Properties of Set Functors ⋆

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Abstract

We prove that any endofunctor on a *class-theoretic* category has a *final coalgebra*. Moreover, we characterize functors on *set-theoretic* categories which are *identical* on objects, and functors which are *constant* on objects.

Key words: categories of sets, partially defined endofunctors, identity functor, constant functor, final coalgebra.

Introduction

In recent years, *set-theoretic categories*, i.e. categories where objects are sets (classes) of a possible non-wellfounded universe and morphisms are set(class)-theoretic functions, have been used as a convenient setting for studying the foundations of the *coalgebraic approach* to coinduction, see [Acz88,AM89,Bar93] [Bar94,BM96,DM97,RT93,RT98,Mos00]. Both among category theorists and among set-theorists however, set-theoretic categories had not received much attention for opposed ideological motivations.

In this paper, we address three questions concerning the structure of endofunctors in set-theoretic categories.

The first question concerns the class of set-theoretic functors which have *final coalgebra*. We show that *all* class-theoretic endofunctors, i.e. endofunctors on a category whose objects are classes and whose morphisms are functional classes, have final coalgebra. This strengthens earlier results of Aczel, Adamek et al. [Acz88,AMV02,AMV03], in *non-wellfounded* Set Theory, see [FH83].

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The latter two questions are basic and concern the constraints which might arise from the object part of a functor onto the morphism part, because of the special nature of sets. In particular:

(i) Are functors *constant* on objects constant on morphisms?

(ii) Are functors *identical* on objects identical on morphisms?

We solve thouroughly these questions, which originate as back as the fundamental book of MacLane, [Lan71].

In particular, we show that:

- any functor on a set-theoretic category $\mathcal{C}$ which has constant cardinality $< \sup_{\text{card}} \mathcal{C}$ on objects, where $\sup_{\text{card}} \mathcal{C}$ is the supremum of the cardinality of objects in $\mathcal{C}$, is naturally isomorphic to a constant functor;
- the above result does not extend to functors which are constantly equal on objects to an object of cardinality $\max_{\text{card}} \mathcal{C}$;
- any functor F on a cartesian closed set-theoretic category which is the identity on objects is naturally isomorphic to the identity functor;
- however, the result above fails on the restriction of set-theoretic categories on *infinite* objects.

A preliminary version of this paper appears in Chapter 3 of [Can03].

Summary.
In Section 1, we recall some definitions and basic facts concerning set theory and set-theoretical categories. In particular, we recall the definitions of *set based* functor and of *inclusion preserving* functor, and Aczel-Mendler Final Coalgebra Theorem. In Section 2, we study some properties of inclusion preserving functors, which will be useful in proving the main results of the paper. In Section 3, we strengthen Aczel-Mendler Theorem, by showing that all class-theoretic functors admit final coalgebras. In Section 4, we study two classes of partially specified endofunctors on set-theoretic categories: functors which are constant on objects and functors which are identical on objects. Directions for future work appear in Section 5.

Notation.
Throughout this paper we omit parentheses whenever no misunderstanding is possible. Moreover we use the following notation.
Let $f : A \to B$ be any function on sets (or classes), and let $A' \subseteq A$, then:

- $gr(f)$ denotes the graph of f;
- $img\ f$ denotes the image of f;
- $f_{A'} : A' \to B$ denotes the function obtained from f by restricting the domain of f to A';

1 Preliminaries

1.1 Set Theory Preliminaries

In this paper we will refer often to *large objects, such as proper classes, or even* very large objects, such as functors over categories whose objects are classes. A foundational formal theory which can accommodate naturally all our arguments is not readily available. A substantial formalistic effort would be needed to "cross all our t's" properly. We shall therefore adopt a pragmatic attitude and freely assume that we have classes and functors over classes at hand. Worries concerning consistency can be eliminated by assuming that our ambient theory is a Set Theory with an inaccessible cardinal κ, and the model of our object theory consists of those sets whose hereditary cardinal is less than κ, V_κ say, the classes of our model are the subsets of V_κ, and functors live at the appropriate ranks of the ambient universe.

1.2 Categorical Preliminaries

We define a *set-theoretic category* as follows:

Definition 1.1 A *set-theoretic category* is a category which is naturally isomorphic to an initial segment of $Card$ ($CARD$).

Typical examples of *set-theoretic categories* are the following, where U is a collection of *Urelementen*:

- $Set(U)$ ($Set^*(U)$) :
  objects: sets belonging to a (non-wellfounded) Universe,
  morphisms: set-theoretic total functions.
- $FinSet(U)$ ($Finset^*(U)$) :
  the subcategory of $Set(U)$ ($Set^*(U)$) of finite sets.
- $\mathcal{C}lass(U)$ ($\mathcal{C}lass^*(U)$) :
  objects: classes of (non-wellfounded) sets,
  morphisms: functional classes.
- $\mathcal{HC}_\kappa(U)$ ($\mathcal{HC}^*_\kappa(U)$) :
  objects: (non-wellfounded) sets whose hereditary cardinal is $< \kappa$, κ inaccessible,
  morphisms: set-theoretic functions.
- $Card$ ($CARD$) :
  objects: cardinals (including Ord),
  morphisms: set-theoretic functions.

Throughout this paper, we shall always assume that set-theoretic functors $F : \mathcal{C} \to \mathcal{C}$, where $\mathcal{C}$ is a generic set-theoretic category, satisfy the property that $F(\emptyset) = \emptyset$ and, for all $f : \emptyset \to A$, $F(f) = \emptyset$. This assumption is not particularly committing since we have the following

Proposition 1.2 *For every set-theoretic functor $F : \mathcal{C} \to \mathcal{C}$, there exists a functor $G : \mathcal{C} \to \mathcal{C}$ such that, for $A \neq \emptyset$, $F(A) = G(A)$ and for all $f : A \to B$, $F(f) = G(f)$, and where $G(\emptyset) = \emptyset$, and $G(f) = \emptyset$ for $f : \emptyset \to A$.*

Proof. One can easily check that, if F is a functor, then also G is a functor, since there exists none function $f : A \to \emptyset$, unless A is empty, and the only function $f : \emptyset \to A$ is the empty one. □

A well-known fact, which will be useful in the sequel, is the following:

Lemma 1.3 *Let $F : \mathcal{C} \to \mathcal{C}$. Then*

i) *if $f : A \to B$ is injective, then $F(f) : F(A) \to F(B)$ is injective;*

ii) *if $f : A \to B$ is surjective, then $F(f) : F(A) \to F(B)$ is surjective.*

Now we recall some definitions on set-theoretic functors. In the literature, these latter have been originally given only for functors defined on $\mathcal{C}lass$, or $\mathcal{C}lass^*$. However, they can be suitably modified for any set-category.

Definition 1.4 ([Acz88,AM89]) Let $F : \mathcal{C}lass^* \to \mathcal{C}lass^*$ be a functor.

