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Research Article

A New Result Concerning the Solvability of a Class of General Systems of Variational Equations with Nonmonotone Operators: Applications to Dirichlet and Neumann Nonlinear Problems

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A new result of solvability for a wide class of systems of variational equations depending on parameters and governed by nonmonotone operators is found in a Banach real and reflexive space with applications to Dirichlet and Neumann problems related to nonlinear elliptic systems.

1. Introduction

Many researchers (e.g., [1–9]) have devoted (and are still devoting) their studies to the solvability and the investigation of multiple and positive solutions of nonlinear elliptic problems.

A class of general systems of variational equations has been studied in [10]. The Dirichlet and Neumann problems investigated in [11] belong to this class.

In this paper, we prove a further existence theorem related to the problem of [10] in the homogeneous case, by using the Lagrange multipliers and the "algebraic" approach which is based on the fibering method [12]. This theorem and the ones of [10] include the results of [8] and some results of [13–15]. Now, let us recall the problem studied in [10].

Let $(W_1, \|\cdot\|_1), \ldots, (W_n, \|\cdot\|_n)$ $(n \geq 1)$ be Banach reflexive and real spaces. Let $W = \bigtimes_{\ell=1}^{n} W_\ell$ with $\|v\| = \sum_{\ell=1}^{n} \|v_\ell\|_\ell$ $\forall v = (v_1, \ldots, v_n) \in W$. Let $\langle\cdot,\cdot\rangle_\ell$ [resp., $\langle\langle\cdot,\cdot\rangle\rangle$] be the duality between W_ℓ^* dual space of W_ℓ [resp., W^* dual space of W] and W_ℓ [resp., W]. Let us denote by "∂" Fréchet differential operator and by "∂_{u_ℓ}" Fréchet differential operator with respect to u_ℓ. Let $A \not\equiv 0$ and $D_j \not\equiv 0$ $(j = 1, \ldots, m;\ m \geq 1)$ be real functionals defined in W and let B_ℓ and $\widehat{B}_\ell$ $(\ell = 1, \ldots, n)$ be real functionals defined in W_ℓ satisfying the following conditions:

(i_{11}) A is lower weakly semicontinuous in W and $C^1(W \setminus \{0\})$, B_ℓ and $\widehat{B}_\ell$ are weakly continuous in W_ℓ and $C^1(W_\ell)$, $\exists p > 1 : A(tv) = t^p A(v)$ $\forall t \geq 0$ and $\forall v \in W$, $B_\ell(tv_\ell) = t^p B_\ell(v_\ell)$ and $\widehat{B}_\ell(tv_\ell) = t^p \widehat{B}_\ell(v_\ell)$ $\forall t \geq 0$ and $\forall v_\ell \in W_\ell$.

(i_{12}) D_j is weakly continuous in W and $C^1(W \setminus \{0\})$, $\exists q_j > 1 : D_j(tv) = t^{q_j} D_j(v)$ $\forall t \geq 0$, and $\forall v \in W$, $1 < q_1 < \cdots < q_m$ if $m > 1$.

Let us set the following:

$$
\begin{aligned}
H_{\lambda\mu}(v) &= A(v) - \sum_{\ell=1}^{n}\left[\lambda_\ell B_\ell(v_\ell) + \mu_\ell \widehat{B}_\ell(v_\ell)\right],\\
E(v) &= H_{\lambda\mu}(v) - \sum_{j=1}^{m} D_j(v),\\
&\qquad \forall v = (v_1,\ldots,v_n) \in W, \text{ where } \lambda = (\lambda_1,\ldots,\lambda_n),\ \mu = (\mu_1,\ldots,\mu_n) \in R^n;\\
S_{\lambda\mu} &= \left\{v \in W : H_{\lambda\mu}(v) = 1\right\},\\
V^-_{\lambda\mu} &= \left\{v \in W : H_{\lambda\mu}(v) < 0\right\},\\
S(D_1) &= \{v \in W : D_1(v) = -1\},\\
V^+\left(D_1,\ldots,D_{m_1}\right) &= \left\{v \in W : \sum_{j=1}^{m_1} D_j(v) > 0\right\} \quad \text{as } m_1 = 1,\ldots,m.
\end{aligned} \tag{1}
$$

Let us consider the following problem.

Problem (P). Find $u = (u_1,\ldots,u_n) \in W \setminus \{0\}$ such that

$$
\begin{aligned}
\left\langle \partial_{u_i} A(u), v_i\right\rangle_i &= \lambda_i \langle \partial B_i(u_i), v_i\rangle_i + \mu_i \left\langle \partial \widehat{B}_i(u_i), v_i\right\rangle_i\\
&\quad + \sum_{j=1}^{m} \left\langle \partial_{u_i} D_j(u), v_i\right\rangle_i\\
&\qquad \forall i \in \{1,\ldots,n\},\ \forall v_i \in W_i.
\end{aligned} \tag{2}
$$

In Section 2, we present new cases ((c_1)–(c_4)) in which Problem (P) is solvable. In these cases, we introduce one of the following hypotheses:

(i_{13}) $\exists c(\lambda,\mu) > 0 : \|v\|^p \le c(\lambda,\mu) H_{\lambda\mu}(v)\ \forall v \in W$.

(i_{14}) $\exists c(\lambda,\mu) > 0 : \|v\|^p \le c(\lambda,\mu) H_{\lambda\mu}(v)\ \forall v \in V^+(D_1)$ (if $V^+(D_1) \ne \varnothing$).

Theorem 1 assures the existence of at least one solution. It is possible to get the existence of a second solution from a result of [10, Theorem 2.2]. This result is based in particular on the following assumption:

(i_{15}) $V^-_{\lambda\mu} \cap S(D_1)$ is not empty and bounded in W.

The applications to Dirichlet problems in Section 3 [resp., Neumann problems in Section 4] (whose variational form is included in Problem (P)), for the sake of brevity, deal with the first case of solvability, since in this case we can use at the same time Theorems 1 and 4. When $n > 1$, thanks to Propositions 2 and 5, we have got sufficient conditions so that the components of the found solutions are not identically equal to zero.

2. Solvability of Problem (P)

Let us consider the following cases:

(c_1) $m > 1$, $q_1 < p$, $V^+(D_1) \ne \varnothing$, $D_j \le 0\ \forall j \in \{2,\ldots,m\}$, and (i_{14}) holds.

(c_2) $m > 1$, $\exists m_1 \in \{2,\ldots,m\} : q_{m_1} < p$, $D_j \ge 0\ \forall j \in \{1,\ldots,m_1\}$, $D_j \le 0\ \forall j \in \{m_1+1,\ldots,m\}$ if $m_1 < m$, and (i_{13}) holds.

(c_3) $m > 1$, $q_m < p$, $D_j \ge 0\ \forall j \in \{1,\ldots,m-1\}$, $\sum_{j=1}^{m-1} D_j(v) > 0\ \forall v \in W \setminus \{0\}$, D_m changes sign, and (i_{13}) holds.

(c_4) $m > 1$, $p < q_1$, D_1 changes sign, $D_j \ge 0\ \forall j \in \{2,\ldots,m\}$, $\sum_{j=2}^{m} D_j(v) > 0\ \forall v \in W \setminus \{0\}$, and (i_{13}) holds.

Let us introduce the open set $\mathscr{A}$ of the space W:

$$
\begin{aligned}
\mathscr{A} &= V^+(D_1) \quad \text{in } (c_1),\\
\mathscr{A} &= V^+\left(D_1,\ldots,D_{m_1}\right) \quad \text{in } (c_2),\\
\mathscr{A} &= W \setminus \{0\} \quad \text{in } (c_3),(c_4),
\end{aligned} \tag{3}
$$

and let us set

$$
\widetilde{E}(t,v) = E(tv) = t^p H_{\lambda\mu}(v) - \sum_{j=1}^{m} t^{q_j} D_j(v), \tag{4}
$$

$$
\psi(t,v) = \frac{\partial \widetilde{E}}{\partial t}(t,v), \qquad \forall t \ge 0,\ \forall v \in \mathscr{A}. \tag{5}
$$

It is easy to control that, for each $v \in \mathscr{A}$, the equation

$$
\psi(t,v) = 0 \tag{6}
$$

has only one positive root $t(v)$ and it results in

$$\frac{\partial\psi}{\partial t}(t(v),v) > 0,\quad \widetilde{E}(t(v),v) = \min_{t\geq 0}\widetilde{E}(t,v) < 0 \quad \text{in } (c_1)-(c_3), \tag{7}$$

$$\frac{\partial\psi}{\partial t}(t(v),v) < 0,\quad \widetilde{E}(t(v),v) = \max_{t\geq 0}\widetilde{E}(t,v) > 0 \quad \text{in } (c_4). \tag{8}$$

The implicit function theory assures the functional $t(v)$ belongs to $C^1(\mathscr{A})$. Then, the functional $\widetilde{\widetilde{E}}(v) = \widetilde{E}(t(v),v)$ belongs to $C^1(\mathscr{A})$ and it results in

$$\begin{aligned}
&\langle\!\langle \partial\widetilde{\widetilde{E}}(u), v\rangle\!\rangle \\
&= \left[p(t(u))^{p-1}H_{\lambda\mu}(u) - \sum_{j=1}^{m} q_j (t(u))^{q_j-1} D_j(u)\right] \\
&\cdot \langle\!\langle \partial t(u), v\rangle\!\rangle + (t(u))^p \langle\!\langle \partial H_{\lambda\mu}(u), v\rangle\!\rangle \\
&- \sum_{j=1}^{m} (t(u))^{q_j} \langle\!\langle \partial D_j(u), v\rangle\!\rangle = (t(u))^p \\
&\cdot \langle\!\langle \partial H_{\lambda\mu}(u), v\rangle\!\rangle - \sum_{j=1}^{m} (t(u))^{q_j} \langle\!\langle \partial D_j(u), v\rangle\!\rangle \\
&\qquad \forall u \in \mathscr{A},\ \forall v \in W.
\end{aligned} \tag{9}$$

It is important to remark that (9) can also be written as follows:

$$\begin{aligned}
&\langle\!\langle \partial\widetilde{\widetilde{E}}(u), v\rangle\!\rangle = (t(u)) \\
&\cdot \left[\langle\!\langle \partial H_{\lambda\mu}(t(u)u), v\rangle\!\rangle - \sum_{j=1}^{m} \langle\!\langle \partial D_j(t(u)u), v\rangle\!\rangle\right] \\
&\qquad \forall u \in \mathscr{A},\ \forall v \in W.
\end{aligned} \tag{10}$$

Theorem 1. *Under assumptions* (i_{11}), (i_{12}), *in cases* (c_1)–(c_4), *one has*

$$\exists v^0 = \left(v_1^0,\dots,v_n^0\right) \in S_{\lambda\mu}\cap\mathscr{A}: \quad \widetilde{\widetilde{E}}\left(v^0\right) = \inf\left\{\widetilde{\widetilde{E}}(v) : v \in S_{\lambda\mu}\cap\mathscr{A}\right\}, \tag{11}$$

$$u^0 = t\left(v^0\right)v^0 \text{ is solution of Problem } (P). \tag{12}$$

Proof. Let us set

$$\begin{aligned}
&\underline{e} = \inf\left\{\widetilde{\widetilde{E}}(v) : v \in S_{\lambda\mu}\cap\mathscr{A}\right\}, \\
&\widetilde{E}_1(t,v) = t^p - \sum_{j=1}^{m} t^{q_j} D_j(v), \\
&\psi_1(t,v) = \frac{\partial\widetilde{E}_1}{\partial t}(t,v) \qquad \forall t\geq 0,\ \forall v\in\mathscr{A}, \\
&\widetilde{\widetilde{E}}_1(v) = \widetilde{E}_1(t_1(v),v) \quad \forall v\in\mathscr{A},
\end{aligned} \tag{13}$$

where $t_1(v)$ is the positive root of equation $\psi_1(t,v) = 0$. Pointing out that $\widetilde{\widetilde{E}}_1 \in C^1(\mathscr{A})$ and

$$\widetilde{\widetilde{E}}(v) = \widetilde{\widetilde{E}}_1(v) \quad \forall v \in S_{\lambda\mu}\cap\mathscr{A}, \tag{14}$$

let us prove that

$$\exists v^0 \in \mathscr{A}: \quad \widetilde{\widetilde{E}}_1\left(v^0\right) = \underline{e}. \tag{15}$$

Let $\{v^k\} \subseteq S_{\lambda\mu}\cap\mathscr{A}$ such that $\widetilde{\widetilde{E}}_1(v^k) \to \underline{e}$; that is,

$$\left(t_1\left(v^k\right)\right)^p - \sum_{j=1}^{m}\left(t_1\left(v^k\right)\right)^{q_j} D_j\left(v^k\right) \longrightarrow \underline{e}. \tag{16}$$

We note that

$$p\left(t_1\left(v^k\right)\right)^{p-1} - \sum_{j=1}^{m} q_j\left(t_1\left(v^k\right)\right)^{q_j-1} D_j\left(v^k\right) = 0; \tag{17}$$

moreover,

$$\left\|v^k\right\|^p \leq c(\lambda,\mu) \tag{18}$$

since (i_{14}) holds in (c_1) and (i_{13}) holds in (c_2)–(c_4).

Inequality (18) implies that $v^0 \in W$ exists such that (within a subsequence) $v^k \to v^0$ weakly in W. Then,

$$0 \leq H_{\lambda\mu}\left(v^0\right) \leq 1, \tag{19}$$

$$D_j\left(v^k\right) \longrightarrow D_j\left(v^0\right) \quad \text{as } j = 1,\dots,m. \tag{20}$$

Let us verify that

$$\sup t_1\left(v^k\right) < +\infty. \tag{21}$$

In (c_1)–(c_3), let $\delta > 0$ such that $|D_j(v)| \leq \delta \forall v \in S_{\lambda\mu}\cap\mathscr{A}$ and $\forall j \in \{1,\dots,m\}$. We set $\forall t \geq 0$, $\Phi(t) = pt^{p-1} - \delta q_1 t^{q_1-1}$, in (c_1), and $\Phi(t) = pt^{p-1} - \delta\sum_{j=1}^{m_1} q_j t^{q_j-1}$ in (c_2), $\Phi(t) = pt^{p-1} - \delta\sum_{j=1}^{m} q_j t^{q_j-1}$ in (c_3).

Let us denote by t_0 the positive root of equation $\Phi(t) = 0$. Since $\psi_1(t,v) \geq \Phi(t)$ $\forall t \geq 0$ and $\forall v \in S_{\lambda\mu}\cap\mathscr{A}$, we have $t_1(v^k) \leq t_0$ $\forall k \in N$.

