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Function space topologies between the uniform topology and the Whitney topology

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ABSTRACT

This paper, dedicated to new function space topologies between the uniform topology and the Whitney topology also in the setting of the ω_μ-metric spaces, splits in two parts. In the former, where X is a Tychonoff space and (Y, d) is a non-discrete metric space, we explore suggestive uniformizable function space topologies on $C(X, Y)$, the set of all continuous functions from X to Y, located between the uniform topology and the Whitney topology. In the Whitney uniformity, whose natural associated topology is the Whitney topology, any continuous function from X to the positive reals gives a measure of closeness between functions in $C(X, Y)$. But, a less stringent and, by the way, efficient uniform control can be performed equally well by limiting, as for example at a first glance, to the measures deriving from all continuous positive functions continuously extendable to a T_2-compactification of X. And next, when X is a local proximity space, i.e. densely embedded in a natural T_2 local compactification $l(X)$, by limiting to the positive ones in $C(l(X), \mathbb{R})$. We investigate two classes of Tychonoff spaces. That of locally compact ones splittable in two essentially different cases: X hemicompact or not. And, that of spaces densely embedded in a locally compact one. We prove that, whenever X is hemicompact, then any weak Whitney topology relative to a T_2-compactification of X agrees with the classical one. Whenever X is locally compact but not hemicompact, then the weak Whitney topology associated with its one-point compactification reduces just to the uniform topology. In the case X is locally compact, paracompact but not hemicompact, thus the free union of an uncountable family of open σ-compact subsets, then, between the uniform topology and the Whitney topology there is a great variety of weak Whitney topologies relative to T_2-compactifications of X. Also, whenever X is not locally compact, weak Whitney topologies associated with different T_2 local compactifications of X are generally different as is the case if X is the rational Euclidean line. So, weakening the Whitney topology but without renouncing to the uniform convergence, we produce different uniformizable topologies on $C(X, Y)$ related to various significant structures on X. In the latter, since ω_μ-metric spaces, where ω_μ is an ordinal number, fill a large and attractive class of peculiar uniform spaces containing the usual metric ones, we focus our attention on the ω_μ-metric framework. Indeed, we extend the Whitney topology to $C(X, Y)$, where X is again a Tychonoff space but Y is replaced with an ω_μ-metric space. Precisely, the range space Y carries a distance $\rho : Y \times Y \to G$, sharing the usual formal properties with real metrics but valued in an ordered Abelian additive group G, which admits a strictly decreasing ω_μ-sequence converging to zero in the order topology. By a proof strategy essentially based on zero-dimensionality of any ω_μ-metric space with $\mu > 0$, we achieve, among others, the following result:

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Whenever X is an ω_μ-additive and paracompact space and (Y, ρ, G) is an ω_μ-metric space, then the Whitney topology on $C(X,Y)$ is independent of the ω_μ-metric ρ. More precisely, the Whitney topology is a topological character as in the classical metric case, $\mu = 0$, and X paracompact.

1. Introduction

Let X be a Tychonoff space and (Y,d) a non-discrete metric space. The *Whitney topology* or also *fine topology*, or *strong topology* on $C(X,Y)$, the set of all continuous functions from X to Y, is the topology having as a local base at any $f \in C(X,Y)$ the family of all sets of the type:

$$T(f,\boldsymbol{\epsilon}) := \{g \in C(X,Y) : d(f(x),g(x)) < \boldsymbol{\epsilon}(x),\ \forall x \in X\}$$

where $\boldsymbol{\epsilon}$ runs over all continuous functions from X to the real positive numbers [9], [10], [5], [2], [13], [11].

The *Whitney uniformity* on $C(X,Y)$ admits as diagonal basic neighborhoods the sets of the type:

$$\mathcal{U}(\boldsymbol{\epsilon}) := \{(f,g) \in C(X,Y) : d(f(x),g(x)) < \boldsymbol{\epsilon}(x),\ \forall x \in X\}$$

where $\boldsymbol{\epsilon}$ runs over all continuous functions from X to the real positive numbers.

In the Whitney uniformity, whose natural underlying topology is just the Whitney topology, any continuous function from X to the positive reals gives a measure of closeness between functions in $C(X,Y)$. But, a less stringent and, by the way, efficient uniform control can be performed equally well by limiting, as for example at a first glance, to the measures deriving from all continuous positive functions continuously extendable to a T_2-compactification of X. And next, when X is a local proximity space, i.e. densely embedded in a natural T_2 local compactification $l(X)$, by limiting to those ones in $C(l(X),\mathbb{R}^+)$.

When thinking to reduce functions we look for a natural and appropriate way. So, when focusing on this, since the Stone-Čech compactification $\beta(X)$ of X is characterized by the property: any real-valued continuous and bounded function on X continuously extends to $\beta(X)$, it is evident that the Whitney topology on $C(X,Y)$ relates naturally to $\beta(X)$. In view of this, it comes in mind to reduce $C(X,\mathbb{R}^+)$ to the set of those functions which are continuously extendable to a T_2-compactification $\gamma(X)$ of X, which we denote as $C_{\gamma(X)}(X,\mathbb{R}^+)$. And, indeed, the usual Whitney topology on $C(X,Y)$ comes naturally associated with the Stone-Čech compactification of X. Generally, the weak Whitney topologies relative to two distinct T_2-compactifications of X are different. But, it can happen that they all coincide with each other as, for example, in the case of the real Euclidean line.

We investigate two different classes of Tychonoff spaces. The former: the class of locally compact spaces splittable in two essentially different subclasses: hemicompact or not. The latter: the class of Tychonoff spaces densely embedded in a T_2 locally compact space or, in other words, the class of local proximity spaces. We prove that, *whenever X is T_2 locally compact and σ-compact, or equivalently hemicompact, then all weak Whitney topologies relative to a T_2-compactification of X agree with the usual one. Whenever X is T_2 and locally compact but not hemicompact, then the weak Whitney topology associated with the one-point-compactification of X reduces just to the uniform topology. Furthermore, whenever X is T_2 locally compact and paracompact but not hemicompact, thus the free union of an uncountable family of open σ-compact subsets, then there is a great variety of weak Whitney topologies relative to T_2-compactifications of X.*