- F is *inclusion preserving* if

$$\forall A, B.\ A \subseteq B \Longrightarrow\ (F(A) \subseteq F(B)\ \wedge\ F(\iota_{A,B}) = \iota_{F(A),F(B)})\ ,$$

  where $\iota_{A,B} : A \to B$ is the inclusion map from A to B.
- F is *set based* if, for each class A and each $x \in F(A)$, there exists a set $a_0 \subseteq A$ and $x_0 \in F(a_0)$ such that $x = F(\iota_{a_0,A})(x_0)$.

In 1989, Aczel and Mendler proved that any functor on $\mathcal{C}lass^*$ which is set based has final coalgebra.

Theorem 1.5 (Final Coalgebra Theorem, [AM89]) *Every* set based *functor on $\mathcal{C}lass^*$ has a final coalgebra.*

The above theorem holds for any class-theoretic functor.

In Section 3, we will prove that Aczel-Mendler's Final Coalgebra Theorem can be extended to *all* endofunctors on class-theoretic categories.

2 Properties of Inclusion Preserving Functors

In this section, we study properties of inclusion preserving functors. In particular, we show that

- a functor is inclusion preserving if and only if its value on morphisms depends only on the graphs of the morphisms;
- any functor on a set-theoretic category is naturally isomorphic to an inclusion preserving functor.

Throughout this section, let $\mathcal{C}$ range over a generic set-theoretic category.

We start with an easy lemma, which says that inclusion preserving functors preserve images of functions:

Lemma 2.1 *Let $F : \mathcal{C} \to \mathcal{C}$ be an inclusion preserving functor. Then, $F(img\ f) = img\ F(f)$.*

Proof. Let $f : A \to B$. Then $F(f) : F(A) \to F(B)$. But $F(f) = F(\iota_{imgf,B} \circ f_{|imgf}) = \iota_{F(imgf),FB} \circ F(f_{|imgf})$, since F is inclusion preserving. Therefore, $img\ F(f) = img\ F(f_{|imgf}) = F(img\ f)$, since $f_{|imgf}$ is surjective and by Lemma 1.3 F preserves surjective functions. □

Proposition 2.2 *Let $F : \mathcal{C} \to \mathcal{C}$. Then F is inclusion preserving if and only if its value on any morphism depends only on the graph of the morphism and not on the target, i.e. for all A, B and for all $f : A \to B$, $f' : A \to B'$, $gr(f) = gr(f') \Rightarrow gr(F(f)) = gr(F(f'))$.*

Proof.

$\Rightarrow$) Let $f : A \to B$, $f' : A \to B'$ be such that $gr(f) = gr(f')$. Then $img(f) = img(f')$, hence $f_{|img\ f} = f'_{|img\ f'}$, $f = \iota_{img\ f,B} \circ f_{|img\ f}$, and $f' = \iota_{img\ f,B'} \circ f_{|img\ f}$. Hence, since F is inclusion preserving, $F(f) = \iota_{F(img\ f),FB} \circ F(f_{|img\ f})$ and $F(f') = \iota_{F(img\ f),FB'} \circ F(f_{|img\ f})$, i.e. $gr(F(f)) = gr(F(f'))$.

$\Leftarrow$) Let $A \subseteq B$. Then $gr(\iota_{A,B}) = gr(id_A)$, and hence $gr(F(\iota_{A,B})) = gr(F(id_A)) = gr(id_{FA})$. Therefore, $F(A) \subseteq F(B)$ and $gr(F(\iota_{A,B})) = gr(\iota_{FA,FB})$, and hence $F(\iota_{A,B}) = \iota_{FA,FB}$.

□

Trivially, not every functor is inclusion preserving. Just consider any functor obtained by mapping isomorphically the value on a given class into a class which is disjoint from the value of the functor on a subclass.

However, in the next proposition we prove that any functor is naturally isomorphic to an inclusion preserving functor.

Proposition 2.3 *Let $F : \mathcal{C} \to \mathcal{C}$. Then there exists $G : \mathcal{C} \to \mathcal{C}$ inclusion preserving such that G is naturally isomorphic to F.*

Proof. Let $G : \mathcal{C} \to \mathcal{C}$ be the functor defined by: for all A, $G(A) = \widehat{F(\iota_{A,V})}(FA)$, and for all $f : A \to B$, $G(f) = G(A) \to G(B)$, $G(f) : F(\iota_{B,V})_{|imgF(\iota_{B,V})} \circ F(f) \circ (F(\iota_{A,V})_{|imgF(\iota_{A,V})})^{-1}$.

• We prove that G is well defined. By definition, G preserves identities. Now we show that G preserves composition. Let $f : A \to B$ and $g : B \to C$,

$$\begin{aligned} G(g \circ f) &= F(\iota_{C,V})_{|imgF(\iota_{C,V})} \circ F(g \circ f) \circ (F(\iota_{A,V})_{|imgF(\iota_{A,V})})^{-1} \\ &= F(\iota_{C,V})_{|imgF(\iota_{C,V})} \circ F(g) \circ F(f) \circ (F(\iota_{A,V})_{|imgF(\iota_{A,V})})^{-1} \\ &= F(\iota_{C,V})_{|imgF(\iota_{C,V})} \circ F(g) \circ (F(\iota_{B,V})_{|imgF(\iota_{B,V})})^{-1} \circ \\ &\quad \circ F(\iota_{B,V})_{|imgF(\iota_{B,V})} \circ F(f) \circ (F(\iota_{A,V})_{|imgF(\iota_{A,V})})^{-1} \\ &= G(g) \circ G(f) \end{aligned}$$

• We prove that the functor G is naturally isomorphic to F. Let $\tau = \{\tau_A : GA \to FA\}_A$ be the family of bijective functions, which are defined by $\tau_A = (F(\iota_{A,V})_{|imgF(\iota_{A,V})})^{-1}$. We prove that τ is a natural isomorphism. Let $f : A \to B$. We show that $\tau_B \circ G(f) \circ \tau_A^{-1} = F(f)$. By substitution on Gf,

$$\begin{aligned}\tau_B \circ G(f) \circ \tau_A^{-1} &= \tau_B \circ F(\iota_{B,V})_{|imgF(\iota_{B,V})} \circ F(f) \circ (F(\iota_{A,V})_{|imgF(\iota_{A,V})})^{-1} \circ \tau_A^{-1} \\ &= F(f) \text{ by definition of } \tau_A, \tau_B\end{aligned}$$

• It remains to prove that G is inclusion preserving, that is, $G(\iota_{A,B}) = \iota_{GA,GB}$. Let $A \subseteq B$ and let $\iota_{A,B} : A \to B$. Then,

$$\begin{aligned}G(\iota_{A,B}) &= (F(\iota_{B,V}) \circ F(\iota_{A,B}) \circ (F(\iota_{A,V})_{|imgF(\iota_{A,V})})^{-1})_{|imgF(\iota_{B,V})} \\ &= (F(\iota_{B,V} \circ \iota_{A,B}) \circ (F(\iota_{A,V})_{|imgF(\iota_{A,V})})^{-1})_{|imgF(\iota_{B,V})} \\ &= (F(\iota_{A,V}) \circ (F(\iota_{A,V})_{|imgF(\iota_{A,V})})^{-1})_{|imgF(\iota_{B,V})} \\ &= \iota_{GA,GB}\end{aligned}$$