In (c_4), we note that from (17) $(t_1(v^k))^{q_1} D_1(v^k) = (p/q_1)(t_1(v^k))^p - \sum_{j=2}^m (q_j/q_1)(t_1(v^k))^{q_j} D_j(v^k)$; then,

$$\begin{aligned}
&\left(t_1\left(v^k\right)\right)^p - \sum_{j=1}^m \left(t_1\left(v^k\right)\right)^{q_j} D_j\left(v^k\right) \\
&\quad = \left(1 - \frac{p}{q_1}\right)\left(t_1\left(v^k\right)\right)^p \\
&\qquad + \sum_{j=2}^m \left(\frac{q_j}{q_1} - 1\right)\left(t_1\left(v^k\right)\right)^{q_j} D_j\left(v^k\right) \\
&\quad \geq \left(1 - \frac{p}{q_1}\right)\left(t_1\left(v^k\right)\right)^p
\end{aligned} \tag{22}$$

from which (21) follows taking into account (16).

Relation (21) assures that $\omega \in [0, +\infty[$ exists such that (within a subsequence) $t_1(v^k) \to \omega$.

Consequently, from (16), (17), and (20), we obtain

$$\omega^p - \sum_{j=1}^m \omega^{q_j} D_j\left(v^0\right) = \underline{e}, \tag{23}$$

$$p\omega^{p-1} - \sum_{j=1}^m q_j \omega^{q_j - 1} D_j\left(v^0\right) = 0. \tag{24}$$

Let us add that

$$\begin{aligned}
&\omega > 0, \\
&v^0 \in \mathscr{A}.
\end{aligned} \tag{25}$$

In (c_1)–(c_3), since from (7) $\underline{e} < 0$, (23) implies that $\omega > 0$ and $v^0 \neq 0$; then, $v^0 \in \mathscr{A}$ in (c_3), while (24) $\Rightarrow$ $v^0 \in \mathscr{A}$ in (c_1) and (c_2). In case (c_4) (where from (8) $\underline{e} \geq 0$), since from (17) $\sum_{j=1}^m q_j (t_1(v^k))^{q_j - p} D_j(v^k) = p$, we have $\sum_{j=1}^m q_j \omega^{q_j - p} D_j(v^0) = p$ from which we obtain (25).

Evidently, ((24), (25)) $\Rightarrow$ $\omega = t_1(v^0)$; then, (23) can be written in the form $(t_1(v^0))^p - \sum_{j=1}^m (t_1(v^0))^{q_j} D_j(v^0) = \underline{e}$; that is, $\tilde{\tilde{E}}_1(v^0) = \underline{e}$. Then, (15) holds.

Let us prove that

$$H_{\lambda\mu}\left(v^0\right) = 1 \quad \text{in (19)}. \tag{26}$$

In fact, taking into account that (i_{14}) holds in (c_1) and (i_{13}) holds in (c_2)–(c_4), we obtain

$$\begin{aligned}
&v^0 \in \mathscr{A} \Longrightarrow \\
&\left\| v^0 \right\| > 0 \Longrightarrow \\
&H_{\lambda\mu}\left(v^0\right) > 0.
\end{aligned} \tag{27}$$

On the other hand, since $v^0 \in \mathscr{A} \Rightarrow \theta v^0 \in \mathscr{A}$ $\forall \theta > 0$, we have

$$\begin{aligned}
\frac{d}{d\theta}\tilde{\tilde{E}}_1\left(\theta v^0\right) &= \left[p\left(t_1\left(\theta v^0\right)\right)^{p-1} \right. \\
&\quad \left. - \sum_{j=1}^m q_j \left(t_1\left(\theta v^0\right)\right)^{q_j - 1} D_j\left(\theta v^0\right) \right] \frac{d}{d\theta} t_1\left(\theta v^0\right) \\
&\quad - \sum_{j=1}^m q_j \left(t_1\left(\theta v^0\right)\right)^{q_j} \theta^{q_j - 1} D_j\left(v^0\right) \\
&= -\theta^{-1} p\left(t_1\left(\theta v^0\right)\right)^p \quad \forall \theta > 0.
\end{aligned} \tag{28}$$

Then, if $H_{\lambda\mu}(v^0) < 1$, we get the contradiction

$$\begin{aligned}
&\left(H_{\lambda\mu}\left(v^0\right)\right)^{-1/p} v^0 \in S_{\lambda\mu} \cap \mathscr{A}, \\
&\tilde{\tilde{E}}_1\left(\left(H_{\lambda\mu}\left(v^0\right)\right)^{-1} v^0\right) < \tilde{\tilde{E}}_1\left(v^0\right) = \underline{e}.
\end{aligned} \tag{29}$$

Relations (14), (15), and (26) allow (11).

From (11), a Lagrange multiplier τ exists such that

$$\left\langle\!\left\langle \partial\tilde{\tilde{E}}\left(v^0\right), v \right\rangle\!\right\rangle = \tau \left\langle\!\left\langle \partial H_{\lambda\mu}\left(v^0\right), v \right\rangle\!\right\rangle \quad \forall v \in W. \tag{30}$$

Setting in (30) $v = v^0$, we get $\tau = 0$, since $\langle\!\langle \partial H_{\lambda\mu}(v^0), v^0 \rangle\!\rangle = pH_{\lambda\mu}(v^0) = p$, and from (9) $\langle\!\langle \partial\tilde{\tilde{E}}(v^0), v^0 \rangle\!\rangle = (t(v^0))^p \langle\!\langle \partial H_{\lambda\mu}(v^0), v^0 \rangle\!\rangle - \sum_{j=1}^m (t(v^0))^{q_j} \langle\!\langle \partial D_j(v^0), v^0 \rangle\!\rangle = p(t(v^0))^p - \sum_{j=1}^m q_j (t(v^0))^{q_j} D_j(v^0) = 0$. Then, (30), taking into account (10), implies $\langle\!\langle \partial H_{\lambda\mu}(t(v^0)v^0), v \rangle\!\rangle = \sum_{j=1}^m \langle\!\langle \partial D_j(t(v^0)v^0), v \rangle\!\rangle$ $\forall v \in W$; then, (12) holds. □

Proposition 2. *Let $n > 1$. Let v^0 as in Theorem 1 and $\mathscr{F} \subseteq S_{\lambda\mu} \cap \mathscr{A}$. Let us suppose the following:*

(i_0^h) *$\forall v \in \mathscr{F} \tilde{v}_h \in W_h \setminus \{0\}$ and a function $\Phi(\cdot, v, \tilde{v}_h) \to W$ belonging to $C^o(]s_0, 1]) \cap C^1(]s_0, 1[)$ $(0 \leq s_0 < 1)$ [resp., $C^o([1, s_0[) \cap C^1(]1, s_0[)$ $(1 < s_0 \leq +\infty)$] exist such that $\Phi(1, v, \tilde{v}_h) = v$, $\tilde{v}(s) = \Phi(s, v, \tilde{v}_h) \in S_{\lambda\mu} \forall s \in]s_0, 1]$ [resp., $\forall s \in [1, s_0[$] and for some $s_1 \in]s_0, 1[$ [resp., $s_1 \in]1, s_0[$] $\sum_{j=1}^m (t(\tilde{v}(s)))^{q_j} (d/ds) D_j(\tilde{v}(s)) < 0$ [resp., > 0] $\forall s \in]s_1, 1[$ [resp., $\forall s \in]1, s_1[$].*

Then, $v^0 \notin \mathscr{F}$.

Proof. In fact, with v and $\tilde{v}(s)$ as in $(i_0^h) s_2 \in]s_1, 1[$ [resp., $s_2 \in]1, s_1[$] exists such that $\tilde{v}(s) \in S_{\lambda\mu} \cap \mathscr{A}$ and $(d/ds)\tilde{\tilde{E}}(\tilde{v}(s)) = -\sum_{j=1}^m (t(\tilde{v}(s)))^{q_j} (d/ds) D_j(\tilde{v}(s)) > 0$ [resp., < 0] $\forall s \in]s_2, 1[$ [resp., $\forall s \in]1, s_2[$]; then, $\tilde{\tilde{E}}(\tilde{v}(s)) < \tilde{\tilde{E}}(v)$ $\forall s \in]s_2, 1[$ [resp., $\forall s \in]1, s_2[$]. □

Remark 3. Condition (i_0^h) with $\mathscr{F} \subseteq S_{\lambda\mu}$ includes condition (i_{16}^h) introduced in [10] and it implies the conclusion of Theorem 2.2 of [10].

In the case

(c_5) $m > 1$, (i_{15}) holds, $q_m < p$, and $D_j \le 0\ \forall j \in \{2,\dots,m\}$,

for each $v \in V^-_{\lambda\mu} \cap S(D_1)$, (6) has only one positive root $t(v)$ and we have $(\partial\psi/\partial t)(t(v),v) \ne 0$.

Set $\tilde{\tilde{E}}(v) = \tilde{E}(t(v),v)\ \forall v \in V^-_{\lambda\mu} \cap S(D_1)$, in [10, Theorem 4.2], the following result has been proved.

Theorem 4. *Under conditions (i_{11}), (i_{12}), in case (c_5), one has the following:*

$$\exists \bar{v} = (\bar{v}_1,\dots,\bar{v}_n) \in V^-_{\lambda\mu} \cap S(D_1):$$
$$\tilde{\tilde{E}}(\bar{v}) = \inf\left\{\tilde{\tilde{E}}(v) : v \in V^-_{\lambda\mu} \cap S(D_1)\right\}, \tag{31}$$
$$\bar{u} = t(\bar{v})\,\bar{v} \text{ is solution of Problem } (P).$$

We add the following proposition.

Proposition 5. *Let $n > 1$. Let $\bar{v}$ as in Theorem 4 and $\mathcal{F} \subseteq V^-_{\lambda\mu} \cap S(D_1)$. Let one suppose the following:*

($\underline{i}^h$) *$\forall v \in \mathcal{F}$ $\tilde{v}_h \in W_h \setminus \{0\}$ and a function $\Phi(\cdot,v,\tilde{v}_h) \to W$ belonging to $C^o(]s_0,1]) \cap C^1(]s_0,1[)$ $(0 \le s_0 < 1)$ [resp., $C^o([1,s_0[) \cap C^1(]1,s_0[)$ $(1 < s_0 \le +\infty)$] exist such that $\Phi(1,v,\tilde{v}_h) = v$, $\tilde{v}(s) = \Phi(s,v,\tilde{v}_h) \in S(D_1)$ $\forall s \in]s_0,1]$ [resp., $\forall s \in [1,s_0[$] and for some $s_1 \in]s_0,1[$ [resp., $s_1 \in]1,s_0[$] $(t(\tilde{v}(s)))^p (d/ds) H_{\lambda\mu}(\tilde{v}(s)) - \sum_{j=1}^m (t(\tilde{v}(s)))^{q_j}(d/ds)D_j(\tilde{v}(s)) > 0$ [resp., < 0] $\forall s \in]s_1,1[$ [resp., $\forall s \in]1,s_1[$].*

Then, $\bar{v} \notin \mathcal{F}$.

Proof. Let v and $\tilde{v}(s)$ be as in ($\underline{i}^h$). Since $t(v) > 0$, $\psi(t(v),v) = 0$ and $(\partial\psi/\partial t)(t(v),v) \ne 0$; then, an open ball $\tilde{B}$ exists with center v included in $V^-_{\lambda\mu}$ and a unique functional $t^*(w)$ belonging to $C^1(\tilde{B})$ such that $t^*(w) > 0$ and $\psi(t^*(w),w) = 0\ \forall w \in \tilde{B}$. Evidently, $t^*(w) = t(w)\ \forall w \in \tilde{B} \cap S(D_1)$. Set $s_2 \in]s_1,1[$ [resp., $s_2 \in]1,s_1[$] such that $\tilde{v}(s) \in \tilde{B}\ \forall s \in]s_2,1]$ [resp., $\forall s \in [1,s_2[$]; we have

$$\begin{aligned}&\frac{d}{ds}\left[(t^*(\tilde{v}(s)))^p H_{\lambda\mu}(\tilde{v}(s)) - \sum_{j=1}^m (t^*(\tilde{v}(s)))^{q_j} D_j(\tilde{v}(s))\right]\\ &= (t(\tilde{v}(s)))^p \frac{d}{ds} H_{\lambda\mu}(\tilde{v}(s)) - \sum_{j=1}^m (t(\tilde{v}(s)))^{q_j} \frac{d}{ds} D_j(\tilde{v}(s)) > 0\\ &\quad [\text{resp.}, < 0]\quad \forall s \in]s_2,1[\quad [\text{resp.}, \forall s \in]1,s_2[].\end{aligned} \tag{32}$$

Consequently, $\tilde{\tilde{E}}(\tilde{v}(s)) < \tilde{\tilde{E}}(v)\ \forall s \in]s_2,1[$ [resp., $\forall s \in]1,s_2[$]. □

Remark 6. Let W_ℓ $(\ell = 1,\dots,n)$ be a vector lattice. Let v^0 [resp., $\bar{v}$] be as in Theorem 1 [resp., Theorem 4]. If $H_{\lambda\mu}(v_1,\dots,v_n) = H_{\lambda\mu}(|v_1|,\dots,|v_n|)$ and $D_j(v_1,\dots,v_n) = D_j(|v_1|,\dots,|v_n|)\ \forall v = (v_1,\dots,v_n) \in W$ and $\forall j \in \{1,\dots,m\}$, then $(|v_1^0|,\dots,|v_n^0|) \in S_{\lambda\mu} \cap \mathcal{A}$, $t(|v_1^0|,\dots,|v_n^0|) = t(v^0)$, $\tilde{\tilde{E}}(|v_1^0|,\dots,|v_n^0|) = \tilde{\tilde{E}}(v^0)$ [resp., $(|\bar{v}_1|,\dots,|\bar{v}_n|) \in V^-_{\lambda\mu} \cap S(D_1)$, $t(|\bar{v}_1|,\dots,|\bar{v}_n|) = t(\bar{v})$, $\tilde{\tilde{E}}(|\bar{v}_1|,\dots,|\bar{v}_n|) = \tilde{\tilde{E}}(\bar{v})$].

Then, reasoning as in Theorem 1 proof's final step [resp., Theorem 4.2 of [10]], we see that $\pm t(v^0)(|v_1^0|,\dots,|v_n^0|)$ [resp., $\pm t(\bar{v})(|\bar{v}_1|,\dots,|\bar{v}_n|)$] are solutions of Problem (P). Consequently, we can assume that $v_\ell^0 \ge 0$; that is, $u_\ell^0 \ge 0$ [resp., $\bar{v}_\ell \ge 0$ i.e., $\bar{u}_\ell \ge 0$]as $\ell = 1,\dots,n$.

3. Dirichlet Problems

We assume $W = (W_0^{1,p}(\Omega))^n$ $(n \ge 1)$ with $\|v\| = (\sum_{\ell=1}^n \int_\Omega |\nabla v_\ell|^p dx)^{1/p}$ $\forall v = (v_1,\dots,v_n) \in W$, $B_\ell(v_\ell) = p^{-1}\int_\Omega b_\ell |v_\ell|^p dx$ $\forall v_\ell \in W_0^{1,p}(\Omega)$, and $\widehat{B}_\ell \equiv 0$, where $1 < p < \infty$, $\Omega \subseteq R^N$ is open, bounded, connected and $C^{2,\beta}$ set $(0 < \beta \le 1)$, $b_\ell \in L^\infty(\Omega) \setminus \{0\}$ with $b_\ell \ge 0$.