The notion of *local proximity space* is a combination of proximity with boundedness with some natural reciprocal compatibility conditions [12]. By embedding the underlying space of a local proximity space X in the T_2 local compactification $l(X)$ naturally associated with it, we can apply to the local proximity case most of the results achieved in the locally compact one. Any given T_2 local compactification $l(X)$ of X takes up two features of X. The former one is the separated EF-proximity on X induced by the one-point compactification of $l(X)$, i.e. two subsets of X are far iff their closures in $l(X)$ don't intersect and one of them is compact, [14], [6]. The latter one is the *bornology* [12], [7], done by all subsets of X whose closure in $l(X)$ is compact, which are called *bounded*. In the proximal case the uniform control with a consequential appropriate way to topologize can be operated by limiting to the functions in $C(l(X), \mathbb{R}^+)$. In the case the bornology has a countable base or, equivalently, $l(X)$ is hemicompact, as we show in Theorem 4.1, we achieve as issue the following: *Whichever is Y, a sequence of continuous functions $\{f_n : X \to Y,\ n \in \mathbb{N}^+\}$ converges in the weak Whitney topology associated with $l(X)$ to a function $f \in C(X, Y)$ iff it uniformly converges to f and there is a bounded set B and a positive number n_0 so that f_n coincides with f outside B and for each $n > n_o$.*

In the examined cases, the players in substitution of $C(X, \mathbb{R}^+)$ are $C_{\gamma(X)}(X, \mathbb{R}^+)$ and $C(l(X), \mathbb{R}^+)$, with $\gamma(X)$, $l(X)$ running over all T_2-compactifications, T_2 local compactifications of X, respectively. In every case, all examined weak Whitney topologies are uniformizable.

Finally, we can enlarge the range Y to run inside all ω_μ-metric spaces.

In [15], independently of [11], Sikorski introduced the concept of ω_μ-metric space as a set X equipped with a distance $\rho : Y \times Y \to G$ valued in a totally ordered Abelian additive group G, which admits a strictly decreasing ω_μ-sequence converging to zero in the order topology, sharing the usual formal properties with real metrics. The ρ-balls, defined as usual, determine the natural topology τ_ρ associated with (Y, ρ, G) which has some peculiar properties giving us a powerful tool for achieving our results. The topology τ_ρ is T_2, paracompact, ω_μ-additive, i.e. every intersection of $|\alpha|$ many open sets is open for each $\alpha < \omega_\mu$, and in the uncountable case, $\mu > 0$, of covering dimension 0.

The proof technique in the ω_μ-metric case when $\mu > 0$ is really different from the usual metric one, $\mu = 0$, and essentially based on the property: Any ω_μ-additive paracompact space, and any ω_μ-metric space is of this type, is zero-dimensional in the sense that every open cover admits a refinement of pairwise disjoint clopen sets, whenever $\mu > 0$. From zero-dimensionality of any ω_μ-metric space with $\mu > 0$, we achieve, among others, the following result: *Whenever X is an ω_μ-additive and paracompact space and (Y, ρ, G) is an ω_μ-metric space, then the Whitney topology on $C(X, Y)$ is independent of the ω_μ-metric ρ.* More precisely, the Whitney topology is a topological character as in the classical metric case, $\mu = 0$, and X paracompact [10].

2. Some basic facts

In order to give some useful background and for more exhaustive information, the definitions, the terminology and the results quoted below are drawn by [6], [14], [18], [8].

We summarize here a number of basic facts. We start by introducing the *classical Whitney topology* [9], [10]. Let X be a Tychonoff space and (Y, d) a non-discrete metric space. The *Whitney topology* or also *fine topology* [10], or *strong topology* [9] on $C(X, Y)$, the set of all continuous functions from X to Y, is the topology having as a local base at any $f \in C(X, Y)$ the family of all sets of the type:

$$T(f, \boldsymbol{\epsilon}) := \{g \in C(X, Y) : d(f(x), g(x)) < \boldsymbol{\epsilon}(x),\ \forall x \in X\}$$

where $\boldsymbol{\epsilon}$ runs over all continuous functions from X to the real positive numbers. The Whitney neighborhood $T(f, \boldsymbol{\epsilon})$ is usually referred as the *tube centered at f with radius $\boldsymbol{\epsilon}$.*

The *Whitney uniformity* on $C(X, Y)$ admits as diagonal basic neighborhoods the sets of the type:

$$\mathcal{U}(\boldsymbol{\epsilon}) := \{(f, g) \in C(X, Y) : d(f(x), g(x)) < \boldsymbol{\epsilon}(x),\ \forall x \in X\}$$

where $\boldsymbol{\epsilon}$ runs over all continuous functions from X to the real positive numbers.

Firstly, we underline the following basic facts in the metric setting:

• For a Tychonoff space X the Whitney topology on $C(X,\mathbb{R})$ (or more generally on $C(X,Y)$, where Y is a non-discrete metric space) is finer than the topology of uniform convergence and reduces to it iff the space X does not admit any real-valued positive continuous function whose range admits zero as its greatest lower bound, or, in other words, iff X is pseudo-compact.

• Whenever X is an n-dimensional topological manifold, i.e. a second countable T_2 topological space for which every point has a neighborhood homeomorphic to the Euclidean space $\mathbb{R}^n$, then a sequence $\{f_n : n \in \mathbb{N}^+\}$ in $C(X,\mathbb{R})$ converges in the Whitney topology to a function f iff it converges to f in the uniform topology and there are a positive integer n_0 and a compact K so that $f_n(x) = f(x)$ for each $n > n_0$ and each x outside K, see chapter 2: Function spaces in [9].

Trying in the next section to extend this result to the class of T_2 locally compact (non-compact) spaces which are first countable at infinity we remind that: a space X is *hemicompact* if it is a union of countably many compact sets K_n so that $K_n \subset int(K_{n+1})$ for each n. In T_2, hemicompactness is equivalent the one-point compactification being first countable at infinity. Again in T_2, hemicompactness is equivalent to local compactness plus σ-compactness; a space is *σ-compact* if it is a union of countably many compact subsets. And, once again in T_2, hemicompactness is equivalent to local compactness plus the Lindelöf property as well. Furthermore, it is worth mentioning that in T_2 hemicompact pseudo-compact spaces are compact [17]. Finally, any T_2 locally compact and paracompact space is a free union of open σ-compact subspaces and, in particular, any T_2 locally compact topological group can be seen as a free union of open σ-compact topological groups (not subgroups).

• Whenever X is paracompact, then the Whitney topology is independent of the metric d. More precisely, it is a topological character [10].

• The Whitney topology is not metrizable in general [10].

• Let $\mathbb{Q}$ be the rational Euclidean line. Then, any real-valued positive continuous function can be minorized by a locally constant function [5].