□

Corollary 2.4 *Every set based functor F is naturally isomorphic to a standard functor.*

3 Strengthening the Final Coalgebra Theorem

For simplicitly, in this section, we work in a universe satisfying the *Axiom N*, i.e. all proper classes are in one-to-one correspondence with *Ord*. In the sequel, we refer to this assumption as the *"blanket assumption"*. Under this hypothesis, we prove the strong result that every inclusion preserving functor is set based. Therefore, by the Final Coalgebra Theorem of [AM89], we can derive that every inclusion preserving functor has a final coalgebra. By Proposition 2.3 of Section 2, using the fact that the property of having final coalgebras reflect under isomorphism, we can prove a very strong final coalgebra theorem, ensuring that all functors on $\mathcal{C}$ admit final coalgebra.

In this section, we let $\mathcal{C}$ range over a class-theoretic category, unless differently stated.

We start by proving some instrumental results.

Lemma 3.1 *Let $F : \mathcal{C} \to \mathcal{C}$ be inclusion preserving. If there exists A such that $\bigcup\{F(a) \mid a \in A \wedge a \text{ set}\} \subset F(A)$* [5]*, then for all B we have $\bigcup\{F(b) \mid b \in B \wedge b \text{ set}\} \subset F(B)$.*

Proof. We proceed by contradiction. We assume that A be a class such that $\bigcup\{F(a) \mid a \in A \wedge a \text{ set}\} \subset F(A)$ whereas B is a class such that $\bigcup\{F(b) \mid b \in B \wedge b \text{ set}\} = F(B)$. By the blanket assumption, there exists a bijective function $\sigma : B \to A$. Then, for all $x \in F(A)$, since F preserves

[5] This symbol denotes strict inclusion.

isomorphisms, there exists $y \in F(B)$ such that $x = F(\sigma)(y)$. Moreover, since $\bigcup\{F(b) \mid b \in B \wedge b \text{ set}\} = F(B)$, there exists a set $b \subseteq B$ such that $y \in F(b)$. But then $x \in imgF(\sigma)_{|Fb}$. Now, one can easily check that, since F is inclusion preserving, $F(\sigma_{|b}) = F(\sigma)_{|Fb}$. Hence $x \in imgF(\sigma_{|b})$, i.e., by Lemma 2.1, $x \in F(img\sigma_{|b})$, and $img\sigma_{|b}$ is a subset of A. Contradiction.

□

Definition 3.2 Let $F : \mathcal{C} \to \mathcal{C}$ be a functor and let A be a proper class. Then $x \in F(A)$ is unreachable if $x \notin \bigcup\{F(a) \mid a \in A \wedge a \text{ set}\}$.

Lemma 3.3 *Let F be inclusion preserving. If $x \in F(A)$ is unreachable, then there does not exist $f : a \to A$, for a set a, such that $x \in img\ F(f)$.*

Proof. We proceed by contradiction. We assume that for set a, there exists a function $f : a \to A$ such that $x \in imgF(f)$, Let $a_0 = img\ (f)$, then by Lemma 2.2, $F(f)_{|imgF(f)} : F(a) \to F(a_0)$ But if $x \in F(f)$, then $x \in F(a_0)$, which contradicts the hypothesis x unreachable. □

In Lemma 3.4 below, we exploit the assumption that V is isomorphic to Ord. If we consider the branches of a binary tree of height Ord, then we obtain 2^{Ord} injective functions, whose domain is Ord and which pairwise coincide on a non-empty set. Hence we have:

Lemma 3.4

i) *There exist 2^{Ord} injective functions $f_\alpha : V \to V$ such that $img(f_\alpha)$ is a proper class and for all $\alpha \neq \beta$, $img(f_\alpha) \cap img(f_\beta)$ is a non-empty set.*

ii) *There exist 2^{Ord} proper classes $\{A\}_\alpha$ such that $\alpha \leq 2^{Ord}$ and $A_\alpha \cap A_\beta$ is a non-empty set, for all $\alpha \neq \beta$.*

Proposition 3.5 *Let $F : \mathcal{C} \to \mathcal{C}$ be an inclusion preserving functor such that if $A \cap B$ is a set, then $F(A) \cap F(B)$ is included into the image of a set. Then, F is set based.*

Proof. We proceed by contradiction. We assume that F is not set based.
By Lemma 3.4.*ii*, there exist 2^{Ord} proper classes A_α such that $A_\alpha \cap A_\beta$ is a set for all $\alpha \neq \beta$. By Lemma 3.1, for each class A_α there exists an unreachable element $x_\alpha \in A_\alpha$.
But since for all α, β, $A_\alpha \cap A_\beta$ is a set, then $x_\alpha \notin A_\alpha \cap A_\beta$, for all $\alpha \neq \beta$, otherwise x_α would not be unreachable, by using the fact that $F(A_\alpha) \cap F(A_\beta)$ is a set by hypothesis.
Therefore there exist 2^{Ord} distinct unreachable elements. This contradicts the fact that $|V| = Ord$.

□

Now we are in the position of proving the following crucial result.

Proposition 3.6 *Let $F : \mathcal{C} \to \mathcal{C}$ be an inclusion preserving functor, and therefore set based.*

Proof. We proceed by contradiction. We assume F inclusion preserving and F is not set based.
By Lemma 3.4.i, there are 2^{Ord} injective functions $f_\alpha : V \to V$ such that $img f_\alpha$ is a proper class and $img(f_\alpha) \cap img(f_\beta)$ is a set, for all $\alpha \neq \beta$. Now we define 2^{Ord} functions $g_\alpha : V \to V$ such that

(a) $img(g_\alpha \circ f_\alpha)$ is a proper class and

(b) $img(g_\alpha \circ f_\beta)$ is a set, for all $\alpha \neq \beta$.
Let $g_\alpha : V \to V$ be defined by

$$g_\alpha(x) = \begin{cases} x & \text{if } \ x \in img f_\alpha \\ \emptyset & \text{otherwise .} \end{cases}$$

By (b) and Lemma 2.1, we obtain that $F(img(g_\alpha \circ f_\beta)) = img F(g_\alpha \circ f_\beta)$ doesn't contain any unreachable.
By (a) and Lemma 2.1, we obtain that $F(img(g_\alpha \circ f_\alpha)) = img F(g_\alpha \circ f_\alpha)$. Moreover, by Lemma 3.1, $img F(g_\alpha \circ f_\alpha)$ contains an unreachable. Hence, for all α, there exists $x_\alpha \in img F(f_\alpha)$, whose image by $F(g_\alpha)$, in the sequel noted by $\bar{x}_\alpha$, is an unreachable. Moreover, $x_\alpha \notin img F(f_\beta)$ for all $\beta \neq \alpha$, because x_α should be otherwise in the image of the set $img(g_\alpha \circ f_\beta)$. Hence, there exists 2^{Ord} distinct unreachable elements. This contradicts the fact that $|V| = Ord$. □

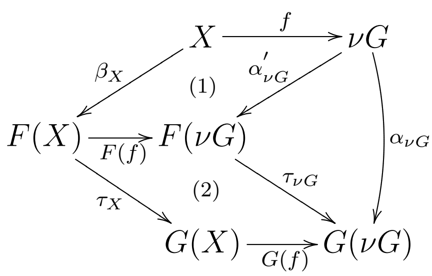
By Proposition 3.6, we have the following result, which generalizes the Final Coalgebra Theorem [AM89].

