Worldline quantization of field theory, effective actions and L_∞ structure

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Abstract: We formulate the worldline quantization (a.k.a. deformation quantization) of a massive fermion model coupled to external higher spin sources. We use the relations obtained in this way to show that its regularized effective action is endowed with an L_∞ symmetry. The same result holds also for a massive scalar model.

Contents

1 Introduction

It is a widespread conviction, and arguments in favor of it are not lacking [1], that, for a consistent quantum theory of gravity and matter, an infinite number of fields is needed. This is so, of course, in the case of (super)string theories, where infinite towers of higher-spin excitations conspire to regulate the singular high-energy behavior present in perturbatively quantized Einstein gravity. Other higher spin theories exist in a four-dimensional and lower space-time, see [2–5]. Very likely these are not the only possibilities. But then a question arises: what are the requirements to be satisfied in order for these theories to make sense? In particular, how can a high energy behavior like in (super)string theories be guaranteed? In the latter this is tied to the short distance behavior and has to do with the finite string size. So it is related to the mild form of non-locality in string theory. In general, what is the right amount of non-locality? All these are very general questions for which answers are not yet available. For the time being we have to content ourselves with the taxonomy of higher spin models.

Recently we have revisited and generalized a method based on effective actions to determine the classical dynamics of higher spin fields, [6–8]. The basic idea is to exploit the one-loop effective actions of elementary free field theories coupled via conserved currents to external higher spin sources, in order to extract information about the (classical) dynamics

of the latter. We focused on massive scalar and Dirac fermion models, but, no doubt, the same method can be applied to other elementary fields. In the cited papers we computed the two-point correlators of conserved currents, which allowed us to reconstruct the quadratic effective action for the higher spin fields coupled to the currents. We were able to show that such effective actions are built out of the Fronsdal differential operators [10, 11], appropriate for those higher spin fields, in the general non-local form discussed in [12, 13].

The method we used in [6–8] is the standard perturbative approach based on Feynman diagrams. This method is ultra-tested and very effective for two-point correlators. For instance, as we have seen in [8], it preserves gauge and diff-invariance (it respects the relevant Ward identities). We have no reason to doubt that this will be the case also for higher order correlators, in particular for the crucial three-point ones. But the burden to guess what the gauge transformations beyond the lowest level are is left to us. In this regard there exists an alternative quantization method which can come to our help, the worldline quantization method,[1] which we wish to discuss in this paper.

The worldline quantization of field theory is based on the Weyl quantization of a particle in quantum mechanics, where the coordinates in the phase space are replaced by position and momentum operator and observables are endowed with a suitable operator ordering. In order to achieve second quantization one, roughly speaking, replaces the field dependence on the position and the field derivatives by the corresponding position and momentum operators, respectively, and relies on the Weyl quantization for the latter. The effective action is then defined. The important thing is that this procedure comes with a bonus, the precise form of the gauge symmetry. This has a remarkable consequence, as we will show in this paper: without doing explicit calculations, it is possible to establish the symmetry of the full (not only the local part of) effective action and demonstrate its L_∞ symmetry. The latter is a symmetry that characterizes many (classical) field theories, including closed string field theory (a good introduction to L_∞ algebras and field theory is [28]).

In section 2 we will carry out the worldline quantization of free Dirac fermions coupled to external sources (the case of a scalar field has already been worked out in [17]) and derive heuristic rules, similar to the Feynman diagrams, to compute amplitudes. In section 3 we will uncover the L_∞ structure of the corresponding effective action. Section 4 is devoted to a summary and discussion of our results.

2 Worldline quantization of a fermion model

2.1 Fermion linearly coupled to higher spin fields

Let us consider a free fermion theory

$$S_0 = \int d^d x \, \overline{\psi}(i\gamma \cdot \partial - m)\psi, \tag{2.1}$$

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[1]The literature on the worldline quantization is large. Here we refer in particular to the calculation of effective actions via the worldline quantization in relation to higher spin theories, [15–17]. The first elaboration of this method is probably in [14], to which many others followed, see for instance [18–27].

coupled to external sources. We second-quantize it using the Weyl quantization method for a particle worldline. The full action is expressed as an expectation value of operators as follows

$$S = \langle \overline{\psi} | - \gamma \cdot (\widehat{P} - \widehat{H}) - m | \psi \rangle \tag{2.2}$$

Here $\widehat{P}_\mu$ is the momentum operator whose symbol is the classical momentum p_μ. $\widehat{H}$ is an operator whose symbol is $h(x, p)$, where

$$h^\mu(x, p) = \sum_{n=0}^{\infty} \frac{1}{n!} \, h_{(s)}^{\mu\mu_1\cdots\mu_n}(x) \, p_{\mu_1} \ldots p_{\mu_n} \tag{2.3}$$

$s = n + 1$ is the spin and the tensors are assumed to be symmetric. We recall that a quantum operator $\widehat{O}$ can be represented with a symbol $O(x, p)$ through the Weyl map

$$\widehat{O} = \int d^d x \, d^d y \frac{d^d k}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} \, O(x, p) \, e^{ik \cdot (x - \widehat{X}) - iy \cdot (p - \widehat{P})} \tag{2.4}$$

where $\widehat{X}$ is the position operator.

