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String Field Theory: a short introduction

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This is a short introduction to open string field theory. Its purpose is to allow the reader to understand and appreciate the recently found analytic solutions of the equation of motion of the theory.

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1. Introduction

The most authoritative candidate to represent the UV completion of our low energy field theory models, in particular the standard model and Einstein-Hilbert gravity, is (super)string theory. Superstring theory seems to have all the ingredients needed to describe the fundamental physical interactions. In particular it provides a consistent quantization of gravity. However how the various physical phenomena are accommodated in the string theory framework, starting from the elementary particle physics and ending with the physics of extreme large distances and times, is still an open problem, to say the least, notwithstanding the strenuous endeavor of string theorists and undeniable, but partial, progress made in many separate applications of string theory. Although the picture is still not precise, it is plausible that the latter can accommodate the physics of the standard model of elementary particles as well as a description of the evolution of the universe; it does also shed light on the black hole physics. However many questions remain unanswered, the identification of the vacuum to start with.

The panorama of superstring theory, on one side, is the same as about twenty years ago. There are five consistent superstring theories in 10D, one open-closed (type I) and four closed (IIA, IIB and two heterotic ones). In addition we have another consistent theory in 11D (M theory), whose low energy limit is 11D supergravity. There exist also other consistent non-supersymmetric theories, but the attention has been mostly focused on the supersymmetric ones. The latter are connected by dualities and appear as limiting cases of a unique theory, characterized by a large moduli space, when the relevant moduli take on specific limiting values. It is this unique theory that people understand when generically referring to superstring theory. The ordinary way to extract low energy information is to compactify the extra dimensions or else to consider configurations of branes. In a way or another it is possible, for instance, to reproduce the spectrum and various qualitative features of the standard model and to produce effective models for the evolution of the universe, describing for instance inflation.

On the other hand in this panorama a different point of view was introduced by Maldacena with his idea of the AdS/CFT correspondence. A stack of D3-branes in type IIB superstring theory generate an AdS geometry that splits physics into two separated systems, a supersymmetric gauge theory and a supergravity theory. However, since the theory is unique, the two systems must be related in a one-to-one way. This argument is the basis of the correspondence. The latter is a duality of the strong-weak coupling type, so that it can be directly verified only in the presence of supersymmetry: the original case refers to N=4 conformal gauge theory in 4D and a supergravity theory in 10D; such an amount of supersymmetry guarantees the persistence of many properties while going from weak to strong coupling. AdS/CFT has been nevertheless hypothesized also in the case of reduced or no supersymmetry, or for non-conformal theories. The basic idea is the holographic correspondence between a gauge theory on the boundary of an AdS space and a (super)gravity theory that lives on the bulk of the latter. This brings into the game a new concept: gauge theories and gravity theories seem to be *complementary* rather than *distinct*, they complete each other rather than being two separate entities. They seem to describe in different ways the same basic underlying physics.

The AdS/CFT correspondence, as it is commonly used, relates two field theories, but it should not be forgotten that, in the original case, it is formulated in the framework of superstring theory and it requires at least type IIB theory on $AdS_5 \times S^5$ for the full duality to work. In other words the natural framework for this kind of correspondence is string theory. And since the low energy effective theory of the open strings on a stack of D-branes is a gauge theory, while gravity is generated by closed strings, one is naturally led to think that the basic duality is the one between open and closed strings. Even more, since closed strings source D-branes, it seems to be unlikely that open and closed strings can be treated as separate entities (except for closed strings at the perturbative level).

A logical conclusion of the previous reasoning is that a full understanding of holographic dualities can be acquired only in the framework of string theory, and that the underlying duality to be considered is the open-closed string duality. Now the question is: what is the best context in which these problems can be analyzed? The abovementioned (super)string theories are first quantized theories, and although one can go a long way even without a second quantized string theory, there seems to be insurmountable difficulties if one tries to draw a completely satisfactory picture of the theory. For instance, while there are no obstructions in constructing on-shell perturbative amplitudes of a given string theory, there is no unambiguous guide in constructing off-shell amplitudes.

Thus it is extremely desirable, if not compulsory, to have a full-fledged second quantized string theory. In this regard, the present situation is as follows. We have a covariant formulation (à la Witten), [1], of second quantized bosonic open string theory (OSFT) with a cubic interaction term, which is well defined and consistent (see below). Witten formulated also a boundary SFT, a theory of 2D theories so to speak, defined on a unit disk with perturbations on the boundary, which, however, has serious renormalization problems. As for bosonic closed string theories, their second quantized version can be formulated in analogy to the OSFT, but the cubic interaction term is not enough to cover the moduli space (see below), so one is obliged to introduce infinite many interaction terms, ending up with a nonpolynomial theory; as a consequence perhaps this theory cannot be properly called a field theory. Coming to the second quantized superstring theories (OSSFT), there is the analog of the bosonic OSFT, also proposed by Witten; this theory however has contact singularities. A successful alternative is Berkovits' open superstring field theory, modelled on the WZW model, which passes many significant tests. Recently strong arguments have been put forward to show that the original Witten's OSSFT is a sort of singular gauge limit of Berkovits' OSSFT. The basic drawback of both approaches is that they have been formulated only for the NS sector while the R sector (the fermionic one) is at present missing.

Summarizing, second quantized superstring theory is still waiting for a complet formulation. On the other hand a bosonic closed SFT does not seem to be (at least) technically viable. If this is so one is obliged to conclude that at present the only consistent SFT at our disposal is Witten's open SFT. Does it make sense to focus on this theory and take it seriously? It should be pointed out that, as a matter of principle, we have no a priori reason to believe that a unique and complete SFT theory exists at all. On the other hand the old objections against Witten's OSFT (the tachyon, the tadpoles contributions) seem by now to be obsolete: the recent successes of this theory indicate that these problems are not intrinsic to the theory but rather to the way we solve it. The traditional motivation: OSFT is an extremely useful playground while waiting for the its consistent full supersymmetric version may be diminishing and perhaps misleading. We will return to this point at the end of the paper. These lecture notes focus on those aspects of OSFT à la Witten that are instrumental in guiding the reader to understand and appreciate the analytic solutions of the SFT equation of motion derived in recent years.

The reader should be made aware that other reviews and lecture notes in SFT exist in the literature, [2, 3, 4, 5, 6, 7, 8]. They cover other aspects not covered here while, sometimes, partially overlapping with the present notes, The reader is invited to consult them.

2. The bosonic open SFT

This is a short summary of first quantized open bosonic theory.

First quantized open string theory in the critical dimension D=26 is formulated in terms of quantum oscillators α_n^μ, $-\infty < n < \infty$, $\mu = 0,1,\dots,25$, which come from the mode expansion of the string scalar field

$$X^\mu(z) = x^\mu - 2ip^\mu \ln z + i\sqrt{2}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n} z^{-n}$$

having set the characteristic square length of the string $\alpha' = 1$. They satisfy the algebra $[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{n+m,0}$, $\eta^{\mu\nu}$ being the space–time Minkowski metric. The vacuum is defined by $\alpha_n^\mu|0\rangle = 0$ for $n > 0$ and $p^\mu|0\rangle = 0$. The states of the theory are constructed by applying to the vacuum the remaining quantum oscillators $\alpha_n^{\mu\dagger} = \alpha_{-n}^\mu$, with $n > 0$. Any such state $|\phi\rangle$ is given momentum k^μ by multiplying it by the eigenfunction e^{ikx}. This state with momentum will be denoted by $|\phi,k\rangle$. In order for such states to be physical they must satisfy the conditions

$$L_n^{(X)}|\phi,k\rangle = 0, \quad n > 0, \qquad (L_0^{(X)} - 1)|\phi,k\rangle = 0 \tag{2.1}$$

where $L_n^{(X)}$ are the matter Virasoro generators

$$L_n^{(X)} = \frac{1}{2}\sum_{k=-\infty}^{\infty} :\alpha_{n-k}^\mu \alpha_k^\nu: \eta_{\mu\nu} \tag{2.2}$$

Here $\alpha_0 = p$ and :: denotes normal ordering. The conditions (2.1) are the quantum translation of the classical on-shell vanishing of the energy–momentum tensor. The $L_n^{(X)}$ are the moments of the energy-momentum tensor of the theory, and the constraints (2.2) are the most stringent one can impose compatible with the Virasoro algebra (2.4) below. These constraints, when $D = 26$, eliminate all the negative norm states of the Fock space and define a physical Hilbert space.

In particular, by means of (2.1), we can identify the physical spectrum of the theory (in D=26). All the states are ordered according to the level, the level being a natural number specified by the eigenvalue of $L_0^{(X)} + L_0^{(gh)} - \frac{p^2}{2}$. The lowest lying state (level 0) is the tachyon represented by the vacuum with momentum k, $|0,k\rangle$, with $p^\mu|0,k\rangle = k^\mu|0,k\rangle$. Its square mass $M^2 = -1$. The next (level 1) is the massless vector state $\zeta_\mu\, \alpha_{-1}^\mu|0,k\rangle$ with $k^2 = 0$ and $\zeta \cdot k = 0$, and is identified with a gauge field. The other states are all massive, with increasing masses proportional to the Planck mass.

String theory is a particular example of 2d conformal field theory. The state-operator correspondence, characteristic of conformal field theory, allow us to associate a 2d field to any state of the spectrum. They are the vertex operators. For instance, to the tachyon we associate $V_t(k) =: e^{ik\cdot X}$:; to the vector state $V_A(k,\zeta) =: \zeta \cdot \dot{X} e^{ik\cdot X}$:, where the dot on top of X denotes the tangent derivative with respect to the world–sheet boundary (the real axis in the z UHP); and so on. In this way the rules of conformal field theory allow us to calculate any kind of amplitude of these operators $\langle V_1(k_1)\dots V_N(k_N)\rangle$, as far as these amplitudes are *on shell*. At low energy, $\alpha' \to 0$, such amplitudes reproduce those of the corresponding field theory (for instance, the amplitudes of V_A reproduce the amplitudes of a Maxwell field theory). If we want to compute *off–shell* amplitudes the above rules are insufficient and in general we have to resort to a field theory of strings. This was a major motivation for introducing string field theories.

So far we have ignored ghosts. Indeed the $b(z), c(z)$ ghosts, which come from the gauge fixing of reparametrization invariance via the Faddeev–Popov recipe, play a minor role in perturbative string theory. They play a much more important role in SFT. We expand them as well in modes c_n and b_n and construct the corresponding Virasoro generators

$$L_n^{(gh)} =: \sum_k (2n+k)\, b_{-k}c_{k+n} : \tag{2.3}$$

Both (2.2) and (2.3) obey the same Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n,0} \tag{2.4}$$

The central charge c equals the number of X fields in the matter case (i.e. 26), while it equals -26 in the case of the b, c ghosts. So the total central charge (i.e. the central charge of $\mathscr{L}_n = L_n^{(X)} + L_n^{(gh)}$) vanishes in D=26. This guarantees the absence of any trace anomaly, and therefore consistency of the bosonic string theory as a gauge theory. From now on we concentrate only on this case.