Corollary 3.7 *Any inclusion preserving functor has final coalgebra.*

The following is a simple, but useful, proposition, which allows us to reflect properties of final coalgebras between pairs of functors, and to derive the main result of this section.

Proposition 3.8 *Let $\mathcal{C}$ be a set-theoretic category, let $F, G : \mathcal{C} \longrightarrow \mathcal{C}$, and let $\tau : F \stackrel{\cdot}{\longrightarrow} G$ be a natural transformation. If $(\nu G, \alpha_{\nu G})$ is a final G-coalgebra, and $\tau_{\nu G}$ is a bijection, then $(\nu G, \tau^{-1}_{\nu G} \circ \alpha_{\nu G})$ is a final F-coalgebra.*

Proof. For sake of brevity, we mean by $\alpha'_{\nu G}$ the function given by $\tau^{-1}_{\nu G} \circ \alpha_{\nu G}$, as shown in the following diagram.



Let (X, β_X) be a F-coalgebra. We first show that there exists a F-coalgebra morphism from (X, β_X) into $(\nu G, \alpha'_{\nu G})$. Since $(X, \tau_X \circ \beta_X)$ is a G-coalgebra and
$(\nu G, \alpha_{\nu G} : \nu G \to G(\nu G))$ is a final coalgebra, then a unique function $f : X \to (\nu G)$ exists such that $\alpha_{\nu G} \circ f = G(f) \circ \tau_X \circ \beta_X \quad (*)$.
Since τ_X is a natural transformation, $G(f) \circ \tau_X = \tau_{\nu(G)} \circ F(f)$.
By substitution, the equation (*) becomes: $\alpha_{\nu G} \circ f = \tau_{\nu(G)} \circ F(f) \circ \beta_X$.
Since $\tau_{\nu G}$ is bijective, the function $\tau_{\nu G}^{-1}$ exists. Therefore $\tau_{\nu G}^{-1} \circ \alpha_{\nu G} \circ f = \tau_{\nu G}^{-1} \circ \tau_{\nu G} \circ F(f) \circ \beta_X$, i.e. $\tau_{\nu G}^{-1} \circ \alpha_{\nu G} \circ f = F(f) \circ \beta_X$, and hence f is a F-coalgebra morphism from (X, β_X) into $(\nu G, \alpha'_{\nu G})$, that is, existence for $(\nu G, \tau_{\nu G}^{-1} \circ \alpha_{\nu G})$ coalgebra.
Now we assume by contradiction that there exists another F-coalgebra morphism
$f' : (X, \beta_X) \to (\nu G, \alpha'_{\nu G})$. Then, $F(f') \circ \beta_X = \tau_{\nu G}^{-1} \circ \alpha_{\nu G} \circ f' \quad (***)$
Since τ is a natural transformation, $G(f') \circ \tau_X = \tau_{\nu G} \circ F(f')$.
By bijectivity of τ, we have $(\tau_{\nu G}^{-1} \circ G(f')) \circ \tau_X = (\tau_{\nu G}^{-1} \circ \tau_{\nu G}) \circ F(f')$,
i.e. $(\tau_{\nu G}^{-1} \circ G(f')) \circ \tau_X = F(f')$.
By substitution of $F(f')$ in (***), $(\tau_{\nu G}^{-1} \circ G(f')) \circ \tau_X \circ \beta_X = (\tau_{\nu G}^{-1} \circ \alpha_{\nu G}) \circ f'$,
i.e. $G(f') \circ \tau_X \circ \beta_X = \alpha_{\nu G} \circ f'$, which contradicts the final G-coalgebra $(\nu G, \alpha_{\nu G})$.

□

Finally, by Proposition 3.8 and 3.6, we have the following strong result.

Theorem 3.9 *Let F be an endofunctor on a class-theoretic category $\mathcal{C}$. Then F has final coalgebra.*

Clearly, the above theorem does not hold in a generic set-theoretic category, e.g. it does not hold for the powerset functor in *Set*.

4 Functors partially specified by their value on objects

In this section, we study two special kinds of functors on set-theoretic categories: the functors which are constants on objects, and the functors which are the the identity on objects. First we prove that if F is a class-theoretic functor which has constant cardinality $< Ord$ on objects, then F is naturally isomorphic to an inclusion preserving functor which is constant on objects and constantly equal to the identity on functions. However, the last property does not hold for functors, which are constantly equal to a class on objects. Suitable adaptations of the above results hold in a generic set-theoretic category. As far as functors which are the identity on objects, we prove that any such functor on a set-theoretic category which is cartesian closed, is naturally isomorphic to the functor which is the identity both on objects and functions. However, the above result does not hold on the categories obtained by restricting any set theoretic category $\mathcal{C}$ to its infinite objects, $Inf_{\mathcal{C}}$.

Throughout this section, let $\mathcal{C}$ range over a set-theoretic category.

4.1 Constant Functor

We call *constant* any functor $F : \mathcal{C} \to \mathcal{C}$ which is constant on objects, i.e. there exists $\bar{A}$ such that $F(A) = \bar{A}$ for all A, and it is constantly equal to identity on functions, i.e. $F(f) = id_{\bar{A}}$ for all function $f : A \to B$.

The main result of this subsection is that any functor on a set-theoretic category $\mathcal{C}$ which has constant cardinality $< \sup_{\text{card}} \mathcal{C}$ on objects, where $\sup_{\text{card}} \mathcal{C}$ is the supremum of the cardinality of objects in $\mathcal{C}$, is naturally isomorphic to a constant functor. For simplicity, in this subsection we focus on the case of class-theoretic categories, the other cases being similar.

The core of this subsection is to prove that if F is a functor which has constant set-cardinality on objects, then F is naturally isomorphic to an inclusion preserving functor G, which is constant. To this end, we need the following Lemmata 4.1 and 4.2.