Next we insert this into the r.h.s. of (2.2), where we also insert two completenesses $\int d^d x |x\rangle\langle x|$, and make the identification $\psi(x) = \langle x|\psi\rangle$. Expressing S in terms of symbols we find

$$S = S_0 + \int \frac{d^d q}{(2\pi)^d} \, d^d x \, d^d z \, e^{iq \cdot z} \, \overline{\psi}\Big(x + \frac{z}{2}\Big) \, \gamma \cdot h(x, q) \, \psi\Big(x - \frac{z}{2}\Big) \tag{2.5}$$

$$= S_0 + \sum_{n=0}^{\infty} \int d^d x \, \frac{i^n}{n!} \frac{\partial}{\partial z^{\mu_1}} \cdots \frac{\partial}{\partial z^{\mu_n}} \overline{\psi}\Big(x + \frac{z}{2}\Big) \gamma_\mu h^{\mu\mu_1\cdots\mu_n}(x) \, \psi\Big(x - \frac{z}{2}\Big)\Big|_{z=0}$$

$$= S_0 + \sum_{s=1}^{\infty} \int d^d x \, J_{\mu_1\ldots\mu_s}^{(s)}(x) \, h_{(s)}^{\mu_1\ldots\mu_s}(x)$$

We see that the symmetric tensor field $h^{\mu\mu_1\cdots\mu_n}$ is linearly coupled to the HS (higher spin) current

$$J_{\mu\mu_1\ldots\mu_{s-1}}^{(s)}(x) = \frac{i^{s-1}}{(s-1)!} \frac{\partial}{\partial z^{(\mu_1}} \cdots \frac{\partial}{\partial z^{\mu_{s-1}}} \overline{\psi}\Big(x + \frac{z}{2}\Big) \gamma_{\mu)} \psi\Big(x - \frac{z}{2}\Big)\Big|_{z=0}. \tag{2.6}$$

For instance, for $s = 1$ and $s = 2$ one obtains

$$J_\mu^{(1)} = \overline{\psi} \gamma_\mu \psi \tag{2.7}$$

$$J_{\mu\mu_1}^{(2)} = \frac{i}{2} \left(\partial_{(\mu_1} \overline{\psi} \gamma_{\mu)} \psi - \overline{\psi} \gamma_{(\mu} \partial_{\mu_1)} \psi \right) \tag{2.8}$$

The HS currents are on-shell conserved in the free theory (2.1)

$$\partial_\mu J_{(s)}^{\mu\mu_1\cdots\mu_{s-1}} = 0 \tag{2.9}$$

which is a consequence of invariance of $S_0[\psi]$ on global (rigid) transformations

$$\delta_n \psi(x) = -\frac{(-i)^{n+1}}{n!} \, \varepsilon_{(n)}^{\mu_1\cdots\mu_n} \partial_{\mu_1} \ldots \partial_{\mu_n} \psi(x) \tag{2.10}$$

We shall next show that for the full action (2.5) this extends to the local symmetry. The consequence is that the currents are still conserved, with the HS covariant derivative substituting ordinary derivative in (2.9).

Notice that these currents are conserved even without symmetrizing μ with the other indices. But in the sequel we will suppose that they are symmetric.

2.2 Symmetries

The action (2.2) is trivially invariant under the operation

$$S = \langle\overline{\psi}|\widehat{O}\widehat{O}^{-1}\widehat{G}\widehat{O}\widehat{O}^{-1}|\psi\rangle \tag{2.11}$$

where $\widehat{G} = -\gamma\cdot(\widehat{P} - \widehat{H}) - m$. So it is invariant under

$$\widehat{G} \longrightarrow \widehat{O}^{-1}\widehat{G}\widehat{O}, \qquad |\psi\rangle \longrightarrow \widehat{O}^{-1}|\psi\rangle \tag{2.12}$$

Writing $\widehat{O} = e^{-i\widehat{E}}$ we easily find the infinitesimal version.

$$\delta|\psi\rangle = i\widehat{E}|\psi\rangle, \qquad \delta\langle\overline{\psi}| = -i\langle\overline{\psi}|\widehat{E}, \tag{2.13}$$

and

$$\delta\widehat{G} = i[\widehat{E}, \widehat{G}] = i[\gamma\cdot(\widehat{P} - \widehat{H}), \widehat{E}] = \gamma\cdot\delta\widehat{H} \tag{2.14}$$

Let the symbol of $\widehat{E}$ be $\varepsilon(x, p)$, then the symbol of $[i\gamma\cdot\widehat{P}, \widehat{E}]$ is

$$\int d^d y\langle x - \frac{y}{2}|[i\gamma\cdot\widehat{P}, \widehat{E}]|x + \frac{y}{2}\rangle e^{iy\cdot p} \tag{2.15}$$

An easy way to make this explicit is to use the fact that the symbol of the product of two operators is given by the Moyal product of the symbols. Thus

$$\text{Symb}\big([\gamma\cdot\widehat{P}, \widehat{E}]\big) = [\gamma\cdot p \overset{*}{,} \varepsilon(x, p)] = \gamma\cdot p\, e^{-\frac{i}{2}\overset{\rightarrow}{\partial_x}\cdot\overset{\leftarrow}{\partial_p}}\varepsilon(x, p) - \varepsilon(x, p)\, e^{\frac{i}{2}\overset{\leftarrow}{\partial_x}\cdot\overset{\rightarrow}{\partial_p}}\gamma\cdot p$$
$$= -i\gamma\cdot\partial_x\varepsilon(x, p) \tag{2.16}$$

Similarly

$$\text{Symb}\big([\widehat{H}^\mu, \widehat{E}]\big) = [h^\mu(x, p) \overset{*}{,} \varepsilon(x, p)] \tag{2.17}$$

where $[a \overset{*}{,} b] \equiv a * b - b * a$. Therefore, in terms of symbols,

$$\delta_\varepsilon h^\mu(x, p) = \partial_x^\mu \varepsilon(x, p) - i[h^\mu(x, p) \overset{*}{,} \varepsilon(x, p)] \equiv \mathcal{D}_x^{*\mu}\varepsilon(x, p) \tag{2.18}$$

where we introduced the covariant derivative defined by

$$\mathcal{D}_x^{*\mu} = \partial_x^\mu - i[h^\mu(x, p) \overset{*}{,} \quad] \tag{2.19}$$

This will be referred to hereafter as HS transformation, and the corresponding symmetry HS symmetry.

