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Transportation Systems with Autonomous Vehicles: models and algorithms for equilibrium assignment

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Abstract

Technologies for connected, automated or autonomous vehicles (AVs) are fast developing, so that they seem ready for substituting in the near future privately owned non-autonomous traditional vehicles (TVs) and further supporting the spread of shared vehicles both for person and good transportation. On the other hand, it may easily be anticipated that the time needed to turn the existing stock of TVs into AVs will last several years during which mixed traffic is expected. A change so great may be not technology-driven only, but also requires a carefully analysis of its several impact through well designed enhancements of tools already available to the transportation systems modelers and planners. Such enhanced tools may be casted in the general framework of multi-user class assignment to transportation networks, concerning: (i) transportation network analysis, through level-of-service models distinguishing between non-autonomous vs. autonomous vehicles, presumably sharing same infrastructure; (ii) travel demand analysis, through behavioral choice modeling paradigms, including choice between AVs vs. TVs, owned vs. shared, as well as route choice behavior; (iii) steady-state equilibrium assignment. This paper describes models and algorithms to deal with steady-state equilibrium assignment; they are used to show to which extent existing methods can still be applied as well as which issues remain still open and worth of further research efforts.

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1. Introduction

Technologies for connected, automated or autonomous vehicles (AVs) are fast developing, so that they seem ready for substituting in the near future privately owned non-autonomous traditional vehicles (TVs) and further supporting the spread of shared vehicles both for person and good transportation. On the other hand, it may easily be anticipated that the time needed to turn the existing stock of TVs into AVs will last several years during which mixed traffic is expected. A change so great may be not technology-driven only, but also requires a carefully analysis of its several impact through well designed enhancements of tools of Traffic and Transportation Theory (TTT) already available to the transportation systems modelers and planners.

TTT studies the interactions between the level of service provided by transportation systems and the results of several types of user choice behavior, which may regard in a hierarchical order:

- driving, concerning interactions between users travelling on the same facility and their effects on travel time, ... ;
- routing, concerning connections between origin and destination of the journey, possibly departing time, … ;
- travelling, concerning transportation mode, time-of-day, destination, frequency, … ;
- mobility, concerning car ownership, driving license acquisition, … .

On top of the above hierarchy there are the kinds of user behavior addressed by land-use/transport interaction theories.

Tools of the Traffic and Transportation Theory have reached a very advanced and sophisticated level, and large-scale applications are current practice (a wide presentation with commented references in Cascetta, 2009). Most of these tools are based on explicitly behavioral modeling approaches, which grant clear interpretation of parameters, and may be referred to two main classes:

- traffic analysis and control [traditionally called traffic engineering], these methods include modeling user driving behavior only;
- transportation systems analysis and design, these methods include modeling of routing user choice behavior, and possibly other choice dimensions as mentioned above.

[Methods based on so-called data-driven (also referred to as soft computing or machine learning, or …) approaches are mainly used for specific tasks, such short-term traffic forecasting, incident detection, … and will not be discussed in this paper.]

Main tools for transportation systems analysis (following a macroscopic approach) are based on methods for *travel demand assignment to a transportation network*, or just *assignment* for short, commonly used to support transportation project assessment and evaluation.

In the following we will discuss to which extent existing methods based can still be applied as such or with straightforward enhancements for the analysis transportation systems with AVs. For brevity's sake kinds of choice behavior others than routing and driving will be not be explicitly considered. Moreover we will assume steady state equilibrium conditions and will not discussed advanced methods aiming at dealing with within-day or day-to-day dynamics.

In this paper we will mainly focus on mathematical features of models and solution algorithms for steady-state equilibrium assignment. Main emphasis is on private road transportation systems for travelers.

The paper is structured as follows: section 2 discusses main assumptions, definitions, notations and equations for modeling a transportation system with AVs; section 3 presents fixed-point models and solution algorithms based on materials in section 2; some concluding remarks and research perspectives are discussed in section 4.