The previous results about ghosts and critical dimension, can be usefully reformulated in terms of BRST symmetry and its charge Q. Q is defined by

$$Q = \sum_n : c_{-n}\left(L_n^{(X)} + \frac{1}{2}L_n^{(gh)} - \delta_{n,0}\right) : \tag{2.5}$$

It is hermitean $Q^\dagger = Q$ and its basic property is nilpotency, $Q^2 = 0$, (only) in critical dimension. The study of the physical spectrum can be reformulated in terms of the cohomology of Q. First of all the vacuum $|0\rangle$ is understood to be the $SL(2,R)$ invariant vacuum, i.e. the vacuum annihilated by $\mathscr{L}_n$ with $n \geq -1$. It is normalized as follows

$$\langle 0, k| c_{-1}c_0c_1 |0, k'\rangle = \delta^{(26)}(k, k')$$

We apply all possible bosonic and ghost creation operators to $c_1|0\rangle$, and split the so obtained states according to the level l. Then *the physical states of perturbative string theory at level l are the states (with momentum) belonging to that level that are annihilated by Q*, defined up to states obtained by acting with Q on any state of the same level. I.e. *the physical states are identified with the non-trivial cohomology classes of Q*. They can be represented by the old physical states $|\phi, k\rangle$ tensored with the ghost factor $c_1|0\rangle_g$ where $|0\rangle_g$ is the $SL(2,R)$ invariant ghost vacuum.

With this at hand we can now turn to string field theory.

3. OSFT

The open string field theory action proposed by Witten, [1], is defined in D=26 by the action

$$\mathscr{S}(\Psi) = -\frac{1}{g_o^2}\int\left(\frac{1}{2}\Psi * Q\Psi + \frac{1}{3}\Psi * \Psi * \Psi\right). \tag{3.1}$$

This action is clearly reminiscent of the Chern–Simons action in 3D. g_o is the open string coupling. The BRST charge Q is the one introduced above for the first quantized string theory. Later on we will explain what $\Psi, \int$ and $\star$ mean. For the time being let us state the rules they must satisfy:

$$\begin{aligned}
&a) \quad Q^2 = 0, \\
&b) \quad \int Q\Psi = 0, \\
&c) \quad Q(\Psi_1 * \Psi_2) = (Q\Psi_1) * \Psi_2 + (-1)^{\varepsilon(\Psi_1)} \Psi_1 * (Q\Psi_2), \qquad \text{Q is a derivation} \\
&d) \quad \int \Psi_1 * \Psi_2 = (-1)^{\varepsilon(\Psi_1)\varepsilon(\Psi_2)} \int \Psi_2 * \Psi_1, \qquad \text{cyclicity} \\
&e) \quad (\Psi_1 * \Psi_2) * \Psi_3 = \Psi_1 * (\Psi_2 * \Psi_3), \qquad \text{associativity}
\end{aligned} \tag{3.2}$$

where $\varepsilon(\Psi)$ is the Grassmannality of the string field Ψ, which, for bosonic strings, coincides with the ghost number. The action (3.1) is invariant under the BRST transformation

$$\delta\Psi = Q\Lambda + \Psi * \Lambda - \Lambda * \Psi. \tag{3.3}$$

Finally, the ghost numbers of the various objects $Q, \Psi, \Lambda, *, \int$ are $1, 1, 0, 0, -3$, respectively.

It is very often convenient to express the action in a more abstract way. The integral therein can be thought of as a bilinear form $\langle \cdot , \cdot \rangle$:

$$\mathscr{S}(\Psi) = -\frac{1}{g_o^2}\left[\frac{1}{2}\langle \Psi, Q\Psi\rangle + \frac{1}{3}\langle \Psi, \Psi * \Psi\rangle\right]. \tag{3.4}$$

While the properties $a), e)$ in (3.2) remain the same, in terms of $\langle \cdot , \cdot \rangle$ the other properties can be written

$$\begin{aligned}
&c') \quad \langle Q\Psi, \Phi\rangle = -(-1)^{\varepsilon(\Psi)}\langle \Psi, Q\Phi\rangle \\
&d') \quad \langle \Psi, \Phi\rangle = (-1)^{\varepsilon(\Psi)\varepsilon(\Phi)}\langle \Phi, \Psi\rangle \\
&f) \quad \langle \Psi, \Phi \star \Xi\rangle = \langle \Psi \star \Phi, \Xi\rangle
\end{aligned} \tag{3.5}$$

The last property is a consequence of the star product associativity and of identifying the bilinear form with integration. The analog of property b) is not explicitly stated. It is a consequence of the existence of the identity string field I, which is defined by $\Psi \star I = I \star \Psi = \Psi$. I has 0 ghost number and Grassmannality. Using that Q is a derivation we get for any Ψ

$$Q\Psi = Q(I \star \Psi) = QI \star \Psi + I \star Q\Psi = QI \star \Psi + Q\Psi,$$

Thus $QI = 0$, which implies

$$\int Q\Psi = \int Q\Psi \star I = \langle Q\Psi, I\rangle = (-1)^{\varepsilon(\Psi)}\langle \Psi, QI\rangle = 0$$

From now on we understand the identification $\langle \Psi, \Phi\rangle = \int \psi \star \Phi$. The bilinear form can be identified with and extends the inner product in the Fock space

$$\langle A, B\rangle = \langle \mathrm{bpz}(A)|B\rangle \tag{3.6}$$

where, according to the state operator correspondence in CFT, $|B\rangle = B(0)|0\rangle$ and $\langle \mathrm{bpz}(A)| = \lim_{z\to\infty} z^{2h}\langle 0|A(-\frac{1}{z})$, where h is the conformal weight of $A(z)$. By definition

$$\mathrm{bpz}|0\rangle = \langle 0|, \qquad \mathrm{bpz}(A(z)) = \mathfrak{i}\circ A(z) = \frac{1}{z^{2h}}A(-\frac{1}{z}) \tag{3.7}$$

where $\mathfrak{i}(z) = -\frac{1}{z}$. It follows in particular that

$$\mathrm{bpz}(a_n) = (-1)^{n+1}a_{-n}, \qquad \mathrm{bpz}(c_n) = (-1)^{n+1}c_{-n} \qquad \mathrm{bpz}(b_n) = (-1)^n b_{-n}$$

and $\mathrm{bpz}(\mathscr{L}_n) = (-1)^n\mathscr{L}_{-n}$. Here we have adopted the notation: $a_n = \frac{\alpha_n}{\sqrt{n}}$).

Let us next explain in turn what $\psi, \star$ and $\int$ are.

3.1 The string field

In (3.1) Ψ is the string field. It can be understood either as a classical functional of the open string configurations $\Psi[x^\mu(\sigma), c(\sigma), b(\sigma)]$, where $\sigma = \Im\ln(z)$, or as a vector in the Fock space of states of the open string theory. In the sequel we will consider essentially this second point of view. In the field theory limit it makes sense to represent Ψ as a superposition of Fock space states with ghost number 1, with coefficient represented by (infinite many) local fields,

$$|\Psi\rangle = \int d^{26}p\left[(\tilde{\phi}(p) + \tilde{A}_\mu(p)a_1^{\mu\dagger} + \ldots\right] c_1|0\rangle. \tag{3.8}$$

3.2 Star product and integral

One of the most fundamental ingredients is the star product. Physically it represents the string interaction, that is the process of two strings coming together to form a third string. More precisely the product of two string fields Ψ, Φ represents the process of identifying the right half of the first string with the left half of the second string and integrating over the overlapping degrees of freedom, to produce a third string which corresponds to $\Psi * \Phi$. This can be implemented in different ways, either by using the classical string functional, conformal field theory or by means of the oscillator formalism.

Consider two classical string fields $\Psi[x(\sigma)], \Phi[x(\sigma)]$ (for simplicity we ignore the ghost dependence). Then their star product is defined by

$$(\Psi\star\Phi)[z(\sigma)] = \int \prod_{0\le\tau'\le\frac{\pi}{2}} dy(\tau')\,dx(\pi-\tau') \prod_{\frac{\pi}{2}\le\tau\le\pi} \delta(x(\tau)-y(\pi-\tau))\Psi[x(\tau)]\Phi[y(\tau)] \tag{3.9}$$

where $z(\sigma) = x(\sigma)$ for $0 \le \sigma \le \frac{\pi}{2}$ and $z(\sigma) = y(\sigma)$ for $\frac{\pi}{2} \le \sigma \le \pi$. The delta function clearly reproduces the overlapping alluded to above.

The integration in (3.1) corresponds to bending the left half of the string over the right half and integrating over the corresponding degrees of freedom in such a way as to produce a number:

$$\int \Psi = \int \prod_{0\le\tau'\le\pi} dx(\tau') \prod_{0\le\tau\le\frac{\pi}{2}} \delta(x(\tau)-x(\pi-\tau))\Psi[x(\tau)] \tag{3.10}$$

The meaning of these two formulas is rather clear, but they are not very practical. A very practical definition is instead provided by embedding the problem in CFT. Let us start from the integral. As we said above we can interpret it as the inner product in the Fock space. Let us make it clear with an example. Consider the string field (a constant tachyon) $|\Psi\rangle \equiv T = tc_1|0\rangle$, where t is a constant. Then we can compute (disregarding an infinite volume factor)

$$\int T \star QT = \langle T, QT\rangle = t^2\langle 0|c_{-1}, Qc_1|0\rangle = -t^2\langle 0|c_{-1}c_0c_1|0\rangle = -t^2 \tag{3.11}$$

since $[Q, c_1]_+ = -c_0c_1$.

The important point is that the inner product $\langle A|B\rangle$ can be interpreted as a two-points correlator. Consider the maps

$$w_1(z) = \frac{1+iz}{1-iz}, \qquad w_2(z) = \frac{z-i}{z+i} \tag{3.12}$$

The first maps the unit semidisk to a unit semidisk rotated 90^o in the anticlockwise direction, so that the string midpoint $z = i$ is mapped to the origin (see fig.1), and the second maps the unit semidisk to a unit semidisk rotated by 90^o in clockwise sense. If we fit the two final semidisks into a unit disk they represent two strings overlapping the left half of one with the right half of the other and forming a third string which bends on itself so that the two halves overlap (the integral), see fig.1. Now we map the so obtained unit disk to the ζ UHP by means of the map

$$\zeta = h^{-1}(w) = -i\frac{w-1}{w+1} \tag{3.13}$$

and define the maps $\mathrm{j}_i = h^{-1} \circ w_i$, that is

$$\mathrm{j}_1(z) = z, \qquad \mathrm{j}_2(z) = \mathrm{i}(z) = -\frac{1}{z} \tag{3.14}$$

Then we can write

$$\langle A, B\rangle = \langle \mathrm{bpz}(A)|B\rangle = \langle \mathrm{j}_2 \circ A(0)\, \mathrm{j}_1 \circ B(0)\rangle \tag{3.15}$$

As an example let us apply this to (3.11). Starting from $Qc(z) = c\partial c(z)$ and using the correlator

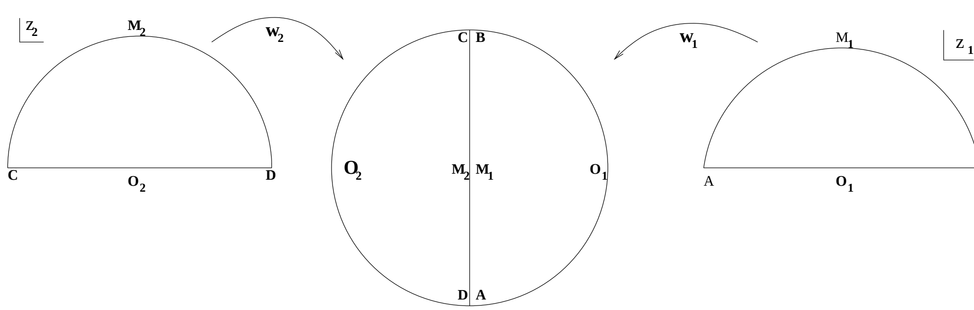

Figure 1: *The conformal maps from the two unit semi-disks to the unit disk*

$$\langle c(z_1)c(z_2)c(z_3)\rangle = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3) \tag{3.16}$$

one gets

$$\langle \mathrm{j}_2 \circ T(0)\, \mathrm{j}_1 \circ QT(0) \rangle = t^2 \, \frac{\langle c(\mathrm{j}_2(0))\, c\partial c(\mathrm{j}_1(0)) \rangle}{\mathrm{j}_2'(0)\mathrm{j}_1'(0)} = -t^2 \tag{3.17}$$

which coincides with $\langle T, QT \rangle$ calculated above, (3.11).