• Even when the space X is first countable at infinity it can happen that a net converges in the Whitney topology without being eventually constant outside a compact set, as proven in the following example.

Example 2.1. Let X be the set of non-negative real numbers. The set $\mathbb{S}$ of decreasing sequences of distinct real positive numbers $\{\epsilon_n : n \geq 0\}$ converging to zero equipped with the relation $<$ defined as $\{\epsilon_n : n \geq 0\} < \{\eta_n : n \geq 0\}$ iff $\epsilon_n > \eta_n, \forall n \geq 0$, is preordered and filtered. For any given sequence $\{\epsilon_n : n \geq 0\}$ in $\mathbb{S}$, let us denote as $f_{\{\epsilon_n : n \geq 0\}} : X \to \mathbb{R}$ the piecewise linear function whose graph relative to the interval $[n, n+1]$ is the segment joining the point (n, ϵ_n) to $(n+1, \epsilon_{n+1})$. Actually, we have a net, $\{f_{\{\epsilon_n : n \geq 0\}} : \{\epsilon_n : n \geq 0\} \in \mathbb{S}\}$ that converges in the Whitney topology to the null function. But, any $f_{\{\epsilon_n : n \geq 0\}}$ is different from the null function at each point of X. Namely, $\{f_{\{\epsilon_n : n \geq 0\}} : \{\epsilon_n : n \geq 0\} \in \mathbb{S}\}$ uniformly converges to the null function. For each positive $\epsilon > 0$ it happens that $f_{\{\epsilon_n : n \geq 0\}}(x) < \epsilon$ for each sequence $\{\epsilon_n : n \geq 0\}$ in $\mathbb{S}$ for which ϵ_0 is less than ϵ and each point x in X. Next, let $g : X \to \mathbb{R}^+$ be a continuous function having zero as its greatest lower bound. The function g has a positive minimum m_n on each of the intervals $[n, n+1], n \geq 0$. Consequently, there are decreasing sequences of distinct real positive numbers $\{\epsilon_n : n \geq 0\}$ converging to zero with $\epsilon_n < m_n$ for each $n \geq 0$. And, for each of them and for each $n \geq 0$ and more for each x in $[n, n+1]$ it happens that $f_{\{\epsilon_n : n \geq 0\}}(x) < m_n < g(x)$. And the result is obtained.

And moreover, the following ones in the ω_μ-metric setting.

In [15], Sikorski introduced the concept of ω_μ-metric space as a set X equipped with a distance $\rho : Y \times Y \to G$ valued in a totally ordered Abelian additive group G, which admits a strictly decreasing ω_μ-sequence converging to zero in the order topology, sharing the usual formal properties, i.e. positiveness, symmetry and triangle inequality, with real metrics. Recall that if $(G, +, <)$ is a totally ordered Abelian group, whose neutral element 0 is not isolated in the order topology, the *character* of G is the minimal ordinal number ω_μ for which there is a strictly decreasing ω_μ-sequence convergent to 0 [4]. The ρ-balls, defined as usual, determine the natural topology τ_ρ associated with (Y, ρ, G) which has some peculiar properties. The topology τ_ρ is T_2, paracompact, ω_μ-additive, i.e. every $|\alpha|$ intersection of many open sets is open for each $\alpha < \omega_\mu$, and in the uncountable case $\mu > 0$ of covering dimension 0, that is, any open covering of X admits a refinement of pairwise disjoint clopen sets of X.

• Furthermore, naturally attached to the topology τ_ρ there is the totally ordered (with respect to the usual inclusion) diagonal uniformity $\mathcal{U}_\rho$ having as a base $\{\mathcal{U}_\alpha : \alpha < \omega_\mu\}$, where $\{a_\alpha : \alpha < \omega_\mu\}$ is a positive strictly decreasing ω_μ-sequence converging to 0 and $\mathcal{U}_\alpha := \{(x, y) : \rho(x, y) < a_\alpha\}$ for all α [1]. But, the converse implication is also true. In association with any diagonal uniformity with a totally ordered base $\{\mathcal{U}_\alpha : \alpha < \omega_\mu\}$, i.e. $\alpha < \beta \;\Leftrightarrow\; \mathcal{U}_\beta \subset \mathcal{U}_\alpha$, of cardinality $\aleph_\mu$, there is an ω_μ-metric structure on X, of which it is a compatible uniformity, based on the group J_μ of the ω_μ-sequences of integers carrying the pointwise addition and lexicographic order [16].

• Any totally ordered Abelian group G of character ω_μ becomes naturally an ω_μ-metric space when equipped with the *absolute value ω_μ-metric* ρ defined as $\rho(a, b) = |a - b|, a, b \in G$, after introducing as the *absolute value* $|a| = max\{a, -a\}, a \in G$ [4].

3. The locally compact case

In this section we investigate the locally compact case. In particular, the hemicompact case, the locally compact but not hemicompact case, the locally compact plus paracompact case.

Let X be a Tychonoff space, $\gamma(X)$ a T_2-compactification of X and (Y, d) a non-discrete metric space. We introduce on $C(X, Y)$ the *weak Whitney topology associated with* $\gamma(X)$, $\tau_{W_{\gamma(X)}}$, as the topology having as a local base at any $f \in C(X, Y)$ the family of all sets of the type:

$$T(f, \boldsymbol{\epsilon}) := \{g \in C(X, Y) : d(f(x), g(x)) < \boldsymbol{\epsilon}(x),\ \forall x \in X\}$$

where $\boldsymbol{\epsilon}$ runs over $C_{\gamma(X)}(X, \mathbb{R}^+)$, the set of all continuous functions from X to the positive reals continuously extendable to $\gamma(X)$.

Theorem 3.1. *Let X be a T_2 locally compact but non-compact space. Then, X admits a real-valued positive continuous function continuously extendable to the one-point compactification of X having zero as its greatest lower bound iff X is first countable at infinity, or, equivalently, hemicompact.*

Proof. Suppose X hemicompact. So, X can be written as a union of countably many compact sets K_n with $K_n \subset int(K_{n+1}),\ n \geq 1$. There are two possibilities. The former: $Fr(K_n) \neq \emptyset$ for each n. In this case, after choosing a sequence $\{\epsilon_n : n \geq 1\}$ of positive real numbers strictly decreasing to zero, we select for each $n \geq 1$ a continuous function $f_n : K_{n+1} \setminus int(K_n) \to [\epsilon_{n+1}, \epsilon_n]$ so that $f_n(Fr(K_n)) = \epsilon_n,\ f_n(Fr(K_{n+1})) = \epsilon_{n+1}$ and as f_1 the constant function on K_1 associated with ϵ_1. Then, we can join together all f_n to each other and to f_1 in a unique global positive continuous function f. Of course, f admits zero as its greatest lower bound and, moreover, is continuously extendable at infinity. In fact, for each ϵ greater than 0 and greater than ϵ_n, f is less than ϵ outside K_n.