Lemma 4.1 *Let $\mathcal{C}$ be a class-theoretic category. Let $G : \mathcal{C} \to \mathcal{C}$ be an inclusion preserving functor, which has constant cardinality $\kappa < Ord$, on objects. Then there exists $\bar{A}$ such that $G(B) = G(\bar{A})$, for all $B \supseteq \bar{A}$.*

Proof. Let A be a non empty set. We define an increasing chain as follows: $A_0 = A$, $A_{\alpha+1} = (A_\alpha)^+$ for any cardinality α, and $A_\lambda = \bigcup_{\gamma<\lambda} A_\gamma$ for any limit cardinality, where, for any α, $(A_\alpha)^+$ is an arbitrary element such that $(A_\alpha)^+ \supset A_\alpha$ and $G(A_\alpha)^+ \supset G(A_\alpha)$, if such a set exists, otherwise $(A_\alpha)^+ = A_\alpha$. Since $|G(A)| = \kappa$ for all A, at most after 2^k steps the chain is definitely constant. Let $\bar{A}$ be the last element of the chain, that is $G(\bar{A}) = G(B)$ for all $B \supseteq \bar{A}$.

□

Lemma 4.2 *Let $\mathcal{C}$ be a class-theoretic category. Let $G : \mathcal{C} \to \mathcal{C}$ be an inclusion preserving functor, which has constant cardinality $\kappa < Ord$ on objects. Then, for any infinite cardinality μ, there exists a family $\mathcal{A}_0$ of at least 2^{2^κ} disjoint sets of cardinality μ, such that*

(i) G is constant on $\mathcal{A}_0$, i.e. $G(A) = A_0$, for all $A \in \mathcal{A}_0$, moreover, for all $A, B \in \mathcal{A}_0$ such that $A \neq B$, for all $f : A \to B$, $G(f) = id_{A_0}$;

(ii) for all $A \in \mathcal{A}_0$, for all $f : A \to A$, $G(f) = id_{A_0}$;

(iii) for all $A \in \mathcal{A}_0$, for all B such that $B \subseteq A$, $G(B) = A_0$;

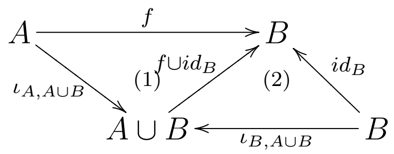
(iv) for all $A \in \mathcal{A}_0$ and for all B such that $|B| = |A|$ and $f : B \to B$, $G(B) = A_0$ and $G(f) = id_{A_0}$;

(v) for all C such that $|C| \leq \mu$, $G(C) = A_0$, moreover for all $f : A \to B$ such that $|A|, |B| \leq \mu$, $G(f) = id_{A_0}$.

Proof.

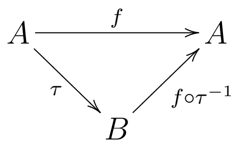
(i) By Lemma 4.1, there exists $\bar{A}$ such that $G(B) = G(\bar{A})$, for all $B \supseteq \bar{A}$. We define $\mathcal{A}$ to be any family of 2^{2^k} disjoint sets of infinite cardinality μ. Since for any $A \in \mathcal{A}$, there exists $B \supseteq \bar{A} \cup A$, then the value of G on elements of $\mathcal{A}$ is a subset of $G(\bar{A})$. Therefore since $|G(\bar{A})| = \kappa$, G is constant at least on 2^{2^κ} elements of $\mathcal{A}$. Let $\mathcal{A}_0$ be the family of elements of $\mathcal{A}$ on which G is constantly equal on objects to, say, A_0.

Now we show that, for all $A, B \in \mathcal{A}_0$ such that $A \neq B$, and $f : A \to B$, $G(f) = id_{A_0}$. The following diagrams straightforwardly commutes.



Hence, if we apply G to all diagrams, these still commute. Since G is inclusion preserving and the diagram (2) commutes, $G(f \cup id_B) = id_{GB}$. Hence by diagram (1), also $G(f) = id_{G(B)}$, i.e. $G(f) = id_{A_0}$.

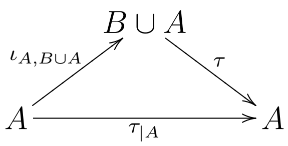
(ii) Let $A \in \mathcal{A}_0$, $f : A \to A$, Since all elements of $\mathcal{A}_0$ are disjoint and have the same cardinality, there exists $B \in \mathcal{A}_0$, such that $B \cap A = \emptyset$ and an isomorphism $\tau : A \to B$. The following diagram straightforwardly commutes



The diagram commutes also when we apply G. Hence, $G(f) = G(f \circ \tau^{-1}) \circ G(\tau)$. But by *(i)*, both $G(\tau) = id_{A_0}$ and $G(f \circ \tau^{-1}) = id_{A_0}$. Therefore, $G(f) = id_{A_0}$.

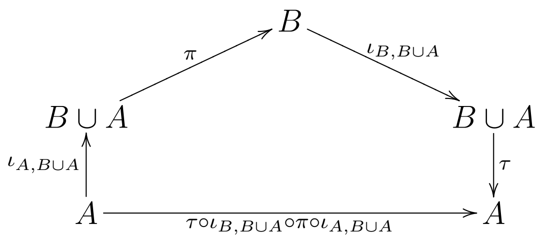
(iii) Let $A \in \mathcal{A}_0$, and $B \subsetneq A$. Let $\pi : A \to B$ be such that $\pi_{|B} = id_B$. Then, by *(ii)*, $\iota_{B,A} \circ \pi : A \to A$ is such that $G(\iota_{B,A} \circ \pi) = id_{A_0}$. Moreover, since $gr(\pi) = gr(\iota_{B,A} \circ \pi)$, by Proposition 2.2, also $gr(G(\pi)) = id_{A_0}$. Therefore, $G(A) = G(B)$.

(iv) Let B be such that $|B| = |A|$ and $A \in \mathcal{A}_0$. We prove that $G(B) = A_0$. Since A and B have the same infinite cardinality, also $|B \cup A| = |A|$. Hence, there exists an isomorphism $\tau : B \cup A \to A$. The following diagram straightforwardly commutes.



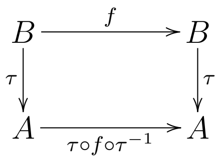
Then, we apply G to the diagram above. By *(ii)*, $G(\tau_{|A}) = id_{A_0}$. Moreover G is inclusion preserving, and, therefore, $G(\iota_{A,B \cup A}) = \iota_{GA, G(B \cup A)}$. Hence, since $G(\tau)$ is bijective, $G(B \cup A) = G(A)$ and $G(\tau) = id_{A_0}$.

In order to conclude, we still need to show that $G(B \cup A) = G(B)$.
Let $\pi : B \cup A \to B$ be a function such that $\pi_{|B} = id_B$. The following diagram trivially commutes.



We apply G to the diagram above. Since G is inclusion preserving, and $G(A) = G(B \cup A)$, then $G(\iota_{A,B\cup A}) = id_{A_0}$. Moreover, by *(ii)*, $G(\tau \circ \iota_{B,B\cup A} \circ \pi \circ \iota_{A,B\cup A}) = id_{A_0}$ and, by above, $G(\tau) = id_{A_0}$. Hence, $G(\iota_{B,B\cup A}) \circ G(\pi) = id_{A_0}$. As a result, $G(\pi)$ need to be injective and not only surjective. Moreover, since G is inclusion preserving, $G(\pi) = id_{A_0}$. Therefore, $G(B \cup A) = G(B)$.