Let us consider the functional A (as in (i_{11})) such that $A(v) \ge p^{-1}\tilde{c}\|v\|^p$ $\forall v \in W$ ($\tilde{c}$ = const. > 0).

Let us use the following notations: $\tilde{p} = Np/(N-p)$ if $N > p$, $\tilde{p} = \infty$ else; $|\cdot|_N$ is the Lebesgue measure in R^N; H_λ, S_λ, V_λ^- are instead of $H_{\lambda\mu}$, $S_{\lambda\mu}$, $V^-_{\lambda\mu}$; λ_ℓ^* is the first eigenvalue and u_ℓ^* is the first eigenfunction of the problem $u_\ell \in W_0^{1,p}(\Omega)$: $-\tilde{c}\,\mathrm{div}(|\nabla u_\ell|^{p-2}\nabla u_\ell) = \theta b_\ell |u_\ell|^{p-2}u_\ell$ in Ω [16].

We present some results about the validity of assumptions (i_{13})–(i_{15}). To this aim, let us set $I = \{1,\dots,n\}$ and, for each I^* $(\ne \emptyset) \subseteq I$, $V^* = \{v \in W : v_\ell \equiv 0$ if $\ell \in I \setminus I^*, v_\ell = c_\ell u_\ell^*$ if $\ell \in I^*$ with $c_\ell \in R$ and $c_\ell \ne 0$ as some $\ell \in I^*\}$, let us introduce the following hypothesis:

(i_3) $I^* \subseteq I$ exists: $D_1(v) < 0$ for every $v \in V^*$.

($\underline{i}_3$) $I^* \subseteq I$ exists such that $D_1(v) < 0$ and $A(v) = \tilde{c}p^{-1}\sum_{\ell\in I^*}\int_\Omega |\nabla v_\ell|^p dx$ for every $v \in V^*$.

Proposition 7. *If $\lambda_\ell < \lambda_\ell^*$ for each $\ell \in \{1,\dots,n\}$, then (i_{13}) holds. When $V^+(D_1) \ne \emptyset$ and (i_3) holds with $I^* \ne I$, then, with $\lambda_\ell < \lambda_\ell^*$ as $\ell \in I\setminus I^*$, $\exists\delta^* > 0 : (i_{14})$ holds if $\lambda_\ell < \lambda_\ell^* + \delta^*$ as $\ell \in I^*$. When $V^+(D_1) \ne \emptyset$ and (i_3) holds with $I^* = I$, then $\exists\delta^* > 0 : (i_{14})$ holds if $\lambda_\ell < \lambda_\ell^* + \delta^*$ as $\ell \in I$.*

Proof. The first statement is obvious. We can prove the second and the third ones as in [11, Propositions 2.3 and 2.4]. □

Proposition 8. *When $(\underline{i}_3)$ holds with $I^* \ne I$, then with $\lambda_\ell < \lambda_\ell^*$ as $\ell \in I\setminus I^*$ $\exists\delta^* > 0 : (i_{15})$ holds if $\lambda_\ell \in [\lambda_\ell^*, \lambda_\ell^* + \delta^*[$ $\forall \ell \in I^*$ and $\lambda_\ell > \lambda_\ell^*$ as some ℓ. When $(\underline{i}_3)$ holds with $I^* = I$, then $\exists\delta^* > 0 : (i_{15})$ holds if $\lambda_\ell \in [\lambda_\ell^*, \lambda_\ell^* + \delta^*[$ $\forall \ell \in I$ and $\lambda_\ell > \lambda_\ell^*$ as some ℓ.*

Proof. See [11, Propositions 2.5 and 2.6]. □

Let us now investigate Problem (P) in two concrete cases where $\widetilde{c} = 1$.

Application 1. Let $n > 1$ and, for each $v = (v_1, \dots, v_n) \in W$,

$$A(v) = p^{-1} \int_\Omega \left(\sum_{\ell=1}^{n} |\nabla v_\ell|^\gamma \right)^{p/\gamma} dx,$$
$$D_1(v) = \left(\int_\Omega \rho_1 |v_1|^{q_{11}} dx \right) \int_\Omega \left(\sum_{\ell=1}^{n} |v_\ell|^{\gamma_1} \right)^{q_{12}/\gamma_1} dx, \quad (33)$$
$$D_j(v) = q_j^{-1} \int_\Omega \rho_j \left| d_{j1} |v_1|^{\gamma_j} + d_{jj} |v_j|^{\gamma_j} \right|^{q_j/\gamma_j} dx \quad \text{as } j = 2, \dots, n,$$

where

$$\begin{aligned} &1 < \gamma < p, \\ &q_{11} > 1, \\ &1 < \gamma_1 < q_{12}, \\ &1 < \gamma_j < q_j \\ &\text{as } j = 2, \dots, n, \ q_1 = q_{11} + q_{12} < q_2 < \dots < q_n < \widetilde{p}, \ q_1 < p; \\ &\rho_1 \in L^\infty(\Omega); \\ &\rho_j \in L^\infty(\Omega), \\ &\rho_j < 0, \\ &d_{j1}, d_{jj} \in L^\infty(\Omega) \setminus \{0\} \\ &\text{as } j = 2, \dots, n. \end{aligned} \quad (34)$$

Let us consider the following system:

$$\begin{aligned} &-\operatorname{div}\left[\left(\sum_{\ell=1}^{n} |\nabla u_\ell|^\gamma \right)^{(p/\gamma)-1} |\nabla u_1|^{\gamma-2} \nabla u_1 \right] \\ &= \lambda_1 b_1 |u_1|^{p-2} u_1 \\ &+ q_{11} \left(\int_\Omega \left(\sum_{\ell=1}^{n} |u_\ell|^{\gamma_1} \right)^{(q_{12}/\gamma_1)} dx \right) \rho_1 |u_1|^{q_{11}-2} u_1 \\ &+ q_{12} \left(\int_\Omega \rho_1 |u_1|^{q_{11}} dx \right) \left(\sum_{\ell=1}^{n} |u_\ell|^{\gamma_1} \right)^{(q_{12}/\gamma_1)-1} \\ &\cdot |u_1|^{\gamma_1-2} u_1 + \sum_{j=2}^{n} \rho_j \left| d_{j1} |u_1|^{\gamma_j} + d_{jj} |u_j|^{\gamma_j} \right|^{(q_j/\gamma_j)-2} \\ &\cdot \left(d_{j1} |u_1|^{\gamma_j} + d_{jj} |u_j|^{\gamma_j} \right) d_{j1} |u_1|^{\gamma_j-2} u_1 \\ &\qquad \text{in } \Omega, \\ &-\operatorname{div}\left[\left(\sum_{\ell=1}^{n} |\nabla u_\ell|^\gamma \right)^{(p/\gamma)-1} |\nabla u_i|^{\gamma-2} \nabla u_i \right] \\ &= \lambda_i b_i |u_i|^{p-2} u_i + q_{12} \left(\int_\Omega \rho_1 |u_1|^{q_{11}} dx \right) \left(\sum_{\ell=1}^{n} |u_\ell|^{\gamma_1} \right)^{(q_{12}/\gamma_1)-1} \\ &\cdot |u_i|^{\gamma_1-2} u_i + \rho_i \left| d_{i1} |u_1|^{\gamma_i} + d_{ii} |u_i|^{\gamma_i} \right|^{(q_i/\gamma_i)-2} \\ &\cdot \left(d_{i1} |u_1|^{\gamma_i} + d_{ii} |u_i|^{\gamma_i} \right) d_{ii} |u_i|^{\gamma_i-2} u_i \\ &\qquad \text{in } \Omega \text{ as } i = 2, \dots, n, \\ &u_i = 0 \quad \text{on } \partial\Omega \text{ as } i = 1, \dots, n. \end{aligned} \quad (35)$$

Let us introduce the following conditions:

$$\rho_1^+ \not\equiv 0 \left(\Longrightarrow V^+(D_1) \neq \varnothing \right), \quad (36)$$
$$\int_\Omega \rho_1 (u_1^*)^{q_{11}} dx < 0 \left(\Longrightarrow D_1(c_1 u_1^*, \dots, c_n u_n^*) < 0 \text{ iff } c_1 \neq 0 \right). \quad (37)$$

Then (Propositions 7 and 8),

$$(36) \Longrightarrow \left(\text{with } \lambda_\ell < \lambda_\ell^* \text{ as } \ell = 1, \dots, n \ (i_{14}) \text{ holds} \right), \quad (38)$$
$$(36), (37) \Longrightarrow \left(\text{with } \lambda_\ell < \lambda_\ell^* \text{ as } \ell = 2, \dots, n \ \exists \delta_1^* > 0 : (i_{14}) \text{ holds if } \lambda_1 < \lambda_1^* + \delta_1^* \right), \quad (39)$$
$$(37) \Longrightarrow \left(\text{with } \lambda_\ell < \lambda_\ell^* \text{ as } \ell = 2, \dots, n \ \exists \delta_2^* > 0 : (i_{15}) \text{ holds if } \lambda_1 \in]\lambda_1^*, \lambda_1^* + \delta_2^*[\right). \quad (40)$$

Proposition 9 (Theorems 1 and 4, Remark 6,). *Under conditions (34), one has the following:*

When (36) holds [resp., (36) and (37) hold], with $\lambda_1, \dots, \lambda_n$ as in (38) [resp., (39)], system (35) has at least two weak solutions u^0 and $-u^0$ ($u^0 = \tau^0 v^0, \tau^0 = const. > 0, v^0 \in S_\lambda \cap V^+(D_1)$), and one has $u_i^0 \geq 0$ as $i = 1, \dots, n$, $u_1^0 \not\equiv 0$.

When $q_n < p$ and (37) holds, with $\lambda_1, \dots, \lambda_n$ as in (40), system (35) has at least two weak solutions $\overline{u}$ and $-\overline{u}$ ($\overline{u} = \overline{\tau}\overline{v}$, $\overline{\tau} = const. > 0$, $\overline{v} \in V_\lambda^- \cap S(D_1)$), and one has $\overline{u}_i \geq 0$ as $i = 1, \dots, n$, $\overline{u}_1 \not\equiv 0$.

Consequently, when $q_n < p$ and (36), (37) hold, with $\lambda_\ell < \lambda_\ell^*$ as $\ell \geq 2$ and $\lambda_1 \in]\lambda_1^*, \lambda_1^* + \min\{\delta_1^*, \delta_2^*\}[$, system (35) has at least four different weak solutions.

Proposition 10. *If*

$$\text{either } \gamma_1 < \gamma \le \gamma_j \quad \text{as } j = 2, \dots, n \tag{41}$$

$$\text{or } d_{j1} d_{jj} < 0, \quad \gamma_j < \gamma \quad \text{as } j = 2, \dots, n, \tag{42}$$

then $u_i^0 \not\equiv 0$ *as* $i = 2, \dots, n$.

Proof. It is sufficient (Proposition 2) to verify the following.

As $h = 2, \dots, n$ (i_0^h) holds with $\mathcal{F} = \{v \in S_\lambda \cap V^+(D_1) : v_\ell \ge 0 \text{ as } \ell = 1, \dots, n,\ v_1 \not\equiv 0,\ v_h \equiv 0\} \ne \varnothing$.

Let, for example, $h = 2$. Let $v \in \mathcal{F}$. First of all,

$$\int_\Omega \rho_1 v_1^{q_{11}} dx > 0, \quad \lambda_2 < \lambda_2^* \Longrightarrow \quad \delta = \int_\Omega |\nabla v_1|^p \, dx - \lambda_2 \int_\Omega b_2 v_1^p dx > 0. \tag{43}$$

Let us introduce the function $g(s, \tau) = H_\lambda(sv_1, \tau v_1, \dots, sv_n) = p^{-1}[\int_\Omega (\tau^\gamma |\nabla v_1|^\gamma + s^\gamma \sum_{\ell \ne 2} |\nabla v_\ell|^\gamma)^{p/\gamma} dx - \tau^p \lambda_2 \int_\Omega b_2 v_1^p dx - s^p \sum_{\ell \ne 2} \lambda_\ell \int_\Omega b_\ell v_\ell^p dx]$ $\forall s, \tau \ge 0$. Let us note that

$$g(1,0) = 1, \quad \frac{\partial g}{\partial \tau}(s,\tau) \ge \delta \tau^{p-1} > 0 \quad \forall s \ge 0,\ \forall \tau > 0, \quad g(s,0) = s^p < 1 \quad \forall s \in]0,1[, \quad \lim_{\tau \to +\infty} g(s,\tau) = +\infty \quad \forall s \ge 0; \tag{44}$$

then,

$$\text{only one } \tau(s) > 0 \ (\tau(s) > 0 \text{ if } s < 1,\ \tau(1) = 0) \text{ exists such that } g(s, \tau(s)) = 1, \quad \forall s \in]0,1]. \tag{45}$$

About the function $\tau(s)$, we have

$$\lim_{\tau \to 1^-} \tau(s) = 0. \tag{46}$$

In fact, with $\{s_n\} \subseteq]0,1[$ and $\lim s_n = 1$, since $g(s_n, \tau(s_n)) = 1$, $\{\tau(s_n)\}$ is necessarily bounded. Then (within a subsequence), $\lim \tau(s_n) = \omega$ with $g(1, \omega) = 1$, from which $\omega = 0$. Then, $\tau(s)$ belongs to $C^0(]0,1])$.

Let us add that $\tau(s)$ belongs to $C^1(]0,1[)$ and it results in

$$\tau'(s) = -\frac{1}{(\tau(s))^{\gamma-1}} \tilde{g}(s, \tau(s)) \quad \forall s \in]0,1[, \quad \lim_{s \to 1^-} \tilde{g}(s, \tau(s)) \in]0, +\infty[. \tag{47}$$

Then, with $\tilde{v}(s) = (sv_1, \tau(s)v_1, \dots, sv_n)$, we have

$$\tilde{v}(1) = v, \quad \tilde{v}(s) \in S_\lambda \quad \forall s \in]0,1], \tag{48}$$

and, moreover, taking into account the first one of (43),

$$\begin{aligned}
&\lim_{s \to 1^-} \frac{d}{ds} D_1(\tilde{v}(s)) = -\infty, \\
&\lim_{s \to 1^-} \frac{d}{ds} D_j(\tilde{v}(s)) \in R \quad \text{as } j \ge 2 \text{ when (41) holds}, \\
&\lim_{s \to 1^-} \frac{d}{ds} D_1(\tilde{v}(s)) < +\infty, \\
&\lim_{s \to 1^-} \frac{d}{ds} D_2(\tilde{v}(s)) = -\infty, \\
&\lim_{s \to 1^-} \frac{d}{ds} D_j(\tilde{v}(s)) \in R \quad \text{as } j > 2 \text{ (if } n > 2) \text{ when (42) holds},
\end{aligned} \tag{49}$$

from which $\lim_{s \to 1^-} \sum_{j=1}^n (t(\tilde{v}(s)))^{q_j} (d/ds) D_j(\tilde{v}(s)) = -\infty$. □

Proposition 11. *If* $d_{j1} d_{jj} < 0$ *and* $\gamma_j < \gamma_1 \le \gamma$ *as* $j = 2, \dots, n$, *then* $\overline{u}_i \not\equiv 0$ *as* $i = 2, \dots, n$.

