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Canonical forms and discrete Liouville–Green asymptotics for second-order linear difference equations

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Abstract

Liouville–Green (WKB) asymptotic approximations are constructed for some classes of linear second-order *difference* equations. This is done starting from certain "canonical forms" for the three-term linear recurrence. Rigorous explicit bounds are established for the error terms in the asymptotic approximations of recessive as well as dominant solutions. The asymptotics with respect to parameters affecting the equation is also discussed. Several illustrative examples are given.

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1 Introduction

In this paper we first derive certain "canonical forms" for the three-term linear recurrence, and then we use them to obtain asymptotic approximations of the Liouville–Green (LG, for short, or WKB) type for their solutions. More precisely, we establish asymptotic representations for two linearly independent solutions, providing explicit rigorous bounds for the error terms.

A typical feature of the classical LG approximation is its "double asymptotic nature" since it holds when both, the independent variable and some parameter, tend to certain limits [9, 14]. We extend the results achieved in [14] to some other classes of difference equations; see also [8, 19, 21]. In Sect. 2, we derive the aforementioned canonical forms, and in Sect. 3 we construct the corresponding LG approximations, equipped with rigorous bounds for the error terms. Some details, lengthy though rather elementary, are relegated to Appendices. Several examples are given in Sect. 4. It is possible to proceed similarly in case of matrix difference equations or, equivalently, for systems, but we will not do it in this paper; see, e.g., [4, 11, 12]. Finally, we summarize the high points of the paper in the short concluding Sect. 5.

2 Canonical forms

Consider the three-term linear recurrence

$$u_{n+2} + A_n u_{n+1} + B_n u_n = 0, \qquad n \in \mathbf{Z}_\nu, \tag{1}$$

where $\mathbf{Z}_\nu := \{n \in \mathbf{Z} : n \geq \nu\}$ for some fixed $\nu \in \mathbf{Z}$, the sequences $\{A_n\}$ and $\{B_n\}$ are scalar (either real- or complex-valued), and $\Delta u_n := u_{n+1} - u_n$, $\Delta^2 := \Delta(\Delta)$. We can show that the following forms

(A)

$$\Delta^2 y_n + q_n y_n = 0, \quad n \in \mathbf{Z}_\nu,$$

(B)

$$\Delta^2 y_n + r_n y_{n+1} = 0, \quad n \in \mathbf{Z}_\nu,$$

(C)

$$\Delta(c_n \Delta y_n) + r_n y_n = 0, \quad n \in \mathbf{Z}_\nu,$$

(D)

$$\Delta(c_n \Delta y_n) + r_n y_{n+1} = 0, \quad n \in \mathbf{Z}_\nu,$$

are *canonical forms* for recurrence (1), in the sense that the latter can be recast (with some exceptions) into such forms. Only if $A_n = -2$ for every n, equation (1) can be immediately rewritten as $\Delta^2 u_n + Q_n u_n = 0$, with $Q_n := A_n + B_n + 1$.

Note that equations of cases (C) and (D) are in self-adjoint form. Equations (B) and (D) seem to have been preferred in oscillation theory [5, 6, 10]. All the previous forms are also reminiscent of the analogous ones for linear second-order *differential* equations (sometimes called Sturm–Liouville or Jacobi forms), and hence some inspiration can be drawn from such similarity. Below, we show how a general recurrence like (1) can be taken into the various canonical forms (A)–(D) (with some exceptions).

Case (A). In [14], an equation like that in (A) was obtained setting

$$u_n = \alpha_n y_n \tag{2}$$

in (1), and choosing

$$\alpha_n := \alpha_{\nu+1} \prod_{k=\nu}^{n-2} \left(-\frac{A_k}{2} \right) \quad \text{for } n \geq \nu + 2, \tag{3}$$

with α_ν and $\alpha_{\nu+1} \neq 0$ arbitrary constants. The canonical form (A) is thus obtained with

$$q_n := -1 + \frac{4B_n}{A_n A_{n-1}} \quad \text{for } n \geq \nu + 1, \tag{4}$$

while

$$q_\nu = -1 - 2\frac{B_\nu}{A_\nu} \frac{\alpha_\nu}{\alpha_{\nu+1}},$$

provided that $A_n \neq 0$ for all $n \geq \nu$.

Case (B). The same transformation (2) also allows to take equation (1) into the form (B), choosing α_n and r_n such that

$$-1 + B_n \frac{\alpha_n}{\alpha_{n+2}} \equiv 0, \qquad 2 + A_n \frac{\alpha_{n+1}}{\alpha_{n+2}} \equiv r_n \quad \text{for } n \geq \nu.$$

The first of this, i.e., $\alpha_{n+2} = B_n \alpha_n$, can be solved explicitly, but we have to distinguish the four cases according to n and ν being even or odd. We obtain

(i) for n and ν both even or both odd,

$$\alpha_n = B_{n-2}B_{n-4} \cdots B_{\nu+2}B_\nu \quad \text{for } n \geq \nu \tag{5}$$

and

$$r_n = 2 + A_n \frac{\prod_{k=0}^{(n-2-\nu)/2} B_{n-2k-1}}{\prod_{k=0}^{(n-\nu)/2} B_{n-2k}} \quad \text{for } n \geq \nu; \tag{6}$$

(ii) for n even and ν odd, or n odd and ν even,

$$\alpha_n = B_{n-2}B_{n-4} \cdots B_{\nu+3}B_{\nu+1} \quad \text{for } n \geq \nu + 1, \tag{7}$$

and

$$r_n = 2 + A_n \frac{\prod_{k=0}^{(n-1-\nu)/2} B_{n-2k-1}}{\prod_{k=0}^{(n-1-\nu)/2} B_{n-2k}} \quad \text{for } n \geq \nu + 1. \tag{8}$$

All this can be done provided that $B_n \neq 0$ for all $n \geq \nu$, or at least for n sufficiently large.

Other than cases (A) and (B), given the two coefficients A_n and B_n of equation (1), the latter can immediately be written in the form (C) or (D) merely rearranging its terms, since now there are *two* "degrees of freedom", represented by the possibility of choosing arbitrarily the two coefficients c_n and r_n. In fact

Case (C). Imposing

$$-1 - \frac{c_n}{c_{n+1}} \equiv A_n, \quad \text{and} \quad \frac{c_n + r_n}{c_{n+1}} \equiv B_n,$$

i.e., choosing

$$c_n := c_\nu \prod_{k=\nu}^{n-1} \left(\frac{-1}{A_k + 1} \right) \quad \text{for } n \geq \nu + 1, \tag{9}$$

and

$$r_n := c_\nu (A_n + B_n + 1) \prod_{k=\nu}^{n} \left(\frac{-1}{A_k + 1} \right) \quad \text{for } n \geq \nu, \tag{10}$$

for any arbitrary constant $c_\nu \neq 0$, we obtain the form **C** with $y_n = u_n$, except when $A_n = -1$ for all n (but it will hold true if $A_n \neq -1$ for all n sufficiently large values of n).

Case (D). Imposing instead

$$-1+\frac{r_n-c_n}{c_{n+1}}\equiv A_n, \quad \text{and} \quad \frac{c_n}{c_{n+1}}\equiv B_n,$$

i.e.,

$$c_n := c_\nu \prod_{k=\nu}^{n-1}\bigl(B_k^{-1}\bigr) \quad \text{for } n\geq \nu+1, \tag{11}$$

and

$$r_n = c_\nu(A_n+B_n+1)\prod_{k=\nu}^{n}\bigl(B_k^{-1}\bigr) \quad \text{for } n\geq \nu, \tag{12}$$

for any arbitrary constant $c_\nu \neq 0$, we obtain the form (D), again with $y_n = u_n$, with the exception now that $B_n = 0$ for all (or for sufficiently large) values of n.

Note that we could also use transformation (2) to obtain equations like (C) or (D), having now *three* degrees of freedom, represented by α_n, c_n, and r_n, for any two given sequences A_n and B_n. This requires choosing the former quantities from the conditions

$$\frac{c_n}{c_{n+1}} = -1 - A_n\frac{\alpha_{n+1}}{\alpha_{n+2}},$$
$$r_n = c_{n+1}\left(B_n\frac{\alpha_n}{\alpha_{n+2}} - \frac{c_n}{c_{n+1}}\right)$$

to obtain the form (C), and

$$\frac{c_n}{c_{n+1}} = B_n\frac{\alpha_n}{\alpha_{n+2}},$$
$$r_n = 1 + \frac{c_n}{c_{n+1}} + A_n\frac{\alpha_{n+1}}{\alpha_{n+2}} = 1 + \frac{A_n\alpha_{n+1}+B_n\alpha_n}{\alpha_{n+2}}$$

to obtain (D).

In the next section, we formulate and prove two theorems, yielding the asymptotic representations of the LG (or WKB) type for a basis of solutions to the model equations (C) and (D). The corresponding results for (A) and (B) are special cases of (C) and (D), respectively; besides, case (A) was already considered in [14].

3 The main results

In this section, we develop an LG (or WKB) asymptotic approximation theory for the three classes of linear second-order difference equations (B), (C), and (D), along the same lines followed for the form (A) in [14]. We observe that case (A) is a special instance of (C), while case (B) is a special instance of (D), and hence we will consider only cases (C) and (D).

Our spirit here is that of the LG approximation, put on rigorous grounds by F.W.J. Olver for second-order *differential* equations [9], in that we wish to obtain *rigorous* and *explicit* computable error bounds.

Starting from the canonical forms to establish LG asymptotic approximations is convenient, since we are able to exploit their similarity with second-order *differential* equations with no first-order derivative terms (Jacobi or Sturm–Liouville forms). This approach parallels closely the original one followed by F.W.J. Olver for differential equations [9].

Below, we assume that the coefficients of equations (C) and (D) have always the form

$$r_n = a + g_n, \qquad c_n = c + h_n, \tag{13}$$

with a real, $c > 0$, and that the "perturbations" g_n and h_n are in ℓ^1. We choose further $c = 1$, without any loss of generality, to simplify the notation, but confine ourselves to $a \neq 0$, $a \neq -1$. In fact, in case (C), e.g., $a + c = a + 1 = 0$ for $a = -1$, and this lets the lowest order term of the unperturbed equation vanish. This case is in some sense pathological, since in this occurrence the unperturbed equation is no longer of the second order.

In fact, equation (1) is *properly* called a *second-order* difference equation only if $B_n \neq 0$ (for all $n > \nu$). It then can be written in the form

$$\Delta^2 u_n + a_n \Delta u_n + b_n u_n = 0, \tag{14}$$

where $a_n := A_n + 2$ and $b_n := A_n + B_n + 1$, and conversely equation (14) can be recast into the form of equation (1), setting $A_n = a_n - 2$ and $B_n = b_n - a_n + 1$. But some care should be paid to the form (14) since, other than in the formally analogous case of differential equations, it is not always guaranteed that equation (14) is of the second order, that is, the vector space of all solutions has dimension 2. However, the formal similarity of equation (14) with the linear second-order *differential* equation suggests the possibility to remove the term containing the first difference from (14) using transformation (2), as it can be done for differential equations. Indeed, we end up with the difference equation in (A).

When $a = 0$, the unperturbed equation (31) falls in the case of "finite moments perturbations" [13, 15, 17]. Some other cases, e.g., a complex in equation (A), have been considered elsewhere using the canonical form (A), see, e.g., [16, 18, 20].

3.1 LG asymptotics for case (C)

We define the basic quantities

$$V_n := \sum_{r=n}^{\infty} |g_r|, \qquad W_n := \sum_{r=n}^{\infty} |h_r|, \tag{15}$$

$$Z_n^{\pm} := |\eta_{\pm}|^{-n} \sum_{r=n}^{\infty} |\eta_{\pm}|^r |g_r|, \qquad U_n^{\pm} := |\eta_{\pm}|^{-n} \sum_{r=n}^{\infty} |\eta_{\pm}|^r |h_r|, \tag{16}$$

where

$$\eta_{\pm} := \frac{\lambda_{\pm}}{\lambda_{\mp}} = \frac{1 \pm \sqrt{|a|}}{1 \mp \sqrt{|a|}}, \tag{17}$$

and

$$\lambda_{\pm} := 1 \pm \sqrt{-a} \tag{18}$$

are the characteristic roots of the unperturbed difference equation associated to (C), that is, $\lambda^2 - 2\lambda + (1 + a) = 0$. We begin with the case of equation (C), stating the following.