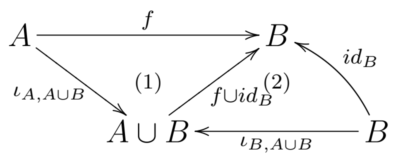
We are left to show that, if $|A| = |B|$ and $f : B \to B$, then $G(f) = id_{A_0}$. The following diagram straightforwardly commutes.



We apply G to the diagram above. By *(ii)*, $G(\tau \circ f \circ \tau^{-1}) = id_{A_0}$. Since the diagram commutes, $G(f) = G(\tau^{-1}) \circ id_{A_0} \circ G(\tau)$. Hence, $G(f) = id_{A_0}$.

(v) Let C be such that $|C| \leq \mu$. Then there exists C_1 such that $C \subseteq C_1$ and $|C_1| = \mu$. By *(iv)*, $G(C_1) = A_0$, then using *(iv)*, with an argument similar to the one used for proving *(iii)*, one can show that $G(C) = A_0$. Here, we only prove that, for all $f : A \to B$, $|A|, |B| \leq \mu$, $G(f) = id_{A_0}$.
There are two cases: either $A \cap B = \emptyset$ or $A \cap B \neq \emptyset$.

`(a)`



Since $A \cap B = \emptyset$, both diagrams, (1) and (2), commute. We apply the functor G to both diagrams. Since by above, $G(A) = G(A \cup B) = G(B)$, then $G(\iota_{A,A\cup B}) = id_{A_0}$. Analogously, $G(\iota_{B,A\cup B}) = id_{A_0}$. By commutativity of diagram (2), $G(id_B) = G(f \cup id_B) \circ G(\iota_{B,A\cup B})$. Hence, $G(f \cup id_B) = id_{A_0}$. Therefore, by commutativity of diagram (1), $Gf = id_{GA}$.

`(b)` Let A' and B' be such that $A \cong A'$ and $B \cong B'$, that is, there exist two isomorphisms $\tau_A : A \to A'$ and $\tau_B : B \to B'$. Let $A' \cap A = \emptyset$, $B' \cap B = \emptyset$, $A' \cap B' = \emptyset$. By `(a)`, $G(\tau_A) = id_{A_0}$, $G(\tau_B) = id_{A_0}$ and $G(\tau_B \circ f \circ \tau_A^{-1}) = id_0$.

Therefore, $G(\tau_B) \circ G(f) \circ G(\tau_A^{-1}) = id_{A_0}$ and hence $G(f) = id_{A_0}$.

□

Proposition 4.3 *Let $\mathcal{C}$ be a class-theoretic category. Any endofunctor G on $\mathcal{C}$ which is inclusion preserving and has constant cardinality on objects $< Ord$ is a constant functor.*

Proof. We proceed by contradiction. We assume that G is not constant everywhere on objects. Then there exist A, B such that $G(A) \neq G(B)$. Let $|A| \geq |B|$. If $|A|$ is infinite, then we immediately have a contradiction by Lemma 4.2.*(v)*. Otherwise, if $|A|$ is finite, then we consider a set A_0 of infinite cardinality μ, such that $A_0 \supseteq A, B$. Then by Lemma 4.2.*(v)* we have a contradiction. Therefore, G must be constant on objects. Moreover, using again Lemma 4.2.*(v)*, one can easily check that G must give the identity on every morphism, i.e. G is a constant functor.

□

Corollary 4.4 *Let $\mathcal{C}$ be a class-theoretic category. Any endofunctor F on $\mathcal{C}$ which has constant cardinality $< Ord$ on objects is naturally isomorphic to a constant functor.*

Proof. By Proposition 2.3, any functor F is naturally isomorphic to an inclusion preserving functor $G : \mathcal{C} \to \mathcal{C}$. Hence also G has constant cardinality on objects. Therefore, by Proposition 4.4, G is a constant functor. □

However, Corollary 4.4 doesn't extend to functors F constantly equal to a class. We show it via a counterexample:

Counterexample.
Let $F : \mathcal{C} \to \mathcal{C}$ be defined as follows. For all A, let $F(A) = \cup_{k \in Ord} A^\kappa$, where A^κ denotes the function space $[\kappa \to A]$, i.e. all sequences of elements of A of length κ. For all $f : A \to B$, let $F(f) : \cup_{\kappa \in Ord} A^k \to \cup_{\kappa \in Ord} B^k$ be such that $F(f)(\boldsymbol{a}) = \boldsymbol{b}$ iff $\forall i \ f(a_i) = b_i$. Then, there exists a functor $G : \mathcal{C} \to \mathcal{C}$ such that, for all A, $G(A) = Ord$, G is naturally isomorphic to F and inclusion preserving. But G cannot be constantly equal to the identity on functions. Moreover, there is no constant functor G' which is inclusion preserving and naturally isomorphic to G.

4.2 Identity Functor

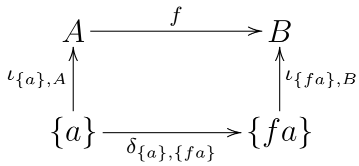
We call *identity* the functor $F : \mathcal{C} \to \mathcal{C}$ which is the identity both on objects, i.e. $F(A) = A$ for all A, and on functions, i.e. $F(f) = f$ for all functions $f : A \to B$.

The main result of this section is that functors on a cartesian closed set-theoretic category which are the identity on objects are naturally isomorphic to the identity functor. We will prove that the function σ, defined below, is the wanted natural isomorphism.

Definition 4.5 Let $F : \mathcal{C} \to \mathcal{C}$ be such that $F(A) = A$ for all A. For all A, we define $\sigma_A : A \to A$ such that $\sigma_A \equiv \lambda a.F(\iota_{\{a\},A})(a)$.

Proposition 4.6 *Let $F : \mathcal{C} \to \mathcal{C}$ be such that $F(A) = A$ for all A. If σ_A is bijective for all A, then F is naturally isomorphic to the identity.*

Proof. Let $f : A \to B$. If $A = \emptyset$, then we have immediately that $F(f) \circ \sigma_A = \sigma_B \circ f$. Therefore, let us assume $A \neq \emptyset$. The following diagram straightforwardly commutes.



We apply F to the diagram above. Hence, for all $a \in A$, $F(f) \circ F(\iota_{\{a\},A})(a) = F(\iota_{\{fa\},B}) \circ F(\delta_{\{a\},\{fa\}})(a)$, i.e. by definition of σ_A and δ_A, $F(f) \circ \sigma_A(a) = \sigma_B \circ f(a)$, for all $a \in A$, hence σ is a natural transformation from Id to F. Moreover, since σ is bijective, it is an isomorphism.