Proof. It is sufficient (Proposition 5) to prove the following.

As $h = 2, \dots, n$ $(\underline{i}^h)$ holds with $\mathcal{F} = \{v \in V_\lambda^- \cap S(D_1) : v_\ell \ge 0 \text{ as } \ell = 1, \dots, n,\ v_1 \not\equiv 0,\ v_h \equiv 0\} \ne \varnothing$.

Let, for example, $h = 2$. Let $v \in \mathcal{F}$. Firstly, we have

$$\int_\Omega \rho_1 v_1^{q_{11}} dx < 0. \tag{50}$$

Let us consider the function $g_1(s, \tau) = D_1(sv_1, \tau v_1, \dots, sv_n) = s^{q_{11}} (\int_\Omega \rho_1 v_1^{q_{11}} dx) \int_\Omega (\tau^{\gamma_1} v_1^{\gamma_1} + s^{\gamma_1} \sum_{\ell \ne 2} v_\ell^{\gamma_1})^{q_{12}/\gamma_1} dx \forall s, \tau \ge 0$. Since

$$g_1(1,0) = -1, \quad \frac{\partial g_1}{\partial \tau}(s,\tau) < 0 \quad \forall s > 0,\ \forall \tau > 0, \quad g_1(s,0) = -s^{q_1} > -1 \quad \forall s \in]0,1[, \quad \lim_{\tau \to +\infty} g_1(s,\tau) = -\infty \quad \forall s > 0, \tag{51}$$

we have the following:

$$\begin{aligned} &\text{only one } \tau(s) \geq 0 \\ &\qquad (\tau(s) > 0 \text{ if } s < 1,\ \tau(1) = 0) \text{ exists such that} \\ &g_1(s, \tau(s)) = -1, \\ &\qquad\qquad \forall s \in\ ]0,1], \end{aligned} \tag{52}$$

the function $\tau(s)$ belongs to $C^0(]0,1]) \cap C^1(]0,1[)$, $\tau'(s) = -(1/(\tau(s))^{\gamma_1 - 1})\tilde{g}_1(s,\tau(s))$, and

$$\lim_{s\to 1^-} \tilde{g}_1(s,\tau(s)) \in\ ]0,+\infty[, \quad \forall s \in\ ]0,1[. \tag{53}$$

Then, set $\tilde{v}(s) = (sv_1, \tau(s)v_1, \ldots, sv_n)$; we get

$$\begin{aligned} &\tilde{v}(1) = v, \\ &D_1(\tilde{v}(s)) = -1 \quad \forall s \in\ ]0,1], \\ &\lim_{s\to 1^-} \frac{d}{ds} H_\lambda(\tilde{v}(s)) \in R, \\ &\lim_{s\to 1^-} \frac{d}{ds} D_2(\tilde{v}(s)) = -\infty, \\ &\lim_{s\to 1^-} \frac{d}{ds} D_j(\tilde{v}(s)) \in R \quad \text{as } j > 2 \ (\text{if } n > 2); \end{aligned} \tag{54}$$

in particular, $\lim_{s\to 1^-}[(t(\tilde{v}(s)))^p (d/ds) H_\lambda(\tilde{v}(s)) - \sum_{j=1}^n (t(\tilde{v}(s)))^{q_j}(d/ds) D_j(\tilde{v}(s))] = +\infty$. □

Application 2. Let $n > 2$ and, for each $v = (v_1, \ldots, v_n) \in W$,

$$\begin{aligned} A(v) &= p^{-1} \sum_{\ell=1}^n \int_\Omega |\nabla v_\ell|^p\, dx, \\ D_1(v) &= q_1^{-1} \left[\int_\Omega \left(d_1 |v_1|^{\gamma_1} + d \sum_{\ell=2}^n |v_\ell|^{\gamma_1} \right)^{q_1/\gamma_1} dx \right. \\ &\quad \left. - \int_\Omega \tilde{d} \left(\sum_{\ell=2}^n |v_\ell|^{\gamma_1} \right)^{q_1/\gamma_1} dx \right], \\ D_j(v) &= q_j^{-1} \sum_{\substack{\ell=1 \\ \ell\neq j}}^n \int_\Omega \rho_j \left| d_{jj} |v_j|^{\gamma_j} + d_{j\ell} |v_\ell|^{\gamma_j} \right|^{q_j/\gamma_j} dx \\ &\qquad\qquad \text{as } j = 2, \ldots, n, \end{aligned} \tag{55}$$

where

$$\begin{aligned} &1 < \gamma_j < q_j \quad \text{as } j = 1, \ldots, n, \\ &q_1 < \cdots < q_n, \\ &q_1 < p, \\ &q_n < \tilde{p}; \\ &d_1, d, \tilde{d} \in L^\infty(\Omega), \quad \text{with } d_1, d > 0,\ d^{q_1/\gamma_1} < \tilde{d}; \\ &\rho_j \in L^\infty(\Omega), \quad \text{with } \rho_j < 0, \\ &d_{j\ell} \in L^\infty(\Omega) \setminus \{0\} \\ &\qquad\qquad \text{as } j = 2, \ldots, n,\ \ell = 1, \ldots, n. \end{aligned} \tag{56}$$

Let us consider the system:

$$\begin{aligned} &-\operatorname{div}\left[|\nabla u_1|^{p-2} \nabla u_1\right] = \lambda_1 b_1 |u_1|^{p-2} u_1 + d_1 \left(d_1 |u_1|^{\gamma_1} \right. \\ &\quad \left. + d \sum_{\ell=2}^n |u_\ell|^{\gamma_1} \right)^{(q_1/\gamma_1)-1} |u_1|^{\gamma_1 - 2} u_1 + \sum_{j=2}^n \rho_j \left| d_{jj} |u_j|^{\gamma_j} \right. \\ &\quad \left. + d_{j1} |u_1|^{\gamma_j} \right|^{(q_j/\gamma_j)-2} \left(d_{jj} |u_j|^{\gamma_j} \right. \\ &\quad \left. + d_{j1} |u_1|^{\gamma_j} \right) d_{j1} |u_1|^{\gamma_j - 2} u_1 \\ &\qquad\qquad \text{in } \Omega, \\ &-\operatorname{div}\left[|\nabla u_i|^{p-2} \nabla u_i\right] = \lambda_i b_i |u_i|^{p-2} u_i + d \left(d_1 |u_1|^{\gamma_1} \right. \\ &\quad \left. + d \sum_{\ell=2}^n |u_\ell|^{\gamma_1} \right)^{q_1/\gamma_1 - 1} |u_i|^{\gamma_1 - 2} u_i - \tilde{d} \left(\sum_{\ell=2}^n |u_\ell|^{\gamma_1} \right)^{q_1/\gamma_1 - 1} \\ &\quad \cdot |u_i|^{\gamma_1 - 2} u_i + \sum_{\substack{\ell=1 \\ \ell\neq i}}^n \rho_i \left| d_{ii} |u_i|^{\gamma_i} + d_{i\ell} |u_\ell|^{\gamma_i} \right|^{(q_i/\gamma_i)-2} \\ &\quad \cdot \left(d_{ii} |u_i|^{\gamma_i} + d_{i\ell} |u_\ell|^{\gamma_i} \right) d_{ii} |u_i|^{\gamma_i - 2} u_i \\ &\quad + \sum_{\substack{j=2 \\ j\neq i}}^n \rho_j \left| d_{jj} |u_j|^{\gamma_j} + d_{ji} |u_i|^{\gamma_j} \right|^{(q_j/\gamma_j)-2} \left(d_{jj} |u_j|^{\gamma_j} \right. \\ &\quad \left. + d_{ji} |u_i|^{\gamma_j} \right) d_{ji} |u_i|^{\gamma_j - 2} u_i \\ &\qquad\qquad \text{in } \Omega \text{ as } i = 2, \ldots, n, \\ &u_i = 0 \quad \text{on } \partial\Omega \text{ as } i = 1, \ldots, n. \end{aligned} \tag{57}$$

Evidently, $V^+(D_1) \neq \varnothing$ and $D_1(0, c_2 u_2^*, \ldots, c_n u_n^*) < 0$ $\forall (c_2, \ldots, c_n) \in R^{n-1} \setminus \{0\}$. Then, set I^* $(\neq \varnothing) \subseteq \{2, \ldots, n\}$; we have (Propositions 7 and 8)

$$\text{with } \lambda_\ell < \lambda_\ell^* \quad \text{as } \ell \in I \setminus I^* \ \exists \delta_1^* > 0 : (i_{14}) \ \text{holds if } \lambda_\ell < \lambda_\ell^* + \delta_1^* \ \forall \ell \in I^*, \tag{58}$$

$$\text{with } \lambda_\ell < \lambda_\ell^* \quad \text{as } \ell \in I \setminus I^* \ \exists \delta_2^* > 0 : (i_{15}) \ \text{holds if } \lambda_\ell \in [\lambda_\ell^*, \lambda_\ell^* + \delta_2^*[\ \ \forall \ell \in I^*, \ \lambda_\ell > \lambda_\ell^* \text{ as some } \ell. \tag{59}$$

Proposition 12 (see Theorems 1 and 4, Remark 6). *Under conditions (56), one has the following:*

With $\lambda_1, \ldots, \lambda_n$ as in (58), system (57) has at least two weak solutions u^0 and $-u^0$ ($u^0 = \tau^0 v^0$, $\tau^0 = const. > 0$, $v^0 \in S_\lambda \cap V^+(D_1)$), and one has $u_i^0 \geq 0$ as $i = 1, \ldots, n$, $u_1^0 \not\equiv 0$.

When $q_n < p$, with $\lambda_1, \ldots, \lambda_n$ as in (59), system (57) has at least two weak solutions $\overline{u}$ and $-\overline{u}$ ($\overline{u} = \overline{\tau}\overline{v}$, $\overline{\tau} = const. > 0$, $\overline{v} \in V_\lambda^- \cap S(D_1)$), and one has $\overline{u}_i \geq 0$ as $i = 1, \ldots, n$.

Consequently, when $q_n < p$, with $\lambda_\ell < \lambda_\ell^*$ as $\ell \in I \setminus I^*$, $\lambda_\ell \in [\lambda_\ell^*, \lambda_\ell^* + \min\{\delta_1^*, \delta_2^*\}[$ $\forall \ell \in I^*$, and $\lambda_\ell > \lambda_\ell^*$ as some ℓ, system (57) has at least four different weak solutions.

Proposition 13. *Let $d_{jj} d_{j\ell} < 0$ as $j = 2, \ldots, n$ and $\ell = 1, \ldots, n$ with $\ell \neq j$. Let $\gamma_j < \gamma_1$ as $j = 2, \ldots, n$. Then,*

$$\begin{aligned} &u_i^0 \not\equiv 0 \quad \text{as } i = 2, \ldots, n; \\ &\overline{u}_i \not\equiv 0 \quad \text{as } i = 1, \ldots, n. \end{aligned} \tag{60}$$

Proof. It is sufficient (Propositions 2 and 5) to prove that

$$\text{As } h = 2, \ldots, n \ \left(i_0^h\right) \ \text{holds with } \mathcal{F} = \{v \in S_\lambda \cap V^+(D_1) : v_\ell \geq 0 \text{ as } \ell = 1, \ldots, n, \ v_1 \not\equiv 0, \ v_h \equiv 0\} \neq \varnothing; \tag{61}$$

$$\left(\underline{i}^1\right) \ \text{holds with } \mathcal{F} = \{v \in V_\lambda^- \cap S(D_1) : v_\ell \geq 0 \text{ as } \ell = 1, \ldots, n, \ v_1 \equiv 0\} \neq \varnothing; \tag{62}$$

$$\text{As } h = 2, \ldots, n \ \left(\underline{i}^h\right) \ \text{holds with } \mathcal{F} = \{v \in V_\lambda^- \cap S(D_1) : v_\ell \geq 0 \text{ as } \ell = 1, \ldots, n, \ v_1 \not\equiv 0, \ v_h \equiv 0\}, \quad \text{if } \mathcal{F} \neq \varnothing. \tag{63}$$

About (61), let $v \in \mathcal{F}$. Let $K \subseteq \Omega$ be a compact set such that $|K|_N > 0$ and $v_1 > 0$ in K. Let [11, Proposition A.1] $(\varphi_\varepsilon)_{0<\varepsilon<\varepsilon_0} \subseteq C_0^\infty(\Omega)$ with $0 \leq \varphi_\varepsilon \leq 1$ such that

$$\begin{aligned} &\varphi_\varepsilon \longrightarrow \chi \quad \text{strongly in } L^s(\Omega), \\ &\int_\Omega |\nabla \varphi_\varepsilon|^s \, dx \longrightarrow +\infty \\ &\qquad \text{as } \varepsilon \longrightarrow 0^+ \ \forall s \in [1, +\infty[, \end{aligned} \tag{64}$$

where χ is the characteristic function of K. We choose ε such that

$$\begin{aligned} &\delta = p^{-1} \left[\int_\Omega |\nabla \varphi_\varepsilon|^p \, dx - \lambda_h \int_\Omega b_h \varphi_\varepsilon^p dx \right] > 0, \\ &\int_\Omega \rho_h |d_{h1}|^{(q_h/\gamma_h)-2} v_1^{q_h - \gamma_h} d_{h1} d_{hh} \varphi_\varepsilon^{\gamma_h} dx > 0 \end{aligned} \tag{65}$$

and let us set $\widetilde{v}_h = \delta^{-1/p} \varphi_\varepsilon$. With $\widetilde{v}(s) = (s^{1/p} v_1, \ldots, (1-s)^{1/p} \widetilde{v}_h, \ldots, s^{1/p} v_n)$, we have

$$\begin{aligned} &\widetilde{v}(1) = v, \\ &\widetilde{v}(s) \in S_\lambda \quad \forall s \in [0, 1]. \end{aligned} \tag{66}$$

It is easy to verify that

$$\begin{aligned} &\frac{d}{ds} D_1(\widetilde{v}(s)) = -(1-s)^{\gamma_1/p - 1} f_1(s) + \widetilde{f}_1(s) \\ &\quad \forall s \in [0, 1[\ \text{with} \ \lim_{s \to 1^-} f_1(s) \in R, \ \lim_{s \to 1^-} \widetilde{f}_1(s) \in R, \\ &\frac{d}{ds} D_h(\widetilde{v}(s)) \sim -c_h (1-s)^{\gamma_h/p - 1} \\ &\qquad \text{as } s \longrightarrow 1^- \ (c_h = \text{const.} > 0), \\ &\frac{d}{ds} D_j(\widetilde{v}(s)) = c_j (1-s)^{(q_j \backslash p) - 1} + \widetilde{f}_j(s) \\ &\quad \text{as } j \in \{2, \ldots, n\} \setminus \{h\} \ \forall s \in [0, 1[\ \text{with } c_j = \text{const.} > 0, \\ &\quad \lim_{s \to 1^-} \widetilde{f}_j(s) \in R \quad \text{if } v_j \widetilde{v}_h \equiv 0, \\ &\lim_{s \to 1^-} \frac{d}{ds} D_j(\widetilde{v}(s)) = -\infty \quad \text{if } v_j \widetilde{v}_h \not\equiv 0. \end{aligned} \tag{67}$$

Then, $\lim_{s \to 1^-} \sum_{j=1}^n (t(\widetilde{v}(s)))^{q_j} (d/ds) D_j(\widetilde{v}(s)) = -\infty$.

