] \to Z$ satisfies the following conditions*:

(i) *$u \neq Q(u,s)$ for each $u \in V \setminus U$ and all $s \in [0,1]$;*

(ii) *there exists $\rho \in \Gamma$ such that, for each $s \in [0,1]$ and all $u,v \in V$, we have*

$$d\bigl(Q(u,s),Q(v,s)\bigr) \le \rho\bigl(d(u,v)\bigr); \tag{10}$$

(iii) *there exists a continuous function $f : [0,1] \to \mathbb{R}$ such that*

$$d\bigl(Q(u,t),Q(u,s)\bigr) \le \bigl|f(t) - f(s)\bigr|$$

for all $t,s \in [0,1]$ and every $u \in V$.

Then $Q(\cdot,1)$ has a fixed point if $Q(\cdot,0)$ has a fixed point.

Proof Assume that $Q(\cdot,0)$ has a fixed point and consider the following set:

$$A := \bigl\{s \in [0,1] : u = Q(u,s) \text{ for some } u \in U\bigr\}.$$

By (i) we get $0 \in A$ and this implies that A is a non-empty set. We claim that A is both open and closed in $[0,1]$. Since $[0,1]$ is connected, we deduce that $A = [0,1]$.

First, we prove that A is a closed subset of $[0,1]$. Let $\{s_j\}$ be a sequence in A and assume that $s_j \to s_0 \in [0,1]$ as $j \to +\infty$. Now, we establish that $s_0 \in A$. The definition of the set A ensures that there exists $u_j \in U$ such that $u_j = Q(u_j, s_j)$ for all $j \in \mathbb{N}$. Then we get

$$\begin{aligned} d(u_i, u_j) &= d\big(Q(u_i, s_i), Q(u_j, s_j)\big) \\ &\leq d\big(Q(u_i, s_i), Q(u_i, s_j)\big) + d\big(Q(u_i, s_j), Q(u_j, s_j)\big) \\ &\leq \big|f(s_i) - f(s_j)\big| + \rho\big(d(u_i, u_j)\big). \end{aligned}$$

Thus

$$d(u_i, u_j) - \rho\big(d(u_i, u_j)\big) \leq \big|f(s_i) - f(s_j)\big|$$

for all $i, j \in \mathbb{N}$. Letting $i, j \to +\infty$ in the previous inequality, we get

$$\lim_{i,j \to +\infty} \big[d(u_i, u_j) - \lambda\big(d(u_i, u_j)\big)\big] = 0,$$

and by (9), we get $d(u_i, u_j) \to 0$ as $i, j \to +\infty$. This relation ensures that $\{u_j\}$ is a Cauchy sequence. Since Z is a complete metric space, there exists some $x \in V$ such that $u_j \to u$. From

$$\begin{aligned} d\big(u_j, Q(x, s_0)\big) &= d\big(Q(u_j, s_j), Q(x, s_0)\big) \\ &\leq d\big(Q(u_j, s_j), Q(u_j, s_0)\big) + d\big(Q(u_j, s_0), Q(x, s_0)\big) \\ &\leq \big|f(s_j) - f(s_0)\big| + \rho\big(d(u_j, x)\big) \\ &\leq \big|f(s_j) - f(s_0)\big| + d(u_j, x), \end{aligned}$$

letting $j \to +\infty$, we obtain

$$d\big(x, Q(x, s_0)\big) = \lim_{j \to +\infty} d\big(u_j, Q(x, s_0)\big) = 0.$$

This implies that $x = Q(x, s_0)$ and by (i), we deduce that $x \in U$. Consequently, $s_0 \in A$ and hence A is a closed subset of $[0,1]$.

Now, we prove that A is an open subset of $[0,1]$. If $s_0 \in A$, then there exists $u_0 \in U$ such that $u_0 = Q(u_0, s_0)$. Since U is open in Z, there exists $\sigma > 0$ such that $\overline{B}(u_0, \sigma) = \{u \in Z : d(u_0, v) \leq \sigma\} \subset U$. Because f is continuous at s_0, in correspondence to $\delta = \sigma - \rho(\sigma) > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that $|f(s) - f(s_0)| < \delta$ for all $s \in]s_0 - \varepsilon, s_0 + \varepsilon[$. Let $s \in]s_0 - \varepsilon, s_0 + \varepsilon[$ and $u \in \overline{B}(u_0, \sigma)$, we have

$$\begin{aligned} d\big(Q(u, s), u_0\big) &= d\big(Q(u, s), Q(u_0, s_0)\big) \\ &\leq d\big(Q(u, s), Q(u, s_0)\big) + d\big(Q(u, s_0), Q(u_0, s_0)\big) \\ &\leq \big|f(s) - f(s_0)\big| + \lambda\big(d(u, u_0)\big) \\ &\leq \sigma - \rho(\sigma) + \rho\big(d(u, u_0)\big) \leq \sigma. \end{aligned}$$

Therefore, $Q(\cdot,s)$ is a self-mapping on $\overline{B}(u_0,\sigma)$ for every fixed $s \in]s_0-\varepsilon, s_0+\varepsilon[$. From (10), we deduce that $Q(\cdot,s)$ is an R-ξ-contraction on $\overline{B}(u_0,\sigma)$ with respect to the R-function $\eta : [0,+\infty[\times [0,+\infty[\to \mathbb{R}$ defined by $\eta(t,s) = \rho(s) - t$ for all $s,t \geq 0$ and the function $\xi \in \Lambda$ defined by $\xi(u) = 0$ for all $u \in \overline{B}(u_0,\sigma)$.

This ensures that $Q(\cdot,s)$ has a fixed point in $\overline{B}(u_0,\sigma)$ and hence in U, since all hypotheses of Theorem 3.1 are satisfied. So $]s_0-\varepsilon, s_0+\varepsilon[\subset A$ and thus we see that A is an open subset of $[0,1]$. □

6 Conclusions

Fixed point theory in various metric settings is largely studied as a useful tool for solving problems arising in mathematics and the other sciences. Here, we proved existence and uniqueness of fixed point by using the notion of an R-function in metric and partial metric spaces. This kind of result is helpful to cover existing theorems in the literature from a unifying point of view. An homotopy result for certain operators supports the new theory.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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