Generalized fractional master equation for self-similar stochastic processes modelling anomalous diffusion

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Abstract: The Master Equation approach to model anomalous diffusion is considered. In particular, the formulation is extended to the time-stretching generalization on the basis of the superposition mechanism of processes with different diffusion coefficients distributed according to a spectrum function. When this superposition is applied to the time-fractional diffusion process, the resulting Master Equation emerges to be the governing equation of the *Erdélyi–Kober fractional diffusion* that is the Master Equation of the *generalized grey Brownian motion* (*ggBm*). The generalized grey Brownian motion is a parametric class of stochastic processes that provides models for both fast and slow anomalous diffusion. This class is made up of self-similar processes with stationary increments and depends on two real parameters: $0 < \alpha \leq 2$ and $0 < \beta \leq 1$. It includes the fractional Brownian motion when $0 < \alpha \leq 2$ and $\beta = 1$, the time-fractional diffusion stochastic processes when $0 < \alpha = \beta < 1$, and the standard Brownian motion when $\alpha = \beta = 1$. In the *ggBm* framework, the M-Wright function (known also as Mainardi function) emerges as a natural generalization of the Gaussian distribution recovering the same key role of the Gaussian density for the standard and the fractional Brownian motion.

Keywords: Anomalous diffusion, fractional derivatives, self-similar stochastic processes, Brownian motion, Wright function, Mainardi function.

1. INTRODUCTION

Statistical description of diffusive processes can be performed both at the microscopic and at the macroscopic level. The microscopic-level description concerns the simulation of the particle trajectories by opportune stochastic models. The macroscopic-level description indeed concerns the derivation of the evolution equation of the probability density function of the particle displacement, which is connected to the microscopic trajectories, such equation is named Master Equation. The problem of microscopic and macroscopic description of physical systems and their connection is addressed and discussed in a number of cases by Balescu (1997).

The most common examples of this microscopic-to-macroscopic formalism are the Brownian motion coupled with the diffusion equation and the Ornstein–Uhlenbeck coupled with the Fokker–Planck equation, see e.g. (Risken, 1989; Gardiner, 1990), but the same coupling occurs for several applications of the random walk method at the microscopic level and the resulting macroscopic Master Equation for the probability density function (Weiss, 1994).

In some cases the classical local flux-gradient relationship does not hold and it is necessary to determine a non-local relationship so that an anomalous diffusion arises. Anomalous diffusion is referred to as *fast diffusion*, when the variance of the particle spreading grows with a power law with exponent greater than 1, and it is referred to as *slow diffusion*, when that exponent is lower than 1. It is well-known that a useful mathematical tool for the macroscopic investigation and description of anomalous diffusion is Fractional Calculus.

A fractional differential approach has been successfully used for modelling in several different disciplines as for example statistical physics (Metzler and Klafter, 2004), neuroscience (Lundstrom et al., 2008), economy (Scalas, 2006), control theory (Vinagre et al., 2000) and combustion science (Pagnini, 2011a,b). Further applications of the fractional approach are recently introduced and discussed by Tenreiro Machado (2011).

Moreover, under the physical point of view, when there is no separation of time-scale between the microscopic and the macroscopic level of the process the randomness of the microscopic level is transmitted to the macroscopic level and the correct description of the macroscopic dynamics has to be in terms of the Fractional Calculus for space variable (Grigolini et al, 1999). Moreover, fractional integro/differential equations in the time variable are related to phenomena with fractal properties (Rocco and West, 1999).

Here the correspondence microscopic-to-macroscopic for anomalous diffusion is considered in the framework of Fractional Calculus.