The transformations of ψ are somewhat different. They can also be expressed as Moyal product of symbols

$$\delta_\varepsilon\tilde{\psi}(x, p) = i\varepsilon(x, p) * \tilde{\psi}(x, p) \tag{2.20}$$

provided we use the partial Fourier transform

$$\tilde{\psi}(x,p) = \int d^d y\, \psi\left(x - \frac{y}{2}\right) e^{iy\cdot p}. \tag{2.21}$$

and finally we antitransform back the result. Alternatively we can proceed as follows. We compute

$$\langle x|\widehat{E}|\psi\rangle = \int \frac{d^d k}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} d^d x' d^d y'\, \varepsilon(x',p)\, \langle x|e^{ik\cdot(x'-\widehat{X})-iy'\cdot(p-\widehat{P})}|\psi\rangle \tag{2.22}$$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} d^d x' d^d y'\, \varepsilon(x',p')\, e^{ik\cdot(x'-x)-iy'\cdot p}\langle x|e^{iy'\widehat{P}}|\psi\rangle e^{-\frac{i}{2}y'\cdot k}$$

Next we insert a momentum completeness $\int d^d q |q\rangle\langle q|$ to evaluate $\langle x|e^{iy'\widehat{P}}|\psi\rangle$ and subsequently a coordinate completeness to evaluate $\langle q|\psi\rangle$ using the standard relation $\langle x|p\rangle = e^{ip\cdot x}$. Then we produce two delta functions by integrating over k and q. In this way we get rid of two coordinate integrations. Finally we arrive at

$$\delta_\varepsilon \psi(x) = i\langle x|\widehat{E}|\psi\rangle = i\int \frac{d^d p}{(2\pi)^d} d^d z\, \varepsilon\left(x + \frac{z}{2}, p\right) e^{-ip\cdot z}\, \psi(x+z) \tag{2.23}$$

$$= i\sum_{n=0}^{\infty} \int \frac{d^d p}{(2\pi)^d} d^d z\, \frac{e^{-ip\cdot z}}{n!} (-i\partial_z)^n \cdot \left(\varepsilon_{(n)}\left(x + \frac{z}{2}\right) \psi(x+z)\right)$$

$$= \sum_{n=0}^{\infty} \frac{i}{n!} (-i\partial_z)^n \cdot \left(\varepsilon_{(n)}\left(x + \frac{z}{2}\right) \psi(x+z)\right)\Big|_{z=0}$$

$$= i\varepsilon_{(0)}(x)\, \psi(x) + \varepsilon_{(1)}^\mu(x)\, \partial_\mu \psi(x) + \frac{1}{2}\partial_\mu \varepsilon_{(1)}^\mu(x)\, \psi(x)$$

$$- \frac{i}{2}\left(\varepsilon_{(2)}^{\mu\nu}\, \partial_\mu \partial_\nu \psi + \partial_\mu \varepsilon_{(2)}^{\mu\nu}\, \partial_\nu \psi + \frac{1}{4}\partial_\mu \partial_\nu \varepsilon_{(2)}^{\mu\nu}\, \psi\right)(x) + \dots$$

where a dot denote the contraction of upper and lower indices. The first method leads to the same result.

Now we want to understand the conservation law ensuing from the HS symmetry of the interacting classical action (2.5)

$$0 = \delta_\varepsilon S[\psi,h] = \int d^d x\left(\frac{\delta S}{\delta \psi(x)}\, \delta_\varepsilon \psi(x) + \delta_\varepsilon \overline{\psi}(x)\frac{\delta S}{\delta \overline{\psi}(x)} + \int d^d p\, \frac{\delta S}{\delta h^\mu(x,p)}\, \delta_\varepsilon h^\mu(x,p)\right)$$

Now we evaluate this expression on the classical solution, in which case the first two terms vanish (remember that h is the background field). We are left with

$$0 = \int d^d x \int d^d p\, J_\mu(x,p)\, \delta_\varepsilon h^\mu(x,p) \qquad (\text{on} - \text{shell}) \tag{2.24}$$

where

$$J_\mu(x,p) \equiv \int \frac{d^d z}{(2\pi)^d}\, e^{ip\cdot z}\, \overline{\psi}\left(x + \frac{z}{2}\right) \gamma_\mu\, \psi\left(x - \frac{z}{2}\right) \tag{2.25}$$

Using (2.18), partially integrating and using the following property of the Moyal product

$$\int d^d x \int d^d p\, a(x,p)[b(x,p) \stackrel{*}{,} c(x,p)] = \int d^d x \int d^d p\, [a(x,p) \stackrel{*}{,} b(x,p)]\, c(x,p) \tag{2.26}$$

we obtain

$$0 = \int d^d x \int d^d p \, \varepsilon(x,p) \, \mathcal{D}_x^{*\mu} J_\mu(x,p) \qquad (\text{on} - \text{shell}) \tag{2.27}$$

From this follows the conservation law in the classical interacting theory

$$\mathcal{D}_x^{*\mu} J_\mu(x,p) = 0 \qquad (\text{on} - \text{shell}) \tag{2.28}$$

It is not hard to shaw that for $h^\mu(x,p) = 0$ this becomes equivalent to (2.10).