About (62), let $v \in \mathcal{F}$. Set

$$\begin{aligned} g_1(s, \tau) &= D_1(\tau u_1^*, s v_2, \ldots, s v_n) \\ &= q_1^{-1} \left[\int_\Omega \left(d_1 \tau^{\gamma_1} (u_1^*)^{\gamma_1} + d s^{\gamma_1} \left(\sum_{\ell=2}^n v_\ell^{\gamma_1} \right) \right)^{q_1 \backslash \gamma_1} dx \right. \\ &\quad \left. - \int_\Omega \widetilde{d} s^{q_1} \left(\sum_{\ell=2}^n v_\ell^{\gamma_1} \right)^{q_1 \backslash \gamma_1} dx \right] \quad \forall s, \tau \geq 0; \end{aligned} \tag{68}$$

we have

$$g_1(1,0) = -1,$$
$$\frac{\partial g_1}{\partial \tau}(s,\tau) > 0 \quad \forall s \geq 0,\ \forall \tau > 0,$$
$$g_1(s,0) = -s^{q_1} < -1 \quad \forall s > 1,$$
$$\lim_{\tau \to +\infty} g_1(s,\tau) = +\infty \quad \forall s \geq 0. \tag{69}$$

Relations (69) imply that

$$\text{only one } \tau(s) > 0 \ (\tau(s) > 0 \text{ if } s > 1,\ \tau(1) = 0) \text{ exists such that } g_1(s,\tau(s)) = -1, \quad \forall s \geq 1 \tag{70}$$

and the function $\tau(s)$ belongs to $C^0([1,+\infty[) \cap C^1(]1,+\infty[)$ and it results in

$$\tau'(s) = -\frac{1}{(\tau(s))^{\gamma_1 - 1}} \tilde{g}_1(s,\tau(s)) \quad \forall s > 1,\ \lim_{s \to 1^+} \tilde{g}_1(s,\tau(s)) \in]-\infty, 0[. \tag{71}$$

Then, with $\tilde{v}(s) = (\tau(s)u_1^*, sv_2, \ldots, sv_n)$, we have

$$\tilde{v}(1) = v,$$
$$\tilde{v}(s) \in S(D_1) \quad \forall s \geq 1; \tag{72}$$

moreover,

$$\lim_{s \to 1^+} \frac{d}{ds} H_\lambda(\tilde{v}(s)) \in]-\infty, 0[,$$
$$\lim_{s \to 1^+} \frac{d}{ds} D_j(\tilde{v}(s)) = +\infty \quad \text{if } v_j \not\equiv 0,$$
$$\lim_{s \to 1^+} \frac{d}{ds} D_j(\tilde{v}(s)) \in R \quad \text{if } v_j \equiv 0, \quad \text{as } j = 2, \ldots, n \tag{73}$$

from which $\lim_{s \to 1^+}[(t(\tilde{v}(s)))^p (d/ds) H_\lambda(\tilde{v}(s)) - \sum_{j=1}^n (t(\tilde{v}(s)))^{q_j} (d/ds) D_j(\tilde{v}(s))] = -\infty$ since $v_j \not\equiv 0$ as some $j \geq 2$.

Finally, let us verify (63), for example, when $h = 2$. Let $v \in \mathcal{F}$. Since

$$v_1 \not\equiv 0 \Longrightarrow$$
$$v_j \not\equiv 0 \quad \text{as some } j > 2, \tag{74}$$

we can suppose $v_3 \sim 0$. Set $\tilde{v}(s) = (v_1, (1-s)^{1/\gamma_1} v_3, s^{1/\gamma_1} v_3, \ldots, v_n)$; we have

$$\tilde{v}(1) = v,$$
$$\tilde{v}(s) \in S(D_1) \quad \forall s \in [0,1]; \tag{75}$$

we add that

$$\lim_{s \to 1^-} \frac{d}{ds} H_\lambda(\tilde{v}(s)) \in R,$$
$$\lim_{s \to 1^-} \frac{d}{ds} D_2(\tilde{v}(s)) = -\infty,$$
$$\lim_{s \to 1^-} \frac{d}{ds} D_3(\tilde{v}(s)) = -\infty,$$
$$\lim_{s \to 1^-} \frac{d}{ds} D_j(\tilde{v}(s)) < +\infty, \quad \text{as } j > 3 \ (\text{if } n > 3) \tag{76}$$

from which $\lim_{s \to 1^-}[(t(\tilde{v}(s)))^p (d/ds) H_\lambda(\tilde{v}(s)) - \sum_{j=1}^n (t(\tilde{v}(s)))^{q_j} (d/ds) D_j(\tilde{v}(s))] = +\infty$. □

4. Neumann Problems

We assume $W = (W^{1,p}(\Omega))^n$ ($n \geq 1$, $1 < p < \infty$, $\Omega \subseteq R^N$ is an open, bounded, connected and $C^{0,1}$ set, $\|v\| = (\sum_{\ell=1}^n \int_\Omega [|\nabla v_\ell|^p + |v_\ell|^p] dx)^{1/p}$ $\forall v = (v_1, \ldots, v_n) \in W$), $B_\ell(v_\ell) = p^{-1} \int_\Omega b_\ell |v_\ell|^p dx$ $\forall v_\ell \in W^{1,p}(\Omega)$ with $b_\ell \in L^\infty(\Omega) \setminus \{0\}$, and $\widehat{B}_\ell(v_\ell) = p^{-1} \int_{\partial\Omega} \widehat{b}_\ell |v_\ell|^p d\sigma$ $\forall v_\ell \in W^{1,p}(\Omega)$ with $\widehat{b}_\ell \in L^\infty(\partial\Omega) \setminus \{0\}$ (σ is the measure on $\partial\Omega$, $v_\ell = \gamma_0(v_\ell)$, where $\gamma_0 : W^{1,p}(\Omega) \to W^{(p-1)/p,p}(\partial\Omega)$ is the trace operator).

Let us consider the functional A (as in (i_{11})) such that $A(v) \geq p^{-1} \tilde{c} \sum_{\ell=1}^n \int_\Omega |\nabla v_\ell|^p dx$ $\forall v \in W$ ($\tilde{c} = \text{const.} > 0$). Moreover, $|\cdot|_N$ and $\tilde{p}$ have been settled as in Section 3, ν is the outward orthogonal unitary vector to $\partial\Omega$, $\widehat{p} = (N-1)p/(N-p)$ if $p < N$, and $\widehat{p} = \infty$ if $p \geq N$.

About the validity of assumptions (i_{13})–(i_{15}), we set $C^* = \{c = (c_1, \ldots, c_n) \in R^n : c_\ell = 0 \text{ if } \ell \in I \setminus I^*, c_\ell \neq 0 \text{ as some } \ell \in I^*\}$ ($I = \{1, \ldots, n\}$, $I^* (\neq \varnothing) \subseteq I$) and we introduce the following assumptions:

(i_4) $I^* \subseteq I$ exists: $D_1(c) < 0$ for every $c \in C^*$.

$(\underline{i}_4)$ $I^* \subseteq I$ exists: $D_1(c) < 0$ and $A(c) = 0$ for every $c \in C^*$.

Proposition 14. *When $b_\ell \geq 0$ and $\widehat{b}_\ell \geq 0$ as $\ell \in I$, then (i_{13}) holds if $\lambda_\ell \leq 0$, $\mu_\ell \leq 0$, and $\lambda_\ell + \mu_\ell < 0$ as $\ell \in I$. When (i_4) holds with $I^* \neq I$, $b_\ell \geq 0$ and $\widehat{b}_\ell \geq 0$ as $\ell \in I \setminus I^*$, $V^+(D_1) \neq \varnothing$, then with $\lambda_\ell \leq 0$, $\mu_\ell \leq 0$ and $\lambda_\ell + \mu_\ell < 0$ as $\ell \in I \setminus I^*$ $\exists \delta^* > 0 : (i_{14})$ holds if $|\lambda_\ell|, |\mu_\ell| \leq \delta^*$ $\forall \ell \in I^*$. When (i_4) holds with $I^* = I$ and $V^+(D_1) \neq \varnothing$, then $\exists \delta^* > 0 : (i_{14})$ holds if $|\lambda_\ell|, |\mu_\ell| \leq \delta^*$ $\forall \ell \in I$.*

Proof. We reason by contradiction as in [11, Propositions 3.2 and 3.3]. □

Proposition 15. *When $(\underline{i}_4)$ holds with $I^* \neq I$, $b_\ell \geq 0$ and $\widehat{b}_\ell \geq 0$ as $\ell \in I \setminus I^*$, $\int_\Omega b_\ell dx > 0$ and $\int_{\partial\Omega} \widehat{b}_\ell d\sigma > 0$ as $\ell \in I^*$, then with $\lambda_\ell \leq 0$, $\mu_\ell \leq 0$, and $\lambda_\ell + \mu_\ell < 0$ as $\ell \in I \setminus I^*$ $\exists \delta^* > 0 : (i_{15})$ holds if $\lambda_\ell, \mu_\ell \in [0, \delta^*]$ $\forall \ell \in I^*$ and $\lambda_\ell + \mu_\ell > 0$ for some ℓ. When $(\underline{i}_4)$ holds with $I^* = I$, $\int_\Omega b_\ell dx > 0$ and $\int_{\partial\Omega} \widehat{b}_\ell d\sigma > 0$ as $\ell \in I$, then $\exists \delta^* > 0 : (i_{15})$ holds if $\lambda_\ell, \mu_\ell \in [0, \delta^*]$ $\forall \ell \in I$ and $\lambda_\ell + \mu_\ell > 0$ for some ℓ.*

Proof. See [11, Propositions 3.4 and 3.5]. □

Passing to the applications, we state in advance that the results of [8] for problem

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)=b(x)u^{s-1}\quad\text{in }\Omega,$$
$$|\nabla u|^{p-2}\frac{\partial u}{\partial\nu}=a(x)u^{q-1}\quad\text{on }\partial\Omega \tag{77}$$

can be obtained by using Theorem 1 and Proposition 14, Theorems 2.1 and 2.2 of [10], and Propositions 3.3 and A.4 of [11] (taking into account [17] too).

Application 3. Let $n>1$ and, for each $v=(v_1,\ldots,v_n)\in W$. We set

$$A(v)=p^{-1}\sum_{\ell=1}^{n}\int_{\Omega}|\nabla v_\ell|^p\,dx,$$
$$D_1(v)=q_1^{-1}\left[\int_{\partial\Omega}\left(\sum_{\ell=1}^{n}d_\ell|v_\ell|^{\gamma}\right)^{q_1/\gamma}d\sigma-\sum_{\ell=1}^{n}\int_{\partial\Omega}\widehat{d}_\ell|v_\ell|^{q_1}\,d\sigma\right], \tag{78}$$
$$D_2(v)=-\prod_{\ell=1}^{n}\left|\int_{\partial\Omega}\widehat{\rho}_\ell|v_\ell|^{\gamma_\ell}\,d\sigma\right|^{\gamma'_\ell/\gamma_\ell},$$

where

$$1<\gamma<q_1<p,$$
$$1<\gamma_\ell<\gamma'_\ell,\quad \gamma_\ell<\widehat{p},$$
$$q_1<q_2=\sum_{\ell=1}^{n}\gamma'_\ell; \tag{79}$$
$$d_\ell,\widehat{d}_\ell\in L^\infty(\partial\Omega),\quad d_\ell,\widehat{d}_\ell>0;$$
$$\widehat{\rho}_\ell\in L^\infty(\partial\Omega)\setminus\{0\}.$$

Let us consider the system:

$$-\operatorname{div}\left(|\nabla u_i|^{p-2}\nabla u_i\right)=\lambda_i b_i|u_i|^{p-2}u_i\quad\text{in }\Omega,$$
$$|\nabla u_i|^{p-2}\frac{\partial u_i}{\partial\nu}=\mu_i\widehat{b}_i|u_i|^{p-2}u_i+\left(\sum_{\ell=1}^{n}d_\ell|u_\ell|^{\gamma}\right)^{q_1/\gamma-1}d_i|u_i|^{\gamma-2}u_i-\widehat{d}_i|u_i|^{q_1-2}u_i$$
$$-\left(\prod_{\ell\neq i}\left|\int_{\partial\Omega}\widehat{\rho}_\ell|u_\ell|^{\gamma_\ell}\,d\sigma\right|^{\gamma'_\ell/\gamma_\ell}\right)\cdot\gamma'_i\left|\int_{\partial\Omega}\widehat{\rho}_i|u_i|^{\gamma_i}\,d\sigma\right|^{\gamma'_i/\gamma_i-2}\left(\int_{\partial\Omega}\widehat{\rho}_i|u_i|^{\gamma_i}\,d\sigma\right)\cdot\widehat{\rho}_i|u_i|^{\gamma_i-2}u_i\quad\text{on }\partial\Omega \tag{80}$$

as $i=1,\ldots,n$.

Proposition 16. *Let $u=(u_1,\ldots,u_n)$ be a weak solution of system (80). Then,*

$$\lambda_i\neq0,\quad u_i\not\equiv0\quad\text{for some } i\in\{1,\ldots,n\}\Longrightarrow u_i\text{ is not constant}; \tag{81}$$

$$u_i\geq0\quad\forall i\in\{1,\ldots,n\}\Longrightarrow u_i\in L^\infty(\Omega)\cap C^{1,\alpha_i}_{\ell oc}(\Omega)\quad\forall i\in\{1,\ldots,n\},\quad u_i>0\quad\text{if } u_i\not\equiv0. \tag{82}$$

Proof. Relation (81) is evident. About (82), it is easy to verify that

$$\sum_{i=1}^{n}\int_{\Omega}|\nabla u_i|^{p-2}\nabla u_i\cdot\nabla v_i dx\leq c_1\int_{\Omega}\left(\sum_{i=1}^{n}u_i+1\right)^{p-1}\cdot\left(\sum_{i=1}^{n}v_i\right)dx+\int_{\partial\Omega}\left[c_2+c_3\sum_{\ell=1}^{n}(u_\ell+1)^{\gamma_0-p}\right]\cdot\left(\sum_{i=1}^{n}u_i+1\right)^{p-1}\left(\sum_{i=1}^{n}v_i\right)d\sigma$$
$$\forall v=(v_1,\ldots,v_n)\in\left(W^{1,p}(\Omega)\cap L^\infty(\Omega)\right)^n\text{ with } v_i\geq0, \tag{83}$$

where c_1–c_3 are positive constants and $\gamma_0=\max\{\gamma_1,\ldots,\gamma_n\}$. Consequently [11, Proposition A.4], $u_i\in L^\infty(\Omega)$ from which, since $-\operatorname{div}(|\nabla u_i|^{p-2}\nabla u_i)=\lambda_i b_i|u_i|^{p-2}u_i$ in Ω, we deduce that $u_i\in C^{1,\alpha_i}_{\mathrm{loc}}(\Omega)$ [17] and $u_i>0$ if $u_i\not\equiv0$ [18]. □

Let us introduce the conditions:

$$d_1^{q_1/\gamma}>\widehat{d}_1\ (\Longrightarrow V^+(D_1)\neq\varnothing), \tag{84}$$

$$\left(\sum_{\ell=2}^{n}d_\ell\right)^{q_1/\gamma}<\min\{\widehat{d}_2,\ldots,\widehat{d}_n\}\quad\left(\Longrightarrow D_1(0,c_2,\ldots,c_n)<0\ \forall(c_2,\ldots,c_n)\in R^{n-1}\setminus\{0\}\right), \tag{85}$$

$$b_1\geq0,\quad \widehat{b}_1\geq0, \tag{86}$$

$$\int_{\Omega}b_\ell dx>0,\quad \int_{\partial\Omega}\widehat{b}_\ell d\sigma>0 \tag{87}$$

as $\ell=2,\ldots,n$.