Theorem 3.1 *Let equation* (C) *be given for* $n \geq \nu$, *for some fixed* $\nu \in \mathbf{Z}$, *with the coefficients as in* (13), *with* a *real,* $a \neq 0, -1$, *and* $c = 1$, g_n *and* h_n *real or complex. Then,*

(i) *if* $a > 0$, *and assuming that*

$$V_n < \infty, \qquad W_n < \infty, \tag{19}$$

two linearly independent solutions of the form

$$y_n^{\pm} = (\lambda_{\pm})^n \bigl(1 + \varepsilon_n^{\pm}\bigr), \quad n \geq n_0, \tag{20}$$

exist for some integer $n_0 \geq \nu$, *and the error terms* $\varepsilon_n^{\pm}$ *can be estimated as*

$$\bigl|\varepsilon_n^{\pm}\bigr| < \mathcal{V}_n, \tag{21}$$

the error control function $\mathcal{V}_n$ *being defined as*

$$\mathcal{V}_n := \frac{1}{\sqrt{a(a+1)}} \bigl\{ V_n + (\sqrt{a+1} + 1)[\sqrt{a+1}\, W_{n+1} + W_{n+2}] \bigr\} \tag{22}$$

for all $n \geq n_0$, *where*

$$n_0 := \min\{n \geq \nu : \mathcal{V}_n < 1\}. \tag{23}$$

(ii) *If* $a < 0$ (*but* $a \neq -1$), *a solution* y_n^- (*as in* (20)) *exists under conditions* (19) *for any fixed* $n \geq n_0^-$, *and the corresponding error term is estimated as*

$$\bigl|\varepsilon_n^-\bigr| < \mathcal{V}_n^-, \tag{24}$$

with

$$\mathcal{V}_n^- := \begin{cases} \frac{1}{2\sqrt{|a|}(1-\sqrt{|a|})} [V_n + (2 - \sqrt{|a|}) W_n + (1 - \sqrt{|a|})(2 - \sqrt{|a|}) W_{n+1}] \\ \quad \text{if } |a| < 1, \\ \frac{1}{\sqrt{|a|}(\sqrt{|a|}-1)} [V_n + W_n + (|a| - 1) W_{n+1} + (\sqrt{|a|} - 1)|h_n|] \\ \quad \text{if } |a| > 1, \end{cases} \tag{25}$$

for all $n \geq n_0^-$, *where we set*

$$n_0^{\pm} := \min\bigl\{n \geq \nu : \mathcal{V}_n^- < 1\bigr\}, \tag{26}$$

while, under the convergence conditions

$$Z_n^+ < +\infty, \qquad U_n^+ < +\infty, \tag{27}$$

for any fixed $n \geq n_0^+$, stronger than those in (19) (since $|\eta_+| > 1$), a second, linearly independent solution, y_n^+ (as in (20)) exists for $n \geq n_0^+$, and the corresponding error term, when $|a| < 1$, is estimated as

$$|\varepsilon_n^+| < \mathcal{V}_n^+, \tag{28}$$

with

$$\begin{aligned}\mathcal{V}_n^+ := \frac{1}{2\sqrt{|a|}(1+\sqrt{|a|})}\bigl\{&V_n + W_n + \bigl(1+\sqrt{|a|}\bigr)\bigl(3+\sqrt{|a|}\bigr)W_{n+1}\\&+\bigl(1+\sqrt{|a|}\bigr)\bigl(1+\sqrt{|a|}+\eta_+\bigr)\eta_+ U_{n+1}^+ + (\eta_+)^2\bigl(1+\sqrt{|a|}+\eta_+\bigr)U_{n+2}^+\\&+(\eta_+)^3 Z_{n+2}^+ + \eta_+\bigl(1+\sqrt{|a|}+\eta_+\bigr)|h_{n+1}|\\&+\bigl[\bigl(1+\sqrt{|a|}\bigr)(1+\eta_+)+\eta_+\bigr]|h_{n+1}| + (\eta_+)^2|g_{n+1}| + \eta_+|g_n|\bigr\}.\end{aligned} \tag{29}$$

When $|a| > 1$, instead, we have for the second solution the estimate

$$\begin{aligned}\mathcal{V}_n^+ := \frac{1}{2\sqrt{|a|}(1+\sqrt{|a|})}\bigl\{&V_{n+2} + \bigl(1+\sqrt{|a|}\bigr)\bigl(2+\sqrt{|a|}\bigr)W_{n+1} + \bigl(2+\sqrt{|a|}\bigr)W_{n+2}\\&+\bigl(1+\sqrt{|a|}\bigr)\bigl[1+\sqrt{|a|}+|\eta_+|\bigr]|\eta_+|U_{n+1}^+ + |\eta_+|^2\bigl[1+\sqrt{|a|}+|\eta_+|\bigr]U_{n+2}^+\\&+|\eta_+|^3 Z_{n+2}^+ + \bigl(1+|\eta_+|\bigr)\bigl(2+\sqrt{|a|}\bigr)|h_n| + \bigl[1+\sqrt{|a|}\\&+|\eta_+|\bigl(1+\sqrt{|a|}+|\eta_+|\bigr)\bigr]|h_{n+1}| + \bigl(1+|\eta_+|^2\bigr)|g_{n+1}| + \bigl(1+|\eta_+|\bigr)|g_n|\bigr\}.\end{aligned} \tag{30}$$

Proof of Theorem 3.1 The proof follows closely that made for the case of equation (A) in Theorem 2.1 of [14], that is, it is based on transforming the difference equation for the error term into a "discrete integral" equation for it. Considering first the "unperturbed" equation associated to it, namely that obtained setting $h_n \equiv 0$, $g_n \equiv 0$ in (13),

$$\Delta^2 y_n + a y_n = 0, \quad a \in \mathbf{R} \setminus \{0, -1\}, \tag{31}$$

the corresponding "characteristic equation" (obtained looking for solutions to (31) of the form $y_n = \lambda^n$,

$$\lambda^2 - 2\lambda + (1+a) = 0, \tag{32}$$

has the "characteristic roots" $\lambda_\pm := 1 \pm \sqrt{-a}$, also written as

$$\lambda_\pm = 1 \pm i\sqrt{a} \quad \text{when } a > 0. \tag{33}$$

When $a = 0$, the characteristic equation has the double root $\lambda = 1$, and hence two linearly independent solutions to (31) are 1 and n, but we will not consider this case here.

Looking now for solutions to (C) of the form $y_n = \lambda^n(1+\varepsilon_n)$, with either $\lambda = \lambda_+$ or $\lambda = \lambda_-$, we obtain, after a little algebra, the error equation

$$\lambda^2 \varepsilon_{n+2} - 2\lambda\varepsilon_{n+1} + (1+a)\varepsilon_n = \chi_n, \tag{34}$$

where we set

$$\chi_n := -\lambda^2 h_{n+1}(1+\varepsilon_{n+2}) + \lambda(h_n + h_{n+1})(1+\varepsilon_{n+1}) - (h_n + g_n)(1+\varepsilon_n). \tag{35}$$

Note that χ_n also depends on the choice of either λ_+ or λ_- (as it happens for ε_n).

We can "solve" equation (34) treating χ_n as known, constructing a *discrete integral equation* (also called *summary difference equation*) for ε_n, rather than an explicit solution. We can proceed following the discrete analogue of the method of *variation of parameters*. To this purpose, we first set $\chi_n \equiv 0$ in (34), obtaining

$$\lambda^2 \Delta^2 \varepsilon_n + \beta \Delta \varepsilon_n = 0, \tag{36}$$

where relation (32) has been used, having set

$$\beta := 2\left(1 - \frac{1}{\lambda}\right). \tag{37}$$

Equation (36) can also be written as

$$\Delta[\Delta\varepsilon_n + \beta\varepsilon_n] = 0,$$

which suggests that two linearly independent solutions can be promptly obtained, one given by $\varepsilon_n \equiv 1$, and the other constructed solving

$$\Delta\varepsilon_n + \beta\varepsilon_n = 0,$$

which yields

$$\varepsilon_n = \eta^n, \quad \eta := 1 - \beta = \frac{2}{\lambda} - 1.$$

Note that here

$$\eta \equiv \eta_\mp = \frac{2 - \lambda_\pm}{\lambda_\pm} = \frac{1 \mp \sqrt{-a}}{1 \pm \sqrt{-a}} = \frac{\lambda_\mp}{\lambda_\pm} \tag{38}$$

for either $a > 0$ or $a < 0$.

The general solution to (36) has the form

$$\varepsilon_n = C + D\eta^n, \tag{39}$$

where C and D are arbitrary constants. To obtain a representation for the solutions to the full equation (34), we look for a solution of the form

$$\varepsilon_n = C_n + D_n\eta^n, \tag{40}$$

where C_n and D_n are two sequences to be determined. We evaluate first

$$\Delta\varepsilon_n = \Delta C_n + D_{n+1}\eta^{n+1} - D_n\eta^n = \Delta C_n + \eta^{n+1}\Delta D_n + (\eta - 1)\eta^n D_n,$$

and claim that

$$\Delta C_n + \eta^{n+1}\Delta D_n \equiv 0, \tag{41}$$

which is the same result that we would obtain if the sequences C_n and D_n would be constant with n (note the strict analogy with the method of variation of parameters for differential equations). Thus, we are left with

$$\Delta \varepsilon_n = -\beta\eta^n D_n, \tag{42}$$

wherefrom we evaluate

$$\Delta^2 \varepsilon_n = -\beta\eta^n(\eta D_{n+1} - D_n), \tag{43}$$

and hence, inserting (42) and (43) in (34) (rewritten in terms of Δ operators), we obtain by a little algebra and using again (32)

$$-\lambda^2\beta\eta^{n+1}\Delta D_n = \chi_n,$$

i.e.,

$$\Delta D_n = -\frac{\eta^{-n-1}}{\lambda^2\beta}\chi_n. \tag{44}$$

Note that

$$\lambda^2\beta = 2\lambda(\lambda - 1). \tag{45}$$

The next step is to "integrate" (44), i.e., to sum up both sides from $k = n$ to ∞, assuming (as a "constant of integration") that $D_n \to 0$ as $n \to \infty$. We obtain

$$\sum_{r=n}^{\infty} \Delta D_r = -D_n = -\frac{1}{\lambda^2\beta}\sum_{r=n}^{\infty}\eta^{-r-1}\chi_r,$$

and hence

$$D_n = \frac{1}{\lambda^2\beta}\sum_{r=n}^{\infty}\eta^{-r-1}\chi_r. \tag{46}$$

Finally, we determine C_n from (41). We have first

$$\Delta C_n = -\eta^{n+1}\Delta D_n = \frac{\chi_n}{\lambda^2\beta},$$

and then, assuming $C_n \to 0$ as $n \to \infty$,

$$\sum_{r=n}^{\infty}\Delta C_r = -C_n = \frac{1}{\lambda^2\beta}\sum_{r=n}^{\infty}\chi_r.$$

Hence,

$$C_n = -\frac{1}{\lambda^2\beta}\sum_{r=n}^{\infty}\chi_r. \tag{47}$$

To be more precise, reintroducing $\lambda = \lambda_\pm$, we have

$$C_n \equiv C_n^\pm = -\frac{1}{\lambda_\pm^2\beta_\pm}\sum_{r=n}^{\infty}\chi_r^\pm, \tag{48}$$

and thus, from (39),

$$\varepsilon_n^\pm = C_n^\pm + \eta_\pm^n D_n^\pm = -\frac{1}{2\lambda_\pm(\lambda_\pm - 1)}\sum_{r=n}^{\infty}\bigl[1-(\eta_\pm)^{r-n+1}\bigr]\chi_r^\pm. \tag{49}$$

We conclude with the *summary equation* of the Volterra type

$$\varepsilon_n^\pm = c^\pm\sum_{r=n}^{\infty}K_{n,r}^\pm\bigl(1+\varepsilon_r^\pm\bigr), \tag{50}$$

where we set

$$c^\pm := -\frac{1}{2\lambda_\pm(\lambda_\pm - 1)} = \mp\frac{1}{2i\sqrt{a}(1\pm i\sqrt{a})}, \tag{51}$$

and

$$K_{n,r}^\pm := \begin{cases} \lambda_\pm^2 a_{n,r-2}^\pm h_{r-1} - \lambda_\pm a_{n,r-1}^\pm(h_{r-1}+h_r) + a_{n,r}^\pm(g_r+h_r), & r \ge n+2, \\ -\lambda_\pm a_{n,n}^\pm(h_n+h_{n+1}) + a_{n,n+1}^\pm(g_{n+1}+h_{n+1}) \quad \text{for } r = n+1, \\ a_{n,n}^\pm(g_n+h_n) \quad \text{for } r = n, \end{cases} \tag{52}$$

with

$$a_{n,r}^\pm := 1-(\eta_\pm)^{r-n+1} \quad \text{for } r \ge n. \tag{53}$$

We now prove that (50) has two linearly independent solutions and establish some estimates for them. Writing for convenience $\varepsilon^\pm(n)$ in place of $\varepsilon_n^\pm$, equation (50) becomes

$$\varepsilon^\pm(n) = c^\pm\sum_{r=n}^{\infty}K_{n,r}^\pm\bigl(1+\varepsilon^\pm(r)\bigr). \tag{54}$$