2. Modelling transportation supply and travel demand

This section presents the main assumptions, definitions, notations and equations for modeling assignment to a transportation system with AVs. We will assume

HYP1: *constant demand flows*, which means that choice behavior others than routing and driving are not explicitly considered, thus the origin-destination demand flows are given, and only car mode is considered.

Thus the general *Six Equation Assignment Modeling* approach (SEAM) for travel demand assignment to transportation networks can be followed (cfr Cantarella and Watling, 2016), as briefly reviewed below.

(i) Preliminary stages. Once trip origins and destinations have been single out, itineraries through each pair of origin and destination can be defined, and each itinerary can be broken down into links, each link being a stretch of street (railway, airway, …) with common characteristics. Then, the connections between trip origins and destinations are described by an oriented *graph*, such that each link is described by an oriented *arc* between two nodes and each itinerary is described by a *route*, usually a path (or a hyperpath or a pair of departure time and a path / hyperpath).

(ii) Variables. According to SEAM assignment can be described through six vector variables:

- arc flows,
- arc costs,
- route costs,
- route utilities,
- route choice proportions,
- route flows.

(iii) Equations. According to SEAM assignment can be described through six equations grouped into

- the *transportation supply* model, describing how user choice behaviour affects network level of service:

- arc-route flow consistency relation,
- arc cost(-flow) function, modeling effects of driving behavior (through a macroscopic approach), say congestion,
- route-arc cost consistency relation;

- the *travel demand* model, describing how network level of service affects user choice behaviour:

- route utility function, between route utility and costs,
- route choice function, between route utilities and route choice proportions, modeling the user route choice behavior together with the utility function,
- route-demand flow consistency relation, involving demand flows and route choice proportions.

The arc cost(-flow) function, derived from the traffic flow theory, is highly non-linear as well as the route choice function, derived from discrete choice modeling theories such as the most often used random utility theory. The utility function is specified through an affine transformation almost always in research analysis as well as in practical applications. The route-demand flow consistency equation is linear in any case, under the assumption of constant demand flows.

Within-day dynamics mainly affects the transportation supply model leading to non-linear consistency relations, whilst under steady state conditions both these two relations can be specified through affine transformations. Day-to-day dynamics mainly affects the travel demand model, in this case both the utility function and choice function are specified through recursive equations in the most general case, whilst neglecting the evolution over time and looking for equilibrium states only leads to non-recursive equations.

In this paper we will apply SEAM assuming that:

HYP2: *steady-state equilibrium* is the relevant state to describe the system.

2.1. Basic elements

In this section, main assumptions definitions, notations are reviewed according to the assumption of steady-state equilibrium with utility function specified through an affine transformation. As said above demand flows are assumed constant and one transportation mode is considered, hence route choice is the only user choice behavior affected by network performances, or more properly by congestion.

Users are distinguished with respect to o-d pair they are travelling from/to, user category (users with common socio-economic and behavioral features) and type of used vehicle (traditional, connected, automatic, autonomous, …). Transportation supply is modeled through a flow network, say a graph with a transportation cost and a flow associated to each arc. (Node costs can be considered by duly modifying the graph).

All costs are assumed measured by a common unit, usually travel time or money, through duly homogenization of different attributes. Demand flows are assumed measured by a common unit, traditional vehicles per unit of time, through duly defined equivalent coefficients for different vehicle types (this procedure is not explicitly described in the following); this way all arc and route flows are measured in the same unit.

Main notations are presented below in alphabetical order for easy reference.