Then (Propositions 14 and 15),

$$(84)-(86) \Longrightarrow \Big(\text{with } \lambda_1 \le 0,\ \mu_1 \le 0,\ \lambda_1+\mu_1<0\ \exists\delta_1^* > 0 : (i_{14}) \text{ holds if } |\lambda_\ell|, |\mu_\ell| \le \delta_1^*\ \forall \ell \in \{2,\dots,n\}\Big), \tag{88}$$

$$(85)-(87) \Longrightarrow \Big(\text{with } \lambda_1 \le 0,\ \mu_1 \le 0 \text{ and } \lambda_1+\mu_1<0\ \exists\delta_2^* > 0 : (i_{15}) \text{ holds if } \lambda_\ell, \mu_\ell \in [0,\delta_2^*]\ \forall \ell \in \{2,\dots,n\} \text{ and } \lambda_\ell+\mu_\ell>0 \text{ as some } \ell\Big). \tag{89}$$

Proposition 17 (Theorems 1 and 4, Remark 6, Proposition 16). *Under conditions (79), one has the following:*

When (84)–(86) hold, with λ_ℓ, μ_ℓ as in (88), system (80) has at least two weak solutions u^0 and $-u^0$ ($u^0 = \tau^0 v^0$, $\tau^0 = const. > 0$, $v^0 \in S_{\lambda\mu} \cap V^+(D_1)$), and one has $u_i^0 \in L^\infty(\Omega) \cap C^{1,\alpha_i^0}_{\ell oc}(\Omega)$, $u_i^0 \ge 0$ as $i = 1,\dots,n$, $u_i^0 > 0$ if $u_i^0 \not\equiv 0$.

When $q_2 < p$ and (85)–(87) hold, with λ_ℓ, μ_ℓ as in (89), system (80) has at least two weak solutions $\overline{u}$ and $-\overline{u}$ ($\overline{u} = \overline{\tau}\,\overline{v}$, $\overline{\tau} = const. > 0$, $\overline{v} \in V^-_{\lambda\mu} \cap S(D_1)$), and one has $\overline{u}_i \in L^\infty(\Omega) \cap C^{1,\overline{\alpha}_i}_{\ell oc}(\Omega)$, $\overline{u}_i \ge 0$ as $i = 1,\dots,n$, $\overline{u}_i > 0$ if $\overline{u}_i \sim 0$.

Consequently, when $q_2 < p$ and (84)–(87) hold, with λ_ℓ, μ_ℓ as in (89) and $\min\{\delta_1^*, \delta_2^*\}$ instead of δ_2^*, system (80) has at least four different weak solutions.

Proposition 18. *One gets the following:*

$$u_1^0 \not\equiv 0 \quad on\ \partial\Omega; \tag{90}$$

$$either\ \gamma_h' \ge p \ \ or\ \gamma \le \gamma_h' \Longrightarrow u_h^0 \not\equiv 0 \quad as\ some\ h \in \{2,\dots,n\}. \tag{91}$$

Proof. Relation (90) is evident. About (91), it is sufficient (Proposition 2) to verify that

(i_0^h) holds with $\mathcal{F} = \{v \in S_{\lambda\mu} \cap V^+(D_1) : v_\ell \ge 0$ as $\ell = 1,\dots,n,\ v_h \equiv 0\} \neq \varnothing$.

Let us check, for example, the case $h = 2$. Let $v \in \mathcal{F}$; then, $\Gamma \subseteq \partial\Omega$ exists such that $\sigma(\Gamma) > 0$ and $\sum_{\ell\neq 2} d_\ell v_\ell^\gamma > 0$ on Γ. Let $k \subseteq \Omega$ be a compact set with positive measure and Ω' an open set such that $\overline{\Omega'} \subseteq \Omega$, $k \subseteq \Omega'$. Thanks to [11, Propositions A.1, A.2], a compact set $\widehat{\Gamma} \subseteq \Gamma$ with $\sigma(\widehat{\Gamma}) > 0$, $(\varphi_{1\varepsilon})_{0<\varepsilon<\varepsilon_0} \subseteq C_0^\infty(\Omega)$, and $(\varphi_{2\varepsilon})_{0<\varepsilon<\varepsilon_0} \subseteq C_0^\infty(R^N)$ exist such that

$$\begin{gathered}
0 \le \varphi_{1\varepsilon} \le 1, \\
\operatorname{supp} \varphi_{1\varepsilon} \subseteq \Omega', \\
\varphi_{1\varepsilon} \longrightarrow \chi \text{ strongly in } L^s(\Omega), \\
\int_\Omega |\nabla\varphi_{1\varepsilon}|^s\,dx \longrightarrow +\infty \quad \text{as } \varepsilon \longrightarrow 0^+\ \forall s \in [1,+\infty[, \\
0 \le \varphi_{2\varepsilon} \le 1, \\
\operatorname{supp} \varphi_{2\varepsilon} \subseteq R^N \setminus \overline{\Omega}', \\
\varphi_{2\varepsilon} \longrightarrow \widehat{\chi} \text{ strongly in } L^s(\partial\Omega), \\
\int_{R^N} \varphi_{2\varepsilon}^s\,dx \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0^+\ \forall s \in [1,+\infty[,
\end{gathered} \tag{92}$$

where χ [resp., $\widehat{\chi}$] is the characteristic function of K [resp., $\widehat{\Gamma}$]. Set $\varphi_\varepsilon = \varphi_{1\varepsilon} + \varphi_{2\varepsilon}$; let us fix ε such that

$$\begin{gathered}
\delta = p^{-1}\left[\int_\Omega |\nabla\varphi_\varepsilon|^p\,dx - \lambda_2 \int_\Omega b_2\varphi_\varepsilon^p\,dx - \mu_2 \int_{\partial\Omega} \widehat{b}_2\varphi_\varepsilon^p\,d\sigma\right] > 0, \\
\int_{\partial\Omega} \left(\sum_{\ell\neq 2} d_\ell v_\ell^\gamma\right)^{q_1/\gamma - 1} d_2\varphi_\varepsilon^\gamma\,d\sigma > 0.
\end{gathered} \tag{93}$$

Then, with $\widetilde{v}(s) = (s^{1/p}v_1, (1-s)^{1/p}\delta^{-1/p}\varphi_\varepsilon, \dots, s^{1/p}v_n)$, we have

$$\begin{gathered}
\widetilde{v}(1) = v, \\
\widetilde{v}(s) \in S_{\lambda\mu} \quad \forall s \in [0,1], \\
\frac{d}{ds} D_1(\widetilde{v}(s)) \sim -\widehat{c}_1 (1-s)^{(\gamma/p)-1} \quad \text{as } s \longrightarrow 1^- \ (\widehat{c}_1 = \text{const.} > 0), \\
D_2(\widetilde{v}(s)) = -\widehat{c}_2 s^{(q_2-\gamma_2')/p} (1-s)^{\gamma_2'/p} \quad \forall s \in [0,1] \ (\widehat{c}_2 = \text{const.} \ge 0) \\
\Longrightarrow \lim_{s\to 1^-} \sum_{j=1}^{2} (t(\widetilde{v}(s)))^{q_j} \frac{d}{ds} D_j(\widetilde{v}(s)) = -\infty.
\end{gathered} \tag{94}$$

□

Proposition 19. *Let for some $h \in \{2,\dots,n\}$ either $\gamma_h' \ge p$ or $\gamma \le \gamma_h'$. Let $b_h \ge 0$, $\lambda_h > 0$, and $\mu_h \int_{\partial\Omega} \widehat{b}_h\,d\sigma \ge 0$. Then, $u_h^0 \not\equiv 0$ on $\partial\Omega$.*

Proof. Since in Proposition 18 $u_h^0 > 0$, it is sufficient (Proposition 2) to prove that

(i_0^h) holds with $\mathcal{F} = \{v \in S_{\lambda\mu} \cap V^+(D_1) : v_\ell \geq 0$ as $\ell = 1, \ldots, n,\ v_h > 0,\ v_h \equiv 0$ on $\partial\Omega\} \neq \varnothing$.

Let us suppose, for example, $h = 2$. Let $v \in \mathcal{F}$ and $c_2 = \text{const.} > 0$. Set $g(s,\tau) = H_{\lambda\mu}(sv_1, sv_2 + \tau c_2, \ldots, sv_n) = p^{-1}[s^p \int_\Omega |\nabla v_2|^p dx - \lambda_2 \int_\Omega b_2(sv_2 + \tau c_2)^p dx - \mu_2 \tau^p c_2^p \int_{\partial\Omega} \widehat{b}_2 d\sigma] + s^p H_{\lambda\mu}(v_1, 0, \ldots, v_n)$ $\forall s, \tau \geq 0$; we have

$$\begin{gathered} g(1,0) = 1, \\ \frac{\partial g}{\partial \tau}(s,\tau) < 0 \quad \forall s \geq 0,\ \forall \tau > 0, \\ g(s,0) = s^p > 1 \quad \forall s > 1, \\ \lim_{\tau \to +\infty} g(s,\tau) = -\infty \quad \forall s \geq 0; \end{gathered} \tag{95}$$

then, $\tau : [1, +\infty[\to [0, +\infty[$ belonging to $C^0([1,+\infty[) \cap C^1(]1,+\infty[)$ exists such that $g(s,\tau(s)) = 1$ $\forall s \geq 1$, $\tau(s) > 0$ if $s > 1$, $\tau(1) = 0$, and $\lim_{s \to 1^+} \tau'(s) \in]0, +\infty[$. Consequently, set $\widetilde{v}(s) = (sv_1, sv_2 + \tau(s)c_2, \ldots, sv_n)$; it results in

$$\begin{gathered} \widetilde{v}(1) = v, \\ \widetilde{v}(s) \in S_{\lambda\mu} \quad \forall s \geq 1, \\ \lim_{s \to 1^+} \frac{d}{ds} D_1(\widetilde{v}(s)) = q_1 D_1(v) > 0, \\ \lim_{s \to 1^+} \frac{d}{ds} D_2(\widetilde{v}(s)) = 0 \Longrightarrow \\ \lim_{s \to 1^+} \sum_{j=1}^{2} (t(\widetilde{v}(s)))^{q_j} \frac{d}{ds} D_j(\widetilde{v}(s)) \in]0, +\infty[. \end{gathered} \tag{96}$$

□

Proposition 20. *If either $\gamma_1' > \gamma$ or $\int_{\partial\Omega} \widehat{\rho}_1 d\sigma = 0$, it results in*

$$\overline{u}_1 \not\equiv 0; \tag{97}$$

$$\overline{u}_1 \not\equiv 0 \quad \text{on } \partial\Omega \text{ when } \lambda_1 = 0. \tag{98}$$

Proof. About (97) [resp., (98)], it is sufficient (Proposition 5) to prove that

$(\underline{i}^1)$ holds with $\mathcal{F} = \{v \in V_{\lambda\mu}^- \cap S(D_1) : v_\ell \geq 0$ as $\ell = 1, \ldots, n,\ v_1 \equiv 0$ [resp., $v_1 \equiv 0$ on $\partial\Omega$]$\} \neq \varnothing$.

Let $v \in \mathcal{F}$ and $c_1 = \text{const.} > 0$. Let $g_1(s,\tau) = D_1(\tau c_1, sv_2, \ldots, sv_n) = q_1^{-1}[\int_{\partial\Omega}(d_1 \tau^\gamma c_1^\gamma + s^\gamma \sum_{\ell \neq 1} d_\ell v_\ell^\gamma)^{q_1/\gamma} d\sigma - \tau^{q_1} c_1^{q_1} \int_{\partial\Omega} \widehat{d}_1 d\sigma - s^{q_1} \sum_{\ell \neq 1} \int_{\partial\Omega} \widehat{d}_\ell v_\ell^{q_1} d\sigma]$ $\forall s \geq 0$ and $\forall \tau \geq 0$. We note that

$$\begin{gathered} g_1(1,0) = -1, \\ \frac{\partial g_1}{\partial \tau}(s,\tau) > 0 \quad \forall s \geq 0,\ \forall \tau > 0, \\ g_1(s,0) = -s^{q_1} < -1 \quad \forall s > 1, \\ \lim_{\tau \to +\infty} g_1(s,\tau) = +\infty \quad \forall s \geq 0. \end{gathered} \tag{99}$$

Then, $\tau : [1, +\infty[\to [0, +\infty[$ ($\tau(s) > 0$ if $s > 1$, $\tau(1) = 0$) belonging to $C^0([1,+\infty[) \cap C^1(]1,+\infty[)$ exists such that

$$\begin{gathered} g_1(s,\tau(s)) = -1 \quad \forall s \geq 1, \\ \tau'(s) = -\frac{1}{(\tau(s))^{\gamma-1}} \widetilde{g}_1(s,\tau(s)) \quad \forall s > 1, \\ \lim_{s \to 1^+} \widetilde{g}_1(s,\tau(s)) \in]-\infty, 0[, \end{gathered} \tag{100}$$

from which with $\widetilde{v}(s) = (\tau(s)c_1, sv_2, \ldots, sv_n)$ [resp.,$\widetilde{v}(s) = (sv_1 + \tau(s)c_1, sv_2, \ldots, sv_n)$] we get

$$\begin{gathered} \widetilde{v}(1) = v, \\ \widetilde{v}(s) \in S(D_1) \quad \forall s \geq 1, \\ \lim_{s \to 1^+} \frac{d}{ds} D_2(\widetilde{v}(s)) = 0, \\ \lim_{s \to 1^+} (t(\widetilde{v}(s)))^p \frac{d}{ds} H_{\lambda\mu}(\widetilde{v}(s)) \in]-\infty, 0[. \end{gathered} \tag{101}$$

□

Evidently, $\overline{u}_2 \not\equiv 0$ on $\partial\Omega$ if $n = 2$.