Defining recursively the s-labeled sequence (of sequences) $\{\varepsilon_s^\pm(n)\}_{s=0}^\infty$ (for every fixed n, $n \ge \nu$) as

$$\varepsilon_0^\pm(n) \equiv 0, \qquad \varepsilon_{s+1}^\pm(n) := c^\pm\sum_{r=n}^{\infty}K_{n,r}^\pm\bigl(1+\varepsilon_s^\pm(r)\bigr), \quad s = 0,1,2,\dots, \tag{55}$$

we can show by induction (on s) that all these series converge for every fixed $n \geq \nu$. In fact, considering

$$\varepsilon_{s+1}^{\pm}(n) - \varepsilon_{s}^{\pm}(n) = c^{\pm} \sum_{r=n}^{\infty} K_{n,r}^{\pm} \bigl[\varepsilon_{s}^{\pm}(r) - \varepsilon_{s-1}^{\pm}(r)\bigr]$$

for $s \geq 1$, we can show inductively on s that

$$\bigl|\varepsilon_{s+1}^{\pm}(n) - \varepsilon_{s}^{\pm}(n)\bigr| \leq (\mathcal{V}_n)^{s+1} \tag{56}$$

for $s = 0, 1, 2, \ldots, n \geq \nu$, where $\mathcal{V}_n$ is defined in (22) for $a > 0$ and is replaced by $\mathcal{V}_n^{\pm}$, defined by (25), (29), when $a < 0$. In fact, we have first that

$$\varepsilon_{1}^{\pm}(n) = c^{\pm} \sum_{r=n}^{\infty} K_{n,r}^{\pm},$$

hence

$$\bigl|\varepsilon_{1}^{\pm}(n)\bigr| \leq \sum_{r=n}^{\infty} \bigl|c^{\pm} K_{n,r}^{\pm}\bigr| \leq \mathcal{V}_n, \tag{57}$$

and then, assuming that (56) is satisfied with s replaced by $s - 1$, we have

$$\begin{aligned} \bigl|\varepsilon_{s+1}^{\pm}(n) - \varepsilon_{s}^{\pm}(n)\bigr| &\leq \sum_{r=n}^{\infty} \bigl|c^{\pm} K_{n,r}^{\pm}\bigr| \bigl|\varepsilon_{s}^{\pm}(r) - \varepsilon_{s-1}^{\pm}(r)\bigr| \leq \sum_{r=n}^{\infty} \bigl|c^{\pm} K_{n,r}^{\pm}\bigr| \mathcal{V}_r^{s} \\ &\leq \mathcal{V}_n^{s} \sum_{r=n}^{\infty} \bigl|c^{\pm} K_{n,r}^{\pm}\bigr| \leq \mathcal{V}_n^{s+1}. \end{aligned} \tag{58}$$

In fact, $\mathcal{V}_n$ *decreases monotonically* (to zero) as n grows (to ∞). Introducing the sequence

$$\varepsilon^{\pm}(n) := \sum_{s=0}^{\infty} \bigl[\varepsilon_{s+1}^{\pm}(n) - \varepsilon_{s}^{\pm}(n)\bigr], \tag{59}$$

which is well defined in view of (58) whenever $\mathcal{V}_n < 1$, a condition satisfied for all $n \geq n_0$, since $\mathcal{V}_n$ decreases, where n_0 is the smallest index not less than ν such that $\mathcal{V}_n < 1$. Similarly, when $\mathcal{V}_n^{\pm}$ replaces $\mathcal{V}_n$, it is understood that $n \geq n_0^{\pm}$.

Moreover, it follows from (58) and (59) that

$$\bigl|\varepsilon^{\pm}(n)\bigr| \leq \frac{\mathcal{V}_n}{1 - \mathcal{V}_n}, \quad n \geq n_0. \tag{60}$$

Finally, we prove that the sequence $\varepsilon^{\pm}(n)$ solves the summary Volterra equation (54). In fact,

$$\begin{aligned} \varepsilon^{\pm}(n) &= \varepsilon_{1}^{\pm}(n) + \sum_{s=1}^{\infty} \bigl[\varepsilon_{s+1}^{\pm}(n) - \varepsilon_{s}^{\pm}(n)\bigr] \\ &= \sum_{r=n}^{\infty} c^{\pm} K_{n,r}^{\pm} + \sum_{s=1}^{\infty} \sum_{r=n}^{\infty} c^{\pm} K_{n,r}^{\pm} \bigl[\varepsilon_{s}^{\pm}(r) - \varepsilon_{s-1}^{\pm}(r)\bigr], \end{aligned}$$

and the proof follows immediately if we can show that it is permissible to interchange the order of summation in the previous relation. This is indeed the case in view of the Lebesgue dominated convergence theorem, since

$$\left|\sum_{s=1}^{S} c^{\pm}K_{n,r}^{\pm}\left[\varepsilon_s^{\pm}(n)-\varepsilon_{s-1}^{\pm}(n)\right]\right| \le \left|c^{\pm}K_{n,r}^{\pm}\right|\frac{\mathcal{V}_{n_0}}{1-\mathcal{V}_{n_0}}$$

for any $S>1$ and $r>n_0$, and using the summability condition (19) (or (27)).

To conclude the proof, the key is establishing in all cases an estimate for $\sum_{r=n}^{\infty}|c^{\pm}K_{n,r}^{\pm}|$ as in (57), with a suitable $\mathcal{V}_n$ [or $\mathcal{V}_n^{\pm}$]. We have first from (52), (53), for $r \ge n+2$,

$$\begin{aligned}
&\left|K_{n,r}^{\pm}\right| \le |\lambda_{\pm}|^2\left|a_{r-2}^{\pm}\right||h_{r-1}| + |\lambda_{\pm}|\left|a_{r-1}^{\pm}\right|\left(|h_{r-1}|+|h_r|\right) + \left|a_r^{\pm}\right|\left(|g_r|+|h_r|\right),\\
&\left|K_{n,n+1}^{\pm}\right| \le |\lambda_{\pm}|\left|a_{n,n}^{\pm}\right|\left(|h_n|+|h_{n+1}|\right) + |a_{n,n+1}|^{\pm}\left(|g_{n+1}|+|h_{n+1}|\right),\\
&\left|K_{n,n}^{\pm}\right| \le \left|a_{n,n}^{\pm}\right|\left(|g_n|+|h_n|\right),
\end{aligned}$$

and then

$$\sum_{r=n}^{S}\left|c^{\pm}K_{n,r}^{\pm}\right| = \left|c^{\pm}\right|\sum_{r=n+2}^{S}\left|K_{n,r}^{\pm}\right| + \left|c^{\pm}K_{n,n+1}^{\pm}\right| + \left|c^{\pm}K_{n,n}^{\pm}\right|.$$

Finally, an estimate like

$$\sum_{r=n}^{S}\left|c^{\pm}K_{n,r}^{\pm}\right| \le \mathcal{V}_n \quad \left[\text{or } \mathcal{V}_n^{\pm}\right]$$

with $\mathcal{V}_n$ [or $\mathcal{V}_n^{\pm}$] given by (22) [or (25), (29)] can be obtained upon lengthy though rather elementary calculations. Details are given in Appendix 1, where the various cases of a real, $a \ne 0$, $a \ne -1$ (i.e., $a>0$, $-1<a<0$, and $a>-1$) are covered. □

Remark 3.1 Note that, W_n being decreasing, $W_n \le W_{n-1}$, in (22) we have also

$$\mathcal{V}_n \le \frac{1}{\sqrt{a(a+1)}}\left(V_n + (\sqrt{a+1}+1)^2 W_n\right). \tag{61}$$

In any case, $\mathcal{V}_n = \mathcal{O}(V_n) + \mathcal{O}(W_{n-1})$. A similar observation can be made also in the case $a<0$.

3.2 The LG asymptotics for case (D)

Here we will use definitions (15), (16), but now with

$$\eta_{\pm} := \frac{b \pm \sqrt{b^2-4}}{b \mp \sqrt{b^2-4}}, \quad b = a-2, \tag{62}$$

instead of (17), and

$$\lambda_{\pm} := -\frac{1}{2}\left(b \pm \sqrt{b^2-4}\right), \tag{63}$$

which are the characteristic roots of the unperturbed equation associated to (D), that is, $\lambda^2 + b\lambda + 1 = 0$ instead of (17) and (18). We can proceed in a strictly similar way for case (D). We report all steps in Appendix 2 for the reader's convenience. We have thus the following.

Theorem 3.2 *Let equation* (D) *be given for* $n \geq \nu$, *for some fixed* $\nu \in \mathbf{Z}$, *with the coefficients as in* (13), *with* a *real*, $a \neq 0, -1$, *and* $c = 1$, g_n *and* h_n *real or complex. Then, under the conditions in* (19) *and, when needed,* (27),

(i) *If* $0 < a < 4$ (*i.e.*, $b^2 < 4$), *two linearly independent solutions of the form* (20) *exist for all* $n \geq n_0$, *for some* $n_0 \geq \nu$, *with*

$$\lambda_\pm = \frac{1}{2}\big[-b \pm i\sqrt{4-b^2}\big], \qquad \lambda_+ = 1/\lambda_- = \overline{\lambda}_-, \qquad |\lambda_\pm| = 1, \qquad |\eta_\pm| = \left|\frac{\lambda_\pm}{\lambda_\mp}\right| = 1,$$

hence

$$\left|a^\pm_{n,r}\right| = \left|1 - (\eta_\pm)^{r-n+1}\right| \leq 2, \tag{64}$$

and

$$\left|K^\pm_{n,r}\right| \leq 2|h_{r-1}| + 2|h_r| + |g_{r-1}|$$

for $r \geq n + 2$, *while*

$$\left|K^\pm_{n,n+1}\right| \leq 2|h_{n+1}| + |h_n| + |g_n|, \quad \textit{and} \quad \left|K^\pm_{n,n}\right| \leq |h_n|.$$

Then

$$\sum_{r=n}^{\infty}\left|c^\pm K^\pm_{n,r}\right| \leq \frac{2}{\sqrt{a(4-a)}}(V_n + 2W_n + 2W_{n+1}) =: \mathcal{V}^\pm_n \equiv \mathcal{V}_n. \tag{65}$$

Details are given in Appendix 3.

When $b^2 > 4$, *we have the two cases*, $a > 4$ ($b > 2$), *and* $a < 0$ ($b < -2$).

(ii) *For* $a > 4$ (*i.e.*, $b > 2$), $|\lambda_+/\lambda_-| < 1$ (*hence* $|\lambda_-/\lambda_+| > 1$), *and thus*

$$|\eta_+| = \left|\frac{\lambda_+}{\lambda_-}\right| = \left|\frac{-b + \sqrt{b^2-4}}{-b - \sqrt{b^2-4}}\right| < 1, \qquad |\eta_-| = \frac{1}{|\eta_+|},$$

and hence

$$\left|a^+_{n,r}\right| = \left|1 - (\eta_+)^{r-n+1}\right| = 1 - (\eta_+)^{r-n+1} < 1,$$
$$\left|a^-_{n,r}\right| = \left|1 - (\eta_-)^{r-n+1}\right| \leq 1 + |\eta_-|^{r-n+1}.$$

Therefore,

$$\left|K^-_{n,r}\right| \leq \lambda_-\big|^2\big[1 + |\eta_-|^{n-r-1}\big]|h_{r-1}| + |\lambda_-|\big[1 + |\eta_-|^{n-r}|\big]\big(|g_{r-1}| + |h_{r-1}| + |h_r|\big)$$
$$+ \big[1 + |\eta_-|^{n-r+1}\big]|h_r|,$$

$$\left|K^-_{n,n+1}\right| \le |\lambda_-|\bigl[1+|\eta_-|\bigr]\bigl(|g_n|+|h_n|+|h_{n+1}|\bigr)+\bigl[1+|\eta_-|^2\bigr]|h_{n+1}|,$$
$$\left|K^-_{n,n}\right| \le \bigl(1+|\eta_-|\bigr)|h_n|.$$

Collecting all these and recalling the definitions of V_n, W_n, Z^-_n, and U^-_n (given in (19), (27)), we obtain upon lengthy calculations

$$\left|\varepsilon^-_n\right| \le \mathcal{V}^-_n,$$

with

$$\begin{aligned}\mathcal{V}^-_n := \left|c^-\right|\bigl[&|\lambda_-|V_{n+1}+\bigl(|\lambda_-|+1\bigr)W_n+|\lambda_-|\bigl(|\lambda_-|+1\bigr)W_{n+1}\\&+|\lambda_-||\eta_-|\bigl(|\lambda_-|+|\eta_-|\bigr)U^-_{n+1}\\&+|\eta_-|^2\bigl(|\lambda_-|+|\eta_-|\bigr)U^-_{n+2}+|\lambda_-||\eta_-|^2Z^-_{n+1}\\&+|\lambda_-|\bigl(1+|\eta_-|\bigr)|g_n|+\bigl(|\lambda_-|+1\bigr)|\eta_-|)|h_n|+|\lambda_-|\bigl(|\lambda_-|+|\eta_-|\bigr)|h_{n+1}|\bigr].\end{aligned} \tag{66}$$

The second solution is estimated simply as

$$\left|\varepsilon^+_n\right| \le \mathcal{V}^+_n$$

with

$$\mathcal{V}^+_n := \left|c^+\right|\bigl(V_n+2W_n+W_{n+1}+|h_{n+1}|\bigr). \tag{67}$$

Details are reported in Appendix 3.