Notations

$\mathbf{B}_{ijm}$	the arc-route incidence matrix for o-d pair i, user category j, vehicle type m, with entries $b_{ak} = 1$ if arc a belongs to route k, $b_{ak} = 0$ otherwise, when routes are paths (presented results still hold with more general definitions be used for hyperpaths, or other representations of routes);
$\mathbf{c} \geq \mathbf{0}$	the vector of arc generic costs, common to vehicle types;
$\mathbf{c}(\bullet)$	the arc cost function;
$\mathbf{c}_{jm} \geq \mathbf{0}$	the vector of arc total costs for user category j, vehicle type m (costs are not distinguished by o-d pair);
$\mathbf{c}_{\mathbf{o},jm} \geq \mathbf{0}$	the vector of arc specific costs for user category j, vehicle type m;
$d_{ijm} \geq 0$	the demand flow for o-d pair i, user category j, vehicle type m;
$\mathbf{d} \geq \mathbf{0}$	the vector of demand flows;
$\mathbf{f}_{ijm} \geq \mathbf{0}$	the vector of arc total flows for o-d pair i, user category j, vehicle type m;
$\mathbf{f} \geq \mathbf{0}$	the vector of arc total flows;
$\mathbf{f}(\bullet)$	the arc flow function;
$\mathbf{f}_{\mathbf{b}} \geq \mathbf{0}$	the vector of arc base flows, arc flows not due to modeled user choice behavior.;
$\mathbf{h}_{ijm} \geq \mathbf{0}$	the vector of route flows for o-d pair i, user category j, vehicle type m;
$\mathbf{p}_{ijm} \geq \mathbf{0}$	the vector of route choice probabilities for o-d pair i, user category j, vehicle type m, with $\mathbf{1}^{\mathrm{T}}\, \mathbf{p}_{ijm} = 1$;
$\mathbf{p}_{ijm}(\bullet)$	the choice function for o-d pair i, user category j, vehicle type m;
$\mathbf{v}_{ijm}$	the vector of route total systematic utilities for o-d pair i, user category j, vehicle type m;
$\mathbf{v}_{\mathbf{o},ijm}$	the vector of route systematic utilities for o-d pair i, user category j, vehicle type m, independent of route costs;
$\mathbf{w}_{ijm} \geq \mathbf{0}$	the vector of route total costs for o-d pair i, user category j, vehicle type m;
$\mathbf{w}_{\mathbf{o},ijm} \geq \mathbf{0}$	the vector of route specific or non-additive costs for o-d pair i, user category j, vehicle type m, such as fees, tolls, ...

In the following equations for supply modeling and for demand modeling are introduced according to the general *Six Equation Assignment Modeling* approach (SEAM) (cfr Cantarella and Watling, 2016), as reviewed below. As shown in the following taking into account of several kinds of vehicles affects both the supply and the demand models, through duly defined parameters, to be calibrated against real data.

2.2. Supply model

This section describes the three equations that specify the *transportation supply* model for a transportation system in steady-state conditions (HYP2) with several types of vehicles.

- ARC-ROUTE FLOW CONSISTENCY RELATION

Under the steady-state assumption (HYP2) the arc flows due to each combination of o-d pair i, user category j and vehicle type m can be obtained from the route flows through a linear transformation from the route space to the arc space defined by the arc-route incidence matrix:

$$\boldsymbol{f}_{ijm} = \boldsymbol{B}_{ijm} \cdot \boldsymbol{h}_{ijm} \qquad \forall i, j, m \tag{1.a}$$

Moreover, having assumed that all flows are measured by the same unit, the arc total flows are given by the sum over all o-d pairs, user categories, vehicle types of arc flows plus the arc base flows:

$$\boldsymbol{f} = \Sigma_{ijm} \boldsymbol{f}_{ijm} + \boldsymbol{f_b} \tag{1.b}$$

Thus the arc-route flow consistency relation is given by combining equations (1.a) and (1.b):

$$\boldsymbol{f} = \Sigma_{ijm} \boldsymbol{B}_{ijm} \cdot \boldsymbol{h}_{ijm} + \boldsymbol{f_b} \tag{1}$$

- ARC COST FUNCTION

The arc costs are generally different with respect to the user category j and the vehicle type m (but not with respect to o-d pair i) to reflect different performances, and we assume that