W.R. Schneider introduced the class of self-similar stochastic processes based on the grey noise theory and named grey Brownian motion (Schneider, 1990, 1992). This class provides stochastic models for the slow anomalous diffusion and the corresponding Master Equation is the time-fractional diffusion equation. This class of self-similar processes has been extended to include stochastic models for both slow and fast anomalous diffusion and it is named *generalized grey Brownian motion* (Mura, 2008; Mura and Mainardi, 2009; Mura and Pagnini, 2008). This is a large class of self-similar stochastic processes whose Master Equation is a fractional differential equation in the Erdélyi–Kober sense, so the resulting diffusion process is named *Erdélyi–Kober fractional diffusion* (Pagnini, 2012).

The rest of the paper is organized as follows. In Section 2, the Master Equation approach is briefly described with the aim to introduce the time-stretched generalization of the non-Markovian formulation. In Section 3, the relationship between a ME in terms of the Erdélyi–Kober fractional derivative operator and the generalized grey Brownian motion is highlighted. Finally, in Section 4 Conclusions are given.

2. THE MASTER EQUATION APPROACH

2.1 The Master Equation and its generalization

The equation governing the evolution in time of the probability density function (*pdf*) of particle displacement $P(x;t)$, where $x \in \mathcal{R}$ is the location and $t \in \mathcal{R}_0^+$ the observation instant, is named Master Equation (ME). Here the time t has to be interpreted as a parameter such that the normalization condition $\int P(x;t)\,dx = 1$ holds for any t. In this respect, the ME approach describes the system under consideration at the macroscopic level because it is refereed to en ensemble of trajectories rather than a single trajectory.

The most simple and more famous ME is the parabolic diffusion equation which describes Normal diffusion. Normal diffusion, or Gaussian diffusion, is a stochastic process whose *pdf* is given by the equation

$$\frac{\partial P}{\partial t} = \mathcal{D}\,\frac{\partial^2 P}{\partial x^2}\,, \tag{1}$$

with initial condition $P(x;0) = P_0(x)$, where $\mathcal{D}$ is the diffusion coefficient. The fundamental solution of (1), which is named also Green function, and corresponding to the initial condition $P(x;0) = P_0(x) = \delta(x)$, is the Gaussian density

$$f(x;t) = \frac{1}{\sqrt{4\pi\mathcal{D}t}}\exp\left\{-\frac{x^2}{4\mathcal{D}t}\right\}\,. \tag{2}$$

The variance of dispersion of the Gaussian diffusion (1) grows linearly in time, i.e., $\langle x^2\rangle = \int_{-\infty}^{+\infty} x^2 f(x;t)\,dx = 2\,\mathcal{D}\,t$.

The density function $P(x;t)$ with general initial condition $P(x;0) = P_0(x)$ is related to the fundamental solution $f(x;t)$ by the following convolution integral

$$P(x;t) = \int_{-\infty}^{+\infty} f(\xi;t)P_0(x-\xi)\,d\xi\,. \tag{3}$$

Actually, diffusion equation (1) is a special case of the Fokker−Planck equation Risken (1989)

$$\frac{\partial P}{\partial t} = \left[-\frac{\partial}{\partial x}D_1(x) + \frac{\partial^2}{\partial x^2}D_2(x)\right]P(x;t)\,, \tag{4}$$

where coefficients $D_1(x)$ and $D_2(x) > 0$ are called the drift and the diffusion coefficient, respectively.

Processes described by (1) and (4) are both Markovian, because density $P(x;t)$ is completely determined by the *pdf* at the initial instant $t = 0$. A more general case for Markovian processes is the ME obtained by the differential form of the Chapman–Kolmogorov equation (Gardiner, 1990), which includes both the pure jump processes and the Fokker–Planck equation.