Proposition 21. *When $n > 2$ and for some $h \in \{2, \ldots, n\}$ $\int_{\partial\Omega} \widehat{\rho}_h d\sigma = 0$, $b_h \geq 0$, $\mu_h > 0$, then*

$$\overline{u}_h \not\equiv 0 \quad \text{on } \partial\Omega. \tag{102}$$

Proof. Let us prove (102) as $h = 2$. Reasoning by contradiction, let $\overline{u}_2 \equiv 0$ on $\partial\Omega$; that is, $\overline{v}_2 \equiv 0$ on $\partial\Omega$. Let $c_2 = \text{const.} > 0$. As $g_1(s,\tau) = D_1(s\overline{v}_1, \overline{v}_2 + \tau c_2, \ldots, s\overline{v}_n) = q_1^{-1}[\int_{\partial\Omega}(d_2 \tau^\gamma c_2^\gamma + s^\gamma \sum_{\ell \neq 2} d_\ell \overline{v}_\ell^\gamma)^{q_1/\gamma} d\sigma - \tau^{q_1} c_2^{q_1} \int_{\partial\Omega} \widehat{d}_2 d\sigma - s^{q_1} \sum_{\ell \neq 2} \int_{\partial\Omega} \widehat{d}_\ell \overline{v}_\ell^{q_1} d\sigma]$ $\forall s, \tau \geq 0$, we have

$$g_1(s,0) = -s^{q_1} > -1 \quad \forall s \in]0,1[, \tag{103}$$

$$\lim_{\tau \to +\infty} g_1(s,\tau) = -\infty \quad \forall s \geq 0. \tag{104}$$

Thanks to (103), (104), for each $s \in]0,1[$, it is possible to choose $\tau(s) > 0$ such that $g_1(s,\tau(s)) = -1$. Then, set $\widetilde{v}(s) = (s\overline{v}_1, \overline{v}_2 + \tau(s)c_2, \ldots, s\overline{v}_n)$; it results in

$$D_1(\widetilde{v}(s)) = -1 \quad \forall s \in]0,1[; \tag{105}$$

moreover, since

$$\begin{aligned} H_{\lambda\mu}(\widetilde{v}(s)) &= p^{-1}\left[\int_\Omega |\nabla \overline{v}_2|^p dx \right. \\ &\quad - \lambda_2 \int_\Omega b_2 (\overline{v}_2 + \tau(s) c_2)^p dx \\ &\quad \left. - \mu_2 (\tau(s))^p c_2^p \int_{\partial\Omega} \widehat{b}_2 d\sigma \right] + s^p H_{\lambda\mu}(\overline{v}_1, 0, \ldots, \overline{v}_n) \\ &< p^{-1}\left[\int_\Omega |\nabla \overline{v}_2|^p dx - \lambda_2 \int_\Omega b_2 \overline{v}_2^p dx\right] \\ &\quad + s^p H_{\lambda\mu}(\overline{v}_1, 0, \ldots, \overline{v}_n) \longrightarrow H_{\lambda\mu}(\overline{v}) < 0 \\ &\qquad \text{as } s \longrightarrow 1^-, \end{aligned} \tag{106}$$

$\bar{s} \in]0,1[$ exists such that

$$H_{\lambda\mu}(\tilde{v}(s)) < 0 \quad \forall s \in]\bar{s},1[. \tag{107}$$

Relations (105), (107) imply that (Theorem 4)

$$\tilde{\tilde{E}}(\tilde{v}(s)) \geq \tilde{\tilde{E}}(\bar{v}) \quad \forall s \in]\bar{s},1[. \tag{108}$$

Since $D_2(\bar{v}) = D_2(\tilde{v}(s)) = 0$, we have

$$\begin{aligned} p(t(\bar{v}))^{p-1} H_{\lambda\mu}(\bar{v}) + q_1(t(\bar{v}))^{q_1-1} &= 0, \\ p(t(\tilde{v}(s)))^{p-1} H_{\lambda\mu}(\tilde{v}(s)) + q_1(t(\tilde{v}(s)))^{q_1-1} &= 0 \\ &\forall s \in]0,1[, \end{aligned} \tag{109}$$

from which

$$\begin{aligned} \tilde{\tilde{E}}(\bar{v}) &= \delta \left|H_{\lambda\mu}(\bar{v})\right|^{-q_1/(p-q_1)}, \\ \tilde{\tilde{E}}(\tilde{v}(s)) &= \delta \left|H_{\lambda\mu}(\tilde{v}(s))\right|^{-q_1/(p-q_1)} \quad \forall s \in]0,1[, \end{aligned} \tag{110}$$

where $\delta = (q_1/p)^{q_1/(p-q_1)} - (q_1/p)^{p/(p-q_1)} > 0$.

From (108), (110), we get

$$H_{\lambda\mu}(\bar{v}) \leq H_{\lambda\mu}(\tilde{v}(s)) \quad \forall s \in]\bar{s},1[. \tag{111}$$

We add that since

$$\begin{aligned} &\frac{\partial g_1}{\partial \tau}(s,\tau) = \tau^{\gamma-1}\tilde{g}(s,\tau) \quad \forall s,\tau \geq 0, \\ &\lim_{\substack{s\to 1^- \\ \tau \to 0^+}} \tilde{g}(s,\tau) \in]0,+\infty[, \end{aligned} \tag{112}$$

then $s_0 \in]\bar{s},1[$ and $\tau_0 \in]0,1[$ exist such that

$$\frac{\partial g_1}{\partial \tau}(s,\tau) > 0 \quad \forall s \in]s_0,1[, \ \forall \tau \in]0,\tau_0[, \tag{113}$$

from which

$$\tau(s) \geq \tau_0 \quad \forall s \in]s_0,1[\tag{114}$$

thanks to (103). Relations (111), (114) imply that

$$\begin{aligned} H_{\lambda\mu}(\bar{v}) &\leq -p^{-1}(\tau(s))^p c_2^p \mu_2 \int_{\partial\Omega} \hat{b}_2 d\sigma \\ &\quad + p^{-1}\left[\int_\Omega |\nabla \bar{v}_2|^p dx - \lambda_2 \int_\Omega b_2 \bar{v}_2^p dx\right] \\ &\quad + s^p H_{\lambda\mu}(\bar{v}_1, 0, \ldots, \bar{v}_n) \\ &\leq -p^{-1}\tau_0^p c_2^p \mu_2 \int_{\partial\Omega} \hat{b}_2 d\sigma \\ &\quad + p^{-1}\left[\int_\Omega |\nabla \bar{v}_2|^p dx - \lambda_2 \int_\Omega b_2 \bar{v}_2^p dx\right] \\ &\quad + s^p H_{\lambda\mu}(\bar{v}_1, 0, \ldots, \bar{v}_n) \quad \forall s \in]s_0,1[, \end{aligned} \tag{115}$$

from which, as $s \to 1^-$, we get the contradiction

$$\begin{aligned} H_{\lambda\mu}(\bar{v}) &\leq -p^{-1}\tau_0^p c_2^p \mu_2 \int_{\partial\Omega} \hat{b}_2 d\sigma + H_{\lambda\mu}(\bar{v}) \\ &< H_{\lambda\mu}(\bar{v}). \end{aligned} \tag{116}$$

□

Application 4. Let $n > 1$ and, for each $v = (v_1, \ldots, v_n) \in W$, we set

$$\begin{aligned} A(v) &= p^{-1}\int_\Omega \left(\sum_{\ell=1}^n |\nabla v_\ell|^\gamma\right)^{p/\gamma} dx, \\ D_1(v) &= q_1^{-1}\int_\Omega \rho_1 \left(\sum_{\ell=1}^n d_\ell |v_\ell|^{\gamma_1}\right)^{q_1/\gamma_1} dx, \\ D_j(v) &= q_j^{-1}\left[\int_\Omega \left(\sum_{\ell=1}^n d_{j\ell} |v_\ell|^{\gamma_j}\right)^{q_j/\gamma_j} dx \right. \\ &\quad \left. - \sum_{\ell=1}^n \int_\Omega \tilde{d}_{j\ell} |v_\ell|^{q_j} dx\right] \quad \text{as } j = 2, \ldots, m, \end{aligned} \tag{117}$$

where

$$\begin{aligned} &1 < \gamma < p, \\ &1 < \gamma_j < q_j \quad \text{as } j = 1, \ldots, m, \\ &q_1 < p, \\ &q_1 < q_2 < \cdots < q_m < \tilde{p}; \\ &\rho_1 \in C^0(\Omega) \cap L^\infty(\Omega), \\ &d_\ell = \text{const.} > 0; \\ &d_{j\ell}, \tilde{d}_{j\ell} \in L^\infty(\Omega), \\ &d_{j\ell}, \tilde{d}_{j\ell} > 0, \\ &\left(\sum_{\ell=1}^n d_{j\ell}\right)^{q_j/\gamma_j} \leq \min\left\{\tilde{d}_{j1}, \ldots, \tilde{d}_{jn}\right\}. \end{aligned} \tag{118}$$

Let us consider the system:

$$\begin{aligned} &-\operatorname{div}\left[\left(\sum_{\ell=1}^n |\nabla u_\ell|^\gamma\right)^{p/\gamma-1} |\nabla u_i|^{\gamma-2}\nabla u_i\right] = \lambda_i b_i |u_i|^{p-2} u_i \\ &+ \rho_1 \left(\sum_{\ell=1}^n d_\ell |u_\ell|^{\gamma_1}\right)^{q_1/\gamma_1-1} d_i |u_i|^{\gamma_1-2} u_i \\ &+ \sum_{j=2}^m \left[\left(\sum_{\ell=1}^n d_{j\ell} |u_\ell|^{\gamma_j}\right)^{q_j/\gamma_j-1} d_{ji} |u_i|^{\gamma_j-2} u_i\right. \end{aligned}$$

$$- \tilde{d}_{ji} |u_i|^{q_j-2} u_i \Bigg] \quad \text{in } \Omega,$$

$$\left(\sum_{\ell=1}^{n} |\nabla u_\ell|^\gamma \right)^{(p/\gamma)-1} |\nabla u_i|^{\gamma-2} \frac{\partial u_i}{\partial \nu} = \mu_i \hat{b}_i |u_i|^{p-2} u_i$$

$$\text{on } \partial\Omega \text{ as } i = 1, \ldots, n. \tag{119}$$

Proposition 22. *Let either $q_m < \hat{p}$ or $\mu_\ell = 0$ $\forall \ell \in \{1, \ldots, n\}$. If $u = (u_1, \ldots, u_n)$ is a weak solution of system (119) with all nonnegative components, then $u_i \in L^\infty(\Omega)$ $\forall i \in \{1, \ldots, n\}$.*

Proof. The statement is true in virtue of [11, Proposition A.4, Remark A.5], and of relation

$$\begin{aligned} &\sum_{i=1}^{n} \int_\Omega \left(\sum_{\ell=1}^{n} |\nabla u_\ell|^\gamma \right)^{(p/\gamma)-1} |\nabla u_i|^{\gamma-2} \nabla u_i \cdot \nabla v_i dx \\ &\quad \le \int_\Omega \left[c_1 + c_2 \left(\sum_{\ell=1}^{n} u_\ell + 1 \right)^{q_m - p} \right] \left(\sum_{i=1}^{n} u_i + 1 \right)^{p-1} \\ &\quad \cdot \left(\sum_{i=1}^{n} v_i \right) dx + c_3 \max_\ell |\mu_\ell| \int_{\partial\Omega} \left(\sum_{i=1}^{n} u_i + 1 \right)^{p-1} \\ &\quad \cdot \left(\sum_{i=1}^{n} v_i \right) d\sigma \end{aligned} \tag{120}$$

$$\forall v = (v_1, \ldots, v_n) \in \left(W^{1,p}(\Omega) \cap L^\infty(\Omega) \right)^n \text{ with } v_i \ge 0,$$

where c_1–c_3 are positive constants. □

Let us introduce the conditions:

$$\rho_1^+ \not\equiv 0 \left(\Longrightarrow V^+(D_1) \neq \varnothing \right), \tag{121}$$

$$\int_\Omega \rho_1 dx < 0 \left(\Longrightarrow D_1(c) < 0 \ \forall c \in R^n \setminus \{0\} \right), \tag{122}$$

$$\begin{aligned} &\int_\Omega b_\ell dx > 0, \\ &\int_{\partial\Omega} \hat{b}_\ell d\sigma > 0 \\ &\qquad \text{as } \ell = 1, \ldots, n. \end{aligned} \tag{123}$$

We have (Propositions 14 and 15)

$$\begin{aligned} &(121), (122) \Longrightarrow \\ &\left(\exists \delta_1^* > 0 : (i_{14}) \text{ holds if } |\lambda_\ell|, |\mu_\ell| \le \delta_1^* \ \forall \ell \right. \\ &\quad \left. \in \{1, \ldots, n\} \right), \end{aligned} \tag{124}$$

$$\begin{aligned} &(122), (123) \Longrightarrow \\ &\left(\exists \delta_2^* > 0 : (i_{15}) \text{ holds if } \lambda_\ell, \mu_\ell \in [0, \delta_2^*] \ \forall \ell \right. \\ &\quad \left. \in \{1, \ldots, n\}, \ \lambda_\ell + \mu_\ell > 0 \text{ as some } \ell \right). \end{aligned} \tag{125}$$

Proposition 23 (see Theorems 1 and 4, Remark 6). *Under conditions (118), one has the following:*

> *When (121) and (122) hold, with λ_ℓ, μ_ℓ as in (124), system (119) has at least two weak solutions u^0 and $-u^0$ ($u^0 = \tau^0 v^0$, $\tau^0 = const. > 0$, $v^0 \in S_{\lambda\mu} \cap V^+(D_1)$), and one has $u_i^0 \ge 0$ as $i = 1, \ldots, n$.*
>
> *When $q_m < p$ and (122), (123) hold, with λ_ℓ, μ_ℓ as in (125), system (119) has at least two weak solutions $\overline{u}$ and $-\overline{u}$ ($\overline{u} = \overline{\tau} \overline{v}$, $\overline{\tau} = const. > 0$, $\overline{v} \in V_{\lambda\mu}^- \cap S(D_1)$), and one has $\overline{u}_i \ge 0$ as $i = 1, \ldots, n$.*

Consequently, when $q_m < p$ and (121)–(123) hold, with λ_ℓ, μ_ℓ as in (125) and $\min\{\delta_1^*, \delta_2^*\}$ instead of δ_2^*, system (119) has at least four different weak solutions.

From Propositions 22 and 23, we deduce the following.

Proposition 24. *We have $\overline{u}_i \in L^\infty(\Omega)$ as $i = 1, \ldots, n$. If either $q_m < \hat{p}$ or $\mu_\ell = 0$ $\forall \ell \in \{1, \ldots, n\}$, then $u_i^0 \in L^\infty(\Omega)$ as $i = 1, \ldots, n$.*

Proposition 25. *Let $\Omega_0 = \{x \in \Omega : \rho_1(x) = 0\}$. If $\gamma > \gamma_1$ and $|\Omega_0|_N = 0$, then $u_i^0 \not\equiv 0$ as $i = 1, \ldots, n$.*

Proof. It is sufficient (Proposition 2) to show that

> as $h = 1, \ldots, n$ (i_0^h) holds with $\mathcal{F} = \{v \in S_{\lambda\mu} \cap V^+(D_1) : v_\ell \ge 0$ as $\ell = 1, \ldots, n$, $v_h \equiv 0\} \neq \varnothing$.