It is easy to apply the same idea to the cubic term in the action. Let us consider three unit semi-disks in the upper half z_a ($a = 1,2,3$) plane. Each one represents the string freely propagating in semicircles from the origin (world-sheet time $\tau = \Re \ln z = -\infty$) to the unit circle $|z_a| = 1$ ($\tau = 0$), where the interaction takes place. We map each unit semi-disk to a 120° wedge of the complex w plane via the following conformal maps:

$$g_a(z_a) = \alpha^{2-a} g(z_a)\,,\; a = 1,2,3, \tag{3.18}$$

where

$$g(z) = \left(\frac{1+iz}{1-iz} \right)^{\frac{2}{3}}. \tag{3.19}$$

Here $\alpha = e^{\frac{2\pi i}{3}}$.

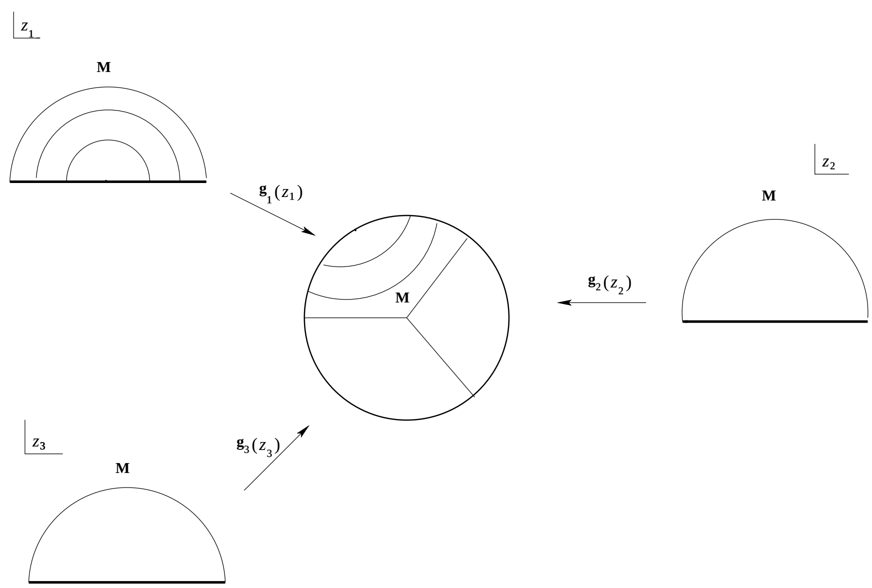

Figure 2: *The conformal maps from the three unit semi-disks to the three-wedges unit disk*

In this way the three semi-disks are mapped to non-overlapping (except along the edges) regions in such a way as to fill up a unit disk centered at the origin. The curvature is zero everywhere except at the center of the disk, which represents the common midpoint of the three strings in interaction. It is clear that this geometry simulates precisely the joining of two strings to form a third string, as explained above. To complete the process we map the unit disk to the ζ UHP via the map h^{-1} and define $f_i = h^{-1} \circ g_i$, $i = 1,2,3$. On this basis the second term in (3.1) can be interpreted as a UHP correlator defined by the above geometry with insertions of Ψ at the origin of each semidisk, which are mapped to the points $-\sqrt{3}, 0, \sqrt{3}$ of the UHP, respectively (after all Ψ is nothing but a - possibly infinite - combination of vertex operators). In general for any three string fields A, B and C, whose total ghost number is 3, the integral of the star product is given by the correlation function on the disk in the following way

$$\int A * B * C = \langle f_1 \circ A(0)\, f_2 \circ B(0)\, f_3 \circ C(0) \rangle \tag{3.20}$$

So, calculating the star product amounts to evaluating a three point function on the UHP.

Let us see an example. Suppose $|\Psi\rangle \equiv T = tc_1|0\rangle$ as above. We have

$$f_1 \circ c(0) = \frac{c(f_1(0))}{f_1'(0)} = \frac{3}{8} c(\sqrt{3}), \qquad f_2 \circ c(0) = \frac{2}{3} c(0), \qquad f_3 \circ c(0) = \frac{3}{8} c(-\sqrt{3})$$

Using (3.16) one finally gets (again forgetting an infinite volume factor)

$$\int T \star T \star T = \frac{81\sqrt{3}}{64} t^3 \tag{3.21}$$

For later use let us record that the action for the string field $T = tc_1|0\rangle$ (constant tachyon) per unit volume, is

$$\int(T) = \frac{1}{g_o^2} \left(\frac{1}{2} t^2 - \frac{27\sqrt{3}}{64} t^3 \right) \tag{3.22}$$

(3.20) suggests how to define the star product of two string fields A and B. It is defined by (3.20) for any string field C.

3.3 The two-strings and three-strings vertex

There is a third way to represent both (3.15) and (3.20). This third way leads to an explicit representation of the star product. It is based on the two-strings and three-strings vertex, which can be explicitly represented in terms of oscillators. The defining relations are

$$\langle A, B \rangle = \int A \star B = \langle \mathscr{V}_2 || A \rangle_1 | B \rangle_2 \tag{3.23}$$

$$\langle A, B, C \rangle = \int A \star B \star C = \langle \mathscr{V}_3 || A \rangle_1 | B \rangle_2 | C \rangle_3 \tag{3.24}$$

for any three string fields A, B, C. $\langle \mathscr{V}_2|$ is defined in the tensor product of two Fock spaces and the labels $1, 2$ in (3.23) refer to the latter. Likewise $\langle \mathscr{V}_3|$ is defined in the triple tensor product of Fock spaces. The ansatz for these vertices (at zero momentum) is as follows

$$\begin{aligned} \langle \mathscr{V}_2| \;=\; & \mathscr{N}_2 (\langle 0|c_{-1})^{(2)} (\langle 0|c_{-1})^{(1)} \left(c_0^{(1)} + c_0^{(2)} \right) e^{-\frac{1}{2}\sum_{r,s=1}^{2} \sum_{n,m\geq 1} \alpha_m^{(r)} M_{mn}^{rs} \alpha_n^{(s)}} \\ & \cdot e^{-\frac{1}{2}\sum_{r,s=1}^{2} \sum_{m,n\geq 1} b_m^{(r)} Y_{mn}^{rs} c_n^{(s)}} \end{aligned} \tag{3.25}$$

$$\begin{aligned} \langle \mathscr{V}_3| \;=\; & \mathscr{N}_3 (\langle 0|c_{-1}c_0)^{(3)} \langle 0|c_{-1}c_0)^{(2)} (\langle 0|c_{-1}c_0)^{(1)} e^{-\frac{1}{2}\sum_{r,s=1}^{3} \sum_{n,m\geq 1} \alpha_m^{(r)} N_{mn}^{rs} \alpha_n^{(s)}} \\ & \cdot e^{-\frac{1}{2}\sum_{r,s=1}^{3} \sum_{m\geq 0, n\geq 1} b_m^{(r)} X_{mn}^{rs} c_n^{(s)}} \end{aligned} \tag{3.26}$$

Here and in the sequel, for simplicity, we understand the Lorentz indices of the α oscillators. The constants $\mathscr{N}_2$ and $\mathscr{N}_3$ can be fixed by normalizing the vertex in such a way that $\langle c, c\partial c\rangle = \langle \mathscr{V}_2||c\rangle_1|c\partial c\rangle_2 = -1$, as in eq.(3.17) ($|c\rangle = c(0)|0\rangle$, etc.), and $\langle c,c,c\rangle = \langle \mathscr{V}_3||c\rangle_1|c\rangle_2|c\rangle_3 = \frac{81\sqrt{3}}{64}$ as required by $\langle c_{-1}c_0c_1\rangle = 1$ and by (3.21). This implies

$$\mathscr{N}_2 = 1, \qquad \mathscr{N}_3 = \beta^3, \qquad \beta = \frac{3\sqrt{3}}{4}$$

Clearly $M^{rs}_{nm} = M^{sr}_{mn}$, $N^{rs}_{nm} = N^{sr}_{mn}$.

A simple method to determine the entries of the matrices M,N,X,Y is to impose that the correlator of two free fields be reproduced by radial ordered product of the two free fields contracted between the vertex and the vacua. In the sequel we will explain this procedure in detail for the two strings vertex at zero momentum. For the matter part of the latter we must have

$$\langle \mathscr{V}_2|\mathscr{R}\left(i\partial X^{(r)}(z)\; i\partial X^{(s)}(z)\right)|0\rangle_1|0\rangle_2 = \langle \mathrm{j}_r \circ i\partial X(z)\; \mathrm{j}_s \circ i\partial X(w)\rangle \tag{3.27}$$

with $r,s = 1,2$. We have dropped, for simplicity, the Lorentz indices in X. For instance, for $|z| > |w|$, the LHS is

$$\langle \mathscr{V}_2| \sum_{n,m<0} \frac{\alpha^{(r)}_n}{z^{n+1}} \frac{\alpha^{(s)}_m}{z^{m+1}}|0\rangle_1|0\rangle_2 = -\sum_{n,m=1} M^{rs}_{mn} z^{n-1}\, w^{m-1} = M^{rs}(z,w) \tag{3.28}$$

$M^{rs}(z,w)$ is referred to as Neumann function. Using the correlator $\langle i\partial X(z)\; i\partial X(w)\rangle = \frac{1}{(z-w)^2}$, the RHS of (3.27) becomes

$$\frac{\mathrm{j}_r'(z)\mathrm{j}_s'(w)}{(\mathrm{j}_r(z)-\mathrm{j}_s(w))^2} = \frac{1}{(1+zw)^2}, \quad r\neq s, \qquad = \frac{1}{(z-w)^2}, \quad r=s$$

It follows that

$$\begin{aligned} M^{12}_{mn} = M^{21}_{mn} &= -\frac{1}{nm}\oint \frac{dz}{2\pi i}\frac{1}{z^m}\oint \frac{dw}{2\pi i}\frac{1}{w^n}\frac{1}{(1+zw)^2} = \frac{(-1)^n}{n}\delta_{n,m} \\ M^{11}_{mn} = M^{22}_{mn} &= -\frac{1}{nm}\oint \frac{dz}{2\pi i}\frac{1}{z^m}\oint \frac{dw}{2\pi i}\frac{1}{w^n}\frac{1}{(z-w)^2} = 0 \end{aligned}$$

As for the ghost part, we use the $b-c$ propagator $\langle b(z)c(w)\rangle = \frac{1}{z-w}$. In order to determine Y^{rs}_{mn}, we equate

$$G^{rs}(z,w) = \langle \mathscr{V}^{(g)}_2|\mathscr{R}\left(b^{(s)}(z)c^{(r)}(w)\right)c^{(1)}_1|0\rangle_1 c^{(2)}_0 c^{(2)}_1|0\rangle_2 \tag{3.29}$$

with

$$\langle \mathrm{j}_s \circ b(z)\; \mathrm{j}_r \circ c(w)\; \mathrm{j}_2 \circ c(0)\; \mathrm{j}_1 \circ c\partial c(0)\rangle \tag{3.30}$$

From the first equation we get

$$Y^{rs}_{mn} = -2\oint \frac{dz}{2\pi i}\frac{1}{z^{n-1}}\oint \frac{dw}{2\pi i}\frac{1}{w^{m+2}}\, G^{rs}(z,w) \tag{3.31}$$