The non-Markovian generalization of the ME follows by introducing memory effects, which means, under the mathematical formulation view point, that the evolution operator on the right-hand side depends also on time, i.e.,

$$\frac{\partial P}{\partial t} = \int_0^t \left[\frac{\partial}{\partial x}D_1(x,t-\tau) + \frac{\partial^2}{\partial x^2}D_2(x,t-\tau)\right]P(x;\tau)\,d\tau. \tag{5}$$

A straightforward non-Markovian generalization is obtained by describing a phase-space (v,x) process, as for example the Kramers equation for the motion of particles with mass m in an external force field $F(x)$, i.e.,

$$\frac{\partial P}{\partial t} = \left[-\frac{\partial}{\partial x}v + \frac{\partial}{\partial v}\left(v - \frac{F(x)}{m}\right) + \frac{\partial^2}{\partial v^2}\right]P(v,x;t)\,. \tag{6}$$

In fact, due to the temporal correlation of particle velocity, eliminating the velocity variable in (6) gives a non-Markovian generalized ME of the following form (Risken, 1989)

$$\frac{\partial P}{\partial t} = \int_0^t K(x,t-\tau)\frac{\partial^2}{\partial x^2}P(x;\tau)\,d\tau\,, \tag{7}$$

where the memory kernel $K(x,t)$ may be an integral operator or contain differential operators with respect to x, or some other linear operator.

If the memory kernel $K(x,t)$ correspond to the Gel'fand–Shilov function

$$K(t) = \frac{t_+^{-\mu-1}}{\Gamma(-\mu)}\,,\quad 0 < \mu < 1\,, \tag{8}$$

where the suffix $+$ is just denoting that the function is vanishing for $t < 0$, then ME (7) results to be

$$\frac{\partial P}{\partial t} = \int\limits_{0_-}^{t^+} \frac{(t-\tau)^{-\mu-1}}{\Gamma(-\mu)} \frac{\partial^2}{\partial x^2} P(x;\tau)\, d\tau = D_t^\mu \frac{\partial^2 P}{\partial x^2}\,, \qquad (9)$$

that is the time-fractional diffusion equation, originally analyzed by Mainardi (1996), where D_t^μ is the Riemann–Liouville fractional differential operator of order μ in its formal definition according to (Gorenflo and Mainardi, 1997, Eq. (1.34)), obatined by using the representation of the generalized derivative of order n of the Dirac delta distribution: $\delta^{(n)}(t) = \frac{t_+^{-n-1}}{\Gamma(-n)}$, with proper interpretation of the quotient as a limit if $t = 0$. Here we reminded that, for a sufficiently well-behaved function $\varphi(t)$, the regularized Riemann–Liouville fractional derivative of non-integer order $\mu \in (n-1, n)$ is

$$D_t^\mu \varphi(t) = \frac{d^n}{dt^n} \left[\frac{1}{\Gamma(n-\mu)} \int\limits_0^t \frac{\varphi(\tau)\, d\tau}{(t-\tau)^{\mu+1-n}} \right] . \qquad (10)$$

For any $\mu = n$ non-negative integer we recover the standard derivative

$$D_t^\mu \varphi(t) = \frac{d^n}{dt^n} \varphi(t)\,.$$

For more details, we refer the reader to (Gorenflo and Mainardi, 1997).

2.2 A physical mechanism for time-stretching generalization

It is well-known that the stretched exponential $\exp(-t^\alpha)$ with $0 < \alpha < 1$, being a completely monotone function, can emerge as a linear superposition of elementary exponential functions with different time scales T. In fact from the well-known formula of the Laplace transform of the unilateral extremal stable density $\mathcal{L}_\alpha^{-\alpha}(\xi)$, see e.g. (Mainardi et al., 2001)

$$\int\limits_0^\infty \mathrm{e}^{-s\xi} \mathcal{L}_\alpha^{-\alpha}(\xi)\, d\xi = \mathrm{e}^{-s^\alpha}\,, \quad s > 0\,, \quad 0 < \alpha < 1\,, \qquad (11)$$

we first obtain

$$\mathcal{L}_\alpha^{-\alpha}(\xi) = \frac{1}{\pi} \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n!} \Gamma(n\alpha) \sin(n\pi\alpha)\, \xi^{-\alpha n - 1}\,. \qquad (12)$$

Then, after the changes of variable $s = t$ and $\xi = 1/T$, it follows that

$$\int\limits_0^\infty \mathrm{e}^{-t/T} \mathcal{L}_\alpha^{-\alpha}\left(\frac{1}{T}\right) \frac{dT}{T^2} = \mathrm{e}^{-t^\alpha}\,, \qquad (13)$$

where $T^{-2} \mathcal{L}_\alpha^{-\alpha}\left(\frac{1}{T}\right)$ is the spectrum of time-scales T.