(iii) *For $a<0$ $[b<-2]$, we obtain upon some algebra*

$$\mathcal{V}^-_n := \left|c^-\right|(V_n+2W_n+2W_{n+1}), \tag{68}$$

while

$$\begin{aligned}\mathcal{V}^+_n := |c_+|\bigl[&|\lambda_+|V_n+\bigl(1+|\lambda_+|\bigr)W_n+|\lambda_+|\bigl(1+|\lambda_+|\bigr)W_{n+1}\\&+\eta_+|\lambda_+|\bigl(\eta_++|\lambda_+|\bigr)U^+_{n+1}+(\eta_+)^2\bigl(\eta_++|\lambda_+|\bigr)U^+_{n+2}\\&+(\eta_+)^2|\lambda_+|Z^+_{n+1}+\eta_+\bigl(1+|\lambda_+|\bigr)|h_n|\\&+\eta_+\bigl(\eta_++|\lambda_+|\bigr)|h_{n+1}|+\eta_+|\lambda_+||g_n|\bigr].\end{aligned} \tag{69}$$

Proof of Theorem 3.2 Referring to the detailed derivation reported in Appendix 3, we end up with the *summary equation* of Volterra type

$$\varepsilon^\pm_n = c^\pm\sum_{r=n}^{\infty}K^\pm_{n,r}\bigl(1+\varepsilon^\pm_r\bigr), \tag{70}$$

where

$$c^\pm := \mp\frac{\lambda_\mp}{\sqrt{b^2-4}}, \qquad \lambda_\pm = -\frac{1}{2}\bigl(b\pm\sqrt{b^2-4}\bigr), \tag{71}$$

and

$$K_{n,r}^{\pm} := \begin{cases} \lambda_{\pm}^2 a_{n,r-2}^{\pm} h_{r-1} + \lambda_{\pm} a_{n,r-1}^{\pm}(g_{r-1} - h_{r-1} - h_r) + a_{n,r}^{\pm} h_r, & r \geq n+2, \\ \lambda_{\pm} a_{n,n}^{\pm}(g_n - h_n - h_{n+1}) + a_{n,n+1}^{\pm} h_{n+1} & \text{for } r = n+1, \\ a_{n,n}^{\pm} h_n & \text{for } r = n, \end{cases} \tag{72}$$

with $a_{n,r}^{\pm} := 1 - (\eta_{\pm})^{r-n+1}$ for $r \geq n$ (as in (53)), being

$$|\eta_{\pm}| = \left| \frac{\lambda_{\pm}}{\lambda_{\mp}} \right| = \left| \frac{b \pm \sqrt{b^2 - 4}}{b \mp \sqrt{b^2 - 4}} \right|, \tag{73}$$

so that $|\eta_+| > 1$ (and $|\eta_-| < 1$) when $a > 0$, and $|\eta_-| > 1$ (and $|\eta_+| < 1$ when $a < 0$.

Similarly to case (C), we can show also for case (D) that two linearly independent solutions $\varepsilon_n^{\pm}$ to (70) exist and can be estimated in a similar way.

The procedure is standard and follows exactly the same lines of the proof of Theorem 3.1, using now the appropriate values of $\mathcal{V}_n$, $\mathcal{V}_n^{\pm}$. □

Remark 3.2 Clearly, when $h_n \equiv 0$, all formulae and error estimates greatly simplify. Summarizing, the definitions of $\mathcal{V}_n$, $\mathcal{V}_n^{\pm}$ given in (22), (25), (29), (65), (66), and (69) reduce, respectively, to the following.

For case (C).

(i) Case (C), $a > 0$:

$$\mathcal{V}_n := \frac{1}{\sqrt{a(a+1)}} V_n. \tag{74}$$

(ii) Case (C), $a < 0$ (with $a \neq -1$), first solution:

$$\mathcal{V}_n^- := \begin{cases} \frac{1}{2\sqrt{|a|}(1-\sqrt{|a|})}(1 - \sqrt{|a|}) V_n, \\ \quad \text{if } |a| < 1, \\ \frac{1}{\sqrt{|a|}(\sqrt{|a|}-1)} V_n, \quad \text{if } |a| > 1. \end{cases} \tag{75}$$

(iii) Case (C), $a < 0$ (with $a \neq -1$), second solution:

$$\mathcal{V}_n^+ := \frac{1}{2\sqrt{|a|}(1+\sqrt{|a|})}\bigl(V_n + |\eta_+| Z_n^+\bigr). \tag{76}$$

In particular, setting $h_n \equiv 0$, we recover the estimate obtained in [14] (as it must be).

For case (D).

(i) Case (D), $0 < a < 4$ [i.e., $b^2 < 4$]:

$$\mathcal{V}_n^{\pm} \equiv \mathcal{V}_n = \frac{2}{\sqrt{a(4-a)}} V_n. \tag{77}$$

(ii) Case (D), $a > 4$ [i.e., $b > 2$], first solution:

$$\mathcal{V}_n^- = \left|c^-\right| \bigl(|\lambda_-| \bigl(1 + |\eta_-|\bigr) V_n + |\lambda_-| |\eta_-|^3 Z_{n+1}^- \bigr), \tag{78}$$

where c^-, λ_-, and $|\eta_-|$ are defined in (71) and (73).

(iii) Case (D), $a > 4$ [i.e., $b > 2$], second solution:

$$\mathcal{V}_n^+ = \left|c^+\right| V_n, \tag{79}$$

where c^+ is defined in (71);

(iv) Case (D), $a < 0$ [i.e., $b < -2$], first solution:

$$\mathcal{V}_n^- = |c_-|(V_n + 2W_n + 2W_{n+1}). \tag{80}$$

(v) Case (D), $a < 0$ [i.e., $b < -2$], second solution (dominant solution):

$$\begin{aligned}\mathcal{V}_n^+ = |c_+|\bigl\{&|\lambda_+|(1+\eta_+)V_n + \bigl(1+|\lambda_+|\bigr)(1+\eta_+)W_n\bigl[|\lambda_+|^2 + |\lambda_+|(1+\eta_+)\\ &+ \bigl(1+(\eta_+)^2\bigr)\bigr]W_{n+1} + W_{n+2} + |\lambda_+|\eta_+\bigl(|\lambda_+|+\eta_+\bigr)\mathcal{U}_{n+1}^+\\ &+ (\eta_+)^2\bigl(|\lambda_+|+\eta_+\bigr)\mathcal{U}_{n+2}^+ + |\lambda_+|(\eta_+)^2 Z_{n+1}^+\bigr\}.\end{aligned} \tag{81}$$

In the next section, we present a few examples to illustrate the results provided by Theorems 3.1 and 3.2.

4 Examples

The validity of Theorems 3.1 and 3.2, generally speaking, rests on the convergence of the series $\sum_{r=n}^{\infty}|g_r|$, $\sum_{r=n}^{\infty}|h_r|$ for at least one of the basis' solutions, the *recessive* solution, while the stronger conditions

$$|\eta_+|^{-n}\sum_{r=n}^{\infty}|\eta_+|^r|g_r| < +\infty, \qquad |\eta_+|^{-n}\sum_{r=n}^{\infty}|\eta_+|^r|h_r| < +\infty,$$

where $|\eta_+| > 1$, are required, correspondingly to the *dominant* solution. Recall that, if there exist two linearly independent solutions, say u_n^- and u_n^+, of a linear second-order difference equation like (1) such that $u_n^-/u_n^+ \to 0$ as $n \to \infty$, then u_n^- is called *recessive* and u_n^+ is called *dominant* solution of (1), see Sect. 6.3, Definition 6.3.1, in [3, Ch. 6, p. 342].

In most examples below, we take either the same sequences $g_n = h_n = \frac{1}{n(n+1)}$ or $g_n = h_n = \frac{A^{-n}}{n^2}$ with suitable $A > 1$. The purpose being of illustration, these choices are made just to allow simple explicit evaluation of all the needed quantities.

Example 4.1 A case of type (C) with $a > 0$.

Consider

$$\Delta\left(\left(1+\frac{1}{n(n+1)}\right)\Delta y_n\right) + \left(1+\frac{1}{n(n+1)}\right)y_n = 0, \quad n \in \mathbf{Z}_1,$$

that is, equations (C)–(13), with $a = c = 1$, $g_n = h_n = \frac{1}{n(n+1)}$, $\nu = 1$. We have from (15) $V_n = W_n = \frac{1}{n}$, and hence, by (22), $\mathcal{V}_n = \mathcal{O}(\frac{1}{n})$. More precisely,

$$\mathcal{V}_n = \frac{1}{\sqrt{2}}\left\{\frac{1}{n} + (\sqrt{2}+1)\left[\sqrt{2}\frac{1}{n+1} + \frac{1}{n+2}\right]\right\},$$

hence, e.g., $\mathcal{V}_{10} \approx 0.42$, reduced to $\mathcal{V}_{100} \approx 0.047$ for $n = 100$. Then, using (33) and (20), we have

$$y_n^{\pm} = (1 \pm i)^n \bigl(1 + \varepsilon_n^{\pm}\bigr),$$

with the error estimates $|\varepsilon_n^{\pm}| \lesssim 0.42$ for all $n \geq 10$ and $|\varepsilon_n^{\pm}| \lesssim 0.047$ for all $n \geq 100$. Note that being $\mathcal{V}_3 \approx 1.18$ and $\mathcal{V}_4 \approx 0.94$, we have $n_0 = 4$, see (23).

Clearly, the smallness of the errors depends on the speed of convergence to zero of the series in (19). In fact, if, for instance, $g_n = h_n = \frac{1}{n^2(n^2+1)}$, we have $V_n = W_n = 1/n^2$ and $\mathcal{V}_{10} \lesssim 0.038$, $\mathcal{V}_{100} \lesssim 3 \times 10^{-4}$.

Example 4.2 A case of type **(C)** with $a < 0$ ($a \neq -1$).

Consider

$$\Delta\left(\left(1 + \frac{A^{-n}}{n^2}\right)\Delta y_n\right) + \left(-2 + \frac{A^{-n}}{n^2}\right)y_n = 0, \quad n \in \mathbf{Z}_1,$$

that is, equations **(C)**–(13) with $a = -2$, and again $c = 1$, $g_n = h_n = \frac{A^{-n}}{n^2}$, where $A = 2|\eta_+| = 2(3 + 2\sqrt{2}) \approx 11.6568$. Thus, there are two linearly independent solutions $y_n^{\pm}$ with

$$y_n^+ = (1 + \sqrt{2})^n \bigl(1 + \varepsilon_n^+\bigr), \qquad y_n^- = (1 - \sqrt{2})^n \bigl(1 + \varepsilon_n^-\bigr)$$

for $n \geq n_0^+$ and $n \geq n_0^-$, respectively, with $n_0^{\pm}$ given by (26), with error terms estimated by $|\varepsilon_n| \leq \mathcal{V}_n^{\pm}$, according to (24), (25) and (28), (29). Note that, being

$$\left|\frac{y_n^-}{y_n^+}\right| \sim \left|\frac{1 - \sqrt{2}}{1 + \sqrt{2}}\right|^n \to 0, \quad \text{as } n \to \infty,$$

the solution y_n^- is recessive and y_n^+ is dominant. Now,

$$V_n = W_n \leq 2\frac{(2|\eta_+|)^{-n}}{n}$$

and

$$\begin{aligned} Z_n^+ = U_n^+ &= |\eta_+|^{-n} \sum_{r=n}^{\infty} |\eta_+|^r \frac{(2|\eta_+|)^{-r}}{r^2} \\ &< |\eta_+|^{-n} \sum_{r=n}^{\infty} \frac{1}{2^r r^2} \leq |\eta_+|^{-n} Li_2(1/2) \approx 0.5822 \times (5.8284)^{-n}, \end{aligned}$$

where we used the *dilogarithm*

$$Li_2(x) := \sum_{r=1}^{\infty} \frac{x^r}{r^2}$$

[7, § 25.15], [2, § 27.7]. It is known that $Li_2(1/2) = \pi^2/12 - (\ln^2 2)/2 \approx 0.5822$ [1]. Therefore, we obtain from (30), by rather tedious but elementary computations, $\mathcal{V}_3^+ \lesssim 0.0296$ and $\mathcal{V}_{10}^+ \lesssim 0.01280 \times 10^{-6}$.

As for the second solution, we have from (25) $\mathcal{V}_3^- \approx 0.15 \times 10^{-2}$. The smallness achieved already for $n = 3$ is due to the fast decay of g_n and h_n. Indeed, if, instead, $g_n = h_n = 1/n^2$, and hence $V_n = W_n < 2/n$, we obtain $\mathcal{V}_3^- \lesssim 3.208$ (that is useless, see (23)), while $\mathcal{V}_{10}^- \lesssim 0.5859$, but the corresponding estimates for $\mathcal{V}_n^+$ blow up.