On the other hand, using the correlator

$$\begin{aligned}\langle b(z)\,c(w)\,c(x)\,c\partial c(w)\rangle \;=\; & -\frac{1}{z-w}(x-y)^2+\frac{1}{z-x}(w-y)^2+\frac{1}{x-y}(w-x)^2\\ & -\frac{1}{(z-y)^2}(w-x)(w-y)(x-y)\end{aligned}$$

(3.30) can be rewritten as

$$\frac{\mathrm{j}_s'(z)^2}{\mathrm{j}_r'(w)}\left(-\frac{1}{\mathrm{j}_s(z)-\mathrm{j}_r(w)}-\frac{\mathrm{j}_r(w)}{\mathrm{j}_s(z)^2}-\frac{1}{\mathrm{j}_s(z)}\right) \tag{3.32}$$

Equating (3.29) and (3.30) yields

$$\begin{aligned}G^{11}(z,w) &= -\frac{1}{z-w}-\frac{w}{z^2}-\frac{1}{z}, \qquad G^{22}(z,w) = -\frac{w^3}{w^3}\frac{1}{z-w}+\frac{w}{z^2}\\ G^{12}(z,w) &= \frac{1}{z^3}\frac{1}{1+zw}-\frac{w}{z^2}+\frac{1}{z^3}, \qquad G^{21}(z,w) = -w^3\frac{1}{1+zw}+\frac{w}{z^2}-\frac{w^2}{z}\end{aligned}$$

Plugging these results in (3.31) we get

$$X^{11}_{mn}=X^{22}_{mn}=0, \qquad X^{12}_{mn}=X^{21}_{mn}=-(-1)^n\delta_{n,m}$$

On the basis of these results the two-strings vertex at zero momentum can be written

$$\langle \mathscr{V}_2| \;=\; (\langle 0|c_{-1})^{(2)}(\langle 0|c_{-1})^{(1)}\left(c_0^{(1)}+c_0^{(2)}\right)\, e^{-(-1)^n\sum_{n\geq 1}\left[a_n^{(1)}a_n^{(2)}+c_n^{(1)}b_n^{(2)}+c_n^{(2)}b_n^{(1)}\right]} \tag{3.33}$$

This is called also *reflector* because it maps any string field into its bpz-conjugate.

The three-strings vertex can be determined in an analogous way. For instance, for the matter part we equate

$$\langle \mathscr{V}_3^{(m)}|\mathscr{R}\left((i\partial X^{(r)}(z)\; i\partial X^{(s)}(w)\right)c_1^{(1)}|0\rangle_1 c_1^{(2)}|0\rangle_2 c_1^{(3)}|0\rangle_2 \tag{3.34}$$

with

$$\mathscr{N}_3\langle f_r\circ i\partial X(z)\;\; f_s\circ i\partial X(w)\rangle \tag{3.35}$$

Using the correlator $\langle i\partial X(z)\; i\partial X(w)\rangle = \frac{1}{(z-w)^2}$ and proceeding as above one gets

$$N^{rs}_{nm} = -\frac{1}{nm}\oint\frac{dz}{2\pi i}\frac{1}{z^m}\oint\frac{dw}{2\pi i}\frac{1}{w^n}\,\frac{f_r'(z)\,f_s'(w)}{(f_r(z)-f_s(w))^2} \tag{3.36}$$

For the ghost part one gets similarly

$$X^{rs}_{nm} = \oint\frac{dz}{2\pi i}\frac{1}{z^{n-1}}\oint\frac{dw}{2\pi i}\frac{1}{w^{m+2}}\,\frac{f_r'(z)^2}{f_s'(w)}\,\frac{-1}{f_r(z)-f_s(w)}\,\frac{\prod_{i=1}^3(f_s(w)-f_i(0))}{\prod_{j=1}^3(f_r(z)-f_j(0))} \tag{3.37}$$

from which one can compute all the entries. It has to be noticed that both in (3.36) and in (3.37) we can replace we can replave f_r with g_r without changing the result. This is due to invariance of the correlators used under the map h^{-1}, more generally under dilatations and $SL(2R)$ transformations.

When considering states carrying nonzero momentum we have to change the matter part of the above vertices. For the two strings vertex the change is rather modest. In (3.25) we use the vacuum with momentum $\langle 0;k|$ instead of the simple vacuum $\langle 0|$ and integrate over the conserved momentum. So

$$\langle \mathscr{V}_2| = \int d^{26}p(\langle 0;p|c_{-1})^{(2)}(\langle 0;-p|c_{-1})^{(1)} \left(c_0^{(1)} + c_0^{(2)}\right) e^{-(-1)^n \sum_{n\geq 1}\left[a_n^{(1)}a_n^{(2)}+c_n^{(1)}b_n^{(2)}+c_n^{(2)}b_n^{(1)}\right]} \tag{3.38}$$

The changes in the three strings vertex are more sizable. The end result turns out to be

$$\begin{aligned}\langle \mathscr{V}_3| = {} & \beta^3 \int d^{26}p^{(3)} \int d^{26}p^{(2)} \int d^{26}p^{(1)}(\langle 0;p|c_{-1}c_0)^{(3)}\langle 0;p|c_{-1}c_0)^{(2)}(\langle 0;p|c_{-1}c_0)^{(1)} \\ & \cdot e^{-\frac{1}{2}\sum_{r,s=1}^3\left[\sum_{n,m\geq 1}\alpha_m^{(r)}V_{mn}^{rs}\alpha_n^{(s)}+\sum_{m\geq 1}2\alpha_m^{(r)}V_{m0}^{rs}p^{(s)}+p^{(r)}V_{00}^{rs}p^{(s)}\right]}\, e^{-\frac{1}{2}\sum_{r,s=1}^3\sum_{m\geq 0,n\geq 1}b_m^{(r)}X_{mn}^{rs}c_n^{(s)}}\,\delta\left(\sum_{r=1}^3 p^{(r)}\right)\end{aligned} \tag{3.39}$$

where the coefficient are as follows.

$$V_{nm}^{rs} = \sqrt{nm}\,N_{nm}^{rs} \tag{3.40}$$

and N_{nm}^{rs} are the Neumann coefficients given above (3.36). The V_{m0}^{rs} are related to the zero mode Neumann coefficients N_{m0}^{rs} defined by

$$N_{0m}^{rs} = -\frac{1}{m}\oint \frac{dw}{2\pi i}\frac{1}{w^m}g_s'(w)\frac{1}{g_r(0)-g_s(w)} \tag{3.41}$$

by the relation

$$V_{n0}^{rs} = (-1)^n\sqrt{2n}\left(N_{0n}^{rs} - \frac{2}{3}\frac{i^n}{n}\right) \tag{3.42}$$

Finally

$$V_{00}^{ab} = \delta_{ab}\ln\frac{27}{16} \tag{3.43}$$

This is motivated by one of the most surprising and mysterious aspects of SFT, namely its underlying integrable structure: the matter Neumann coefficients obey the Hirota equations of the dispersionless Toda lattice hierarchy.

From (3.383.39) one can compute the corresponding ket expressions for the vertices, by taking the *bpz*- conjugate. Using them one can do more than computing the SFT action, one can compute the star product of any two string fields Ψ_1, Ψ_2. To this end compute

$$\langle \Psi_3| = \langle \mathscr{V}_3||\Psi_1\rangle_1|\Psi_2\rangle_2$$

This is the *bpz*-conjugate of $\Psi_1 \star \Psi_2$. Alternatively compute $\langle \Psi_3||\mathscr{V}_2\rangle$.

3.4 Gauge symmetry and gauge fixing

Above we have pointed out that the action (3.1) is invariant under the gauge transformation

$$\delta\Psi = Q\Lambda_0 + \Psi\star\Lambda_0 - \Lambda_0\star\Psi \tag{3.44}$$

In this formula we have emphasized with a subscript 0 the fact that Λ_0 has zero ghost number. We remark that (3.44) is the infinitesimal version of

$$\Phi \to e^{\Lambda_0} (Q + \Psi) e^{\Lambda_0} \tag{3.45}$$

In order to limit the complexity of the formulas we had better fix this huge gauge freedom. This is usually done by choosing the Siegel gauge, i.e. imposing the condition $b_0|\Psi\rangle = 0$.

We will see that this works very well and, in fact, corresponds to the Lorenz gauge in the field theory limit. Here we would like to point out, however, that it cannot be as easily implemented with the Faddeev-Popov method as the Lorenz gauge in field theory, for it is reducible. If we change

$$\Lambda_0 \to \Lambda_0 + \Psi \star \Lambda_{-1} + \Lambda_{-1} \star \Psi$$

$\delta\Psi$ in (3.44) does not change if $Q\Psi + \Psi \star \Psi = 0$. The same is true for any transformation

$$\Lambda_{-n} \to \Lambda_{-n} + \Psi \star \Lambda_{-n-1} + (-1)^n \Lambda_{-n-1} \star \Psi$$

where $-n$ is the ghost number. Therefore, due to reducibility, the traditional FP method of introducing $1 = \Delta_{FP}(\Phi) \int \mathscr{D}\Lambda \delta\left(F\left(\Psi_\Lambda\right)\right)$, where $F(\Psi) = 0$ is a linear gauge fixing, does not work properly because it fixes the gauge freedom only partially. We need the BV approach.

The action (3.1) has indeed been quantized with the BV method. Choosing the Siegel gauge, i.e. imposing the condition $b_0|\Psi\rangle = 0$ to fix the enormous gauge symmetry (3.3), the kinetic term becomes particularly simple and can be easily inverted to produce a free propagator $b_0 L_0^{-1}$. This allows one to define the perturbative series and relevant Feynman rules. 0-th and 1-st order amplitudes for tachyons have been computed. Putting the external legs on shell, they reproduce the corresponding first quantized amplitudes, in particular the Veneziano amplitude. This is an important check, but of course now one has an unambiguous way to compute off-shell expressions for the amplitudes, virtually to any perturbative order. What is more important, one should remember that the first quantized amplitudes are integrated over the moduli space of the appropriate Riemann surfaces corresponding to the given perturbative order. It is far from obvious a priori that the perturbative OSFT reproduces the same procedure. However one of the most remarkable results in this context was the proof that it fully ‘covers’ the moduli space of Riemann surfaces and it does it only once. This is in contrast to the analogous problem in closed string field theory, where a third order interaction is not sufficient to cover the full moduli space, and one is obliged to introduce higher order vertices.