What concerns diffusion processes, the same superposition mechanism can be considered for the particle *pdf*. In fact, anomalous diffusion emerging in complex media can be interpreted as the resulting global effect following on from particles that along their trajectory have experienced the different values assumed by one or more characteristic properties of the crossed medium, e.g, different values of the diffusion coefficient. Then, particles diffuse in a medium that is disorderly layered.

This mechanism can explain, for example, the origin of a time-dependent diffusion coefficient, as it is considered in the fractional Brownian motion, from the classical Gaussian diffusion when different constant diffusion coefficients are experienced by the particles. In fact, let $\rho(\mathcal{D}, x, t)$ be the spectrum of the values of $\mathcal{D}$ and highliting the dependence of the Gaussian density (2) on the diffusion coefficient by adopting the notation $f(x;t) \equiv f(x;t,\mathcal{D})$, then, in analogy with (13),

$$f_*(x;t^\alpha) = \int f(x;t,\mathcal{D}) \rho(\mathcal{D},x,t)\, d\mathcal{D} = \frac{1}{\sqrt{4\pi t^\alpha}} \exp\left\{ -\frac{x^2}{4t^\alpha} \right\}\,, \qquad (14)$$

where

$$\rho(\mathcal{D},x,t) = t^{-(\alpha-1)/2} \mathcal{D}^{-1/2} \frac{x^{2-2/\alpha}}{4^{1-1/\alpha}\mathcal{D}} \mathcal{L}_\alpha^{-\alpha}\left(\frac{x^{2-2/\alpha}}{4^{1-1/\alpha}\mathcal{D}} \right) . \qquad (15)$$

See in Appendix the details for computation of $\rho(\mathcal{D}, x, t)$.

From (14) it follows that it holds

$$\frac{\partial f_*}{\partial t^\alpha} = \frac{1}{\mathcal{D}} \frac{\partial f}{\partial t}\,, \qquad (16)$$

hence the ME for $f_*(x;t)$ is

$$\frac{\partial f_*}{\partial t^\alpha} = \frac{\partial^2 f_*}{\partial x^2}\,. \qquad (17)$$

Since $\frac{\partial f_*}{\partial t^\alpha} = \frac{1}{\alpha\, t^{\alpha-1}} \frac{\partial f_*}{\partial t}$, this is the ME of the fractional Brownian motion, i.e.,

$$\frac{\partial f_*}{\partial t} = \alpha\, t^{\alpha-1} \frac{\partial^2 f_*}{\partial x^2}\,, \qquad (18)$$

that in integral form reads

$$f_*(x;t) = f_{*0}(x) + \int\limits_0^t \alpha\, \tau^{\alpha-1} \frac{\partial^2 f_*}{\partial x^2}\, d\tau\,. \qquad (19)$$

In general, let the ME of a *pdf* $P(x;t,\mathcal{D})$ be written in the non-Markovian form

$$\frac{\partial P}{\partial t} = \mathcal{D} \int\limits_0^t K(x, t-\tau) \frac{\partial^2 P}{\partial x^2}\, d\tau\,, \qquad (20)$$

where the Gaussian diffusion is recovered when the memory kernel is $k(x,t) = \delta(t)$. Assuming, in analogy with (14), that it exists a general spectrum $\rho_G(\mathcal{D},x,t)$ of values of $\mathcal{D}$ such that it holds