$\square$

In the sequel, we prove that, in any set-theoretic category $\mathcal{C}$ which is cartesian closed, σ_A must be injective for all A. To this end, we start by building a non-trivial category, denoted by $\mathcal{C}_F$, which turns out to be cartesian closed with finite coproducts. Then we proceed by contradiction, assuming that for some A, σ_A is not injective, and we show that under this assumption, every morphism $f : C \to C$ in $\mathcal{C}_F$ has a fixed point. Therefore, we get a contradiction since, by [Law69,HP90], every cartesian closed category with coproducts and fixed points is inconsistent, i.e. trivial (see Theorem 4.13).

Definition 4.7 Let $F : \mathcal{C} \to \mathcal{C}$. We define the "image category" $\mathcal{C}_F$ by $Obj(\mathcal{C}_F) = \{(F(A), A) \mid A \in Obj(\mathcal{C})\}$ and by $Mor((F(A), A), (F(B), B)) = \{(F(f), f) \mid f : A \to B\}$.

Notation.

In order to improve readability, we will denote objects $F(A, A)$ and morphisms $(F(f), f)$ of the image category simply by $F(A)$ and $F(f)$, respectively.

Lemma 4.8 *Let $\mathcal{C}$ be a cartesian closed set-theoretic category, and let $F : \mathcal{C} \to \mathcal{C}$. Then, the category $\mathcal{C}_F$ is a non-trivial cartesian closed category with finite coproducts.*

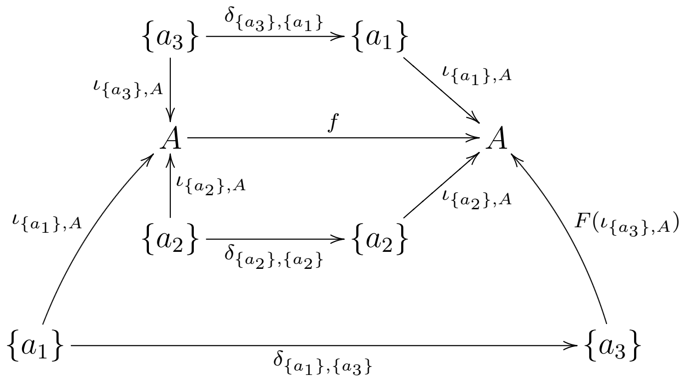
Proof. The *product* $G(A) \tilde{\times} G(B)$ on $\mathcal{C}_F$ can be defined in terms of the product on $\mathcal{C}_F$ by $G(A \times B)$. Projections on the image category are $G(\pi_1), G(\pi_2)$, where π_1, π_2 are the projections in $\mathcal{C}$.

Similarly for *coproduct*.

Moreover, one can check that $\mathcal{C}_F$ is cartesian closed, by defining the exponent object FB^{FA} by $F(B^A)$, the function eval, $\tilde{ev} : F(B^A)\tilde{\times}F(A) \to F(B)$ by $F(ev)$, where $ev : B^A \times A \to B$, and the isomorphism $\tilde{\Lambda}_{FC} : \mathcal{C}(F(C)\tilde{\times}F(A), F(B)) \to \mathcal{C}(F(C), (FB)^{FA})$ by for any $F(f) : F(C \times A) \to F(B)$, $\tilde{\Lambda}_{FC}(Ff) = F(\Lambda_C(f))$ where $\Lambda_C : \mathcal{C}(C \times A, B) \to \mathcal{C}(C, B^A)$. □

Lemma 4.9 *Let $F : \mathcal{C} \to \mathcal{C}$ such that $F(A) = A$ for all A. If σ_A is not injective for some A, then for all $a_1, a_2 \in A$, $\sigma_A(a_1) = \sigma_A(a_2)$.*

Proof. Let $f : A \to A$ be such that $f(a_1) = a_3, f(a_2) = a_2$, and $f(a_3) = a_1$. The following diagrams straightforwardly commute.



We apply F to the above diagrams. By hypothesis and by definition of σ, $F(\iota_{\{a_1\},A})(a_1) = F(\iota_{\{a_2\},A})(a_2)$. Hence, $F(f)(F(\iota_{\{a_1\},A})(a_1)) = F(f)(F(\iota_{\{a_2\},A})(a_2))$. Since the diagram commute

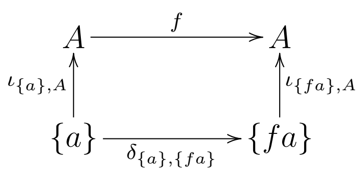
$$F(f)(F(\iota_{\{a_1\},A)}(a_1)) = F(\iota_{\{a_3\},A})(a_3) \text{ and } F(f)(F(\iota_{\{a_2\},A})(a_2)) = F(\iota_{\{a_2\},A})(a_2)\,,$$

hence $F(\iota_{\{a_3\},A})(a_3) = F(\iota_{\{a_2\},A})(a_2)$.

□

Proposition 4.10 *Let $F : \mathcal{C} \to \mathcal{C}$ be such that $F(A) = A$ for all A. If σ_A is not injective for some A then for all $f : A \to A$, $F(f)$ has a fixed point.*

Proof. We show that $F(\iota_{\{a\},A})$ is a fixed point of any $f : A \to A$. The following diagram straightforwardly commutes.

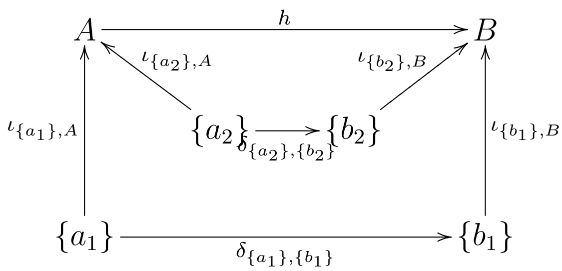


We apply F to the above diagram. Then $F(f)(F(\iota_{\{a\},A})(a)) = F(\iota_{\{fa\},A})(f(a))$, which, by Lemma 4.9, is equal to $F(\iota_{\{a\},A})(a)$, i.e. $F(\iota_{\{a\},A})(a)$ is a fixed point of $F(f)$.

□

Lemma 4.11 *Let $F : \mathcal{C} \to \mathcal{C}$ be such that $F(A) = A$ for all A. If σ_A is not injective for some A, then for all B, if $b_1, b_2 \in B$, then $\sigma_B(b_1) = \sigma_B(b_2)$.*

Proof. Let $b_1, b_2 \in B$. Then, there exists a function $h : A \to B$ such that for some $a_1, a_2 \in A$, $h(a_1) = b_1$ and $h(a_2) = b_2$. The following diagram trivially commutes.



We apply F to the above diagram. Hence, since by hypothesis $F(\iota_{\{a_1\},A})(a_1) = F(\iota_{\{a_2\},A})(a_2)$, also $F(\iota_{\{b_1\},B})(b_1) = F(\iota_{\{b_2\},B})(b_2)$.