In conclusion the OSFT introduced in this section reproduces the results of first quantized string theory. Its added value with respect to the latter is not only that it allows us to compute off-shell amplitudes, but especially that it puts us in the condition to tackle nonperturbative problems. The first and up to now most remarkable result of SFT is the treatment of *tachyon condensation.*

4. The tachyon condensation

Following the rules of the previous section it is possible in favorable cases to explicitly compute the action (3.1). For instance, in the low energy limit, where the string field may be assumed to take the form (3.8), the action becomes an integrated function F of an infinite series of local polynomials (kinetic and potential terms) of the fields involved in (3.8). To limit the number of terms one has to limit once again the gigantic BRST symmetry of OSFT, by choosing a gauge, which is usually the Feynman–Siegel gauge: this means that we limit ourselves to the states that satisfy the condition: $b_0|\Psi\rangle = 0$. As an example, let us write down the first few terms of the most general string field

$$\Psi = \int d^{26}k \left(\tilde{\phi}(k) + \tilde{A}_\mu(k)\alpha^\mu_{-1} + \tilde{\chi}(k)b_{-1}c_0 + \tilde{B}_{\mu\nu}(k)\alpha^\mu_{-1}\alpha^\nu_{-1} + \ldots\right) c_1|0;k\rangle \tag{4.1}$$

The Feynman–Siegel gauge eliminates the χ term. Still the action with all the infinite sets of fields contained in Ψ remains unwieldy. As it turns out, it makes sense to limit the number of fields in Ψ, provided we insert all the fields up to a certain level. This is called *level truncation* and turns out to be an excellent approximation and regularization scheme in SFT. Let us truncate the string field and keep only the first two terms in the RHS. For instance, for the kinetic term of the action we get

$$\frac{1}{2}\langle\mathscr{V}_2|\Psi\rangle_1|Q\Psi\rangle_2 = \int d^{26}k \left(\tilde{\phi}(-k)\frac{p^2-1}{2}\tilde{\phi}(k) + \tilde{A}_\mu(-k)\frac{p^2}{2}A^\mu(k) + \ldots\right)$$

Altogether, after Fourier anti-transforming, one obtains

$$\begin{aligned}\mathscr{S}(\Psi) = \frac{1}{g_o^2}\int d^{26}x \Big(&-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\phi^2 - \frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu - \frac{\beta^3}{3}\hat{\phi}^3 \\ &-\frac{\beta}{2}\left(\partial_\mu\partial_\nu\hat{\phi}\hat{A}^\mu\hat{A}^\nu + \hat{\phi}\partial_\mu\hat{A}^\nu\partial_\nu\hat{A}^\mu - 2\partial_\mu\hat{\phi}\partial_\nu\hat{A}^\mu\hat{A}^\nu\right)\Big)\end{aligned} \tag{4.2}$$

where again $\beta = \frac{3\sqrt{3}}{4}$. One can see the kinetic term and the 'wrong' mass term for the tachyon, as well as the gauge-fixed kinetic term for the gauge field. The fields appearing in the interaction term carry a hat. This means

$$\hat{f}(x) = e^{(\ln\beta)\partial_\mu\partial^\mu} f(x)$$

for any local field f. Incidentally, the fact that the interaction is formulated in terms of hatted fields is a manifestation of the strong (exponential) convergence properties of string theory in the UV.

We would like now to single out the potential in the action (4.2) and study its minimum. For a static configuration the potential coincides with $-$ the action. But this is not enough. We must remember that this theory is supposed to represent the open strings attached to a space–filling D–brane, the D25–brane. So the total energy is the sum of the brane energy plus the energy of the string modes. The brane has its intrinsic energy, whose density is the tension τ, which in our conventional units ($\alpha' = 1$), is given by $\tau = \frac{1}{2\pi^2 g_o^2}$. The string modes are represented by the action and, as we have just said, in a static situation their total energy is given by $-$ the action itself. We wish to study this system in the vacuum. Lorentz invariance requires that only scalars can acquire a VEV. Therefore in (4.2) one must set all the derivatives to 0. Setting $\langle\phi\rangle = t$, what remains of the action (divided by the total volume) can be written in terms of the function $u(t)$ as follows

$$-\frac{S}{V} \equiv \tau\, u(t) = 2\pi^2\,\tau, \left(-\frac{1}{2}t^2 + \frac{1}{3}\beta^3 t^3\right) \tag{4.3}$$

This is the total tachyon potential energy density extracted from the action. It is proportional to (3.22).

The total energy of the system will be given by the sum of (4.3) and the D25–brane tension

$$U(t) = \tau(1+u(t)) \tag{4.4}$$

This potential is cubic, it goes to $+\infty$ for positive large t and to $-\infty$ for negative large t and it has a local maximum and and a local minimum, which are easily determined. The former is at $t=0$, the latter is given by

$$t = t_0 = \frac{1}{\beta^3}, \qquad u(t) \approx -0.684 \tag{4.5}$$

Of course this is a first approximation result. Considering higher level scalar fields (there are infinite many of them) the minimum will be modified. The numerical evaluations performed within the level truncation scheme indicate that the true minimum of the potential corresponds to $u=-1$, i.e. $U=0$. This coincides with the first conjecture by Sen.

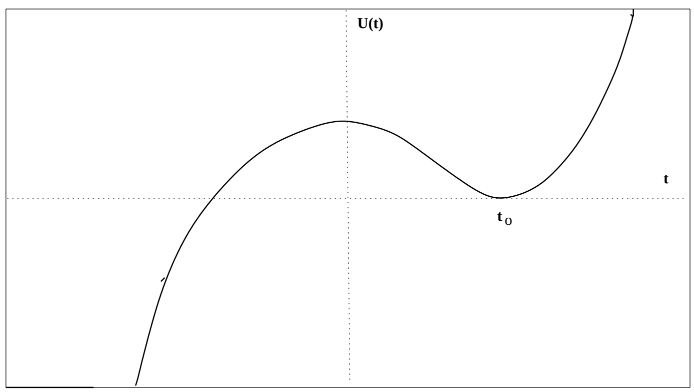

Figure 3: *The tachyon potential*

In order to describe the physics of tachyon condensation Sen [9] formulated three conjectures. The first claims that at the minimum of the potential the theory must be stable, so the energy of the space-filling brane must compensate exactly the energy of the strings. The second conjecture concerns the features of the tachyon condensation vacuum: in this vacuum there cannot be open string modes, i.e. it is the vacuum of an entirely different system, that of closed strings. The third conjecture is a consequence of this statement: one should be able to find in the new vacuum the physics of closed string theory[1].

The numerical results mentioned above were the first evidence that Sen's first conjecture is correct. Also the other conjectures got support from numerical methods or via cousin theories, such as BCFT. After this evidence the real challenge was to find a solution of the SFT equation of motion. The turning point in this field came in 2005 with the first analytic tachyon vacuum solution found by Schnabl [11].

[1]Often in the literature the second and third conjecture are called third and second, respectively. To me this seems to be logically reversed.

5. The analytic solution

The equation of motion derived from (3.1) is

$$Q\Psi + \Psi * \Psi = 0 \tag{5.1}$$

In this section I will explain how the first analytic solution of this equation was found, [11]. This solution is a string state that specifies the (locally) stable vacuum, to be identified as the closed string vacuum. In the oversimplified language of the figure (3) it would correspond to $|\Psi_0\rangle = t_0 c_1|0\rangle$, but it actually identifies the vev of all the infinite many scalar fields that feature in the most general string field.

5.1 The new coordinate and the wedge states

The breakthrough was facilitated by an improvement in the mathematical language of SFT. One can now say, in hindsight, that for many years any progress was thwarted by the complexity of the star product. A simple change of geometrical perspective suddenly made everything easier, the star algebra took up a very simple form (see below). The geometrical improvement consists in the arctan map. This map

$$\xi(z) = \arctan(z)$$

maps the unit semidisk in the z plane to the semi-infinite shaded area in the ξ plane, see fig.4. The complementary part of the semidisk in the upper half z plane is mapped to the unshaded semi-infinite rectangles on the two side of the latter, the two external sides being identified as they correspond to the point at ∞ in the UHP. The resulting figure is a semi-infinite cylinder of circumference π.

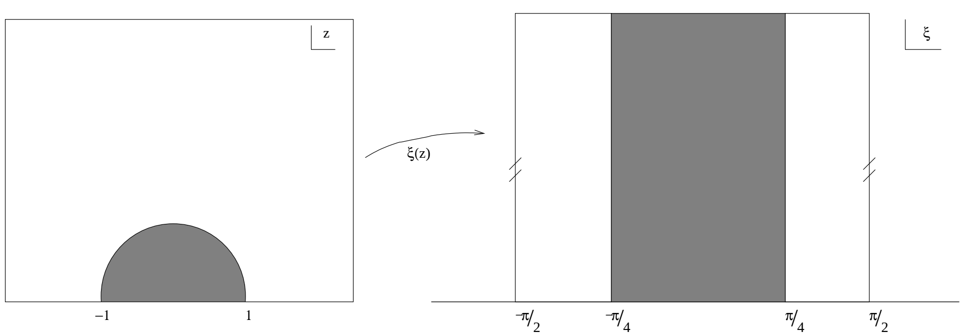

Figure 4: *The arctan map*

The first simple application of this frame is to wedge states. Wedge states are particular surface states. The latter are states defined as follows: take any map f from the UHP to a Riemann surface Σ, for instance the unit disk; we will denote by $\mathscr{R}$ the surface Σ minus the image $\mathscr{H}$ of the unit halfdisk in it. Let us consider any field ϕ and the state $|\phi\rangle = \phi(0)|0\rangle$ in the Fock space of the theory; then the surface state $|S\rangle$ is defined by

$$\langle \phi | S \rangle = \langle I \circ f \circ \phi \rangle_\Sigma \tag{5.2}$$

The definition is implicit and may seem at first not very handy, but one can reduce the calculation to very simple test states $|\phi\rangle$, much in the same way as we have done in calculating the Neumann coefficients for the two and three strings vertices.

It follows that a surface state can be written as a squeezed state represented by a Neumann matrix S^f_{nm}, both for the matter and the ghost part. For instance

$$|S\rangle = e^{-\frac{1}{2}\sum_{m,n\geq 1}\alpha^\dagger_n S^f_{mn}\alpha^\dagger_n}|0\rangle \tag{5.3}$$

where

$$S^f_{mn} = -\frac{1}{mn}\oint\frac{dz}{2\pi i}\frac{1}{z^n}\oint\frac{1}{2\pi i}\frac{1}{w^m}\partial_z\partial_w \ln\left(f(z)-f(w)\right) \tag{5.4}$$

The star product of two surface states has a simple geometrical interpretation. As we have seen, we associate to each state a unit disk with the image of the unit semidisk and the complementary $\mathscr{R}$ patch in it. Remove the patch $\mathscr{R}_2$ from the unit disk Σ_2 associated to the second state, cut Σ_1 along the right half string in the boundary of $\mathscr{H}_1$ and glue it with the left half string of $\mathscr{H}_2$. The right half string of $\mathscr{H}_2$ is glued to $\mathscr{R}_1$ in such a way to form a new unit disk. The resulting surface (and map) defines the star product

Wedges states are particularly simple. Their defining functions are

$$f_r(z) = \tan\left(\frac{2}{r}\arctan(z)\right) = h^{-1}\left(h(z)^{\frac{2}{r}}\right) \tag{5.5}$$

where, for simplicity, we take r to be a positive integer. This map first rotates the unit semidisk anticlockwise by 90^o (see the right hand side of fig.1), then shrinks it to a wedge of angle $\frac{2\pi}{r}$, finally rotates it back clockwise by 90^o. Instead of using f_r, which maps to the UHP, we can stop midway at $w = h(z)^{\frac{2}{r}}$, which maps to a wedge in the unit disk in the w plane. Using $w = h(z)^{\frac{2}{r}}$ it is easy to compute the Neumann matrix of any wedge state.

The star product of wedge states is characterized by an elemenary recursion relation

$$|r\rangle \star |s\rangle = |r+s-1\rangle \tag{5.6}$$

In particular we see that calling $|\Xi\rangle$ the result of taking $r\to\infty$ in $|r\rangle$, we recover $\Xi^2 = \Xi$. This may seem formal, but it can be shown to give rise precisely to the sliver, which is a surface state defined by a wedge of vanishing angle (see below for a more accurate definition). So, in particular, wedge states approximate the sliver.