$$P_*(x;t^\gamma) = \int P(x;t,\mathcal{D}) \rho_G(\mathcal{D},x,t)\, d\mathcal{D}\,, \qquad (21)$$

then the analogous of (16) reads

$$\frac{\partial P_*}{\partial t^\gamma} = \frac{1}{\mathcal{D}} \frac{\partial P}{\partial t}\,, \qquad (22)$$

from which it follows that the time-stretched ME corresponding to (20) is

$$\frac{\partial P_*}{\partial t} = \gamma\, t^{\gamma-1} \int\limits_0^{t^\gamma} K(x, t^\gamma - \tau^\gamma)\, \frac{\partial^2 P_*}{\partial x^2}\, d\tau^\gamma$$

$$= \gamma\, t^{\gamma-1} \int\limits_0^{t} K(x, t^\gamma - \tau^\gamma)\, \frac{\partial^2 P_*}{\partial x^2}\, \gamma \tau^{\gamma-1}\, d\tau\,. \quad (23)$$

Finally, the previous formalism can be furthermore generalized assuming that the time-stretching is described by a smooth and increasing function $g(t)$, with $g(0) = 0$. Since $\frac{\partial P_*}{\partial g(t)} = \frac{1}{\partial g/\partial t}\frac{\partial P_*}{\partial t}$, equations (23) turn out to be

$$\frac{\partial P_*}{\partial t} = \frac{dg}{dt} \int\limits_0^{t} K[x, g(t) - g(\tau)]\, \frac{\partial^2 P_*}{\partial x^2}\, \frac{dg}{d\tau}\, d\tau\,. \quad (24)$$

Choosing memory kernel (8), which guaranties that P be a probability density and that the process be self similar (Mura et al., 2008), and the time-stretching function

$$g(t) = t^{\alpha/\beta}\,, \quad 0 < \alpha \le 2\,, \quad 0 < \beta \le 1\,, \quad (25)$$

equation (24) turns out to be

$$\frac{\partial P_*}{\partial t} = \frac{\alpha}{\beta} t^{\alpha/\beta - 1} \int\limits_{0_-}^{t^+} \frac{(t^{\alpha/\beta} - \tau^{\alpha/\beta})^{-\mu-1}}{\Gamma(-\mu)}\, \frac{\partial^2 P_*}{\partial x^2}\, \frac{\alpha}{\beta} \tau^{\alpha/\beta - 1}\, d\tau$$

$$= \frac{\alpha}{\beta} t^{\alpha/\beta - 1}\, D^{\mu}_{t^{\alpha/\beta}} \frac{\partial^2 P_*}{\partial x^2}\,, \quad (26)$$

that, setting $\mu = 1 - \beta$, is the stretched time-fractional diffusion equation (see Mainardi et al., 2010, Eq. (5.19)).

In terms of the regularized Riemann–Liouville fractional differential operator (10), equation (26) is the ME corresponding to the following integral evolution equation

$$P_*(x,t) = P_{*0}(x) + \frac{1}{\Gamma(\beta)} \frac{\alpha}{\beta} \int\limits_0^t \tau^{\alpha/\beta - 1}\, (t^{\alpha/\beta} - \tau^{\alpha/\beta})^{\beta-1}\, \frac{\partial^2 P(x,\tau)}{\partial x^2}\, d\tau\,, \quad (27)$$

that was originally introduced by Mura (2008) in his PhD Thesis, and later discussed in a number of papers by Mainardi et al. (2010); Mura and Mainardi (2009); Mura and Pagnini (2008); Mura et al. (2008).