HYP3: the arc cost per vehicle type are given by an affine transformation of the arc generic costs

thus:

$$\boldsymbol{c}_{jm} = \chi_{jm}\, \boldsymbol{c} + \boldsymbol{c}_{\boldsymbol{o},jm} \qquad \forall j,m \tag{2.a}$$

where $\chi_m > 0$ is a (dimensionless) vehicle type specific coefficient. Moreover, due to congestion arc generic costs depend on the arc total flows:

$$\boldsymbol{c} = \boldsymbol{c}(\boldsymbol{f};\ \boldsymbol{\kappa},\ \boldsymbol{\mu}) \qquad \forall \boldsymbol{f} \tag{2.b}$$

where $\boldsymbol{\kappa} > \mathbf{0}$ is the vector of *arc capacities* and $\boldsymbol{\mu} > \mathbf{0}$ indicates the vector of any other parameter, both independent of vehicle type (and o-d pair, or user category). Thus, the arc total costs for each vehicle type m can be expressed as a function of the arc total flows combining together equations (2.a) and (2.b):

$$\boldsymbol{c}_{jm} = \chi_{jm} \cdot \boldsymbol{c}(\boldsymbol{f};\ \boldsymbol{\kappa},\ \boldsymbol{\mu}) + \boldsymbol{c}_{\boldsymbol{o},jm} \qquad \forall j,m \tag{2}$$

- ROUTE-ARC COST CONSISTENCY RELATION

Under the steady-state assumption the route costs for each combination of o-d pair i, user category j and vehicle type m can be obtained from the corresponding arc total costs through an affine transformation from the arc space to the route space defined by the transpose of arc-route incidence matrix:

$$\boldsymbol{w}_{ijm} = \boldsymbol{B}_{ijm}{}^{T} \cdot \boldsymbol{c}_{jm} + \boldsymbol{w}_{ijm} \qquad \forall i, j, m \tag{3}$$

2.3. Demand model

This section describes the three equations that specify the *travel demand* model for a transportation system with several types of vehicles assuming constant demand flows (HYP1).

- ROUTE UTILITY FUNCTION

The utility function for o-d pair *i*, user category *j* and vehicle type *m* is specified through an affine transformation almost always in research analysis as well as in practical applications:

$$\boldsymbol{v}_{ijm} = -\beta_{jm}\, \boldsymbol{w}_{ijm} + \boldsymbol{v}_{\boldsymbol{o},ijm} \qquad \forall\, ijm \tag{4}$$

where $\beta_{im} > 0$ is a parameter such that the term $\beta_{im} \cdot \mathbf{w}_{ijm}$ is dimensionless to be consistent with utility.

- ROUTE CHOICE FUNCTION

Route choice behavior for of o-d pair *i*, user category *j* and vehicle type *m* can be modeled by applying any discrete choice modeling theory so that route choice proportions depend on route utility:

$$\boldsymbol{p}_{ijm} = \boldsymbol{p}_{ijm}(\boldsymbol{v}_{ijm};\ \theta_{jm},\ \eta_{jm}) \qquad \forall ijm \tag{5}$$

where $\theta_{jm} \geq 0$ is the utility *scale parameter* and η_{jm} indicates any other parameter; these parameters are usually not distinguished per o-d pair.

Most often Random Utility Theory is applied (Domencich and McFadden, 1974) assuming that (i) each user, travelling between o-d pair *i*, belonging to user category *j* and using vehicle type *m*, associates to each available route a value of *perceived utility*, (ii) the perceived utility is modeled by a continuous random variable, with mean given by the *(total) systematic utility*, due to several sources of uncertainty regarding the users or the modeler, and (iii) chooses the maximum perceived utility route; thus the choice proportion of an alternative is given by the choice probability, say the probability that its perceived utility is equal to maximum among all alternatives.

When the perceived utility co-variance matrix is non singular, *probabilistic route choice functions* are obtained; in this case the scale parameter also plays the role of the *dispersion parameter* related to the perceived utility standard deviation.

All usually adopted probabilistic choice functions give strictly positive probabilities, and are continuous and continuously differentiable with respect to systematic utility; moreover, if the parameters of the perceived utility pdf do not depend on systematic utility values, the resulting choice function, called *invariant*, is monotone increasing with respect to systematic utility with symmetric (semi-definite positive) Jacobian, in this case choice probabilities depend on differences between systematic utility values only (Cantarella, 1997).