Using the $*$-Jacobi identity (it holds also for the Moyal product, because it is associative) one can easily get

$$
\begin{aligned}
(\delta_{\varepsilon_2} \delta_{\varepsilon_1} - \delta_{\varepsilon_1} \delta_{\varepsilon_2}) \, h^\mu(x,p) &= i \left(\partial_x [\varepsilon_1 \overset{*}{,} \varepsilon_2](x,p) - i[h^\mu(x,p) \overset{*}{,} [\varepsilon_1 \overset{*}{,} \varepsilon_2](x,p)]] \right) \\
&= i \, \mathcal{D}_x^{*\mu} [\varepsilon_1 \overset{*}{,} \varepsilon_2](x,p)
\end{aligned}
\tag{2.29}
$$

We see that the HS ε-transform is of the Lie algebra type.

2.3 Perturbative expansion of the effective action

In this subsection we work out (heuristic) rules, similar to the Feynman ones, to compute n-point amplitudes in the above fermion model. The purpose is to reproduce formulas similar to those of [17] for the scalar case. We would like to point out, however, that this is not strictly necessary: the good old Feynman rules are anyhow a valid alternative.

We start from the representation of the effective action as trace-logarithm of a differential operator:

$$W[h] = N \, \mathrm{Tr}[\ln \widehat{G}] \tag{2.30}$$

and use a well-known mathematical formula to regularize it

$$W_{reg}[h,\epsilon] = -N \int_\epsilon^\infty \frac{dt}{t} \, \mathrm{Tr}\left[e^{-t\widehat{G}} \right] \tag{2.31}$$

where ϵ is an infrared regulator. The crucial factor is therefore

$$K[g|t] \equiv \mathrm{Tr}\left[e^{-t\widehat{G}} \right] = \mathrm{Tr}\left[e^{t(\gamma \cdot (\widehat{P} - \widehat{H}) + m)} \right], \tag{2.32}$$

known as the heat kernel, where g is the symbol of $\widehat{G}$. The trace Tr includes both an integration over the momenta and tr, the trace over the gamma matrices,

$$K[g|t] = e^{mt} \int \frac{d^d p}{(2\pi)^d} \, \mathrm{tr}\langle p | e^{t\gamma \cdot (\widehat{P} - \widehat{H})} | p \rangle \tag{2.33}$$

Next we expand

$$e^{t\gamma \cdot (\widehat{P} - \widehat{H})} = e^{t\gamma \cdot \widehat{P}} \sum_{n=0}^\infty (-1)^n \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n \, \gamma \cdot \widehat{H}(\tau_1) \, \gamma \cdot \widehat{H}(\tau_2) \ldots \gamma \cdot \widehat{H}(\tau_n)$$

where $\gamma \cdot \widehat{H}(\tau) = e^{-\tau \, \gamma \cdot \widehat{P}} \gamma \cdot \widehat{H} \, e^{\tau \, \gamma \cdot \widehat{P}}$.

We have

$$\langle p | \gamma \cdot \widehat{H}(\tau) | q \rangle = e^{-\tau \, \gamma \cdot p} \langle p | \gamma \cdot \widehat{H} | q \rangle \, e^{\tau \, \gamma \cdot q} \tag{2.34}$$

Using a formula analogous to (2.22) for $\widehat{H}$ and inserting completenesses one finds

$$\langle p|\gamma\cdot\widehat{H}|q\rangle = \int d^d x \int d^d y \frac{d^d k}{(2\pi)^d}\frac{d^d p'}{(2\pi)^d}\, \gamma\cdot h(x,p')\langle p|e^{ik\cdot(x-\widehat{X})-iy\cdot(p'-\widehat{P})}|q\rangle \qquad (2.35)$$

$$= \int d^d x\, \gamma\cdot h(x,\partial_u)e^{i(q-p)\cdot x+u\cdot\frac{p+q}{2}}\Big|_{u=0}$$

Therefore

$$\langle p|\gamma\cdot\widehat{H}(\tau)|q\rangle = \int d^d x\, e^{-\tau\,\gamma\cdot p}\, \gamma\cdot h(x,\partial_u)\, e^{\tau\,\gamma\cdot q}\, e^{i(q-p)\cdot x+u\cdot\frac{p+q}{2}}\Big|_{u=0} \qquad (2.36)$$

Using this we can write

$$\mathrm{Tr}\left[e^{-t\widehat{G}}\right] = e^{mt}\sum_{n=0}^{\infty}(-1)^n\int\prod_{i=1}^{n}\frac{d^d p_i}{(2\pi)^d}\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2\ldots\int_0^{\tau_{n-1}}d\tau_n$$

$$\times\mathrm{tr}\left(e^{t\,\gamma\cdot p_n}\langle p_n|\gamma\cdot\widehat{H}(\tau_1)|p_1\rangle\langle p_1|\gamma\cdot\widehat{H}(\tau_2)|p_2\rangle\ldots\langle p_{n-1}|\gamma\cdot\widehat{H}(\tau_n)|p_n\rangle\right)$$

$$= e^{mt}\sum_{n=0}^{\infty}(-1)^n\int\prod_{i=1}^{n}d^d x_i\frac{d^d p_i}{(2\pi)^d}\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2\ldots\int_0^{\tau_{n-1}}d\tau_n$$

$$\times\mathrm{tr}\left(e^{(t-\tau_1)\,\gamma\cdot p_n}\gamma^{\mu_1}e^{(\tau_1-\tau_2)\,\gamma\cdot p_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{n-1}}e^{(\tau_{n-1}-\tau_n)\,\gamma\cdot p_{n-1}}\gamma^{\mu_n}e^{\tau_n\gamma\cdot p_n}\right)$$