3. THE GENERALIZED FRACTIONAL MASTER EQUATION FOR SELF–SIMILAR PROCESSES

3.1 The Erdélyi–Kober fractional diffusion: the generalized grey Brownian motion

It is well-known that it exists a relationship between the solutions of a certain class of integral equations that are used to model anomalous diffusion and stochastic processes. In this respect, the density function $P_*(x;t)$ which solves (27) is the marginal particle *pdf*, i.e., the one-point one-time density function of particle dispersion, of the *generalized grey Brownian motion* (*ggBm*) (Mura, 2008; Mura and Mainardi, 2009; Mura and Pagnini, 2008). The *ggBm* is a special class of H-sssi processes of order $H = \alpha$, or Hurst exponent $H = \alpha/2$, where, according to a common terminology, H-sssi means H-self-similar-stationary-increments. The *ggBm* provides non-Markovian stochastic models for anomalous diffusion, of both slow type $0 < \alpha < 1$ and fast type $1 < \alpha < 2$. The *ggBm* includes some well-known processes, so that it defines an interesting general theoretical framework. In fact, the fractional Brownian motion appears for $\beta = 1$, the grey Brownian motion, in the sense of Schneider (1990, 1992), corresponds to the choice $0 < \alpha = \beta < 1$, and finally the standard Brownian motion is recovered by setting $\alpha = \beta = 1$. It is worth noting to remark that only in the particular case of the Brownian motion the stochastic process is Markovian.

Following Pagnini (2012), the integral in the non-Markovian kinetic equation (27) can be expressed in terms of an Erdélyi–Kober fractional integral operator $I_\eta^{\gamma,\mu}$ that, for a sufficiently well-behaved function $\varphi(t)$, is defined as, see (Kiryakova, 1994, Eq. (1.1.17)),

$$I_\eta^{\gamma,\mu}\, \varphi(t) = \frac{\eta}{\Gamma(\mu)} t^{-\eta(\mu+\gamma)} \int\limits_0^t \tau^{\eta(\gamma+1)-1} (t^\eta - \tau^\eta)^{\mu-1} \varphi(\tau)\, d\tau, \quad (28)$$

where $\mu > 0$, $\eta > 0$ and $\gamma \in \mathcal{R}$. Hence equation (27) can be re-written as (Pagnini, 2011b, 2012)

$$P_*(x;t) = P_{*0}(x) + t^\alpha \left[I^{0,\beta}_{\alpha/\beta}\, \frac{\partial^2 P_*}{\partial x^2} \right]. \quad (29)$$

Since the ggBm serves as a stochastic model for the anomalous diffusion, this leads to define the family of diffusive processes governed by the ggBm as *Erdélyi–Kober fractional diffusion* (Pagnini, 2012).

The ME corresponding to (27) is (26). But, since in (26) it is used the Riemann–Liouville fractional differential operator with a stretched time variable, an abuse of notation occurs. Due to the correspondence between (26) and (29), the correct expression for the ME of (27) is obtained by introducing the Erdélyi-Kober fractional differential operator $D_\eta^{\gamma,\mu}$ that is defined, for $n-1 < \mu \le n$, as (Kiryakova, 1994, Eq. (1.5.19))

$$D_\eta^{\gamma,\mu}\, \varphi(t) = \prod_{j=1}^{n} \left(\gamma + j + \frac{1}{\eta} t \frac{d}{dt} \right) (I_\eta^{\gamma+\mu, n-\mu}\, \varphi(t))\,. \quad (30)$$

From definition (10) it follows that the Erdélyi–Kober and the Riemann–Liouville fractional derivatives are related through the formula

$$D_1^{-\mu,\mu}\, \varphi(t) = t^\mu D_t^\mu\, \varphi(t)\,. \quad (31)$$

A further important property of the Erdélyi–Kober fractional derivative is the reduction to the identity operator when $\mu = 0$, i.e.,

$$D_\eta^{\gamma,0} \varphi(t) = \varphi(t)\,. \quad (32)$$

Recently, the notions of Erdélyi–Kober fractional integrals and derivatives have been further extended by Luchko (2004) and by Luchko and Trujillo (2007). Finally, the ME of the *ggBm*, or Erdélyi–Kober fractional diffusion, is (Pagnini, 2012)

$$\frac{\partial P_*}{\partial t} = \frac{\alpha}{\beta} t^{\alpha-1}\, D^{\beta-1,1-\beta}_{\alpha/\beta}\, \frac{\partial^2 P_*}{\partial x^2}\,. \tag{33}$$