- ROUTE-DEMAND FLOW CONSISTENCY RELATION

Demand conservation for o-d pair *i*, user category *j*, vehicle type *m* can be expressed as:

$$\boldsymbol{h}_{ijm} = d_{ijm}\, \boldsymbol{p}_{ijm} \qquad \forall ijm \tag{6}$$

It assures that flows of all routes connecting the o-d pair *i*, for user category *j*, using vehicle type *m* sum up to demand flow, say $\mathbf{1}^{\mathrm{T}}\, \mathbf{h}_{ijm} = 1\ d_{ijm}$.

3. Multi-vehicle equilibrium assignment

Equilibrium assignment searches for mutually consistent arc flows and costs. It was first introduced under steady-state conditions by Wardrop (1952), who named it User Equilibrium (UE), following what we may now call a deterministic utility approach to routing behavior modeling. A more general kind of equilibrium was introduced by Daganzo and Sheffi (1977), who named it Stochastic User Equilibrium (SUE), following a random utility approach.

Equilibrium assignment may effectively be formulated through fixed-point models (introduced by Daganzo, 1983, through the use of inverse cost function) as shown by the general framework in Cantarella (1997); these models can be easily extended to deal with several types of assignment (such as multi-mode and/or with variable demand) and allow for weaker uniqueness conditions when compared with optimization models and/or Wardrop user equilibrium.

In the following the general *Two Equation Assignment Modeling* approach (TEAM) for travel demand assignment to transportation networks can be followed (cfr Cantarella and Watling, 2016) based on the arc cost function, already introduced, and on the arc flow function briefly reviewed below.

3.1. The arc flow function

A relation between arc flows and (common) arc costs can be obtained by combining together equations (1, 2.a, 3 4, 5, 6) leading to the arc flow function:

$$\boldsymbol{f}(\boldsymbol{c};\, \boldsymbol{d}) = \textstyle\sum_{ijm} d_{ijm}\, \boldsymbol{B}_{ijm} \cdot \boldsymbol{p}_{ijm}(-\beta_{jm}\, \chi_{jm}\, \boldsymbol{B}_{ijm}{}^{T} \cdot \boldsymbol{c} - \beta_{jm}\, \chi_{jm}\, \boldsymbol{B}_{ijm}{}^{T} \cdot \boldsymbol{c}_{\boldsymbol{o},jm} - \beta_{im}\, \boldsymbol{w}_{ijm} + \boldsymbol{v}_{\boldsymbol{o},ijm};\ \theta_{jm},\ \eta_{jm}) + \boldsymbol{f_b}$$

Without any loss of generality the above expression can be simplified by

- considering the product $\beta_{jm}\, \chi_{jm}$ as a single parameter and including it in the utility scale parameter θ_{jm},
- dividing the vector $\mathbf{v}_{\mathbf{o},ijm}$ by parameter $-\beta_{jm}$ and including in the vector $\mathbf{w}_{ijm}$,
- including the vector $\chi_{jm}\, \mathbf{B}_{ijm}{}^{\mathrm{T}}\, \mathbf{c}_{\mathbf{o},jm}$ in the vector $\mathbf{w}_{ijm}$.

This way the (so-called standard) arc flow function is obtained:

$$\boldsymbol{f}(\boldsymbol{c};\, \boldsymbol{d}) = \textstyle\sum_{ijm} d_{ijm}\, \boldsymbol{B}_{ijm} \cdot \boldsymbol{p}_{ijm}(-\boldsymbol{B}_{ijm}{}^{T} \cdot \boldsymbol{c} - \boldsymbol{w}_{ijm};\ \theta_{jm},\ \eta_{jm}) + \boldsymbol{f_b} \tag{7}$$

Let n be the number of arcs, the arc flow function gets values in the *feasible arc flow set*: $S_f \subseteq \mathbb{R}^n_+$, which is non-empty (if the network is connected), compact (since closed and bounded), convex. The arc flow function is continuous and continuous differentiable with respect to (common) arc costs if all the choice functions are continuous and continuously differentiable; moreover, it is monotone non decreasing with respect to (common) arc costs with symmetric (semi-definite negative) Jacobian if all the choice functions are monotone increasing with respect to systematic utility with symmetric (semi-definite positive) Jacobian.