The latter: If the set of the empty boundaries is finite we proceed similarly. In the infinite case, for simplicity, X can be considered as the union of K_1 with the family of pairwise disjoint closed sets $\{K_{n+1} \setminus K_n : n \geq 1\}$. The function f, defined as $f(x) = \epsilon_1$ when $x \in K_1$, $\ f(x) = \epsilon_n$, when $x \in K_{n+1} \setminus K_n$ and $n \geq 1$, works well. Being locally constant, f is continuous, also positive and admits zero as its greatest lower bound. Moreover, as above, f is continuously extendable at infinity.

The converse implication is also true. Let $f : X \to [0, 1]$ be a positive continuous function convergent to zero at infinity and $\{\epsilon_n : n \geq 1\}$ a sequence of positive real numbers decreasing to zero. If we denote as f_∞ the continuous extension of f to $X \cup \{\infty\}$, the one-point compactification of X, then $\{f_\infty^{-1}([0, \epsilon_n[)\}$ is a countable family of open neighborhoods of ∞ forming a local base. Indeed, by continuity, on any compact set K of X, f admits a positive minimum ϵ. Consequently, whenever ϵ_n is less than ϵ, K cannot intersect $f_\infty^{-1}([0, \epsilon_n[)$, which is therefore contained in $X \cup \{\infty\} \setminus K$. And this completes the proof. □

Theorem 3.2. *Let X be a T_2 hemicompact space and (Y, d) any non-discrete metric space. Then any weak Whitney topology relative to a T_2-compactification of X on $C(X, Y)$ agrees with the usual one.*

Proof. In the case X is pseudo-compact, since the Whitney topology agrees with the uniform topology, the result is trivially obtained. Otherwise, since X is hemicompact but not pseudo-compact, then X admits a real-valued positive continuous function f which is infinitesimal at infinity, see Theorem 3.1. For that any other real positive continuous function g can be minorized with the function $h = min(f, g)$ that in its turn is infinitesimal at infinity. Consequently, if a net converges with respect to the weak Whitney topology relative to the one-point compactification, it does converge with respect to any other one, as well. □

We give now some results on Whitney convergence for sequences. It is known that for an n-dimensional topological manifold M, see section 2, and chapter 2 in [9], a sequence $\{f_n : M \to \mathbb{R},\ n \in \mathbb{N}^+\}$ converges in the Whitney topology to a limit f iff it converges to f in the uniform topology and f_n agrees eventually with f outside a compact set of M.

In the absence of local compactness as in the rational case a sequence can converge in the Whitney topology without being eventually constant outside a compact set, as we prove with the next example.

Example 3.1. Let r be an irrational number and, for each integer $n \in N^+$, $\sigma_n : \mathbb{Q} \to \mathbb{R}$ so defined: $\sigma_n(q) = -q$, when $q \in]-r/n, r/n[$ and $\sigma_n(q) = q$, otherwise. Any σ_n agrees with the identity function $i_\mathbb{Q}$ except locally around zero where, instead, it agrees with the symmetry $\sigma : \mathbb{Q} \to \mathbb{R}$ such that $\sigma(q) = -q$ for all q. Two local symmetries σ_n, σ_m coincide except on $]-r/m, -r/n[\cup]r/n, r/m[$ when $m > n$. The sequence of the local symmetries $\{\sigma_n : n \in N^+\}$ converges in the Whitney topology to the identity function $i_\mathbb{Q}$. It happens that $|\sigma_{n(q)} - q| = |2q|$ when $q \in]-r/n, r/n[$ and is zero otherwise. Now, let $\boldsymbol{\epsilon} : \mathbb{Q} \to \mathbb{R}^+$ be a continuous function. Since the inequality $|2q| < \boldsymbol{\epsilon}(q)$ holds in zero, then, by continuity, it holds around zero as for example in $]-r/n_0, r/n_0[$. It follows that for each $n \geq n_0$, $|\sigma_n(q) - q|$ is zero or less than $\boldsymbol{\epsilon}(q)$. And convergence in the Whitney topology follows. Next, let K be a compact set in $\mathbb{Q}$. Zero can be an accumulation point of K or not. If not, there is an integer n so that $]-r/n, r/n[$ does not intersect K. But, in $]-r/n, r/n[$ no σ_m agrees with $i_\mathbb{Q}$, when $m \geq n$. In the other case, from compactness of K, zero can be approached by a sequence $\{q_n : n \in N^+\}$ of rationals chosen outside K so that $r/(n+1) < q_n < r/n$. For each integer $n \in N^+$, it happens that $\sigma_{n+k}(q_n) \neq \sigma_n(q_n)$, when $k > 0$.

The following theorem generalizes the already known and above cited result for n-dimensional topological manifolds. We extend that to X being a T_2 hemicompact space and Y any non-discrete metric space.

Theorem 3.3. *Let X be a T_2 hemicompact space and Y a non-discrete metric space. Then, a sequence $\{f_n : n \in \mathbb{N}^+\}$ in $C(X, Y)$ converges to f in the Whitney topology iff it converges to f in the uniform topology and, moreover, there are a compact set K in X and a natural number n_0 so that $f_n(x) = f(x)$ for each $n > n_0$ and x in $X \setminus K$.*

Proof. Consider X as a union of countably many compact sets K_n so that $K_n \subset int(K_{n+1}), n \in \mathbb{N}^+$. Of course, any compact set K is contained in some K_n. Suppose that $\{f_n : n \in \mathbb{N}^+\}$ converges in the Whitney topology to f and, at the same time, for each integer k there exist a positive integer n_k greater than k and a point x_{n_k} outside K_k such that $|f_{n_k}(x_{n_k}) - f(x_{n_k})| = \epsilon_{n_k} > 0$. Since any compact set is contained in some K_n, the sequence $\{x_{n_k} : k \in \mathbb{N}^+\}$ cannot admit any accumulation point in X. Finally, as previously proven in the first part of the Theorem 3.1, the hemicompactness of X allows to see any ϵ_{n_k} as the value taken in x_{n_k} from a positive continuous function on X. And that contradicts the Whitney convergence. The converse implication derives easily from the very basic property: any real-valued continuous function admits minimum on every compact set, so yielding the coincidence of the uniform convergence with the Whitney convergence on any compact set. □

We exhibit the following example to show two different facts:

• Weak Whitney topologies associated with different T_2-compactifications of X are generally different.