The star product of wedge states takes a particularly simple form in the arctan frame. In this new representation a wedge state $|r\rangle$ is represented by a cylinder in the ξ UHP of circumference $\frac{\pi r}{2}$, see fig.5. It is in fact an infinite strip in the imaginary direction of width $r\frac{\pi}{2}$. It is formed by two external strips of width $\frac{\pi}{4}$ each (the ruled strips in the figure), and an internal strip of width $(r-1)\frac{\pi}{2}$. The rightmost and leftmost sides are identified so as to form a cylinder. The star product of two such states is simply obtained by dropping the rightmost ruled strip of the first state and the leftmost ruled strip of the second and gluing the two cut cylinders along the dashed line in fig.5. In this language the wedge state with $r = 2$ corresponds to the vacuum $|0\rangle$ and the state with $r = 1$ to the identity state $|I\rangle$.

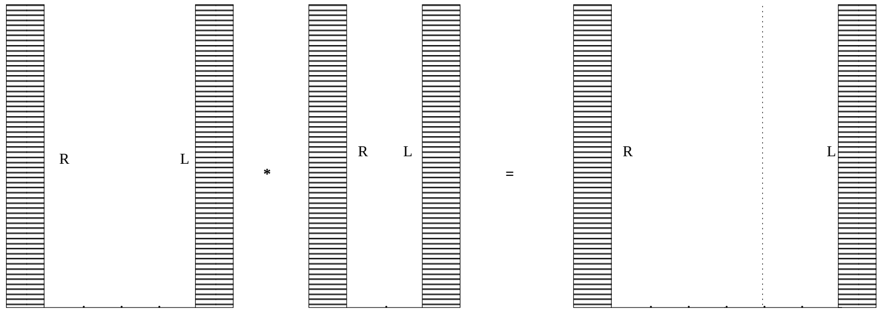

Figure 5: *Star product of two wedge states* $|3\rangle \star |2\rangle = |4\rangle$

Pure wedge states, as we have just described them, are not enough to describe the analytic solution we are looking for. Later on we will need wedge states with insertion of operators in the real diameter of the halfdisk, that is wedge states with the insertion of an operator at some point of the unruled patches. The $|n\rangle$ wedge state itself can be seen as such.

$$|n\rangle = \left(\frac{2}{n}\right)^{\mathscr{L}_0^\dagger} |0\rangle \tag{5.7}$$

where $\mathscr{L}_0$ will be introduced in a moment.

These states will play a major role in what follows. What we need now is exploit the new coordinate $\xi = \arctan z$ to get a few basic definitions and relations. If we map a primary operator $\mathscr{O}^h$ of weight h to the arctan frame we have

$$\tilde{\mathscr{O}}^h(\xi) = f'(\xi)^h \mathscr{O}^h(z), \qquad z = f(\xi) = \tan(\xi)$$

The corresponding modes in the expansion $\mathscr{O}^h(z) = \sum_n \mathscr{O}^h_n z^{-n-h}$ will transform according to

$$\tilde{\mathscr{O}}^h_n = \sum_m \oint \frac{dz}{2\pi i} \xi(z)^{n+h-1} (f'(\xi(z))^{h-1} z^{-m-h} \tag{5.8}$$

For instance, denoting with $\mathscr{L}_n$ the Virasoro generators in the arctan frame, we get

$$\mathscr{L}_n = \oint \frac{dz}{2\pi i} (1+z^2) \left(\arctan(z)\right)^{n+1} T(z) \tag{5.9}$$

In particular

$$\mathscr{L}_{-1} = L_1 + L_{-1} \equiv K_1 \tag{5.10}$$

$$\mathscr{L}_0 = L_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2-1} L_{2k} \tag{5.11}$$

They satisfy $[\mathscr{L}_n, \mathscr{L}_m] = (n-m)\mathscr{L}_{n+m}$. The central charge is obviously zero because we are in critical dimensions.

The hermitean conjugates of $\mathscr{L}_n$ are

$$\mathscr{L}_n^\dagger = \oint \frac{dz}{2\pi i} (1+z^2) \left(\text{arccot}(z)\right)^{n+1} T(z) \tag{5.12}$$

A remarkable commutator is

$$[\mathscr{L}_0, \mathscr{L}_0^\dagger] = \oint \frac{dz}{2\pi i}(1+z^2)\,(\arctan(z)+\mathrm{arccot}(z))\,T(z) = \mathscr{L}_0 + \mathscr{L}_0^\dagger \tag{5.13}$$

We have

$$\arctan(z)+\mathrm{arccot}(z) = \begin{cases} -\frac{\pi}{2} & \Re(z)<0 \\ \frac{\pi}{2} & \Re(z)>0 \end{cases} \tag{5.14}$$

This function has a branch cut along the imaginary axis from $-i$ to i due to $\arctan(z)$ and a branch cut from i to $+\infty$ and from $-i$ to $-\infty$ due to $\mathrm{arccot}(z)$. The step function (5.14) suggests that we split the integration contour in left and right part, as in fig.6. That is we can write

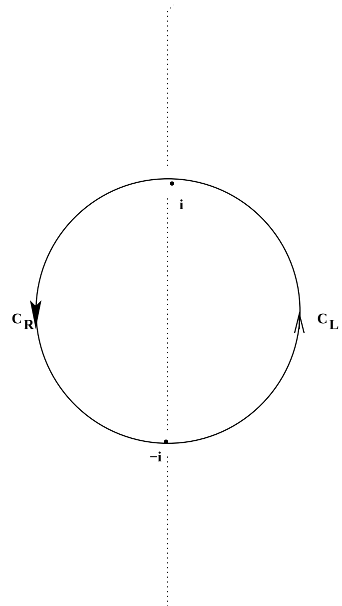

Figure 6: *Star product of two wedge states* $|3\rangle \star |2\rangle = |4\rangle$

$$\hat{\mathscr{L}}_0 = \mathscr{L}_0 + \mathscr{L}_0^\dagger = \frac{\pi}{2}\left(K_1^L - K_1^R\right) \tag{5.15}$$

where

$$K_1^{L,R} = \frac{\pi}{2}\int_{C_{L,R}} \frac{dz}{2\pi i}(1+z^2)T(z) \tag{5.16}$$

These objects have remarkable properties

$$[\mathscr{L}_0, \hat{\mathscr{L}}_0] = \hat{\mathscr{L}}_0, \qquad [\mathscr{L}_0, \mathscr{L}_{-1}] = \mathscr{L}_{-1} \tag{5.17}$$

and, due to the contours where they are defined, $K_1^{L,R}$ are left(right) derivations with respect to the star product

$$K_1^L(\Psi\star\Phi) = (K_1^L\Psi)\star\Phi, \qquad K_1^R(\Psi\star\Phi) = \Psi\star(K_1^R\Phi)$$

In other words K_1^L does not 'feel' the right hand part of the string; the opposite for K_1^R (the left and right part are defined by looking from the point $+i\infty$ in the arctan frame). Here are more explicit expressions for $K_1^{L,R}$

$$K_1^L = \frac{1}{2}K_1 + \frac{1}{\pi}\left(\mathscr{L}_0 + \mathscr{L}_0^\dagger\right)$$
$$K_1^R = \frac{1}{2}K_1 - \frac{1}{\pi}\left(\mathscr{L}_0 + \mathscr{L}_0^\dagger\right)$$

where $K_1 = L_1 + L_{-1}$.

Since in critical dimensions the ghost field $b(z)$ has the same conformal properties as $T(z)$, we can introduces quantities $\mathscr{B}_0, B_1, B_1^L, B_1^R$ analogous to $\mathscr{L}_0, K_1, K_1^L, K_1^R$:

$$\mathscr{B}_0 = b_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2-1} b_{2k}$$
$$B_1 = b_1 + b_{-1}$$

and

$$B_1^L = \frac{1}{2}B_1 + \frac{1}{\pi}\left(\mathscr{B}_0 + \mathscr{B}_0^\dagger\right)$$
$$B_1^R = \frac{1}{2}B_1 - \frac{1}{\pi}\left(\mathscr{B}_0 + \mathscr{B}_0^\dagger\right)$$

B_1^L (B_1^R) are also left (right) derivation with respect to the star product.

By denoting $\tilde{c}$ the ghost field in the arctan frame ($\tilde{c}(\xi) = \frac{1}{1+z^2}c(z)$) one can demonstrate the following commutators

$$[Q, B_1^L] = K_1^L, \qquad [Q, K_1^L] = 0, \qquad [B_1^L, K_1^L] = 0$$
$$[K_1^L, \tilde{c}] = \tilde{\partial}\tilde{c}, \qquad [B_1^L, \tilde{c}(\xi)] = \theta(\xi), \qquad [Q, \tilde{c}] = \tilde{c}\tilde{\partial}\tilde{c} \tag{5.18}$$

where θ represents the step function.

Using these new symbols the wedge states can be written, beside (5.7), also as

$$|n\rangle = e^{\frac{\pi}{2}(n-1)K_1^L}|I\rangle \tag{5.19}$$

From this equation and (5.7) we see that it makes sense to consider n a real variable, and therefore also to differentiate with respect to it. We can also interpret (5.19) by saying that K_1^L acting on $|I\rangle$ generate a cylinder of length $\frac{n\pi}{2}$.

5.2 The solution

To appreciate the subsequent solutions it is useful to consider first pure gauge solutions. A pure gauge solution can be written

$$\Psi_g = \Gamma^{-1}(\Lambda)Q\Gamma(\Lambda) \tag{5.20}$$

where Γ is an invertible expression of the gauge parameter Λ, for instance

$$\Gamma(\Lambda) = \frac{1}{1-\lambda\Lambda} = \sum_{n\geq 0}\lambda^n\Lambda^n \tag{5.21}$$

with λ a numerical parameter. Then one can write

$$\Psi_g = (1-\lambda\Lambda)Q\frac{1}{1-\lambda\Lambda} = \sum_{n=1}^{\infty}\lambda^n\psi_n, \qquad \psi_n = (Q\Lambda)\Lambda^{n-1} \tag{5.22}$$

Now if the series is convergent one has

$$\partial_\lambda S \sim \langle\partial_\lambda\Psi_g(Q\Psi_g+\Psi_g^2)\rangle = 0$$

for $\lambda=0$, since $\Psi_g(\lambda=0)=0$. Thus $S=0$ for any λ, because $Q\Psi_g+\Psi_g^2=0$. Therefore the energy of this solution vanishes, because for a static solution the energy coincides with the negative of the action.

However, as it turns out, the series (5.21) may not converge for $\lambda=1$ and the corresponding Γ may not be interpretable as a gauge transformation. Schnabl's solution was constructed exploiting this fact. First of all the gauge fixing is $\mathscr{B}_0|\Psi\rangle=0$, rather than the Feynman–Siegel one. Then one chooses $\Lambda = B_1^L c_1|0\rangle$. After some calculations one finds

$$\psi_n = (Q\Lambda)\Lambda^{n-1} = c_1|0\rangle\star K_1^L B_1^L|n-1\rangle\star c_1|0\rangle = \frac{d}{dn}\phi_{n-1}, \tag{5.23}$$

where

$$\phi_n = c_1|0\rangle\star B_1^L e^{\frac{\pi}{2}(n-1)K_1^L}|I\rangle\star c_1|0\rangle$$

prime denotes derivative with respect to n. The state ψ_n is made out of wedges states with insertions of the field c and of B. In particular for $n=0$ we have

$$\psi_0 = (cB_1^Lc)(0)|0\rangle, \qquad \psi_0' = (cB_1^LK_1^Lc)(0)|0\rangle$$

Finally the solution is

$$\Psi = \lim_{N\to\infty}\left(\sum_{n=0}^{N}\psi_n' - \psi_N\right) \tag{5.24}$$

The second term $-\psi_N$ is added only for regularization purposes.