Example 4.3 A case of type (D) with $0 < a < 4$ (i.e., $b^2 < 4$).
Consider

$$\Delta\left(\left(1 + \frac{1}{n(n+1)}\right)\Delta y_n\right) + \left(2 + \frac{1}{n\ln^2 n}\right)y_{n+1} = 0, \qquad n \in \mathbf{Z}_2,$$

that is, equations (D)–(13), with $\nu = 2$, $a = 2$, $c = 1$, $g_n = \frac{1}{n\ln^2 n}$, $h_n = \frac{1}{n(n+1)}$. We have

$$y_n^\pm = (\pm i)^n\left(1 + \varepsilon_n^\pm\right),$$

where the error terms can be bounded through (100),

$$\mathcal{V}_n^\pm \equiv \mathcal{V}_n = V_n + 2W_n + 2W_{n+1} < \frac{1}{\ln n} + \frac{2}{n} + \frac{2}{n+1},$$

since we can estimate V_n with $V_n < \int_n^{+\infty} \frac{dx}{x\ln^2 x} = \frac{1}{\ln n}$ for every $n \geq 1$. We then have $\mathcal{V}_{10} \lesssim 0.816$, $\mathcal{V}_{20} \lesssim 0.529$, $\mathcal{V}_{30} \lesssim 0.4251$, $\mathcal{V}_{50} \lesssim 0.3348$, $\mathcal{V}_{100} \lesssim 0.2569$.

Example 4.4 A case of type (D) with $a > 4$ (i.e., $b > 2$).
Consider

$$\Delta\left(\left(1 + \frac{1}{n(n+1)}\right)\Delta y_n\right) + \left(4.1 + \frac{1}{n(n+1)}\right)y_{n+1} = 0, \qquad n \in \mathbf{Z}_1,$$

that is, equations (D)–(13), with $\nu = 1$, $a = 4.1$ (hence $b = 2.1$), $c = 1$, $g_n = h_n = \frac{1}{n(n+1)}$. Therefore, we have

$$y_n^+ = (-0.1266\ldots)^n\left(1 + \varepsilon_n^+\right), \qquad y_n^- = (-1.9733\ldots)^n\left(1 + \varepsilon_n^-\right).$$

These solutions are recessive and dominant, respectively. We have $V_n = W_n = \frac{1}{n}$, and from (101), (103), $|c^+| \approx 1.1398$, $|c^-| \approx 2.1398$, $|\lambda_-| \approx 1.8262$, $|\eta_-| \approx 6.6698$. We have from (104) $\mathcal{V}_{10}^+ \approx 0.4541$, $\mathcal{V}_{100}^+ \approx 0.0455$, while the estimate for $\mathcal{V}_n^-$ blows up.

Taking instead $g_n = h_n = \frac{(2|\eta_-|)^{-n}}{n^2} \approx \frac{1}{n^2(13.3396)^n}$, so that $Z_n^- = U_n^- = |\eta_-|^{-n}\sum_{r=n}^\infty \frac{1}{2^r r^2} \approx 0.5822 \times (6.6698)^{-n}$ (using the dilogarithm), we obtain from (104) $\mathcal{V}_3^+ \lesssim 0.96 \times 10^{-5}$, already very small, and from (102) $\mathcal{V}_3^- \lesssim 0.1561$.

Example 4.5 A case of type (D) with $a < 0$ (i.e., $b < -2$).
Consider finally

$$\Delta\left(\left(1 + \frac{1}{n(n+1)}\right)\Delta y_n\right) + \left(-2 + \frac{1}{n(n+1)}\right)y_{n+1} = 0, \qquad n \in \mathbf{Z}_1,$$

that is, equations (D)–(13), with $\nu = 1$, $a = -2$ (hence $b = -4$), $c = 1$, and compute from (105), (107) $|c^-| = \sqrt{3}/3 - 1/2 \approx 0.0773$, $|c^+| = \sqrt{3}/3 + 1/2 \approx 1.0773$, $|\lambda_+| = 2 + \sqrt{3} \approx 3.7320$,

$|\eta_+| = (2+\sqrt{3})/(2-\sqrt{3}) \approx 13.9282$. Here we have

$$y_n^+ = (3.732\ldots)^n\bigl(1+\varepsilon_n^+\bigr), \qquad y_n^- = (0.02679\ldots)^n\bigl(1+\varepsilon_n^-\bigr).$$

Here again we can see that these solutions are dominant and recessive, respectively. Taking $g_n = h_n = \frac{1}{n(n+1)}$, using (108) and (106), we can evaluate $\mathcal{V}_{10}^- \lesssim 0.0373$, but the estimate for $\mathcal{V}_n^+$ blows up.

Taking $g_n = h_n = \frac{(2|\eta_+|)^{-n}}{n^2} \approx \frac{1}{n^2(27.8564)^n}$ instead, we have $V_n = W_n \approx \sum_{r=n}^{\infty} \frac{1}{r^2(27.8564)^r} \leq \frac{2}{n(27.8564)^n}$, $Z_n^+ = U_n^+ \approx 0.5855 \times |\eta_+|^n \approx 0.5822 \times (13.9282)^n$. We can then evaluate by (108) and (106) $\mathcal{V}_3^- \lesssim 0.7 \times 10^{-5}$, while $\mathcal{V}_3^+ \lesssim 0.0324$.

Remark 4.1 Note that in all estimates established in Theorems 3.1 and 3.2, we have $|g_n| = o(V_n)$, etc., hence the terms like V_p dominate over those containing g_q, and W_p dominate over those with h_q. Also, V_n and W_n being monotonic decreasing functions of n, we have, e.g., $V_{n+1} < V_n$, etc., hence, V_n dominates over V_{n+1}. Note that also the sequences $Z_n^{\pm}$ and $U_n^{\pm}$ decrease monotonically with n.

Remark 4.2 The "double asymptotic nature" of the classical LG approximations [9] can be extended to the present discrete version, as it was done in [14] for equation (A), but not in all cases, in particular when there are perturbations of the coefficient c_n. Indeed, we can see that, in case (C), $\mathcal{V}_n = \mathcal{O}(a^{-1})$ as $a \to +\infty$ (with fixed n), but only *if* $h_n \equiv 0$, otherwise we have merely $\mathcal{V}_n = \mathcal{O}(1)$. The same results are obtained for $\mathcal{V}_n^{\pm}$, when, still in case (C), $a \to -\infty$.

In case (D), we have, as $a \to +\infty$, $\mathcal{V}_n^- = \mathcal{O}(a^{-1})$ and $\mathcal{V}_n^+ = \mathcal{O}(a^{-2})$, while for $a \to -\infty$, $\mathcal{V}_n = \mathcal{O}(|a|^{-1})$ and $\mathcal{V}_n = \mathcal{O}(a^{-2})$.

5 Summary

A Liouville–Green (LG, or WKB) asymptotic theory has been developed for some classes of linear second-order difference equations on the basis of certain canonical forms, into which rather general linear three-term recurrences can be transformed. Both recessive and dominant solutions are included. Rigorous and computable bounds have been derived for the error terms appearing in the asymptotic representations of solutions. These bounds may also be useful for the numerical evaluation of solutions. The results are similar, in general, to those obtained in the analogous cases of differential equations.

Appendix 1

In this appendix we give details, rather elementary but lengthy, pertaining to the proof of Theorem 3.1, that is, to case (C) of equation $\Delta(c_n \Delta y_n) + r_n y_n = 0$. We have

(i) for $a > 0$,

$$\bigl|c^{\pm}\bigr| = \frac{1}{2\sqrt{a(a+1)}}, \qquad |\lambda_{\pm}| = \sqrt{a+1}, \qquad |\eta_{\pm}| = 1,$$

and hence

$$\bigl|a_{n,r}^{\pm}\bigr| \equiv 1 - (\eta_{\pm})^{r-n+1} \leq 2.$$

Thus, for $r \ge n+2$,

$$\begin{aligned}
&\bigl|c^{\pm}K^{\pm}_{n,r}\bigr| \\
&\quad \le \frac{1}{2\sqrt{a(a+1)}}\bigl\{|\lambda_{\pm}|^2|a^{\pm}_{n,r-2}|h_{r-1}| + |\lambda_{\pm}|\bigl|a^{\pm}_{n,r-1}\bigr|\bigl(|h_{r-1}| + |h_r|\bigr) \\
&\qquad + \bigl|a^{\pm}_{n,r}\bigr|\bigl(|g_r| + |h_r|\bigr)\bigr\} \\
&\quad \le \frac{1}{\sqrt{a(a+1)}}\bigl\{(a+1)|h_{r-1}| + \sqrt{a+1}\bigl(|h_{r-1}| + |h_r|\bigr) + |g_r| + |h_r|\bigr\}; \\
&\bigl|c^{\pm}K^{\pm}_{n,n+1}\bigr| \le \frac{1}{2\sqrt{a(a+1)}}\bigl\{|\lambda_{\pm}|\bigl|a^{\pm}_{n,n}\bigr|\bigl(|h_n| + |h_{n+1}|\bigr) + \bigl|a^{\pm}_{n,n+1}\bigr|\bigl(|g_{n+1}| + |h_{n+1}|\bigr)\bigr\} \\
&\qquad \le \frac{1}{\sqrt{a(a+1)}}\bigl\{\sqrt{a+1}\bigl(|h_| + |h_{n+1}|\bigr) + |g_{n+1}| + |h_{n+1}|\bigr\}; \\
&\bigl|c^{\pm}K^{\pm}_{n,n}\bigr| \le \frac{1}{2\sqrt{a(a+1)}}\bigl|a^{\pm}_{n,n}\bigr|\bigl(|g_{n+1}| + |h_{n+1}|\bigr) \le \frac{1}{\sqrt{a(a+1)}}\bigl(|g_{n+1}| + |h_{n+1}|\bigr).
\end{aligned}$$

Collecting all the previous terms, upon some little algebra and recalling the definitions of V_n and W_n, we have

$$\begin{aligned}
&\sum_{r=n}^{\infty}\bigl|c^{\pm}K^{\pm}_{n,r}\bigr| \\
&\quad = \sum_{r=n+2}^{\infty}\bigl|c^{\pm}K^{\pm}_{n,r}\bigr| + \bigl|c^{\pm}K^{\pm}_{n,n+1}\bigr| + \bigl|c^{\pm}K^{\pm}_{n,n}\bigr| \\
&\quad \le \frac{1}{\sqrt{a(a+1)}}\Biggl[(a+1)\sum_{r=n+2}^{\infty}|h_{r-1}| + \sqrt{a+1}\Biggl(\sum_{r=n+2}^{\infty}|h_{r-1}| + \sum_{r=n+2}^{\infty}|h_r|\Biggr) \\
&\qquad + \sum_{r=n+2}^{\infty}|g_r| + \sum_{r=n+2}^{\infty}|h_r| + \sqrt{a+1}\bigl(|h_n| + |h_{n+1}|\bigr) \\
&\qquad + |g_{n+1}| + |h_{n+1}| + |g_n| + |h_n|\Biggr] \\
&\quad = \frac{1}{\sqrt{a(a+1)}}\bigl\{V_n + (\sqrt{a+1}+1)[\sqrt{a+1}W_{n+1} + W_{n+2}]\bigr\} =: \mathcal{V}^{\pm}_n \equiv \mathcal{V}_n.
\end{aligned} \tag{82}$$

Here we used the fact that $\sum_{r=n+2}^{\infty}|h_{r-1}| = \sum_{j=n+1}^{\infty}|h_j| = W_{n+1}$, etc., and that $V_{n+2} + |g_{n+1}| + |g_n| = V_n$ and $W_{n+2} + |h_{n+1}| + |h_n| = W_n$. It is worth noting that W_n *decreases* as n grows, hence, e.g., W_{n+1} could be estimated by W_n.