Let us consider, for example, the case $h = 1$. Let $v \in \mathcal{F}$. Firstly, let us prove that $\tilde{v}_1 \in W^{1,p}(\Omega)$ exists with $\tilde{v}_1 \ge 0$ such that

$$\begin{aligned} \delta &= \left[\left[\int_\Omega |\nabla \tilde{v}_1|^p dx - \lambda_1 \int_\Omega b_1 \tilde{v}_1^p dx - \mu_1 \int_{\partial\Omega} \hat{b}_1 \tilde{v}_1^p d\sigma \right] \right. \\ &> 0, \end{aligned} \tag{126}$$

$$\int_\Omega \left(\sum_{\ell \neq 1} |\nabla v_\ell|^\gamma \right)^{(p/\gamma)-1} |\nabla \tilde{v}_1|^\gamma dx > 0, \tag{127}$$

$$\int_\Omega \rho_1 \left(\sum_{\ell \neq 1} d_\ell v_\ell^{\gamma_1} \right)^{q_1/\gamma_1 - 1} \tilde{v}_1^{\gamma_1} dx > 0. \tag{128}$$

Let $\Omega^+ = \{x \in \Omega : \rho_1(x) > 0\}$ and $\Omega_\ell = \{x \in \Omega : |\nabla v_\ell(x)| > 0\}$ ($\ell = 1, \ldots, n$). Since $v \in V^+(D_1)$, then $\ell_0 \in \{2, \ldots, n\}$ exists such that $|\Omega_{\ell_0}|_N > 0$. In fact, $|\Omega_\ell|_N = 0$ $\forall \ell \in \{2, \ldots, n\} \Rightarrow v_\ell = \text{const.}$ $\forall \ell \in \{2, \ldots, n\} \Rightarrow D_1(v) < 0$.

Firstly, let us suppose $|\Omega_{\ell_0} \cap \Omega^+|_N > 0$. Then, a compact set $K_0 \subseteq \Omega^+$ exists such that $|K_0|_N > 0$ and $|\nabla v_{\ell_0}| > 0$ in K_0. Let [11, Proposition A.1] $(\varphi_\varepsilon)_{0<\varepsilon<\varepsilon_0} \subseteq C_0^\infty(\Omega^+)$ with $0 \le \varphi_\varepsilon \le 1$ such that

$$\begin{aligned} &\varphi_\varepsilon \longrightarrow \chi \quad \text{strongly in } L^s(\Omega), \\ &\int_\Omega |\nabla \varphi_\varepsilon|^s dx \longrightarrow +\infty \\ &\qquad \text{as } \varepsilon \longrightarrow 0^+ \ \forall s \in [1, +\infty[, \end{aligned} \tag{129}$$

where χ is the characteristic function of K_0. Set $\Omega_\varepsilon = \{x \in \Omega_{\ell_0} : |\nabla\varphi_\varepsilon(x)| > 0\}$ and $J = \{\varepsilon \in]0, \varepsilon_0[: |\Omega_\varepsilon|_N > 0\}$; we can consider two cases:

$$\inf J = 0, \tag{130}$$

$$\text{either } J = \varnothing \quad \text{or} \quad \inf J > 0. \tag{131}$$

In case (130), thanks to (129), it is possible to choose ε such that (126)–(128) hold with $\widetilde{v}_1 = \varphi_\varepsilon$. In case (131), also from (129), we can find ε such that $\widetilde{v}_1 = \varphi v_{\ell_0} + \varphi_\varepsilon$ satisfying (126)–(128), where $\varphi \in C_0^\infty(\Omega^+)$, $0 \le \varphi \le 1$, and $\varphi = 1$ in K_0.

Let us suppose now $|\Omega_{\ell_0} \cap \Omega^+|_N = 0$. Since $|\Omega_0|_N = 0$, we have

$$\begin{gathered}\exists \text{ a compact set } K_0 \subseteq \Omega \setminus \overline{\Omega}^+ : \\ |K_0|_N > 0, \\ |\nabla v_{\ell_0}| > 0 \\ \text{in } K_0. \end{gathered} \tag{132}$$

Let us add that

$$\begin{gathered} v \in V^+(D_1) \Longrightarrow \\ \exists \ell_1 \in \{2, \ldots, n\}, \\ \text{a compact set } K_1 \subseteq \Omega^+ : \\ |K_1|_N > 0, \\ v_{\ell_1} > 0 \\ \text{in } K_1. \end{gathered} \tag{133}$$

Let $(\varphi_\varepsilon)_{0<\varepsilon<\varepsilon_0} \subseteq C_0^\infty(\Omega^+)$ such that $0 \le \varphi_\varepsilon \le 1$, $\varphi_\varepsilon \to \chi$ strongly in $L^s(\Omega)$ and $\int_\Omega |\nabla\varphi_\varepsilon|^s dx \to +\infty$ as $\varepsilon \to 0^+ \forall s \in [1, +\infty[$, where χ is the characteristic function of K_1. Evidently, $\widetilde{v}_1 = \eta\varphi v_{\ell_0} + \varphi_\varepsilon$ satisfies (126)–(128) with suitable $\eta > 0$ and $\varepsilon, \varphi \in C_0^\infty(\Omega \setminus \overline{\Omega}^+)$, $0 \le \varphi \le 1$, and $\varphi = 1$ in K_0.

Let us introduce the function $g(s,\tau) = H_{\lambda\mu}(\tau\widetilde{v}_1, sv_2, \ldots, sv_n) = p^{-1}[\int_\Omega(\tau^\gamma|\nabla\widetilde{v}_1|^\gamma + s^\gamma \sum_{\ell\ne1}|\nabla v_\ell|^\gamma)^{p/\gamma}dx - \tau^p(\lambda_1\int_\Omega b_1\widetilde{v}_1^p dx + \mu_1\int_{\partial\Omega}\widetilde{b}_1\widetilde{v}_1^p d\sigma) - s^p\sum_{\ell\ne1}(\lambda_\ell\int_\Omega v_\ell^p dx + \mu_\ell\int_{\partial\Omega}\widetilde{b}_\ell v_\ell^p d\sigma)]\forall s, \tau \ge 0$. Equation (126) implies that

$$\begin{gathered}\lim_{\tau\to+\infty} g(s,\tau) = +\infty \quad \forall s \ge 0, \\ \frac{\partial g}{\partial\tau}(s,\tau) \ge \delta\tau^{p-1} > 0 \quad \forall s \ge 0,\ \forall\tau > 0.\end{gathered} \tag{134}$$

Consequently, since $g(1,0) = 1$ and $g(s,0) = s^p < 1\ \forall s \in]0,1[$, $\exists\tau :]0,1] \to [0,+\infty[$ ($\tau(1) = 0$ and $\tau(s) > 0$ if $s < 1$) belonging to $C^0(]0,1]) \cap C^1(]0,1[)$ such that $g(s,\tau(s)) = 1\ \forall s \in]0,1]$, and we have

$$\tau'(s) = -\frac{1}{(\tau(s))^{\gamma-1}}\widetilde{g}(s,\tau(s)) \quad \forall s \in]0,1[; \tag{135}$$

and then, $\lim_{s\to1^-}\widetilde{g}(s,\tau(s)) \in]0,+\infty[$ by (127).

In conclusion, set $\widetilde{v}(s) = (\tau(s)\widetilde{v}_1, sv_2, \ldots, sv_n)$ and, taking into account (128), (135), it results in

$$\begin{gathered}\widetilde{v}(1) = v, \\ \widetilde{v}(s) \in S_{\lambda\mu} \quad \forall s \in]0,1], \\ \lim_{s\to1^-}\frac{d}{ds}D_1(\widetilde{v}(s)) = -\infty, \\ \lim_{s\to1^-}\frac{d}{ds}D_j(\widetilde{v}(s)) < +\infty \quad \text{as } j \ne 1 \\ \Longrightarrow \lim_{s\to1^-}\sum_{j=1}^m (t(\widetilde{v}(s)))^{q_j}\frac{d}{ds}D_j(\widetilde{v}(s)) = -\infty.\end{gathered} \tag{136}$$

□

Proposition 26. *If for some $j_0 \in \{2,\ldots,m\}$ $\gamma_{j_0} < \gamma \le \gamma_1$, then $u_i^0 \not\equiv 0$ as $i = 1,\ldots,n$.*

Proof. It is sufficient (Proposition 2) to show that

as $h = 1,\ldots,n$ (i_0^h) holds with $\mathcal{F} = \{v \in S_{\lambda\mu} \cap V^+(D_1) : v_\ell \ge 0 \text{ as } \ell = 1,\ldots,n,\ v_h \equiv 0\} \ne \varnothing$.

Let us suppose $h = 1$ and let $v \in \mathcal{F}$. Let us verify that $\widetilde{v}_1 \in W^{1,p}(\Omega)$ exists with $\widetilde{v}_1 \ge 0$ such that (126) and (127) hold. At this aim, we note that

$$\begin{gathered} v \in V^+(D_1) \Longrightarrow \\ \exists \ell_0 \in \{2,\ldots,n\} : \\ |\Omega_{\ell_0}|_N > 0,\end{gathered} \tag{137}$$

where $\Omega_{\ell_0} = \{x \in \Omega : |\nabla v_{\ell_0}(x)| > 0\}$. Let $K \subseteq \Omega$ be a compact set with positive measure. Let $(\varphi_\varepsilon)_{0<\varepsilon<\varepsilon_0} \subseteq C_0^\infty(\Omega)$ such that $0 \le \varphi_\varepsilon \le 1$, $\varphi_\varepsilon \to \chi$ strongly in $L^s(\Omega)$ and $\int_\Omega |\nabla\varphi_\varepsilon|^s dx \to +\infty$ as $\varepsilon \to 0^+\ \forall s \in [1,+\infty[$ (χ = characteristic function of K). With Ω_ε and J as in Proposition 25, it is possible to find ε such that $\widetilde{v}_1 = \varphi_\varepsilon$ [resp., $\widetilde{v}_1 = v_{\ell_0} + \varphi_\varepsilon$] satisfying (126) and (127) in case (130) [resp., (131)]. Then, if $\tau(s)$ and $\widetilde{v}(s)$ are as in Proposition 25, we have

$$\begin{gathered}\widetilde{v}(1) = v, \\ \widetilde{v}(s) \in S_{\lambda\mu} \quad \forall s \in]0,1], \\ \lim_{s\to1^-}\frac{d}{ds}D_{j_0}(\widetilde{v}(s)) = -\infty, \\ \lim_{s\to1^-}\frac{d}{ds}D_j(\widetilde{v}(s)) < +\infty \quad \text{as } j \ne j_0 \\ \Longrightarrow \lim_{s\to1^-}\sum_{j=1}^m (t(\widetilde{v}(s)))^{q_j}\frac{d}{ds}D_j(\widetilde{v}(s)) = -\infty.\end{gathered} \tag{138}$$

□

Proposition 27. *If for some $j_0 \in \{2,\ldots,m\}$ $\gamma_{j_0} < \gamma_1 \le \gamma$, then $\overline{u}_i \not\equiv 0$ as $i = 1,\ldots,n$.*

Proof. It is sufficient (Proposition 5) to show that

as $h = 1, \ldots, n$ ($\underline{i}^h$) holds with $\mathcal{F} = \{v \in V^-_{\lambda\mu} \cap S(D_1) : v_\ell \geq 0$ as $\ell = 1, \ldots, n,\ v_h \equiv 0\}$, if $\mathcal{F} \neq \varnothing$.

Let us suppose, for example, $h = 1$ and let us set $\Omega^- = \{x \in \Omega : \rho_1(x) < 0\}$. Let $K \subseteq \Omega^-$ be a compact set such that $|K|_N > 0$ and $v_{\ell_0} > 0$ in K for some $\ell_0 \in \{2, \ldots, n\}$. Let $\varphi \in C_0^\infty(\Omega^-)$ with $0 \leq \varphi \leq 1$ and $\varphi = 1$ in K. Set $g_1(s, \tau) = D_1(\tau\varphi, sv_2, \ldots, sv_n) = q_1^{-1} \int_\Omega \rho_1 (d_1 \tau^{\gamma_1} \varphi^{\gamma_1} + s^{\gamma_1} \sum_{\ell \neq 1} d_\ell v_\ell^{\gamma_1})^{q_1/\gamma_1} dx\ \forall s, \tau \geq 0$; it results in

$$\begin{aligned} & g_1(1,0) = -1, \\ & \frac{\partial g_1}{\partial \tau}(s,\tau) < 0 \quad \forall s \geq 0,\ \forall \tau > 0, \\ & g_1(s,0) = -s^{q_1} > -1 \quad \forall s \in]0,1[\,, \\ & \lim_{\tau \to +\infty} g_1(s,\tau) = -\infty \quad \forall s \geq 0. \end{aligned} \tag{139}$$

Then, $\exists \tau :]0,1] \to [0,+\infty[$ ($\tau(1) = 0$, $\tau(s) > 0$ if $s < 1$) belonging to $C^0(]0,1]) \cap C^1(]0,1[)$ such that $g_1(s, \tau(s)) = -1\ \forall s \in]0,1]$, and we have

$$\begin{aligned} \tau'(s) = -\frac{1}{(\tau(s))^{\gamma_1 - 1}} \widetilde{g}_1(s, \tau(s)) \\ \forall s \in]0,1[\,,\ \lim_{s \to 1^-} \widetilde{g}_1(s,\tau(s)) \in]0,+\infty[\,. \end{aligned} \tag{140}$$

Consequently, with $\widetilde{v}(s) = (\tau(s)\varphi, sv_2, \ldots, sv_n)$, we get

$$\begin{aligned} & \widetilde{v}(1) = v, \\ & \widetilde{v}(s) \in S(D_1) \quad \forall s \in]0,1]\,, \\ & \lim_{s \to 1^-} \frac{d}{ds} H_{\lambda\mu}(\widetilde{v}(s)) \in]-\infty, 0[\,, \\ & \lim_{s \to 1^-} \frac{d}{ds} D_{j_0}(\widetilde{v}(s)) = -\infty, \\ & \lim_{s \to 1^-} \frac{d}{ds} D_j(\widetilde{v}(s)) < +\infty \quad \text{as } j \neq j_0 \\ & \Longrightarrow \lim_{s \to 1^-} \left[(t(\widetilde{v}(s)))^p \frac{d}{ds} H_{\lambda\mu}(\widetilde{v}(s)) \right. \\ & \qquad \left. - \sum_{j=1}^m (t(\widetilde{v}(s)))^{q_j} \frac{d}{ds} D_j(\widetilde{v}(s)) \right] = +\infty. \end{aligned} \tag{141}$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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