• • In local compactness but without hemicompactness the characterization of the Whitney convergence for sequences, as in Theorem 3.3, does not hold.

Example 3.2. Let X be a free union $X = \cup\{\mathbb{R}_\alpha : \alpha \in \mathcal{A}\}$ of open fibers $\mathbb{R}_\alpha$ each a homeomorphic copy of the Euclidean real line $\mathbb{R}$ and Y be a metric space. The space X is hemicompact iff the number of fibers is finite or countable. Consequently, whichever is Y, in that case all weak Whitney topologies on $C(X,Y)$ associated with T_2-compactifications of X agree each other, see Theorem 3.2. On the other hand, when the family of fibers is uncountable, then the Whitney topology associated with the one-point compactification of X is the topology of uniform convergence, see Theorem 3.1. Moreover, the space X has different types of two-point compactifications admitting different relative weak Whitney topologies. Remind that a T_2 space X admits an n-point compactification iff it is a union of non-empty non-compact pairwise disjoint open sets $A_1, ..., A_n$ with a compact set K [3]. When X splits in the union of two disjoint open sets A, B, each union of fibers, then the free union of their one-point compactifications is a two-point compactification of X. When one of the open set A is a union of countably many fibers, then a sequence $\{f_n : X \to Y,\ n \in \mathbb{N}\}$ converges to f in the weak Whitney topology associated with the relative two-point compactification of X iff it uniformly converges to f and there are a compact set K and a positive integer n_0 so that $f_n(x) = f(x)$ for each $n > n_0$ and each $x \in A \setminus K$. We underline that $A \setminus K$ is not the complement of a compact set in X. Namely, the complement of a compact set in X in the examined case has to contain uncountable families of fibers. If the splitting of X consists of the unions of two disjoint open sets A, B, both union of uncountably many fibers, then the relative weak Whitney topology reduces just to the uniform convergence. Of course, we can proceed by considering n-point compactifications, since X admits topologically distinct n-point compactifications for each n.

The same argument works equally well whenever X is a T_2 locally compact topological group. In fact, any T_2 locally compact topological group can be seen as a free union of open σ-compact subsets.

4. Local proximity spaces

In this section we investigate weak Whitney topologies when X is densely embedded in a T_2 locally compact space.

Let X be a Tychonoff space, $l(X)$ a T_2 local compactification of X and (Y,d) a non-discrete metric space. We introduce on $C(X,Y)$ the *weak Whitney topology associated with* $l(X), \tau_{W_{l(X)}}$, as the topology having as a local base at any $f \in C(X,Y)$ the family of all sets of the type:

$$T(f,\boldsymbol{\epsilon}) := \{g \in C(X,Y) : d(f(x), g(x)) < \boldsymbol{\epsilon}(x),\ \forall x \in X\}$$

where $\boldsymbol{\epsilon}$ runs $\in C(l(X), R^+)$.

Any given T_2 local compactification $l(X)$ of X takes up two features of X. The former one is the separated EF-proximity on X induced by the one-point compactification of $l(X)$, i.e. two sets in X are far iff their closures in $l(X)$ don't intersect and one of them is compact, [14], [6]. The latter one is the *bornology* [12], [7], done by all subsets of X whose closure in $l(X)$ is compact, which are called *bounded*.

In trying to apply results obtained in the locally compact case we focus on the T_2 local compactifications which are hemicompact, or, in other words, to the T_2 local compactifications $l(X)$ whose one-point compactification $l(X) \cup \{\infty\}$ is first countable at ∞. So, first of all, we will look for conditions which guarantee that and see two examples showing:

Theorem 4.1. *A T_2 local compactification $l(X)$ of a Tychonoff space X is first countable at infinity, or, equivalently, hemicompact, iff the natural associated bornology $\mathcal{B}$ admits a countable base, i.e. there is a countable family $\{B_n : n \in \mathbb{N}^+\}$ of bounded sets of X so that any bounded set B is contained in some B_n.*

Proof. First, we show that any compact set K of $l(X)$ is contained in the closure in $l(X)$ of some bounded set of X. Let A be an open set of $l(X)$ with compact closure containing K. By the density of X in $l(X)$, the trace $A \cap X$ of A on X is a non-empty open bounded set of X with the same closure in $l(X)$ of A. From this it follows that K is contained in the $Cl_{l(X)}(A \cap X)$.

Now, suppose there is a countable base $\{B_n : n \in \mathbb{N}^+\}$ of bounded sets. Since, as previously proven, any compact set K of $l(X)$ is contained in the closure in $l(X)$ of a bounded set B of X which is in turn contained in the closure in $l(X)$ of some B_n, it follows that $l(X)$ is union of the countable family of compact sets $\{Cl_{l(X)}(B_n) : n \in \mathbb{N}^+\}$. Thus, $l(X)$, which is T_2 and locally compact, is hemicompact too.

For the converse implication, let $\{K_n : n \in \mathbb{N}^+\}$ be a countable cover of $l(X)$ done of compact sets with non-empty interiors and so that any other compact set of X is contained in some K_n. Then $\{B_n : n \in \mathbb{N}^+\}$, where $B_n := K_n \cap X$, is a countable family of non-empty bounded sets of X which is a base. Indeed, any bounded set B of X has a compact closure in $l(X)$ contained in some K_n. Thus, $B \subseteq Cl_{l(X)}(B) \cap X \subseteq K_n \cap X$. □

When X is a local proximity space, the combination of the results in Theorem 4.1 and Theorem 3.1 gives the following characterization of the weak Whitney convergence for sequences relative to the natural T_2 local compactification $l(X)$ associated with X.