(ii) for $-1 < a < 0$ (i.e., $a < 0$, $|a| < 1$),

$$\bigl|c^{-}\bigr| = \frac{1}{2\sqrt{|a|}(1-\sqrt{|a|})}, \qquad |\lambda_-| = 1 - \sqrt{|a|}, \qquad |\eta_-| < 1 \quad (\eta_- > 0),$$

and hence

$$\bigl|a^{-}_{n,r}\bigr| < 1.$$

Thus, using (52), we have, for $r \geq n+2$,

$$\begin{aligned}
&\left|c^- K^-_{n,r}\right| \\
&\quad < \frac{1}{2\sqrt{|a|}(1-\sqrt{|a|})}\bigl[\bigl(1-\sqrt{|a|}\bigr)^2|h_{r-1}| + \bigl(1-\sqrt{|a|}\bigr)\bigl(|h_{r-1}|+|h_r|\bigr) + |g_r| + |h_r|\bigr]; \\
&\left|c^- K^-_{n,n+1}\right| < \frac{1}{2\sqrt{|a|}(1-\sqrt{|a|})}\bigl[\bigl(1-\sqrt{|a|}\bigr)\bigl(|h_n|+|h_{n+1}|\bigr) + |g_{n+1}| + |h_{n+1}|\bigr]; \\
&\left|c^- K^-_{n,n}\right| < \frac{1}{2\sqrt{|a|}(1-\sqrt{|a|})}\bigl(|g_n| + |h_n|\bigr),
\end{aligned}$$

and then

$$\begin{aligned}
&\sum_{r=n}^{\infty}\left|c^- K^-_{n,r}\right| \\
&\quad < \frac{1}{2\sqrt{|a|}(1-\sqrt{|a|})}\Biggl[\bigl(1-\sqrt{|a|}\bigr)^2\sum_{r=n+2}^{\infty}|h_{r-1}| \\
&\qquad + \bigl(1-\sqrt{|a|}\bigr)\Biggl(\sum_{r=n+2}^{\infty}|h_{r-1}| + \sum_{r=n+2}^{\infty}|h_r|\Biggr) + \sum_{r=n+2}^{\infty}|g_r| + \sum_{r=n+2}^{\infty}|h_r| \\
&\qquad + \bigl(1-\sqrt{|a|}\bigr)\bigl(|h_n|+|h_{n+1}|\bigr) + |g_{n+1}| + |h_{n+1}| + |g_n| + |h_n|\Biggr] \\
&\quad = \frac{1}{2\sqrt{|a|}(1-\sqrt{|a|})}\bigl[V_n + \bigl(2-\sqrt{|a|}\bigr)W_n + \bigl(1-\sqrt{|a|}\bigr)\bigl(2-\sqrt{|a|}\bigr)W_{n+1}\bigr] \\
&\quad =: \mathcal{V}^-_n.
\end{aligned} \tag{83}$$

Similarly (still for $-1 < a < 0$),

$$\begin{aligned}
&\left|c^+\right| = \frac{1}{2\sqrt{|a|}(1+\sqrt{|a|})}, \qquad |\lambda_+| = 1 + \sqrt{|a|} > 1, \\
&|\eta_+| = \left|\frac{1+\sqrt{|a|}}{1-\sqrt{|a|}}\right| > 1 \quad (\eta_+ > 0),
\end{aligned}$$

and hence

$$\left|a^+_{n,r}\right| < 1 + (\eta_+)^{r-n+1}.$$

(Actually, we could do better since $|a^+_{n,r}| = (\eta_+)^{r-n+1} - 1$. We leave this possible improvement to the reader.) Then, for $r \geq n+2$,

$$\begin{aligned}
&\left|c^+ K^+_{n,r}\right| \\
&\quad \leq \frac{1}{2\sqrt{|a|}(1+\sqrt{|a|})} \\
&\qquad \times \bigl[\bigl(1+\sqrt{|a|}\bigr)^2\bigl(1+(\eta_+)^{r-n-1}\bigr)|h_{r-1}| + \bigl(1+\sqrt{|a|}\bigr)\bigl(1+(\eta_+)^{r-n}\bigr)\bigl(|h_{r-1}|+|h_r|\bigr) \\
&\qquad + \bigl(1+(\eta_+)^{r-n+1}\bigr)\bigl(|g_r|+|h_r|\bigr)\bigr];
\end{aligned}$$

$$
\begin{aligned}
&\left|c^+K^+_{n,n+1}\right| \\
&\quad\le \frac{1}{2\sqrt{|a|}(1+\sqrt{|a|})} \\
&\qquad \times \left[\left(1+\sqrt{|a|}\right)(1+\eta_+)\left(|h_n|+|h_{n+1}|\right)+\left(1+(\eta_+)^2\right)\left(|g_{n+1}|+|h_{n+1}|\right]; \\
&\left|c^+K^+_{n,n}\right| \le \frac{1}{2\sqrt{|a|}(1+\sqrt{|a|})}\left[(1+\eta_+)(|g_n|+|h_n|\right].
\end{aligned}
$$

After rather lengthy though elementary calculations, we have

$$
\begin{aligned}
&\sum_{r=n}^{\infty}\left|c^+K^+_{n,r}\right| \\
&\quad\le \frac{1}{2\sqrt{|a|}(1+\sqrt{|a|})}\Bigg[\left(1+\sqrt{|a|}\right)^2\sum_{r=n+2}^{\infty}|h_{r-1}| \\
&\qquad + \left(1+\sqrt{|a|}\right)^2(\eta_+)^{-n-1}\sum_{r=n+2}^{\infty}(\eta_+)^r|h_{r-1}| + \left(1+\sqrt{|a|}\right)\sum_{r=n+2}^{\infty}|h_{r-1}| \\
&\qquad + \left(1+\sqrt{|a|}\right)\sum_{r=n+2}^{\infty}|h_r| + \left(1+\sqrt{|a|}\right)(\eta_+)^{-n}\sum_{r=n+2}^{\infty}(\eta_+)^r|h_{r-1}| \\
&\qquad + \left(1+\sqrt{|a|}\right)(\eta_+)^{-n}\sum_{r=n+2}^{\infty}(\eta_+)^r|h_r| + \sum_{r=n+2}^{\infty}|g_r| \\
&\qquad + \sum_{r=n+2}^{\infty}|h_r| + (\eta_+)^{-n}\sum_{r=n+2}^{\infty}(\eta_+)^r|g_r| + (\eta_+)^{-n-1}\sum_{r=n+2}^{\infty}(\eta_+)^r|h_r| \\
&\qquad + \left(1+\sqrt{|a|}\right)|h_n| + \left(1+\sqrt{|a|}\right)\eta_+|h_n| + \left(1+\sqrt{|a|}\right)|h_{n+1}| \\
&\qquad + \left(1+\sqrt{|a|}\right)\eta_+|h_{n+1}| + |g_{n+1}| + |h_{n+1}| + (\eta_+)^2|g_{n+1}| + (\eta_+)^2|h_{n+1}| \\
&\qquad + |g_n| + |h_n| + \eta_+|g_n| + \eta_+|h_n|\Bigg] \\
&\quad= \frac{1}{2\sqrt{|a|}(1+\sqrt{|a|})}\left\{\left[\left(1+\sqrt{|a|}\right)^2+2\left(1+\sqrt{|a|}\right)\right]W_{n+1}+W_n+V_n\right. \\
&\qquad + \left(1+\sqrt{|a|}\right)\eta_+|h_{n+1}| + (\eta_+)^2|h_{n+1}| + \left(1+\sqrt{|a|}\right)|h_n| + \left(1+\sqrt{|a|}\right)\eta_+|h_n| \\
&\qquad + \eta_+|h_n| + \left[\left(1+\sqrt{|a|}\right)^2\eta_+ + \left(1+\sqrt{|a|}\right)(\eta_+)^2\right]U^+_{n+1} \\
&\qquad \left. + \left[\left(1+\sqrt{|a|}\right)(\eta_+)^2+\right)\left(\eta^3\right)\right]U^+_{n+2} + (\eta_+)^3Z^+_{n+2} + (\eta_+)^2|g_{n+1}| + \eta_+|g_n|\Big\} \\
&\quad= \frac{1}{2\sqrt{|a|}(1+\sqrt{|a|})}\left\{V_n + W_n + \left(1+\sqrt{|a|}\right)\left(3+\sqrt{|a|}\right)W_{n+1}\right. \\
&\qquad + \left(1+\sqrt{|a|}\right)\left(1+\sqrt{|a|}+\eta_+\right)\eta_+U^+_{n+1} + (\eta_+)^2\left(1+\sqrt{|a|}+\eta_+\right)U^+_{n+2} \\
&\qquad + (\eta_+)^3Z^+_{n+2} + \eta_+\left(1+\sqrt{|a|}+\eta_+\right)|h_{n+1}| + \left[\left(1+\sqrt{|a|}\right)(1+\eta_+)+\eta_+\right]|h_{n+1}| \\
&\qquad \left. + (\eta_+)^2|g_{n+1}| + \eta_+|g_n|\right\} \\
&\quad=: \mathcal{V}^+_n.
\end{aligned}
\tag{84}
$$

Here above we used the definitions of V_n, W_n, U^-_n, and Z^-_n given by (15), (16).

(iii) for $a < -1$ (i.e., $a < 0$, $|a| > 1$),

$$\left|c^-\right| = \left|\frac{1}{2\sqrt{|a|}(\sqrt{|a|} - 1)}, \qquad |\lambda_-| = \sqrt{|a|} - 1, \quad |\eta_-| < 1 \quad (\eta_- < 0),\right.$$

and hence

$$\left|a^-_{n,r}\right| = \left|1 - (\eta_-)^{r-n+1}\right| \le 1 + |\eta_-|^{r-n+1} < 2.$$

Thus, for $r \ge n + 2$,

$$\begin{aligned}
\left|c^- K^-_{n,r}\right| &< \frac{1}{\sqrt{|a|}(\sqrt{|a|} - 1)} \\
&\quad \times \left[\left(\sqrt{|a|} - 1\right)^2 |h_{r-1}| + \left(\sqrt{|a|} - 1\right)\left(|h_{r-1}| + |h_r|\right) + |g_r| + |h_r|\right]; \\
\left|c^- K^-_{n,n+1}\right| &< \frac{1}{\sqrt{|a|}(\sqrt{|a|} - 1)}\left[\left(\sqrt{|a|} - 1\right)\left(|h_n| + |h_{n+1}|\right) + |g_{n+1}| + |h_{n+1}|\right]; \\
\left|c^- K^-_{n,n}\right| &< \frac{1}{\sqrt{|a|}(\sqrt{|a|} - 1)}\left(|g_n| + |h_n|\right).
\end{aligned}$$

Summing up, we have

$$\begin{aligned}
&\sum_{r=n}^{\infty}\left|c^- K^-_{n,r}\right| \\
&\quad < \frac{1}{\sqrt{|a|}(\sqrt{|a|} - 1)}\Bigg[\left(\sqrt{|a|} - 1\right)^2 \sum_{r=n+2}^{\infty} |h_{r-1}| \\
&\qquad + \left(\sqrt{|a|} - 1\right)\left(\sum_{r=n+2}^{\infty} |h_{r-1}| + \sum_{r=n+2}^{\infty} |h_r|\right) + \sum_{r=n+2}^{\infty} |g_r| + \sum_{r=n+2}^{\infty} |h_r| \\
&\qquad + \left(\sqrt{|a|} - 1\right)\left(|h_n| + |h_{n+1}|\right) + |g_{n+1}| + |h_{n+1}| + |g_n| + |h_n|\Bigg] \\
&\quad = \frac{1}{\sqrt{|a|}(\sqrt{|a|} - 1)}\left[\left(\sqrt{|a|} - 1\right)^2 W_{n+1} + \left(\sqrt{|a|} - 1\right)(W_{n+1} + W_{n+2}) + V_{n+2}\right. \\
&\qquad \left. + W_{n+2} + \left(\sqrt{|a|} - 1\right)|h_n| + \left(\sqrt{|a|} - 1\right)|h_{n+1}| + |g_{n+1}| + |h_{n+1}| + |g_n| + |h_n|\right] \\
&\quad = \frac{1}{\sqrt{|a|}(\sqrt{|a|} - 1)}\left[V_n + W_n + \left(|a| - 1\right) W_{n+1} + \left(\sqrt{|a|} - 1\right)|h_n|\right] =: \mathcal{V}^-_n.
\end{aligned} \tag{85}$$

Here above we used the fact that $V_{n+2} + |g_{n+1}| + |g_n| = V_n$, $W_{n+2} + |h_{n+1}| = W_{n+1}$, $W_{n+2} + |h_{n+1}| + |h_n| = W_n$, and
$(\sqrt{|a|} - 1)(W_{n+1} + W_{n+2}) + W_{n+2} + (\sqrt{|a|} - 1)|h_{n+1}| + |h_{n+1}| + |h_n| =$
$2(\sqrt{|a|} - 1)W_{n+1} + W_n$.