$$\times\prod_{j=1}^{n}e^{ip_j\cdot\left(x_j-x_{j+1}-i\frac{u_{j+1}+u_j}{2}\right)}h_{\mu_1}\left(x_1,\overleftarrow{\partial}_{u_1}\right)\ldots h_{\mu_n}\left(x_n,\overleftarrow{\partial}_{u_n}\right)\Big|_{u_j=0} \qquad (2.37)$$

where $x_{n+1} = x_1$. Now we can factor out in $K[g,t]$ the terms $h_{\mu_1}\left(x_1,\overleftarrow{\partial}_{u_1}\right)\ldots h_{\mu_n}\left(x_n,\overleftarrow{\partial}_{u_n}\right)$, and write

$$K[g|t] = \sum_{n=0}^{\infty}\langle\langle K^{(n)\mu\ldots\mu}(t)|h_\mu^{\otimes n}\rangle\rangle \qquad (2.38)$$

where the double brackets means integration of the x_i and derivation with respect to the u_i. In turn $K^{(n)\mu\ldots\mu}(t)$ can be written more explicitly as

$$K^{\mu_1\ldots\mu_n}(x_1,u_1,\ldots,x_n,u_n|t) = e^{tm}\int\prod_{j=1}^{n}\frac{d^d p_j}{(2\pi)^d}e^{ip_j\cdot\left(x_j-x_{j+1}-i\frac{u_{j+1}+u_j}{2}\right)}\widetilde{K}^{\mu_1\ldots\mu_n}(p_1,\ldots,p_n|t)$$

$$(2.39)$$

where

$$\widetilde{K}^{\mu_1\ldots\mu_n}(p_1,\ldots,p_n|t) = \frac{(-1)^n}{n}\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2\ldots\int_0^{\tau_{n-1}}d\tau_n \qquad (2.40)$$

$$\times\,\mathrm{tr}\left(\gamma^{\mu_1}e^{(\tau_1-\tau_2)\,\gamma\cdot p_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{n-1}}e^{(\tau_{n-1}-\tau_n)\,\gamma\cdot p_{n-1}}\gamma^{\mu_n}e^{(\tau_n-\tau_1)\,\gamma\cdot p_n}e^{t\gamma\cdot p_n}\right.$$

$$+\gamma^{\mu_2}e^{(\tau_1-\tau_2)\,\gamma\cdot p_2}\gamma^{\mu_3}\ldots\gamma^{\mu_n}e^{(\tau_{n-1}-\tau_n)\,\gamma\cdot p_n}\gamma^{\mu_1}e^{(\tau_n-\tau_1)\,\gamma\cdot p_1}e^{t\gamma\cdot p_1}$$

$$\vdots$$

$$\left.+\gamma^{\mu_n}e^{(\tau_1-\tau_2)\,\gamma\cdot p_n}\gamma^{\mu_1}\ldots\gamma^{\mu_{n-2}}e^{(\tau_{n-1}-\tau_n)\,\gamma\cdot p_{n-2}}\gamma^{\mu_{n-1}}e^{(\tau_n-\tau_1)\,\gamma\cdot p_{n-1}}e^{t\gamma\cdot p_{n-1}}\right)$$

Now, the nested integral can be rewritten in the following way

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n = \int_0^t d\sigma_1 \int_0^{t-\sigma_1} d\sigma_2 \int_0^{t-\sigma_1-\sigma_2} d\sigma_3 \ldots \int_0^{t-\sigma_1-\ldots-\sigma_{n-1}} d\sigma_n$$

$$= \int_0^\infty d\sigma_1 \int_0^\infty d\sigma_2 \ldots \int_0^\infty d\sigma_n \, \theta(t-\sigma_1-\ldots-\sigma_n) \quad (2.41)$$

where $\sigma_i = \tau_{i-1} - \tau_i$, with $\tau_0 = t$. Notice that defining $\sigma_0 = t - \sigma_1 - \ldots - \sigma_n$ we can identify $\sigma_0 = \tau_n$.

Next one uses the following representation of the Heaviside function

$$\theta(t) = \lim_{\epsilon \to 0^+} \int_{-\infty}^\infty \frac{d\omega}{2\pi i} \frac{e^{i\omega t}}{\omega - i\epsilon} = \lim_{\epsilon \to 0^+} \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{i\omega t} \int_0^\infty d\sigma_0 \, e^{-i\sigma_0(\omega - i\epsilon)} \quad (2.42)$$

The ω integration has to be understood as a contour integration. Using this in (2.41) we obtain

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n = \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{i\omega t} \int_0^\infty d\sigma_0 \int_0^\infty d\sigma_1 \ldots \int_0^\infty d\sigma_n \, e^{-i(\sigma_0+\ldots+\sigma_n)(\omega-i\epsilon)}$$

$$(2.43)$$

Replacing this inside (2.40) we get

$$\widetilde{K}^{\mu_1 \cdots \mu_n}(p_1, \ldots, p_n | t) = \frac{(-1)^n}{n} \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{i\omega t} \int_0^\infty d\sigma_0 \int_0^\infty d\sigma_1 \ldots \int_0^\infty d\sigma_n \quad (2.44)$$

$$\times \mathrm{tr}\Big[\gamma^{\mu_1} e^{\sigma_2(\gamma \cdot p_1 - i\omega')} \gamma^{\mu_2} \ldots \gamma^{\mu_{n-1}} e^{\sigma_n(\gamma \cdot p_{n-1} - i\omega')} \gamma^{\mu_n} e^{(\sigma_0+\sigma_1)(\gamma \cdot p_n - i\omega')}$$