Corollary 4.1. *Let $l(X)$ be a hemicompact T_2 local compactification of a Tychonoff space X and Y a metric space. Then, a sequence $\{f_n : n \in \mathbb{N}^+\}$ in $C(X,Y)$ converges to f in the weak Whitney topology relative to $l(X)$ iff it converges to f in the uniform topology and, moreover, there are a bounded set B in X and a positive integer n_0 so that $f_n(x) = f(x)$ for each $n > n_0$ and x in $X \setminus B$.*

Proof. One way is simple. Namely, any continuous function from $l(X)$ to the reals admits minimum on every bounded set of X, so yielding the coincidence of the weak Whitney topology relative to $l(X)$ with the uniform topology. For the converse implication consider $l(X)$ as a union of countably many compact sets K_n with $K_n \subset int(K_{n+1})$ for each $n \in \mathbb{N}^+$. Therefore, the family $\{B_n := K_n \cap X,\ n \in \mathbb{N}^+\}$, is a countable base of the bornology associated with $l(X)$. Suppose that $\{f_n : X \to Y, n \in \mathbb{N}^+\}$ converges to f in the weak Whitney topology on $C(X,Y)$ relative to $l(X)$ and, at the same time, for each positive k there exist an integer n_k greater than k and a point x_{n_k} outside B_k such that $|f_{n_k}(x_{n_k}) - f(x_{n_k})| = \epsilon_{n_k} > 0$. Since any bounded set is contained in some B_n, the sequence $\{x_{n_k} : k \in \mathbb{N}^+\}$ cannot admit any accumulation point in $l(X)$. So, as in Theorem 3.1, we can see ϵ_{n_k} as the value in x_{n_k} of a continuous function from $l(X)$ to the positive reals. And that contradicts the weak Whitney convergence. □

• In general, a weak Whitney topology relative to a T_2 local compactification is finer than the uniform topology. To this end we exhibit the following example:

Example 4.1. Let r be an irrational number and $f_n : \mathbb{Q} \to \mathbb{R}$ be defined as: $f_n(q) = 0$ if $q \in]-r/n, r/n[$ and $f_n(q) = 1/n$ otherwise, for each $n \in \mathbb{N}^+$. The sequence $\{f_n : n \in N^+\}$ converges uniformly to the null function but not in the weak Whitney topology relative to the Euclidean real line $\mathbb{R}$ regarded as a T_2 local compactification of the rational Euclidean line $\mathbb{Q}$ with relative bornology that of the bounded sets of $\mathbb{Q}$ in the Euclidean metric. That is because there is no bounded set in $\mathbb{Q}$ outside of which f_n eventually agrees with the null function.

When X is a local proximity space whichever is Y, two weak Whitney topologies emerge naturally. That associated with $l(X)$, the natural T_2 local compactification of X, determined from $(Cl(X), \mathbb{R}^+)$ and that associated with the T_2-compactification of X, $l(X) \cup \{\infty\} = \gamma(X)$, determined from $C_{\gamma(X)}(X, \mathbb{R}^+)$. We prove that, in general, they are different. Namely:

• • In general, a weak Whitney topology relative to a T_2 local compactification $l(X)$ is different from the weak Whitney topology relative to the one-point compactification of $l(X)$ and weaker than the usual Whitney topology, as well.

Example 4.2. Regard the Euclidean real line $\mathbb{R}$ as the T_2 local compactification of the Euclidean rational line $\mathbb{Q}$ naturally associated with the bornology of bounded sets and the proximity both induced by the Euclidean metric. Furthermore, consider the unit circle S^1 in the Euclidean plane as the one-point compactification of $\mathbb{R}$. Let r be an irrational number and, for each $n \in N^+$, $f_n : \mathbb{Q} \to \mathbb{R}$ the function taking $1/n$ as value on $]r-1, r+1[$ and 0 otherwise. Then, the sequence $\{f_n : n \in N^+\}$ converges to the null function in the weak Whitney topology associated with $\mathbb{R}$ since it is uniformly convergent to the null function and, moreover, outside the bounded set $]r-1, r+1[$ all f_n agree with the null function. But, it does not converge in the weak Whitney topology associated with S^1. That is because there exists a continuous real-valued positive function $\boldsymbol{\epsilon}$ on $\mathbb{Q}$ continuously extendable to S^1 admitting zero as its greatest lower bound when restricted to $]r-1, r+1[\cap \mathbb{Q}$ and a sequence of rationals $\{q_n : n \in N^+\}$ converging to an irrational so that $\{\boldsymbol{\epsilon}(q_n) : n \in N^+\}$ tends to zero more rapidly than the sequence $\{1/n : n \in N^+\}$.

5. ω_μ-metric spaces

We now enlarge the range Y to run over ω_μ-metric spaces. Indeed, the previous player (Y, d), a non-dicrete metric space, is replaced by (Y, ρ, G^+) an ω_μ-metric one, the reals by the base group G and $C(X, \mathbb{R}^+)$ by the set of continuous functions from X to the positive cone G^+ of G equipped with the subspace topology induced by the absolute value ω_μ-metric on G.

Let X be a Tychonoff space and (Y, ρ, G) an ω_μ-metric space. We introduce on $C(X, Y)$ the *Whitney topology*, τ_W, as usual, as the topology having as a local base at any $f \in C(X, Y)$ the family of all sets of the type:

$$T(f, \boldsymbol{\epsilon}) := \{g \in C(X, Y) : \rho(f(x), g(x)) < \boldsymbol{\epsilon}(x),\ \forall x \in X\}$$

where $\boldsymbol{\epsilon}$ runs over $C(X, G^+)$.

Of course, in the ω_μ-metric setting too the Whitney convergence implies the uniform convergence. Now, we give conditions for their coincidence.

Proposition 5.1. *Let X be a Tychonoff space and (Y, ρ, G) an ω_μ-metric space. Then, the Whitney topology is strictly finer than the uniform topology iff there is a continuous function from X to G^+ admitting an ω_μ-sequence of its values approaching* 0*, or, equivalently, whose range admits* 0 *as the greatest lower bound.*

Proof. It is trivial that, when any continuous function from X to G^+ admits a positive lower bound, the Whitney convergence reduces to the uniform one. Conversely. Suppose $\boldsymbol{\epsilon} : X \to G^+$ as a continuous function and $\{x_\lambda : \lambda < \omega_\mu\}$ an ω_μ-sequence of points in X so that $\{a_\lambda := \boldsymbol{\epsilon}(x_\lambda), \lambda < \omega_\mu\}$ is a decreasing ω_μ-sequence of values of $\boldsymbol{\epsilon}$ tending to 0. This way, we produce an ω_μ-sequence of constant functions $\{\boldsymbol{\eta}_\lambda : X \to Y,\ \lambda < \omega_\mu\}$, by putting $\boldsymbol{\eta}_\lambda(x) := \boldsymbol{\epsilon}(x_\lambda), \forall \lambda < \omega_\mu, \forall\ x \in X$, that converges to the null function in the uniform topology but not in the Whitney topology. □

Next, we give conditions for an ω_μ-additive space to satisfy the properties stated in Proposition 5.1. We remind that a space is ω_μ-compact iff any ω_μ-sequence has an accumulation point.