As for the other solution (still pertaining to the case $a < -1$), we have

$$\left|c^+\right| = \left|\frac{1}{2\sqrt{|a|}(\sqrt{|a|} + 1)}, \qquad |\lambda_+| = \sqrt{|a|} + 1, \qquad |\eta_+| > 1 \quad (\eta_+ < 0),\right.$$

and hence

$$\left|a^+_{n,r}\right| \le 1 + |\eta_+|^{r-n+1}.$$

Thus, for $r \ge n+2$,

$$\begin{aligned}
\left|c^+K^+_{n,r}\right| &\le \frac{1}{2\sqrt{|a|}(\sqrt{|a|}+1)}\Big[\big(1+\sqrt{|a|}\big)^2\big(1+|\eta_+|^{r-n-1}\big)|h_{r-1}| \\
&\quad + \big(\sqrt{|a|}+1\big)\big(1+|\eta_+|^{r-n}\big)\big(|h_{r-1}|+|h_r|\big) + \big(1+|\eta_+|^{r-n+1}\big)\big(|g_r|+|h_r|\big)\Big]; \\
\left|c^+K^+_{n,n+1}\right| &\le \frac{1}{2\sqrt{|a|}(\sqrt{|a|}+1)}\Big[\big(\sqrt{|a|}+1\big)\big(1+|\eta_+|\big)\big(|h_n|+|h_{n+1}|\big) \\
&\quad + \big(1+|\eta_+|^2\big)\big(|g_{n+1}|+|h_{n+1}|\big)\Big]; \\
\left|c^+K^+_{n,n}\right| &\le \frac{1}{2\sqrt{|a|}(\sqrt{|a|}+1)}\big(1+|\eta_+|\big)\big(|g_n|+|h_n|\big),
\end{aligned}$$

and finally,

$$\begin{aligned}
&\sum_{r=n}^{\infty}\left|c^+K^+_{n,r}\right| \\
&\quad\le \frac{1}{2\sqrt{|a|}(\sqrt{|a|}+1)}\Bigg\{\big(1+\sqrt{|a|}\big)^2\Bigg[\sum_{r=n+2}^{\infty}|h_{r-1}| + \sum_{r=n+2}^{\infty}|\eta_+|^{r-n-1}|h_{r-1}|\Bigg] \\
&\quad\quad + \big(\sqrt{|a|}+1\big)\Bigg[\sum_{r=n+2}^{\infty}|h_{r-1}| + \sum_{r=n+2}^{\infty}|h_r| + \sum_{r=n+2}^{\infty}|\eta_+|^{r-n}|h_{r-1}| + \sum_{r=n+2}^{\infty}|\eta_+|^{r-n}|h_r|\Bigg] \\
&\quad\quad + \sum_{r=n+2}^{\infty}|g_r| + \sum_{r=n+2}^{\infty}|h_r| + \sum_{r=n+2}^{\infty}|\eta_+|^{r-n+1}|g_r| + \sum_{r=n+2}^{\infty}|\eta_+|^{r-n+1}|h_r| \\
&\quad\quad + \big(\sqrt{|a|}+1\big)\big(1+|\eta_+|\big)\big(|h_n|+|h_{n+1}|\big) \\
&\quad\quad + \big(1+|\eta_+|^2\big)\big(|g_{n+1}|+|h_{n+1}|\big) + \big(1+|\eta_+|\big)\big(|g_n|+|h_n|\big)\Bigg\} \\
&\quad= \frac{1}{2\sqrt{|a|}(\sqrt{|a|}+1)}\Big\{\big(1+\sqrt{|a|}\big)^2\big[W_{n+1}+|\eta_+|U^+_{n+1}\big] + \big(1+\sqrt{|a|}\big)\big[W_{n+1}+W_{n+2} \\
&\quad\quad + |\eta_+|^2U^+_{n+1} + |\eta_+|^2U^+_{n+2}\big] + V_{n+2} + W_{n+2} + |\eta_+|^3Z^+_{n+2} + |\eta_+|^3U^+_{n+2} \\
&\quad\quad + \big(1+\sqrt{|a|}\big)\big(1+|\eta_+|\big)\big(|h_n|+|h_{n+1}|\big) + \big(1+|\eta_+|^2\big)\big(|g_{n+1}|+|h_{n+1}|\big) \\
&\quad\quad + \big(1+|\eta_+|\big)\big(|g_n|+|h_n|\big)\Big\} \\
&\quad= \frac{1}{2\sqrt{|a|}(\sqrt{|a|}+1)}\Big\{V_{n+2} + \big(1+\sqrt{|a|}\big)\big(2+\sqrt{|a|}\big)W_{n+1} + \big(2+\sqrt{|a|}\big)W_{n+2} \\
&\quad\quad + \big(1+\sqrt{|a|}\big)\big[1+\sqrt{|a|}+|\eta_+|\big]|\eta_+|U^+_{n+1} + |\eta_+|^2\big[1+\sqrt{|a|}+|\eta_+|\big]U^+_{n+2} + |\eta_+|^3Z^+_{n+2} \\
&\quad\quad + \big(1+|\eta_+|\big)\big(2+\sqrt{|a|}\big)|h_n| + \big[1+\sqrt{|a|}+|\eta_+|\big(1+\sqrt{|a|}+|\eta_+|\big)\big]|h_{n+1}| \\
&\quad\quad + \big(1+|\eta_+|^2\big)|g_{n+1}| + \big(1+|\eta_+|\big)|g_n|\Big\} =: \mathcal{V}^+_n. \qquad (86)
\end{aligned}$$

Here we used the definitions of V_n, W_n, U^-_n, and Z^-_n given in (15), (16).

Appendix 2

In this appendix, we report all steps of the proof of Theorem 3.2, that is, for case ((D). They follow closely the corresponding ones for case (C).

The characteristic equation of the associated unperturbed equation is now

$$\lambda^2 + b\lambda + 1 = 0, \quad b := a - 2 \tag{87}$$

(which coincides with that of (B)), so that we have the characteristic roots

$$\lambda_\pm := -\frac{1}{2}\big(b \pm \sqrt{b^2 - 4}\big). \tag{88}$$

Hence, we have, for $a > 0$ [$b > -2$],

$$|\eta_+| \equiv \left|\frac{\lambda_+}{\lambda_-}\right| = \left|\frac{b + \sqrt{b^2 - 4}}{b - \sqrt{b^2 - 4}}\right| > 1,$$

and for $a < 0$ [$b < -2$],

$$|\eta_-| \equiv \left|\frac{\lambda_-}{\lambda_+}\right| > 1.$$

The error equation is

$$\lambda^2 \varepsilon_{n+2} + b\lambda\varepsilon_{n+1} + \varepsilon_n = \chi_n, \tag{89}$$

where

$$\chi_n := -\big\{\lambda^2 h_{n+1}(1 + \varepsilon_{n+2}) + \lambda(g_n - h_n - h_{n+1})(1 + \varepsilon_{n+1}) + h_n(1 + \varepsilon_n)\big\}. \tag{90}$$

Here above relation (87) has been used. Note that χ_n also depends on the choice of either λ_+ or λ_-.

Setting first $\chi_n \equiv 0$ in (89), we obtain

$$\lambda\Delta^2\varepsilon_n + (2\lambda + b)\Delta\varepsilon_n = 0, \tag{91}$$

which can be written as

$$\Delta\big[\lambda\Delta\varepsilon_n + (2\lambda + b)\varepsilon_n\big] = 0.$$

This suggests that two linearly independent solutions are given by

$$\varepsilon_n \equiv 1, \quad \text{and} \quad \varepsilon_n = \eta^n, \qquad \eta := -1 - \frac{b}{\lambda}, \tag{92}$$

where the latter was obtained by solving

$$\lambda\Delta\varepsilon_n + (2\lambda + b)\varepsilon_n = 0.$$

Note that, for $\lambda = \lambda_+$, we have $\eta = \eta_- = \lambda_-/\lambda_+$, while for $\lambda = \lambda_-$, we have $\eta = \eta_+ = \lambda_+/\lambda_-$, and that $\lambda_+ = 1/\lambda_-$ for any value of b (i.e., for any a), see (87).

The general solution to (91) has the form $\varepsilon_n = C + D\eta^n$, where C and D are arbitrary constants as long as $\lambda_+ \neq \lambda_-$, that is, for any $b \neq \pm 2$ (i.e., $a \neq 0$, $a \neq 4$). [If $a = 4$, then $\lambda_+ = \lambda_-$, $\eta = 1$, and the general solution should be $\varepsilon_n = C + Dn$.]

To obtain a representation for the solutions to the full equation (89), following the method of *variation of parameters*, we look for a solution of the form

$$\varepsilon_n = C_n + D_n\eta^n, \tag{93}$$

where C_n and D_n are two sequences to be determined. We evaluate first

$$\Delta\varepsilon_n = \Delta C_n + \eta^{n+1}\Delta D_n + (\eta - 1)\eta^n D_n,$$

and claim that

$$\Delta C_n + \eta^{n+1}\Delta D_n \equiv 0, \tag{94}$$

as it would be if the sequences C_n and D_n would be constant with n. Thus, it remains

$$\Delta\varepsilon_n = (\eta - 1)\eta^n D_n, \tag{95}$$

wherefrom

$$\Delta^2\varepsilon_n = (\eta - 1)\bigl(\eta^{n+1}D_{n+1} - \eta^n D_n\bigr), \tag{96}$$

and hence, from (95), (96), and (89), rewritten in terms of Δ operators, we obtain using (87)

$$\lambda^2(\eta - 1)\eta^{n+1}\Delta D_n = \chi_n,$$

i.e.,

$$\Delta D_n^{\pm} = \mp\frac{\chi_n^{\pm}}{\sqrt{b^2 - 4}}\lambda_{\pm}(\eta_{\pm})^n, \tag{97}$$

since $\lambda^2\eta = 1$. In fact, $\lambda^2\eta \equiv \lambda_{\pm}^2\eta_{\pm} = \lambda_{\pm}^2\lambda_{\mp}/\lambda_{\pm} = \lambda_{\pm}\lambda_{\mp} = 1$.

"Integrating" (97), i.e., summing up both sides from $k = n$ to ∞, assuming that $D_n \to 0$ as $n \to \infty$, we obtain

$$\sum_{r=n}^{\infty}\Delta D_r^{\pm} = -D_n^{\pm} = \mp\frac{\lambda_{\pm}}{\sqrt{b^2 - 4}}(\eta_{\pm})^n\chi_n^{\pm},$$

and finally

$$D_n^{\pm} = \pm\frac{\lambda_{\pm}}{\sqrt{b^2 - 4}}\sum_{r=n}^{\infty}(\eta_{\pm})^r\chi_r^{\pm}. \tag{98}$$

We can now determine C_n from (94). We have first

$$\Delta C_n^{\pm} = -\eta_{\pm}^{n+1}\Delta D_n^{\pm} = \pm\frac{\lambda_{\mp}}{\sqrt{b^2-4}}\chi_n^{\pm},$$

and then, assuming $C_n^{\pm} \to 0$ as $n \to \infty$,

$$\sum_{r=n}^{\infty}\Delta C_k^{\pm} = -C_n^{\pm} = \pm\frac{\lambda_{\mp}}{\sqrt{b^2-4}}\sum_{r=n}^{\infty}\chi_r^{\pm}.$$

Hence,

$$C_n^{\pm} = \pm\frac{\lambda_{\mp}}{\sqrt{b^2-4}}\sum_{r=n}^{\infty}\chi_r^{\pm}. \tag{99}$$

Appendix 3

In this appendix, we provide details for the proof of Theorem 3.2, which concerns case (D) of equation $\Delta(c_n\Delta y_n) + r_n y_{n+1} = 0$. Recalling that $b = a - 2$, we have

(i) for $0 < a < 4$ $[b^2 < 4]$,

$$\left|c^{\pm}\right| = \frac{1}{\sqrt{4-b^2}} = \frac{1}{\sqrt{a(4-a)}}, \qquad |\lambda_{\pm}| = 1, \qquad |\eta_{\pm}| = 1,$$

and hence

$$\left|a_{n,r}^{\pm}\right| \le \left|1-(\eta_{\pm})^{r-n+1}\right| \le 2.$$

Then (72) yields

$$\left|c^{\pm}K_{n,r}^{\pm}\right| \le \frac{2}{\sqrt{a(4-a)}}\left(2|h_{r-1}| + 2|h_r| + |g_{r-1}|\right) \quad \text{for } r \ge n+2;$$
$$\left|c^{\pm}K_{n,n+1}^{\pm}\right| \le \frac{2}{\sqrt{a(4-a)}}\left(|g_n| + |h_n| + 2|h_{n+1}|\right);$$
$$\left|c^{\pm}K_{n,n}^{\pm}\right| \le \frac{2}{\sqrt{a(4-a)}}|h_n|.$$

Therefore,

$$\begin{aligned}\sum_{r=n}^{\infty}\left|c^{\pm}K_{n,r}^{\pm}\right| &\le \frac{2}{\sqrt{a(4-a)}}\Bigg[2\sum_{r=n+2}^{\infty}|h_{r-1}| + 2\sum_{r=n+2}^{\infty}|h_r| + \sum_{r=n+2}^{\infty}|g_{r-1}| \\ &\qquad + |g_n| + |h_n| + 2|h_{n+1}| + |h_n|\Bigg] \\ &= \frac{2}{\sqrt{a(4-a)}}\left(V_n + W_n + 2W_{n+1} + W_{n+2} + |h_n| + |h_{n+1}|\right) \\ &= \frac{2}{\sqrt{a(4-a)}}(V_n + 2W_n + 2W_{n+1}) =: \mathcal{V}_n^{\pm} \equiv \mathcal{V}_n.\end{aligned} \tag{100}$$

Here we recalled in particular that $W_n = W_{n+2} + |h_{n+1}| + |h_n|$.