3.2. Fixed-point models and algorithms

In this case, equilibrium assignment can effectively be expressed by fixed-point models given by the arc cost function (2.b) and the arc flow function (7):

$$\boldsymbol{c}^* = \boldsymbol{c}(\boldsymbol{f}^*)\ \in \boldsymbol{c}(S_f) \subseteq R^n_+ \tag{8}$$

$$\boldsymbol{f}^* = \boldsymbol{f}(\boldsymbol{c}^*;\, \boldsymbol{d})\ \in S_f \subseteq R^n_+ \tag{9}$$

Other equivalent models can be formulated with respect to route variables. An equivalent formulation with respect to flows (or costs) only is often used in literature (Cantarella, 1997), which can be obtained by explicitly including equation (8) into equation (9):

$$\boldsymbol{f}^* = \boldsymbol{f}(\boldsymbol{c}(\boldsymbol{f}^*; \boldsymbol{\mu}); \boldsymbol{d}, \theta) \in S_f \subseteq \mathbb{R}_+^n \tag{10}$$

or vice versa equation (9) into (8):

$$\boldsymbol{c}^* = \boldsymbol{c}(\boldsymbol{f}(\boldsymbol{c}^*; \boldsymbol{d}, \theta); \boldsymbol{\mu}) \in \boldsymbol{c}(S_f) \subseteq \mathbb{R}_+^n \tag{11}$$

Existence is guaranteed if both the arc cost function and the arc flow function are continuous (and the network is connected), applying Brouwer theorem to model (10). With reference to model (8, 9) for monotone decreasing arc flow function, if the arc cost function is monotone strictly increasing uniqueness is guaranteed. Uniqueness conditions can be weakened for strictly positive invariant probabilistic route choice functions only requiring that arc cost function is monotone increasing (but not necessarily strictly monotone). Anyhow uniqueness of arc flows also guarantees uniqueness of arc costs as well as route flows and costs, as well as of flows and cost per o-d pair, user category, vehicle kind. Weaker (sufficient) conditions for uniqueness have been recently derived; a full discussion of this topic is out the scope of this paper, it suffices mentioning that monotonicity of the arc cost function is not needed to assure uniqueness.

Applying the Method of Successive Averages (MSA) to model (10) the MSA-FA solution algorithm is obtained based on the recursive equation $\mathbf{f}^k = \mathbf{f}(\mathbf{c}(\mathbf{f}^{k-1})$; it may be proved converging if the Jacobian of the arc cost function is symmetric. On the other hand applying the MSA to model (11) the MSA-CA solution algorithm is obtained based on the recursive equation $\mathbf{c}^k = \mathbf{c}(\mathbf{f}(\mathbf{c}^{k-1})$; it may be proved converging if the Jacobian of the arc flow function is symmetric as shown in Cantarella (1997).

4. Concluding remarks

In this paper a general approach for steady-state equilibrium assignment to transportation systems with different kinds of vehicles has been presented, as an extension of existing method for multi-class assignment to large scale systems. The proposed approach may be further extended to also model the user choice behavior among different kind of vehicles, as likely shown in a future paper. The effect on congestion as well as on user choice behavior can be modeled through duly defined parameters, to be calibrated against real data. Implementation calibration and validation issues are out of the scope of this paper since require data collection, laboratory tests and SP surveys, but surely are all worth of further research effort. A relevant research perspective concerns the extension to within-day dynamics, which needs the introduction of departure time among the user choice alternatives and to apply non-linear arc-route consistency relations, which requires an explicit modeling of the (average) number of users riding in each kind of vehicle.

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