Theorem 5.1. *Let X be an ω_μ-additive space of covering dimension zero and (Y, ρ, G) an ω_μ-metric space. Then, there exists a continuous function from X to G^+ admitting an ω_μ-sequence of its values approaching 0 iff X is non-ω_μ-compact.*

Proof. We examine the case $\mu > 0$. One way is simple. If $\boldsymbol{\epsilon} : X \to G^+$ is a continuous function with $\{x_\lambda : \lambda < \omega_\mu\}$ an ω_μ-sequence of points in X for which the ω_μ-sequence of values $\{\boldsymbol{\epsilon}(x_\lambda) : \lambda < \omega_\mu\}$ is decreasing to 0 in G, it happens that, by continuity and positiveness of $\boldsymbol{\epsilon}$, the ω_μ-sequence $\{x_\lambda : \lambda < \omega_\mu\}$ cannot accumulate in X. The converse implication. Let $\{x_\lambda : \lambda < \omega_\mu\}$ be an ω_μ-sequence of distinct points in X with no accumulation point and $\{\epsilon_\lambda : \lambda < \omega_\mu\}$ an ω_μ-sequence in G^+ strictly decreasing to zero. For each x in X there is a neighborhood U_x of x and a tail $T_x = \{x_\lambda, \lambda > \lambda_x\}$ so that $U_x \cap T_x = \emptyset$. Consequently, for each x_λ distinct from x and not in T_x there is a clopen neighborhood H_λ of x not containing x_λ and not intersecting T_x. Now, from ω_μ-additivity of X, $C_x = \cap\{H_\lambda :\ x_\lambda \notin T_x\}$ is a clopen neighborhood of x so that $C_x \setminus \{x\}$ does not contain any point in the ω_μ-sequence. The collection $\{C_x : x \in X\}$ is an open cover of X refinable by covering dimension zero with a cover of pairwise disjoint clopen sets, any of them containing at most one point in the ω_μ-sequence. So, it makes sense to introduce the function $\boldsymbol{\epsilon} : X \to G^+$ by putting $\boldsymbol{\epsilon}(x) = \epsilon_\lambda$ whenever x is in the unique clopen in the refinement containing x_λ and $\boldsymbol{\epsilon}(x) = \epsilon_0$ otherwise. Of course, the function $\boldsymbol{\epsilon}$ is locally constant, hence continuous. □

Corollary 5.1. *Let X be an ω_μ-additive space of covering dimension zero and (Y, ρ, G) an ω_μ-metric space. Then, the Whitney topology agrees with the uniform topology iff X is ω_μ-compact.*

Proof. It is an immediate consequence of Proposition 5.1 and Theorem 5.1. □

Proposition 5.2. *Let X be a non-ω_μ-compact space of covering dimension zero and G an Abelian totally ordered group G with character ω_μ. Then, any continuous function $\boldsymbol{\epsilon}$ from X to G^+ having an ω_μ-sequence of values decreasing to 0 in G can be minorized with a locally constant function $\boldsymbol{\eta} : X \to G^+$ with $2\boldsymbol{\eta} < \boldsymbol{\epsilon}$.*

Proof. Let $\boldsymbol{\epsilon}$ be a continuous function from X to G^+ and $\{x_\lambda : \lambda < \omega_\mu\}$ an ω_μ-sequence in X so that $\{a_\lambda := \boldsymbol{\epsilon}(x_\lambda), \lambda < \omega_\mu\}$ is a decreasing ω_μ-sequence of values of $\boldsymbol{\epsilon}$ converging to 0. For each $\lambda < \omega_\mu$ put $A_\lambda := \{x \in X : \boldsymbol{\epsilon}(x) > 2a_\lambda\}$. If $\lambda < \mu$, then $A_\lambda \subseteq A_\mu$ and each A_λ is different from X. Positiveness and continuity of $\boldsymbol{\epsilon}$ plus convergence of $\{2a_\lambda : \lambda < \omega_\mu\}$ to 0 imply that the family $\{A_\lambda : \lambda < \omega_\mu\}$ is an open cover of X so admitting a refinement $\{C_\gamma : \gamma \in \Gamma\}$ of pairwise disjoint clopen subsets of X. If $\lambda(\gamma)$ is the first λ for which C_γ is contained in A_λ, it happens that $\boldsymbol{\epsilon}(x) > 2a_{\lambda(\gamma)}, \forall x \in C_\gamma$. Thus, by putting $\boldsymbol{\eta}(x) = a_{\lambda(\gamma)}$ for each x in C_γ we construct a locally constant, hence continuous, positive function $\boldsymbol{\eta}$ so that $2\boldsymbol{\eta} < \boldsymbol{\epsilon}$. □

Proposition 5.3. *Let X be a space of covering dimension zero and (Y, ρ, G) an ω_μ-metric space. Then the Whitney topology on $C(X, Y)$ is uniformizable.*

Proof. The family of diagonal neighborhoods of the type:

$$\mathcal{U}(\boldsymbol{\epsilon}) := \{(f, g) \in C(X, Y) : \rho(f(x), g(x)) < \boldsymbol{\epsilon}(x),\ \forall x \in X\},$$

where $\boldsymbol{\epsilon}$ runs in $C(X, G^+)$, is a base for a uniformity on $C(X, Y)$ inducing the Whitney topology. Namely, the minimum of two functions in $C(X, G^+)$ is in its turn in $C(X, G^+)$. Thanks to Proposition 5.2, if both $\boldsymbol{\epsilon}$, $\boldsymbol{\eta}$ are in $C(X, G^+)$ and $2\boldsymbol{\eta} < \boldsymbol{\epsilon}$, then $\mathcal{U}(\boldsymbol{\eta}) \circ \mathcal{U}(\boldsymbol{\eta}) \subset \mathcal{U}(\boldsymbol{\epsilon})$. □

In the ω_μ-metric setting the Whitney topology can be uniformizable but not ω_μ-metrizable. Remind that the metric case is once again in [10].