$$+ \gamma^{\mu_2} e^{\sigma_2(\gamma \cdot p_2 - i\omega')} \gamma^{\mu_3} \ldots \gamma^{\mu_n} e^{\sigma_n(\gamma \cdot p_n - i\omega')} \gamma^{\mu_1} e^{(\sigma_0+\sigma_1)(\gamma \cdot p_1 - i\omega')}$$

$$\vdots$$

$$+ \gamma^{\mu_n} e^{\sigma_2(\gamma \cdot p_n - i\omega')} \gamma^{\mu_1} \ldots \gamma^{\mu_{n-2}} e^{\sigma_n(\gamma \cdot p_{n-2} - i\omega')} \gamma^{\mu_{n-1}} e^{(\sigma_0+\sigma_1)(\gamma \cdot p_{n-1} - i\omega')} \Big]$$

where $\omega' = \omega - i\epsilon$. ϵ in the exponents allows us to perform the integrals,[2] the result being

$$\widetilde{K}^{\mu_1 \cdots \mu_n}(p_1, \ldots, p_n | t) = \frac{(-1)^n}{n} \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{i\omega t} \quad (2.45)$$

$$\times \mathrm{tr}\Big[\gamma^{\mu_1} \frac{-1}{\not{p}_1 - i\omega'} \gamma^{\mu_2} \ldots \gamma^{\mu_{n-1}} \frac{-1}{\not{p}_{n-1} - i\omega'} \gamma^{\mu_n} \frac{1}{(\not{p}_n - i\omega')^2}$$

$$+ \gamma^{\mu_2} \frac{-1}{\not{p}_2 - i\omega'} \gamma^{\mu_3} \ldots \gamma^{\mu_n} \frac{-1}{\not{p}_n - i\omega'} \gamma^{\mu_1} \frac{1}{(\not{p}_1 - i\omega')^2}$$

$$\vdots$$

$$+ \gamma^{\mu_n} \frac{-1}{\not{p}_n - i\omega'} \gamma^{\mu_1} \ldots \gamma^{\mu_{n-2}} \frac{-1}{\not{p}_{n-2} - i\omega'} \gamma^{\mu_{n-1}} \frac{1}{(\not{p}_{n-1} - i\omega')^2} \Big]$$

---

[2]This is evident with the Majorana representation of the gamma matrices, because in such a case the term $\gamma \cdot p$ in the exponent is purely imaginary, the gamma matrices being imaginary. This term therefore gives rise to oscillatory contributions, much like the $i\omega$ term.

We remark that $\frac{1}{(\not{p}-i\omega')^2} = \frac{\partial}{\partial(i\omega)}\frac{1}{\not{p}-i\omega'}$. This allows us, via integration by parts, to simplify (2.45)

$$\widetilde{K}^{\mu_1\cdots\mu_n}(p_1,\ldots,p_n|t) = \frac{t}{n}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{i\omega t}\,\mathrm{tr}\left[\gamma^{\mu_1}\frac{1}{\not{p}_1-i\omega'}\gamma^{\mu_2}\cdots\frac{1}{\not{p}_{n-1}-i\omega'}\gamma^{\mu_n}\frac{1}{\not{p}_n-i\omega'}\right] \quad (2.46)$$

We can also include the factor e^{tm} in (2.39) in a new kernel $\widetilde{K}^{\mu_1\cdots\mu_n}(p_1,\ldots,p_n|m,t)$ which has the same form as $\widetilde{K}^{\mu_1\cdots\mu_n}(p_1,\ldots,p_n|t)$ with all the $\not{p}_i$ replaced by $\not{p}_i+m$:

$$K^{\mu_1\cdots\mu_n}(x_1,u_1,\ldots,x_n,u_n|t) = \prod_{j=1}^{n}e^{ip_j\cdot\left(x_j-x_{j+1}-i\frac{u_{j+1}+u_j}{2}\right)}\widetilde{K}^{\mu_1\cdots\mu_n}(p_1,\ldots,p_n|m,t) \quad (2.47)$$

$$\widetilde{K}^{\mu_1\cdots\mu_n}(p_1,\ldots,p_n|m,t) = \frac{(-1)^n}{n}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{i\omega t} \quad (2.48)$$

$$\times\mathrm{tr}\left[\gamma^{\mu_1}\frac{-1}{\not{p}_1+m-i\omega'}\gamma^{\mu_2}\cdots\gamma^{\mu_{n-1}}\frac{-1}{\not{p}_{n-1}+m-i\omega'}\gamma^{\mu_n}\frac{1}{(\not{p}_n+m-i\omega')^2}\right.$$

$$+\gamma^{\mu_2}\frac{-1}{\not{p}_2+m-i\omega'}\gamma^{\mu_3}\cdots\gamma^{\mu_n}\frac{-1}{\not{p}_n+m-i\omega'}\gamma^{\mu_1}\frac{1}{(\not{p}_1+m-i\omega')^2}$$

$$\vdots$$

$$\left.+\gamma^{\mu_n}\frac{-1}{\not{p}_n+m-i\omega'}\gamma^{\mu_1}\cdots\gamma^{\mu_{n-2}}\frac{-1}{\not{p}_{n-2}+m-i\omega'}\gamma^{\mu_{n-1}}\frac{1}{(\not{p}_{n-1}+m-i\omega')^2}\right]$$

$$=\frac{t}{n}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{i\omega t}\,\mathrm{tr}\left[\gamma^{\mu_1}\frac{1}{\not{p}_1+m-i\omega'}\gamma^{\mu_2}\cdots\frac{1}{\not{p}_{n-1}+m-i\omega'}\gamma^{\mu_n}\frac{1}{\not{p}_n+m-i\omega'}\right]$$

Integrating further as in the scalar model case, [17], is not possible at this stage because of the gamma matrices. One has to proceed first to evaluate the trace over the latter.