5.3 Sen's first and second conjectures

From the equation of motion we get

$$\langle\Psi, Q\Psi\rangle = -\langle\Psi,\Psi\star\Psi\rangle \tag{5.25}$$

This equation has to be explicitly checked over the solution (5.24) – a rather nontrivial task –, because one of the subtleties of SFT is that, even if $|\Psi\rangle$ is a solution to the equation of motion, it is not automatically guaranteed that (5.25) holds.

On the other hand, from the explicit form of the solution one gets

$$\langle\Psi, Q\Psi\rangle = -\frac{3}{\pi^2}$$

Therefore, finally, the total energy of the string modes is (V is the total 26–th dimensional volume):

$$E = -\frac{S}{V} = \frac{1}{g_o^2 V}\left(\frac{1}{2}\langle\Psi, Q\Psi\rangle + \frac{1}{3}\langle\Psi, \Psi\star\Psi\rangle\right) = -\frac{1}{2\pi^2 g_0^2} \tag{5.26}$$

which is precisely the negative of the D25–brane tension τ.

Let us now pass to briefly illustrate the proof of the second conjecture [12]. The purpose is to show that the cohomology about Schnabl's solution is trivial. Relabeling Schnabl's solution as Ψ_0, we are looking now for solutions to (5.1) of the type $\Psi_0+\psi$, linearized on ψ. It is easy to see that the relevant (linearized) equation of motion is

$$\mathscr{Q}\psi \equiv Q\psi + \Psi_0\star\psi - (-1)^{|\psi|}\psi\star\Psi_0 \tag{5.27}$$

This defines a new BRST operator $\mathscr{Q}$ (indeed $\mathscr{Q}^2=0$) and defines the cohomology around Schnabl's solution. The purpose is to prove that this cohomology is empty.

Let us introduce the symbol

$$W_r = |r+1\rangle$$

and define the state

$$A = -\frac{2}{\pi} B \int_0^1 W_r\, dr \tag{5.28}$$

Here we make use of the fact that wedge states can be defined for any real label r, not just for an integral r. It is possible to prove that

$$\mathscr{Q}A = |I\rangle \tag{5.29}$$

where the RHS represents the identity state.

Now suppose that ψ satisfies $\mathscr{Q}\psi = 0$. Then, using the previous results, we get

$$\mathscr{Q}(A\star\psi) = (\mathscr{Q}A)\star\psi - A\star(\mathscr{Q}\psi) = |1\rangle\star\psi = \psi$$

which means that ψ is BRST trivial. This is a very general result. It implies not only that the cohomology of ghost number 1 is trivial (i.e., there is no physical perturbative string mode in the new vacuum), but that the cohomology is trivial for any ghost number state.

5.4 Another analytic tachyon vacuum solution

After the first solution presented above another analytic solution was subsequently found by Erler and Schnabl. This second solution is simpler and opened the way to new developments. For this reason I will describe it in detail. First of all I will introduce a new tool, the K,B,c algebra, whose simplifying virtues will be evident in a moment.

Let us introduce the symbols K,B,c which are obtained by acting with K_1^L, B_1^L and the field c on the identity state $|I\rangle$:

$$K = \frac{\pi}{2}K_1^L|I\rangle, \qquad B = \frac{\pi}{2}B_1^L|I\rangle, \qquad c = c\left(\frac{1}{2}\right)|I\rangle, \tag{5.30}$$

They obey the remarkably simple algebra

$$[K,B]=0, \qquad [K,c]=\partial c, \qquad \{B,c\}=1, \qquad \{B,\partial c\}=0 \tag{5.31}$$

where the product is understood to be the star product and 1 represents $|I\rangle$. In this algebra Q operates as follows

$$QB=K, \qquad Qc=c\partial c \tag{5.32}$$

It is also useful to recall that $|I\rangle = e^{-\frac{\pi}{2}K_1^L}|0\rangle$, so that $|0\rangle = e^K$.

The Erler-Schnabl, [13], solution (ES) is constructed by fully exploiting the simplicity of this algebra with operators. The ansatz is

$$\psi_0 = \frac{1}{1+K}c(1+K)Bc = c - \frac{1}{1+K}Bc\partial c, \tag{5.33}$$

Since, using the K,B,c algebra, it is easy to show that

$$Q\Psi_0 = cKc\frac{1}{K+1} \qquad \text{and} \qquad \psi_0\psi_0 = -cKc\frac{1}{K+1}$$

it is obvious that ψ_0 is a solution to (5.1).

The energy of this solution turns out to be the correct one (1st conjecture)

$$E=-\frac{S}{V}=\frac{1}{g_o^2V}\left(\frac{1}{2}\langle\Psi,Q\Psi\rangle+\frac{1}{3}\langle\Psi,\Psi\star\Psi\rangle\right)=-\frac{1}{2\pi^2g_0^2} \tag{5.34}$$

It is also possible to define a homotopy operator $A=\frac{B}{K+1}$, which satisfies the property $\mathscr{Q}A=1$, where as above

$$\mathscr{Q}\psi \equiv Q\psi+\Psi_0\star\psi-(-1)^{|\psi|}\psi\star\Psi_0$$

is the BRST operator at the tachyon vacuum. As we saw above, this implies that the cohomology around the tachyon vacuum is trivial (2nd conjecture).

It is instructive to compute the energy of the ES solution. Using the equation of motion and the Schwinger representation

$$\frac{1}{K+1}=\int_0^\infty dt\, e^{-t(K+1)} \tag{5.35}$$

we have

$$\begin{aligned} E[\psi_0] &= \frac{1}{6}\langle\psi_0,Q\psi_0\rangle = \frac{1}{6}\langle(c+cKBc)\frac{1}{K+1}cKc\frac{1}{K+1}\rangle \\ &= \frac{1}{6}\int_0^\infty dt_1dt_2e^{-t_1-t_2}\left(\langle ce^{-t_1K}cKc\,e^{-t_2K}\rangle_{C_{t_1+t_2}}-\langle Q\left(Bc\,e^{-t_1K}cKc\,e^{-t_2K}\right)\rangle_{C_{t_1+t_2}}\right)\end{aligned} \tag{5.36}$$

The second term vanishes because it is BRST exact. The first can be rewritten

$$\langle ce^{-t_1K}cKc\,e^{-t_2K}\rangle_{C_{t_1+t_2}}=\langle e^{-(t_1+t_2)K}c(t_1+t_2)cKc(t_2)\rangle_{C_{t_1+t_2}} \tag{5.37}$$

To evaluate this we have to start from the corresponding correlator in the UHP and lift it to the arctan frame. In the UHP we have

$$\langle c(z_1) c\partial c(z_2)\rangle_{UHP} = -(z_1 - z_2)^2$$

Mapping it to the cylinder C_π with the map $z \to \xi = \arctan(z)$ this becomes

$$\langle \tilde{c}(\xi_1)\, \tilde{c}\tilde{\partial}\tilde{c}(\xi_2)\rangle_{C_\pi} \tag{5.38}$$

Mapping this to a cylinder of length ℓ, i.e. $\xi \to \frac{\ell}{\pi}\xi$ and $\tilde{c} \to \hat{c}$, one gets

$$\langle \hat{c}(\xi_1)\, \hat{c}\hat{\partial}\hat{c}(\xi_2)\rangle_{C_\ell} = \left(\frac{\ell}{\pi}\right)^2 \sin^2\left(\pi \frac{\xi_1 - \xi_2}{\ell}\right)$$

from which

$$E[\Psi_0] = -\frac{1}{6}\int_0^\infty dt_1 dt_2 e^{-t_1 - t_2} \frac{(t_1 - t_2)^2}{\pi^2} \sin^2\left(\frac{\pi t_1}{t_1 + t_2}\right)$$

follows. The integration is elementary and one finally obtains (5.34).

It should be noticed that also the ES solution can be formally regarded as a pure gauge solution since

$$\Psi_0 = UQU^{-1}, \qquad U = 1 - \frac{1}{K+1}Bc, \qquad U^{-1} = 1 + \frac{1}{K}Bc$$

However the state $\frac{1}{K}$ is singular due to the zero mode of K_1^L.

6. The third conjecture and the lump solutions

The third conjecture predicts in particular the existence of lower dimensional solitonic solutions or lumps, interpreted as Dp–branes, with $p < 25$. These solutions yield the breaking of translational symmetry and background independence. The evidence for the existence of such solutions collected in the past years is overwhelming. It has been possible to find them with approximate methods or with exact methods in related theories. In the sequel I will present a recently proposed explicit example of analytic lump solution in OSFT.

6.1 Analytic lump solutions

In a recent paper[14], a general method has been proposed to obtain new exact analytic solutions in open string field theory, and in particular solutions that describe inhomogeneous tachyon condensation. The method consists in translating an exact renormalization group (RG) flow generated in a two–dimensional world–sheet theory by a relevant operator, into the language of OSFT. The so-constructed solution is a deformation of the ES solution described above. It has been shown in [14] that, if the operator has suitable properties, the solution will describe tachyon condensation in specific space directions, thus representing the condensation of a lower dimensional brane. In the following, after describing the general method, we will focus on a particular solution, generated by an exact RG flow first analyzed by Witten[15]. On the basis of the analysis carried out in the framework of 2D CFT in [16], we expect it to describe a D24-brane, with the correct ratio of tension with respect to the starting D25 brane.

Let us see first the general recipe to construct such kind of lump solutions. To start with we enlarge the K, B, c algebra by adding a (relevant) matter operator

$$\phi = \phi\left(\frac{1}{2}\right)|I\rangle. \tag{6.1}$$

with the properties

$$[c,\phi] = 0, \qquad [B,\phi] = 0, \qquad [K,\phi] = \partial\phi, \qquad Q\phi = c\partial\phi + \partial c\delta\phi. \tag{6.2}$$

It can be easily proven that

$$\psi_\phi = c\phi - \frac{1}{K+\phi}(\phi - \delta\phi)Bc\partial c \tag{6.3}$$

does indeed satisfy (formally, see below) the OSFT equation of motion

$$Q\psi_\phi + \psi_\phi\psi_\phi = 0 \tag{6.4}$$

It is clear that (6.3) is a deformation of the Erler–Schnabl solution, which can be recovered for $\phi = 1$.

After some algebraic manipulations one can show that

$$\mathscr{Q}_{\psi_\phi}\frac{B}{K+\phi} = Q\frac{B}{K+\phi} + \left\{\psi_\phi, \frac{B}{K+\phi}\right\} = 1.$$

So, unless the string field $\frac{B}{K+\phi}$ is singular, it defines a homotopy operator and the solution has trivial cohomology, which is the defining property of the tachyon vacuum [12]. On the other hand, in order for the solution to be well defined, the quantity $\frac{1}{K+\phi}(\phi - \delta\phi)$ should be well defined. Moreover, in order to be able to show that (6.3) satisfies the equation of motion, one needs $K+\phi$ to be invertible.

In full generality we thus have a new nontrivial solution if

1. $\frac{1}{K+\phi}$ is in some sense singular, but
2. $\frac{1}{K+\phi}(\phi - \delta\phi)$ is regular and
3. $\frac{1}{K+\phi}(K+\phi) = 1$

These conditions seem to be hard to satisfy: for instance, $K+\phi$ may not be invertible, one needs a regularization. It is indeed so without adequate specifications. This problem was discussed in [17, 19], where it was shown that the right framework is distribution theory, which guarantees not only regularity of the solution but also its 'non-triviality', in the sense that if these conditions are satisfied, it cannot fall in the same class as the ES tachyon vacuum solution.