(ii) for $a > 4$ [$b > 2$],

$$\begin{aligned} \left|c^-\right| &= \frac{(a-2)+\sqrt{a(a-4)}}{2\sqrt{a(a-4)}}, \\ |\lambda_-| &= \left|\frac{1}{2}\bigl(-b-\sqrt{b^2-4}\bigr)\right| = \frac{1}{2}\bigl(b+\sqrt{b^2-4}\bigr) > 1, \\ |\eta_-| &= \frac{b+\sqrt{b^2-4}}{b-\sqrt{b^2-4}} > 1 \quad (\eta_- < 0), \end{aligned} \tag{101}$$

and hence

$$\left|a_{n,r}^-\right| \le 1 + |\eta_-|^{r-n+1}.$$

Then we have

$$\begin{aligned} &\left|c^- K_{n,r}^-\right| \\ &\quad\le \left|c^-\right|\bigl[|\lambda_-|^2\left|a_{n,r-2}^-\right||h_{r-1}| + |\lambda_-|\left|a_{n,r-1}^-\right|\bigl(|g_{r-1}| + |h_{r-1}| + |h_r|\bigr) + \left|a_{n,r}^-\right||h_r|\bigr] \\ &\quad\le \left|c^-\right|\bigl\{|\lambda_-|^2\bigl[1+|\eta_-|^{n-r-1}\bigr]|h_{r-1}| + |\lambda_-|\bigl[1+|\eta_-|^{n-r}\bigr]\bigl(|g_{r-1}| + |h_{r-1}| + |h_r|\bigr) \\ &\qquad + \bigl[1+|\eta_-|^{n-r+1}\bigr]|h_r|\bigr\} \quad \text{for } r \ge n+2; \\ &\left|c^- K_{n,n+1}^-\right| \le \left|c^-\right|\bigl(|\lambda_-|\left|a_{n,n}^-\right|\bigl(|g_n| + |h_n| + |h_{n+1}|\bigr) + \left|a_{n,n+1}^-\right||h_{n+1}|\bigr) \\ &\qquad\qquad\qquad \le \left|c^-\right|\bigl[|\lambda_-|\bigl(1+|\eta_-|\bigr)\bigl(|g_n| + |h_n| + |h_{n+1}|\bigr) + \bigl(1+|\eta_-|^2\bigr)|h_{n+1}|\bigr]; \\ &\left|c^- K_{n,n}^-\right| \le \left|c^-\right|\left|a_{n,n}^-\right||h_n| \le \left|c^-\right|\bigl(1+|\eta_-|\bigr)|h_n|. \end{aligned}$$

Collecting all these and recalling the definitions of V_n, W_n, Z_n^-, and U_n^- (given in (15), (16)), we obtain

$$\begin{aligned} &\sum_{r=n}^{\infty}\left|c^- K_{n,r}^-\right| \\ &\quad= \sum_{r=n+2}^{\infty}\left|c^- K_{n,r}^-\right| + \left|c^- K_{n,n+1}^-\right| + \left|c^- K_{n,n}^-\right| \\ &\quad\le \left|c^-\right|\Biggl[|\lambda_-|^2\Biggl(\sum_{r=n+2}^{\infty}|h_{r-1}| + \sum_{r=n+2}^{\infty}|\eta_-|^{r-n-1}|h_{r-1}|\Biggr) \\ &\qquad + |\lambda_-|\Biggl(\sum_{r=n+2}^{\infty}|g_{r-1}| + \sum_{r=n+2}^{\infty}|h_{r-1}| + \sum_{r=n+2}^{\infty}|h_r| + \sum_{r=n+2}^{\infty}|\eta_-|^{r-n}|g_{r-1}| \\ &\qquad + \sum_{r=n+2}^{\infty}|\eta_-|^{r-n}|h_{r-1}| + \sum_{r=n+2}^{\infty}|\eta_-|^{r-n}|h_r|\Biggr) \\ &\qquad + \sum_{r=n+2}^{\infty}|h_r| + \sum_{r=n+2}^{\infty}|\eta_-|^{r-n+1}|h_r| + |\lambda_-|\bigl(1+|\eta_-|\bigr)\bigl(|g_n| + |h_n| + |h_{n+1}|\bigr) \\ &\qquad + \bigl(1+|\eta_-|^2\bigr)|h_{n+1}| + \bigl(1+|\eta_-|\bigr)|h_n|\Biggr] \end{aligned}$$

$$
\begin{aligned}
&= \left|c^-\right| \bigl\{ |\lambda_-|^2 \bigl(W_{n+1} + |\eta_-| U^-_{n+1}\bigr) + |\lambda_-| \bigl[V_{n+1} + W_{n+1} + W_{n+2} + |\eta_-|^2 Z^-_{n+1} \\
&\quad + |\eta_-|^2 U^-_{n+1} + |\eta_-|^2 U^-_{n+2}\bigr] + W_{n+2} + |\eta_-|^3 U^-_{n+2} \\
&\quad + |\lambda_-| \bigl(1 + |\eta_-|\bigr)\bigl(|g_n| + |h_n| + |h_{n+1}|\bigr) + \bigl(1 + |\eta_-|^2\bigr)|h_{n+1}| + \bigl(1 + |\eta_-|\bigr)|h_n| \bigr\} \\
&= \left|c^-\right| \bigl[|\lambda_-| V_{n+1} + \bigl(|\lambda_-| + 1\bigr) W_n + |\lambda_-| \bigl(|\lambda_-| + 1\bigr) W_{n+1} \\
&\quad + |\lambda_-||\eta_-| \bigl(|\lambda_-| + |\eta_-|\bigr) U^-_{n+1} + |\eta_-|^2 \bigl(|\lambda_-| + |\eta_-|\bigr) U^-_{n+2} + |\lambda_-||\eta_-|^2 Z^-_{n+1} \\
&\quad + |\lambda_-| \bigl(1 + |\eta_-|\bigr)|g_n| + \bigl(|\lambda_-| + 1\bigr)|\eta_-||h_n| + |\eta_-| \bigl(|\lambda_-| + |\eta_-|\bigr)|h_{n+1}| \bigr] \\
&=: \mathcal{V}^-_n. \qquad (102)
\end{aligned}
$$

As for the other solution (still pertaining to the case $a > 4$), we have

$$
\begin{aligned}
&\left|c^+\right| = \frac{(a-2) - \sqrt{a(a-4)}}{2\sqrt{a(a-4)}}, \\
&|\lambda_+| = \frac{1}{2}\bigl(b - \sqrt{b^2 - 4}\bigr) < 1, \\
&|\eta_+| = \frac{b - \sqrt{b^2 - 4}}{b + \sqrt{b^2 - 4}} < 1 \quad (\eta_+ > 0),
\end{aligned}
\qquad (103)
$$

and hence

$$
\left|a^+_{n,r}\right| \le 1 + |\eta_-|^{r-n+1} < 2.
$$

Then we have

$$
\begin{aligned}
&\sum_{r=n}^{\infty} \left|c^+ K^+_{n,r}\right| \\
&\quad < \left|c^+\right| \left(2 \sum_{r=n+2}^{\infty} |h_{r-1}| + \sum_{r=n+2}^{\infty} |h_r| + \sum_{r=n+2}^{\infty} |g_{r-1}| + |g_n| + |h_n| + 2|h_{n+1}| + |h_n| \right) \\
&\quad = \left|c^+\right| \bigl(2W_{n+1} + W_{n+2} + V_{n+1} + |g_n| + 2|h_n| + 2|h_{n+1}|\bigr) \\
&\quad = \left|c^+\right| \bigl(V_n + 2W_n + W_{n+1} + |h_{n+1}|\bigr) =: \mathcal{V}^+_n. \qquad (104)
\end{aligned}
$$

(iii) for $a < 0$ $[b < -2]$,

$$
\begin{aligned}
&\left|c^-\right| = \frac{|a| + 2 - \sqrt{a(a-4)}}{2\sqrt{a(a-4)}}, \\
&|\lambda_-| = \frac{1}{2}\bigl(|b| - \sqrt{b^2 - 4}\bigr) < 1, \\
&|\eta_-| = \frac{|b| - \sqrt{b^2 - 4}}{|b| + \sqrt{b^2 - 4}} < 1 \quad (\eta_- > 0),
\end{aligned}
\qquad (105)
$$

and hence

$$
\left|a^-_{n,r}\right| = 1 - (\eta_+)^{r-n+1} < 1.
$$

Then

$$\begin{aligned}
&\sum_{r=n}^{\infty}\left|c^{-}K_{n,r}^{-}\right| \\
&\quad= \sum_{r=n+2}^{\infty}\left|c^{-}K_{n,r}^{-}\right| + \left|c^{-}K_{n,n+1}^{-}\right| + \left|c^{-}K_{n,n}^{-}\right| \\
&\quad\leq \left|c^{-}\right|\left(2\sum_{r=n+2}^{\infty}|h_{r-1}| + 2\sum_{r=n+2}^{\infty}|h_r| + \sum_{r=n+2}^{\infty}|g_{r-1}| + |g_n| + 2|h_n| + 2|h_{n+1}|\right) \\
&\quad= \left|c^{-}\right|(V_n + 2W_n + 2W_{n+1}) =: \mathcal{V}_n^{-}.
\end{aligned} \tag{106}$$

As for the other solution,

$$\begin{aligned}
&\left|c^{+}\right| = \frac{|a| + 2 + \sqrt{a(a-4)}}{2\sqrt{a(a-4)}}, \\
&|\lambda_+| = \frac{1}{2}\left(|b| + \sqrt{b^2-4}\right) > 1, \\
&|\eta_+| = \frac{|b| + \sqrt{b^2-4}}{|b| - \sqrt{b^2-4}} > 1 \quad (\eta_+ > 0),
\end{aligned} \tag{107}$$

and hence

$$\left|a_{n,r}^{+}\right| \leq 1 + (\eta_+)^{r-n+1}.$$

Then we have

$$\begin{aligned}
&\left|c^{+}K_{n,r}^{+}\right| \\
&\quad\leq \left|c^{+}\right|\Big\{|\lambda_+|^2\big[1 + (\eta_+)^{r-n-1}\big]|h_{r-1}| + |\lambda_+|\big[1 + (\eta_+)^{r-n}\big]\big(|g_{r-1}| + |h_{r-1}| + |h_r|\big) \\
&\qquad + \big[1 + (\eta_+)^{r-n+1}\big]|h_r|\Big\} \quad \text{for } r \geq n+2; \\
&\left|c^{+}K_{n,n+1}^{+}\right| \leq \left|c^{+}\right|\Big\{|\lambda_+|[1+\eta_+]\big(|g_n| + |h_n| + |h_{n+1}|\big) + \big[1 + (\eta_+)^2\big]|h_{n+1}|\Big\}; \\
&\left|c^{+}K_{n,n}^{+}\right| \leq \left|c^{+}\right|(1+\eta_+)|h_n|.
\end{aligned}$$

Collecting all these and recalling the usual definitions of V_n, W_n, Z_n^{+}, and U_n^{+}, we obtain upon some algebra

$$\begin{aligned}
&\sum_{r=n}^{\infty}\left|c^{+}K_{n,r}^{+}\right| \\
&\quad\leq |c_+|\Bigg\{|\lambda_+|^2\sum_{r=n+2}^{\infty}|h_{r-1}| + |\lambda_+|^2\sum_{r=n+2}^{\infty}(\eta_+)^{r-n-1}|h_{r-1}| \\
&\qquad + |\lambda_+|\left(\sum_{r=n+2}^{\infty}|g_{r-1}| + \sum_{r=n+2}^{\infty}|h_{r-1}| + \sum_{r=n+2}^{\infty}|h_r|\right) \\
&\qquad + |\lambda_+|\left[\sum_{r=n+2}^{\infty}(\eta_+)^{r-n}|g_{r-1}| + \sum_{r=n+2}^{\infty}(\eta_+)^{r-n}|h_{r-1}| + \sum_{r=n+2}^{\infty}(\eta_+)^{r-n}|h_r|\right]
\end{aligned}$$

$$+\sum_{r=n+2}^{\infty}|h_r|+\sum_{r=n+2}^{\infty}(\eta_+)^{r-n+1}|h_r|+|\lambda_+|(1+\eta_+)\bigl(|g_n|+|h_n|+|h_{n+1}|\bigr)$$
$$+\bigl(1+(\eta_+)^2\bigr)|h_{n+1}|+(1+\eta_+)|h_n|\Biggr\}$$
$$=|c^+|\bigl[|\lambda_+|V_n+\bigl(1+|\lambda_+|\bigr)W_n+|\lambda_+|\bigl(1+|\lambda_+|\bigr)W_{n+1}+\eta_+|\lambda_+|\bigl(\eta_++|\lambda_+|\bigr)U^+_{n+1}$$
$$+(\eta_+)^2\bigl(\eta_++|\lambda_+|\bigr)U^+_{n+2}+(\eta_+)^2|\lambda_+|Z^+_{n+1}+\eta_+\bigl(1+|\lambda_+|\bigr)|h_n|$$
$$+\eta_+\bigl(\eta_++|\lambda_+|\bigr)|h_{n+1}|+\eta_+|\lambda_+||g_n|\bigr]=:\mathcal{V}^+_n. \tag{108}$$

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