Using (2.37) we can write the regularized effective action as

$$W_{reg}[h,\epsilon] = -N\int_{\epsilon}^{\infty}\frac{dt}{t}\,e^{mt}\sum_{n=0}^{\infty}\int\prod_{i=1}^{n}d^dx_i\,\frac{d^dp_i}{(2\pi)^d}\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2\ldots\int_0^{\tau_{n-1}}d\tau_n$$

$$\times\,\mathrm{tr}\left(e^{(t-\tau_1)\cdot p_n}\gamma^{\mu_1}e^{(\tau_1-\tau_2)\cdot p_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{n-1}}e^{(\tau_{n-1}-\tau_n)\cdot p_{n-1}}\gamma^{\mu_n}e^{\tau_n\gamma\cdot p_n}\right)$$

$$\times\prod_{j=1}^{n}e^{ip_j\cdot(x_j-x_{j+1})}\,h_{\mu_1}\left(x_1,\frac{p_1+p_n}{2}\right)\ldots h_{\mu_n}\left(x_n,\frac{p_{n-1}+p_n}{2}\right)$$

$$=-N\int_{\epsilon}^{\infty}dt\sum_{n=0}^{\infty}\frac{1}{n}\int\prod_{i=1}^{n}d^dx_i\,\frac{d^dp_i}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{i\omega t}$$

$$\times\,\mathrm{tr}\left[\gamma^{\mu_1}\frac{1}{\not{p}_1+m-i\omega'}\gamma^{\mu_2}\ldots\gamma^{\mu_{n-1}}\frac{1}{\not{p}_{n-1}+m-i\omega'}\gamma^{\mu_n}\frac{1}{\not{p}_n+m-i\omega'}\right]$$

$$\times\prod_{j=1}^{n}e^{ip_j\cdot(x_j-x_{j+1})}\,h_{\mu_1}\left(x_1,\frac{p_1+p_n}{2}\right)\ldots h_{\mu_n}\left(x_n,\frac{p_{n-1}+p_n}{2}\right) \quad (2.49)$$

2.4 Ward identities and generalized EoM

The general formula for the effective action is

$$W[h] = \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} d^d x_i \, \frac{d^d p_i}{(2\pi)^d} \, \mathcal{W}^{(n)}_{\mu_1,\ldots,\mu_n}(x_1, p_1, \ldots, x_n, p_n, \epsilon) \, h^{\mu_1}(x_1, p_1) \ldots h^{\mu_n}(x_n, p_n) \tag{2.50}$$

where we have discarded the constant 0-point contribution, as we will do hereafter. The effective action can be calculated by various methods, of which (2.49) is a particular example. In the latter case the amplitudes are given by

$$\mathcal{W}^{(n)}_{\mu_1,\ldots,\mu_n}(x_1, p_1, \ldots, x_n, p_n, \epsilon) = -N \frac{n!}{n} \int_{\epsilon}^{\infty} dt \int \prod_{i=1}^{n} \frac{d^d q_i}{(2\pi)^d} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t}$$

$$\times \, \mathrm{tr}\left[\gamma^{\mu_1} \frac{1}{\not{q}_1 + m - i\omega'} \gamma^{\mu_2} \cdots \gamma^{\mu_{n-1}} \frac{1}{\not{q}_{n-1} + m - i\omega'} \gamma^{\mu_n} \frac{1}{\not{q}_n + m - i\omega'} \right]$$

$$\times \prod_{j=1}^{n} e^{iq_j \cdot (x_j - x_{j+1})} \, \delta\left(p_1 - \frac{q_1 + q_n}{2} \right) \ldots \delta\left(p_n - \frac{q_{n-1} + q_n}{2} \right) \tag{2.51}$$

We stress once more, however, that the regularized effective action (2.50) may not be derived only via (2.51), that is via the procedure of section 2.2. It could as well be obtained by means of the ordinary Feynman diagrams.

This amplitude has cyclic symmetry. When saturated with the corresponding h's, as in (2.50), it gives the level n effective action. Here we would like to investigate some general consequences of the invariance of the general effective action under the HS symmetry, codified by eq. (2.18), assuming for the $\mathcal{W}^{(n)}$ the same cyclic symmetry as (2.51). The invariance of the effective action under (2.18) is expressed as[3]

$$0 = \delta_\varepsilon W[h] \tag{2.52}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int \prod_{i=1}^{n} d^d x_i \, \frac{d^d p_i}{(2\pi)^d}$$

$$\times \, \mathcal{W}^{(n)}_{\mu_1,\ldots,\mu_n}(x_1, p_1, \ldots, x_n, p_n, \epsilon) \, \delta_\varepsilon h^{\mu_1}(x_1, p_1) \ldots h^{\mu_n}(x_n, p_n)$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int \prod_{i=1}^{n} d^d x_i \, \frac{d^d p_i}{(2\pi)^d}$$

$$\times \, \mathcal{W}^{(n)}_{\mu_1,\ldots,\mu_n}(x_1, p_1, \ldots, x_n, p_n) \, \mathcal{D}_x^{*\mu_1} \varepsilon(x_1, p_1) \, h^{\mu_2}(x_2, p_2) \ldots h^{\mu_n}(x_n, p_n)$$

In order to expose the L_∞ structure we need the equations of motion (EoM). Here we can talk of *generalized equations of motion*. They are obtained by varying $W[h, \epsilon]$ with respect to $h^\mu(x, p)$:

$$\frac{\delta}{\delta h^\mu(x, p)} W[h] = 0 \tag{2.53}$$

---

[3]Hereafter we assume that the HS symmetry is not anomalous and that there is a regularization procedure leading to a HS invariant effective action. The question of whether the particular effective action (2.49) satisfies (2.52) requires an explicit calculation of (2.51) and is left to future work.