For concreteness we parametrize the worldsheet RG flow, referred to above, with a parameter u, where $u = 0$ represents the UV and $u = \infty$ the IR, and label ϕ by ϕ_u, with $\phi_{u=0} = 0$. Then we require for ϕ_u the following properties under the coordinate rescaling $f_t(z) = \frac{z}{t}$

$$f_t \circ \phi_u(z) = \frac{1}{t}\,\phi_{tu}\left(\frac{z}{t}\right). \tag{6.5}$$

We will consider in the sequel a specific relevant operator ϕ_u and the corresponding SFT solution. This operator generates an exact RG flow studied by Witten in [15], see also [16], and is based on the operator (defined in the cylinder C_T of width T in the arctan frame)

$$\phi_u(s) = u(X^2(s) + 2\ln u + 2A) \tag{6.6}$$

where A is a constant. On the unit disk D we have

$$\phi_u(\theta) = u(X^2(\theta) + 2\ln\frac{Tu}{2\pi} + 2A) \tag{6.7}$$

If we set

$$g_A(u) = \langle e^{-\frac{1}{2\pi}\int_0^{4\pi} d\theta\, u\left(X^2(\theta)+2\ln\frac{u}{2\pi}+2A\right)}\rangle_D$$

we get

$$g_A(u) = Z(2u)e^{-2u(\ln\frac{u}{2\pi}+A)} \tag{6.8}$$

where $Z(u)$ is the partition function of the system on the unit disk computed by[15]. Requiring finiteness for $u \to \infty$ one gets $A = \gamma - 1 + \ln 4\pi$, which implies

$$g_A(u) \equiv g(u) = \frac{1}{2\sqrt{\pi}}\sqrt{2u}\,\Gamma(2u)e^{2u(1-\ln(2u))}, \qquad \lim_{u\to\infty} g(u) = 1 \tag{6.9}$$

Moreover, as it turns out, $\phi_u - \delta\phi_u = u\partial_u\phi_u(s)$

The ϕ_u just introduced satisfies all the requested properties. According to [16], the corresponding RG flow in BCFT reproduces the correct ratio of tension between D25 and D24 branes. Consequently $\psi_u \equiv \psi_{\phi_u}$ is expected to represent a D24 brane solution.

In SFT the most important gauge invariant quantity is of course the energy. Therefore in order to make sure that $\psi_u \equiv \psi_{\phi_u}$ is the expected solution we must prove that its energy equals a D24 brane energy.

The energy expression for the lump solution was determined in [14] by evaluating a three–point function on the cylinder C_T. It equals $-\frac{1}{6}$ times the following expression

$$\begin{aligned}\langle\psi_u\psi_u\psi_u\rangle = &-\int_0^\infty dt_1dt_2dt_3\mathscr{E}_0(t_1,t_2,t_3)u^3g(uT)\Bigg\{\Big(-\frac{\partial_{2uT}g(uT)}{g(uT)}\Big)^3\\ &+\frac{1}{2}\Big(-\frac{\partial_{2uT}g(uT)}{g(uT)}\Big)\Big(G^2_{2uT}(\frac{2\pi t_1}{T})+G^2_{2uT}(\frac{2\pi(t_1+t_2)}{T})+G^2_{2uT}(\frac{2\pi t_2}{T})\Big)\\ &+G_{2uT}(\frac{2\pi t_1}{T})G_{2uT}(\frac{2\pi(t_1+t_2)}{T})G_{2uT}(\frac{2\pi t_2}{T})\Bigg\}\end{aligned} \tag{6.10}$$

Here $T = t_1 + t_2 + t_3$ and $g(u)$ is as above, while $G_u(\theta)$ represents the boundary-to-boundary correlator first determined by Witten[15]:

$$G_u(\theta) = \frac{1}{u} + 2\sum_{k=1}^{\infty}\frac{\cos(k\theta)}{k+u}$$

Finally $\mathscr{E}_0(t_1,t_2,t_3)$ represents the ghost three–point function in C_T.

$$\mathscr{E}_0(t_1,t_2,t_3) = \langle Bc\partial c(t_1+t_2)\partial c(t_1)\partial c(0)\rangle_{C_T} = -\frac{4}{\pi}\sin\frac{\pi t_1}{T}\sin\frac{\pi(t_1+t_2)}{T}\sin\frac{\pi t_2}{T}$$

A remarkable property of (6.10) is that it does not depend on u. In fact u can be absorbed in a redefinition of variables $t_i \to ut_i$, $i = 1,2,3$, and disappears from the expression.

The integral in (6.10) is well defined in the IR (s very large, setting $s = 2uT$) but has an UV ($s \approx 0$) singularity, which must be subtracted away. Once this done, the expression (6.10) can be numerically computed, the result being ≈ 0.069. This is not the expected result, but this is not surprising, for the result depends on the UV subtraction we have made. Therefore we cannot assign to it any physical significance. To get a meaningful result we must return to the very meaning of Sen's third conjecture, which says that *the lump solution is a solution of the theory on the tachyon condensation vacuum*. Therefore we must measure the energy of our solution with respect to the tachyon condensation vacuum. Simultaneously the resulting energy must be a subtraction-independent quantity because only to a subtraction-independent quantity can a physical meaning be assigned. Both requirements have been satisfied in [17] in the following way.

First a new solution to the EOM, depending on a parameter ε, has been introduced

$$\psi_u^\varepsilon = c(\phi_u+\varepsilon) - \frac{1}{K+\phi_u+\varepsilon}(\phi_u+\varepsilon-\delta\phi_u)Bc\partial c. \tag{6.11}$$

in the limit $\varepsilon \to 0$. This limit will be mostly understood from now on. The energy of (6.11) (after the same UV subtraction as in the previous case) is (numerically) 0. Since (unlike the previous case) the presence of the parameter ε prevents the IR transition to a new critical point, it is sensible to assume that $\lim_{\varepsilon\to 0}\psi_u^\varepsilon$ represents the tachyon condensation vacuum solution. In other words it is gauge equivalent to the ES, solution. Using it, *a solution to the EOM at the tachyon condensation vacuum* has been obtained. The equation of motion at the tachyon vacuum is

$$\mathscr{Q}\Phi + \Phi\Phi = 0, \quad \text{where } \mathscr{Q}\Phi = Q\Phi + \psi_u^\varepsilon\Phi + \Phi\psi_u^\varepsilon. \tag{6.12}$$

One can easily show that

$$\Phi_0 = \psi_u - \psi_u^\varepsilon \tag{6.13}$$

is a solution to (6.12). The action at the tachyon vacuum is $-\frac{1}{2}\langle\mathscr{Q}\Phi,\Phi\rangle - \frac{1}{3}\langle\Phi,\Phi\Phi\rangle$. Thus the energy of Φ_0 is

$$\begin{aligned} E[\Phi_0] &= -\frac{1}{6}\langle\Phi_0,\Phi_0\Phi_0\rangle \\ &= -\frac{1}{6}\big[\langle\psi_u,\psi_u\psi_u\rangle - \langle\psi_u^\varepsilon,\psi_u^\varepsilon\psi_u^\varepsilon\rangle - 3\langle\psi_u^\varepsilon,\psi_u\psi_u\rangle + 3\langle\psi_u,\psi_u^\varepsilon\psi_u^\varepsilon\rangle\big]. \end{aligned} \tag{6.14}$$

The UV subtractions necessary for each correlator at the RHS of this equation are always the same, therefore they cancel out and the final result is subtraction-independent. A final bonus of this procedure is that the final result can be derived purely analytically and $E[\Phi_0]$ turns out to be precisely the D24-brane energy. With the conventions of [17], this is

$$T_{D24} = \frac{1}{2\pi^2} \tag{6.15}$$

In [18] the same result was extended to Dp-brane lump solutions for any p.

7. Comments

The three conjectures formulated by Ashoke Sen about fifteen years ago have been demonstrated beyond doubt in the framework of Witten's OSFT. This is certainly a remarkable result, but from the point of view of OSFT it is only a beginning. The correctness of the three conjectures confirms that open string theory knows about closed string theory. As anticipated in the introduction this was somehow expected. Even the first quantized open string theory contains at one loop information about the closed string spectrum. However what we have learnt from OSFT is much richer information. Even at the classical level (tree level of the perturbative expansion) [10], provided we consider exact analytic solutions (which correspond to specific boundary CFT's, i.e. full expansions in the α' parameter), we can get information about closed string theory. The question is now how rich and complete this information is. The example of AdS/CFT suggests that open and closed strings are two different description of the same underlying physics. On the other hand, the study of string field theory seems to suggest that there is some asymmetry between the two descriptions. If field theory is the right language for physics the open string description is favored. OSFT seems to respond to the best expectations and the exciting questions we are left with at the end of this exposition are: how far can we go in the description of closed string theory by means of open strings? is there a way, for instance, to represent black holes in the open string theory language? Even more important, can we recognize in this language bosonized solutions representing fermions? It is clear however that this is possible only if in the solutions we insert matter, as was done in the last section. Without it the spectrum of solutions that we obtain is too poor.

References

[1] E. Witten, *Noncommutative Geometry And String Field Theory*, Nucl. Phys. B **268** (1986) 253.

[2] C. B. Thorn, String Field Theory, Phys.Rep. **175** (1989) 1.

[3] B. Zwiebach, *Closed string field theory: An Introduction*,, (Les Houches Summer School 1992) [hep-th/9305026]

[4] W. Taylor and B. Zwiebach, *D-branes, tachyons, and string field theory*,, TASI 2001, [hep-th/0311017].

[5] K. Ohmori, *A Review on tachyon condensation in open string field theories*, [hep-th/0102085].

[6] A. Sen, *Tachyon dynamics in open string theory*, Int. J. Mod. Phys. A **20** (2005) 5513 [hep-th/0410103].

[7] E. Fuchs and M. Kroyter, *Analytical Solutions of Open String Field Theory*, Phys. Rept. **502** (2011) 89 [arXiv:0807.4722 [hep-th]].

[8] L. Bonora, C. Maccaferri, D. Mamone and M. Salizzoni, *Topics in string field theory*, Proc. Kopaonik Conf. 2002 [hep-th/0304270].

[9] A. Sen, *Tachyon condensation on the brane anti-brane system*, JHEP **9808** (1998) 012, and *BPS D-branes on nonsupersymmetric cycles*, JHEP **9812** (1998) 021.

[10] A. Sen, *Open closed duality: Lessons from matrix model*, Mod. Phys. Lett. A **19** (2004) 841
Open and closed strings from unstable D-branes, Phys. Rev. D **68** (2003) 106003

[11] M. Schnabl, *Analytic solution for tachyon condensation in open string field theory,* Adv. Theor. Math. Phys. **10** (2006) 433.

[12] I. Ellwood and M. Schanabl, *Proof of vanishing cohomology at the tachyon vacuum,* JHEP **0702** (2007) 096.

[13] T. Erler and M. Schnabl, *A Simple Analytic Solution for Tachyon Condensation*, JHEP **0910** (2009) 066.

[14] L. Bonora , C. Maccaferri and D. D. Tolla, *Relevant Deformations in Open String Field Theory: a Simple Solution for Lumps,* JHEP **1111** (2011) 107.

[15] E. Witten, *Some computations in background independent off-shell string theory,* Phys. Rev. D **47** (1993) 3405

[16] D. Kutasov, M. Marino and G. W. Moore, *Some exact results on tachyon condensation in string field theory*, JHEP **0010** (2000) 045 ...

[17] L. Bonora , S. Giaccari and D. D. Tolla, *The energy of the analytic lump solution in SFT,* JHEP **1108** (2011)158.

[18] L. Bonora, S. Giaccari and D. D. Tolla, *Analytic solutions for Dp branes in SFT,* JHEP **1112** (2011)033.

[19] L. Bonora, S. Giaccari *Generalized states in SFT* EPJC **73** (2013) 2644.