Theorem 5.2. *Let X be an ω_μ-additive non-ω_μ-compact space of covering dimension zero and G a totally ordered Abelian group with character ω_μ, $\mu > 0$. Then, the Whitney topology on $C(X, G)$, where G carries the absolute value ω_μ-metric, is not ω_μ-metrizable.*

Proof. Suppose G carrying the absolute value ω_μ-metric ρ [4]. It is enough to show that in the Whitney topology on $C(X, G)$ the null function f on X has no totally ordered basic family of neighborhoods $\{\mathcal{U}_\alpha : \alpha < \omega_\mu\}$ i.e. $\mathcal{U}_\alpha$ contains $\mathcal{U}_\beta$ when $\alpha < \beta$, of cardinality $\aleph_\mu$. If it were so, we might extract from any $\mathcal{U}_\alpha$ a tube $T(f, \boldsymbol{\epsilon}_\alpha)$. Hence, the family $\{T(f, \boldsymbol{\epsilon}_\alpha) : \alpha < \omega_\mu\}$ should be a local base at f. But then there should be a tube centered at f containing no tube in that family; a contradiction. Indeed, any ω_μ-sequence $\{x_\lambda : \lambda < \omega_\mu\}$ of distinct points in X with no accumulation point determines a partition of X consisting of $\aleph_\mu$ non-empty clopen sets. Namely, by ω_μ-additivity and zero-dimensionality of X, proceeding as in the second part of Theorem 5.1, X can be covered with a collection of pairwise disjoint non-empty clopen sets each of them containing at most one point in the ω_μ-sequence. The clopen sets in that collection containing some point in the ω_μ-sequence plus the union of those ones containing no point in the ω_μ-sequence, which is in its turn a clopen set, form a partition of $\aleph_\mu$ non-empty clopen sets of X. So, after splitting the space X in a partition $\{C_\alpha : \alpha < \omega_\mu\}$ of non-empty clopen sets and after choosing for each $\alpha < \omega_\mu$ a continuous function $\boldsymbol{\eta_\alpha} : X \to G^+$ so that $2\boldsymbol{\eta_\alpha} < \boldsymbol{\epsilon_\alpha}$, it should be possible to join together in a unique globally continuous function $\boldsymbol{\eta}$ the restrictions of the $\boldsymbol{\eta_\alpha}$ to C_α. Since, for each α, $T(f, \boldsymbol{\eta_\alpha})$ is included in $T(f, \boldsymbol{\epsilon_\alpha})$ but not in $T(f, \boldsymbol{\eta})$, consequently the tube $T(f, \boldsymbol{\eta})$ should contain no $T(f, \boldsymbol{\epsilon_\alpha})$. □

It is known that for a paracompact space X and (Y, d) a metric space, the usual Whitney topology on $C(X, Y)$ is a topological invariant. The following result generalizes that one proven in [10] relative to the metrizable case.

Theorem 5.3. *Let X be an ω_μ-additive paracompact space and (Y, ρ, G) an ω_μ-metric space. Then, the Whitney topology on $C(X, Y)$ is independent of the ω_μ-metric ρ. More precisely, it is a topological character.*

Proof. First remind that the result in the metrizable case, $\mu = 0$, is in [10]. From the ω_μ-additiveness plus paracompactness of the space X and $\mu > 0$, it follows that the Whitney topology is generated from the base of sets of the type:

$$\mathcal{A} = (\{C_\alpha : \alpha \in A\}, \{V_\beta : \beta \in B\}, \phi) := \{f : X \to Y,\ f(C_\alpha) \subset V_{\phi(\alpha)},\ \forall \alpha \in A\},$$

where $\{C_\alpha : \alpha \in A\}$ is a pairwise disjoint clopen cover of X, $\{V_\beta : \beta \in B\}$ an open cover of Y and ϕ a refinement map from A to B. Let f be in $\mathcal{A}$. For each x in X there is only one C_α containing x and there are balls centered at $f(x)$ contained in $V_{\phi(\alpha)}$. For each a in G^+, let U_a stand for the subset of X sharing with C_α the points x for which there is a ball centered at $f(x)$ contained in $V_{\phi(\alpha)}$ whose radius is greater than a. Any point x belongs to some U_a and any U_a is an open subset in X. In fact, if x is in $C_\alpha \cap U_a$ and $\delta_x > a$ is the radius of a ball centered at $f(x)$ contained in $V_{\phi(\alpha)}$, then $C_\alpha \cap f^{-1}(B_\rho(f(x), \delta_x - a))$ is contained in U_a. Consequently, the open cover $\{U_a : a \in G^+\}$ admits a refinement of pairwise disjoint clopen sets $\{A_\gamma : \gamma \in C\}$ with refinement map ψ. To any point x, if A_{γ_x} is the unique clopen in the refining partition containing x, we can associate $\psi(\gamma_x)$ in G^+, so coherently defining a locally constant, hence continuous, function $\boldsymbol{\psi} : X \to G^+$ so that the tube $T(f, \boldsymbol{\psi})$ is contained in $\mathcal{A}$. Let $g \in T(f, \boldsymbol{\psi})$. If x is in $C_\alpha \cap A_{\gamma_x}$, then, by definition, the ball $B_\rho(f(x), \psi(\gamma_x))$, and thus $g(x)$, has to be contained in $V_{\phi(\alpha)}$. The converse implication. Let $T(f, \boldsymbol{\epsilon})$ be a tube around f. Choose η_x in G^+ so that $4\eta_x < \boldsymbol{\epsilon}(x)$. For each x in X put:

$$V_x := f^{-1}(B_\rho(f(x)), \eta_x) \cap \{z \in X : 2\eta_x < \boldsymbol{\epsilon}(z)\}.$$

The family $\{V_x : x \in X\}$ is an open cover with a refinement of pairwise disjoint clopen sets $\{C_\alpha : \alpha \in A\}$ with refinement map ϕ. Of course, $f \in \mathcal{A} = (\{C_\alpha : \alpha \in A\}, \{B_\rho(f(x), \eta_x) : x \in X\}, f \circ \phi)$. Furthermore, $\mathcal{A} \subseteq T(f, \boldsymbol{\epsilon})$. Indeed, if g is in $\mathcal{A}$, and z is in C_α, then $g(z)$ and $f(z)$ both are in $B_\rho(f(\phi(\alpha)), \eta_{\phi(\alpha)})$ and $\boldsymbol{\epsilon}(z) > 2\eta_{\phi(\alpha)}$. So, $\rho(f(z), g(z)) \leq \rho(f(z), f(\phi(\alpha)) + \rho(g(z), f(\phi(\alpha)) \leq 2\eta_{\phi(\alpha)} < \boldsymbol{\epsilon}(z)$ and g is in $T(f, \boldsymbol{\epsilon})$, just the final result. □

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