3.2 The Green function of the generalized fractional master equation as marginl pdf of the ggBm

The Green function corresponding to (29, 33) is (Mura, 2008; Mura and Mainardi, 2009; Mura and Pagnini, 2008; Mura et al., 2008)

$$\mathcal{G}(x;t) = \frac{1}{2}\,\frac{1}{t^{\alpha/2}} M_{\beta/2}\left(\frac{|x|}{t^{\alpha/2}}\right)\,, \tag{34}$$

where $M_{\beta/2}(z)$ is the M-Wright function of order $\beta/2$, also referred to as Mainardi function (Mainardi, 2010; Podlubny, 1999). For a generic order $\nu \in (0,1)$ it was formerly introduced by Mainardi (1996) by the series representation

$$\begin{aligned} M_\nu(z) &= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma[-\nu n + (1-\nu)]}\,, \\ &= \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!}\,\Gamma(\nu n)\sin(\pi\nu n)\,. \end{aligned} \tag{35}$$

For reviews see (Gorenflo et al., 1999, 2000; Mainardi et al., 2010).

The marginal *pdf* of the *ggBm* process emerges and describes both slow and fast anomalous diffusion. In fact, the variance of Green function (34) is $\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 \mathcal{G}(x;t)\, dx = (2/\Gamma(\beta+1))\, t^\alpha$, then the resulting process turns out to be self-similar with Hurst exponent $H = \alpha/2$ and the variance law is consistent with slow diffusion for $0 < \alpha < 1$ and fast diffusion for $1 < \alpha \le 2$. However, it is worth noting to be remarked also that a linear variance growing is possible, but with non-Gaussian *pdf*, when $\beta \neq \alpha = 1$ (purely random, the increments are uncorrelated), and a Gaussian *pdf* with non-linear variance growing when $\beta = 1$ and $\alpha \neq 1$ (fractional Brownian motion, fBm).

In Figure 1 we present a diagram that allows to identify the elements of the *ggBm* class, referred to as $B_{\alpha,\beta}(t)$. The top region $1 < \alpha < 2$ corresponds to the domain of fast diffusion. with *long-range dependence.* In this domain the increments of the process are positively correlated, so that the trajectories tend to be more regular (*persistent*). It should be noted that long-range dependence is associated to a non-Markovian process which exhibits long-memory properties. The horizontal line $\alpha = 1$ corresponds to processes with uncorrelated increments, which model various phenomena of normal diffusion. For $\alpha = \beta = 1$ we recover the Gaussian process of the standard Brownian motion. The Gaussian process of the fractional Brownian motion is identified by the vertical line $\beta = 1$. The bottom region $0 < \alpha < 1$ corresponds to the domain of slow diffusion. The increments of the corresponding process turn out to be negatively correlated and this implies that the trajectories are strongly irregular (*anti-persistent motion*); the increments form a stationary process which does not exhibit long-range dependence. Finally, the lower diagonal line ($\alpha = \beta$) represents the Schneider grey Brownian motion (gBm) whereas the upper diagonal line indicates the "conjugated" process of gBm.

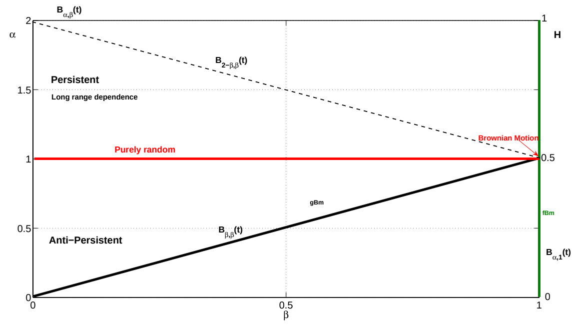

Fig. 1. Parametric class of generalized grey Brownian motion.