□

By Lemma 4.11, we have

Theorem 4.12 *Let $F : \mathcal{C} \to \mathcal{C}$ be such that $F(A) = A$, for all A. If σ_A is not injective for some A, then for all $B \neq \emptyset$, and for all $f : B \to B$, $F(f)$ has fixed point.*

Now we are in the position of stating the main result of this section, i.e. Theorem 4.14 below. This follows from the following inconsistency result of fixed points in a cartesian closed category with coproducts, applied to the image category of Definition 4.7.

Theorem 4.13 ([Law69,HP90]) *Let $\mathcal{C}$ be a cartesian closed category with fixed points and coproducts. Then $\mathcal{C}$ is inconsistent.*

Theorem 4.14 *Let $\mathcal{C}$ be a cartesian closed set-theoretic category, and let $F : \mathcal{C} \setminus \{\emptyset\} \to \mathcal{C} \setminus \{\emptyset\}$ such that $F(A) = A$ for all A. Then for all A, σ_A is injective.*

Proof. By Lemma 4.8 and Theorem 4.12, $(\mathcal{C} \setminus \{\emptyset\})_F$ is cartesian closed with finite coproducts. Hence, by Theorem 4.13, $(\mathcal{C} \setminus \{\emptyset\})_F$ is trivial. Therefore we have a contradiction with Lemma 4.8. □

In particular, if $\mathcal{C} = SetFin$, then, since injectivity implies bijectivity, by Proposition 4.6, we obtain the following

Theorem 4.15 *Let $F : SetFin \setminus \{\emptyset\} \to SetFin \setminus \{\emptyset\}$ be such that $F(A) = A$ for all A. Then F is naturally isomorphic to the identity functor.*

In the next theorem, we generalize Theorem 4.15 to the case of $\mathcal{C}$. In particular, for any inclusion preserving functor G, we prove that σ_A is also surjective for all classes A. We start by proving the following instrumental lemma.

Lemma 4.16 *Let $F : \mathcal{C} \setminus \{\emptyset\} \to \mathcal{C} \setminus \{\emptyset\}$ be such that $F(A) = A$ for all A. Then, for all $f : A \to B$, $F(f)_{|img\sigma_A} \subseteq img(\sigma_B)$.*

Proof. Assume by contradiction that there exist $a, a_1 \in A$ such that $a_1 = \sigma_A(a)$ and such that $F(f)(a_1) \not\subseteq img(\sigma_B)$. The following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow{\scriptstyle \iota_{\{a\},A}} & & \uparrow{\scriptstyle \iota_{\{fa\},B}} \\ \{a\} & \xrightarrow[\delta_{\{a\},\{fa\}}]{} & \{fa\} \end{array}$$

Applying F to the above diagram and by definition of δ, we have $F(f) \circ F(\iota_{\{a\},A})(a) = F(\iota_{\{fa\},B})(f(a))$, i.e. by definition of σ, $F(f)(\sigma_A(a)) = \sigma_B(f(a))$ which contradicts the hypothesis $F(f)(a_1) \notin img(\sigma_B)$.

□

Proposition 4.17 *Let $F : \mathcal{C} \setminus \{\emptyset\} \to \mathcal{C} \setminus \{\emptyset\}$ such that $F(A) = A$ for all A. Then, for all A, σ_A is surjective.*

Proof. We proceed by contradiction, and we assume that there exists A such that σ_A is not surjective. Let $\bar{F} : \mathcal{C} \setminus \{\emptyset\} \to \mathcal{C} \setminus \{\emptyset\}$ be the functor defined by:
- for all A, $\bar{F}(A) = \bar{A} + \{*\}$, and
- for all $f : A \to B$, $\bar{F}(f) : \bar{A} + \{*\} \to \bar{B} + \{*\}$

$$\bar{F}(f)(a) = \begin{cases} * & \text{if } \ a = * \vee \forall a \notin \bar{A} \\ F(f)(a) & \text{otherwise .} \end{cases}$$

Then, since σ_A is not surjective, the $\bar{F}$-image category, denoted by $(\mathcal{C} \setminus \{\emptyset\})_{\bar{\mathcal{F}}}$ is non trivial. Moreover, by construction, $(\mathcal{C} \setminus \{\emptyset\})_{\bar{\mathcal{F}}}$ has fixed points. By Lemma 4.8, $\mathcal{C} \setminus \{\emptyset\})_{\bar{\mathcal{F}}}$ is a closed category with coproducts. Therefore, by Proposition 4.13, $\mathcal{C} \setminus \{\emptyset\})_{\mathcal{F}}$ is trivial. Contradiction.

□

Finally, we have

Theorem 4.18 *Let $\mathcal{C}$ be a cartesian closed category, and let $F : \mathcal{C} \setminus \{\emptyset\} \to \mathcal{C} \setminus \{\emptyset\}$ be such that $F(A) = A$ for all A. Then, F is naturally isomorphic to the identity functor.*

The above argument can be easily extended to prove the following:

Proposition 4.19 *Let $F : \mathcal{C} \setminus \{\emptyset\} \to \mathcal{C} \setminus \{\emptyset\}$ be such that $F(A) = A$ for all A. If the set-theoretic category consisting of those objects whose cardinality is $< sup_{card}\, \mathcal{C}$ is cartesian closed, then F is naturally isomorphic to the identity functor.*

We are confident that, using this approach, the above result holds in any set-theoretic category.

Notice that, however, Theorem 4.18 above fails if we consider the restriction of any set-theoretic category $\mathcal{C}$ to infinite objects. The following is a counterexample:

Counterexample.
Let $Inf_{\mathcal{C}}$ be the category which is obtained from $\mathcal{C}$ by considering only infinite objects.
Let $A \in Inf_{\mathcal{C}}$. We define an injective function $\rho_A : A \to A$ such that $A \setminus img(\rho_A) = \{\bar{a}\}$. Let $F : Inf_{\mathcal{C}} \to Inf_{\mathcal{C}}$ be a functor such that $F(A) = A$ for all $A \in Inf_{\mathcal{C}}$. For all $f : A \to B$, and for all $x \in A$, we define $F(f)$ as follows.

$$F(f)(a) = \begin{cases} \bar{b} & \text{if } \ a = \bar{a} \\ \rho_B \circ f \circ \rho_A^{-1}(a) & \text{if } \ a \neq \bar{a} \end{cases}$$

One can easily check that F is a functor.
However, even if F is the identity on objects, F is not the identity on morphisms. For example, F does not map constant functions into constant functions.

5 Directions For Future Work

The results of this paper should be compared to those in [Kou71,BBT0o] on endofunctors on sets uniquely defined by their value on objects, and results of [AMV02,AMV03] on final coalgebra theorems, and results of [JM95].

We conjecture that if F is constant on morphisms, then either for any class A, FA is a singleton, or $imgF = \{FA \mid A \in ObjClass^*(U)\}$ is a proper class.

It would be interesting to explore to what extent the argument in Subsection 3 can be extended to the case of set-theoretic categories.

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