In general, even if the Green functions are interpreted as one-point *pdf* evolving in time, they cannot in general determine a *unique* (self-similar) stochastic process because this requires the determination of any multi-point *pdf*. But, as far as the *ggBm* is concerned, since its increments are stationary, it emerges to be uniquely determined by its covariance structure (Mura and Mainardi, 2009; Mura and Pagnini, 2008). Then, even if the *ggBm* is not Gaussian in general, it is a valuable example of a process defined only through its first and second moments, which indeed is a remarkable property of the Gaussian processes. Then the *ggBm* is a direct generalization of the Gaussian processes and, in the same way, the Mainardi function M_ν is a generalization of the Gaussian function, and it emerges to be the marginal *pdf* of non-Markovian diffusion processes that describe both slow and fast anomalous diffusion.

Special cases of ME (27) are straightforwardly obtained (Pagnini, 2012). In particular, it reduces to the time-fractional diffusion if $\alpha = \beta < 1$, to the stretched Gaussian diffusion if $\alpha \neq 1$ and $\beta = 1$, and finally to the standard Gaussian diffusion if $\alpha = \beta = 1$.

4. CONCLUSIONS

We have highlighted the relationship between the Erdélyi–Kober fractional operators and the valuable family of stochastic processes generated by the *ggBm*, whose some remarkable properties are reported above.In fact, the particle *pdf* of associated to the *ggBm* is the solution of a fractional integral equation (29), or analogously of a fractional diffusion equation (33), in the Erdélyi–Kober sense and this solution is provided by a transcendental function of the Wright type, also referred to as Mainardi function. Since the governing equation of these processes is a fractional equation in the Erdélyi–Kober sense it is natural to call this family of diffusive processes as *Erdélyi–Kober fractional diffusion.*

APPENDIX

The spectrum $\rho(\mathcal{D}, x, t)$ of values of $\mathcal{D}$ can be computed as it follows. Consider formula (11), then applying the change of variables

$$s = \left(\frac{x^2}{4\,t^\alpha}\right)^{1/\alpha}\,, \quad \xi = \frac{x^{2-2/\alpha}}{4^{1-1/\alpha}\,\mathcal{D}}\,, \tag{A.1}$$

it turns out to be

$$\int_0^\infty \exp\left\{-\frac{x^2}{4\mathcal{D}t}\right\} \frac{x^{2-2/\alpha}}{4^{1-1/\alpha}\mathcal{D}^2}\mathcal{L}_\alpha^{-\alpha}\left(\frac{x^{2-2/\alpha}}{4^{1-1/\alpha}\mathcal{D}}\right) d\mathcal{D} = \exp\left\{-\frac{x^2}{4\,t^\alpha}\right\} . \qquad (A.2)$$

Finally, dividing both sides by $\sqrt{4\,\pi t^\alpha}\,\sqrt{\mathcal{D}\,t}$, it results

$$\int_0^\infty \frac{1}{\sqrt{4\,\pi\,\mathcal{D}\,t}} \exp\left\{-\frac{x^2}{4\,\mathcal{D}\,t}\right\} t^{-(\alpha-1)/2}\mathcal{D}^{-1/2}\,\frac{x^{2-2/\alpha}}{4^{1-1/\alpha}\,\mathcal{D}}\mathcal{L}_\alpha^{-\alpha}\left(\frac{x^{2-2/\alpha}}{4^{1-1/\alpha}\,\mathcal{D}}\right) d\mathcal{D} = \frac{1}{\sqrt{4\,\pi\,t^\alpha}} \exp\left\{-\frac{x^2}{4\,t^\alpha}\right\} , \qquad (A.3)$$

from which spectrum (15) follows.

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