Then, expanding in p, we obtain the generalized EoM's for the components $h^{\mu_1 \cdots \mu_n}(x)$.

The most general EoM is therefore

$$\mathcal{F}_\mu(x,p) = 0 \tag{2.54}$$

where

$$\mathcal{F}_\mu(x,p) \equiv \sum_{n=0}^\infty \frac{1}{n!} \int \prod_{i=1}^n d^d x_i \, \frac{d^d p_i}{(2\pi)^d} \, \mathcal{W}^{(n+1)}_{\mu,\mu_1\ldots,\mu_n}(x,p,x_1,p_1,\ldots,x_n,p_n,\epsilon)$$
$$\times \, h^{\mu_1}(x_1,p_1) \ldots h^{\mu_n}(x_n,p_n) \tag{2.55}$$

Integrating by parts (2.52) and using (2.26) we obtain the off-shell equation

$$\mathcal{D}^{*\mu}_x \, \mathcal{F}_\mu(x,p) \equiv \partial^\mu_x \mathcal{F}_\mu(x,p) - i[h^\mu(x,p) \,\overset{*}{,}\, \mathcal{F}_\mu(x,p)] = 0 \tag{2.56}$$

Taking the variation of this equation with respect to (2.18) we get

$$0 = \delta_\varepsilon(\mathcal{D}^{*\mu}_x \, \mathcal{F}_\mu(x,p)) = \mathcal{D}^{*\mu}_x \, (\delta_\varepsilon \mathcal{F}_\mu(x,p)) - i[\mathcal{D}^{*\mu}_x \varepsilon \,\overset{*}{,}\, \mathcal{F}_\mu(x,p)] \tag{2.57}$$

From (2.56) and (2.57) one can deduce

$$\delta_\varepsilon \mathcal{F}_\mu(x,p) = i[\varepsilon(x,p) \,\overset{*}{,}\, \mathcal{F}_\mu(x,p)] \tag{2.58}$$

A final remark for this section. Using standard regularizations one obtains that in general the effective action contains term linear in HS fields, which gives constant contribution to EoM's of even-spin HS fields of the form $c(s,\epsilon)\,(\eta_{\mu\mu})^{s/2}$, where $c(s,\epsilon)$ are scheme dependent coefficients which need to be renormalized. As this term is a generalization of the lowest-order contribution of the cosmological constant term expanded around flat spacetime, we shall call the part of the effective action that contains the full linear term and is invariant on HS transformations (2.18), *generalized cosmological constant* term. In the next section we shall assume that this term is removed from the effective action.

3 L_∞ structure in higher spin theory

3.1 L_∞ symmetry of higher spin effective actions

In this section we will uncover the L_∞ symmetry of the $W[h]$. To this end we use the general transformation properties derived in the previous subsection, notably eqs. (2.54), (2.58), beside (2.18). We will also introduce a simplification, which is required by the classical form of the L_∞ symmetry. The expansion of the effective action (2.50) is in essence an expansion around a flat background. As a flat background is not a solution when the generalized cosmological constant term is present, consistency requires that we take this term out of an effective action (or, in other words, renormalize the cosmological constant to zero). This will be assumed from now on. Technically, this means that we now assume that the sum in (2.50) starts from $n=2$, and the sum in (2.55) starts from $n=1$, while all other relations from subsection 2.4 are the same.

To start with let us recall that an L_∞ structure characterizes closed string field theory.[4] This fact first appeared in [29, 30], see also [31], as a particular case of a general mathematical structure called strongly homotopic algebras (or SH algebras), see the introduction for physicists [32, 33]. It became later evident that this kind of structure characterizes not only closed string field, but other field theories as well [34], in particular gauge field theories [35–37], Chern-Simons theories, Einstein gravity and double field theory [28]. For other more recent applications, see [38–40].

For the strongly homotopic algebra L_∞ we closely follow the notation and definitions of [28]. L_∞ is determined by a set of vector spaces X_i, $i = \ldots, 1, 0, -1, \ldots$, with degree i and multilinear maps (products) among them L_j, $j = 1, 2, \ldots$, with degree $d_j = j - 2$, satisfying the following quadratic identities:

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_\sigma (-1)^\sigma \epsilon(\sigma; x)\, L_j(L_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0 \quad (3.1)$$

In this formula σ denotes a permutation of the entries so that $\sigma(1) < \ldots \sigma(i)$ and $\sigma(i+1) < \ldots \sigma(n)$, and $\epsilon(\sigma; x)$ is the Koszul sign. To define it consider an algebra with product $x_i \wedge x_j = (-1)^{x_i x_j} x_j \wedge x_i$, where x_i is the degree of x_i; then $\epsilon(\sigma; x)$ is defined by the relation

$$x_1 \wedge x_2 \wedge \ldots \wedge x_n = \epsilon(\sigma; x)\, x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge \ldots \wedge x_{\sigma(n)} \quad (3.2)$$

In our case, due to the structure of the effective action and the equation of motion, we will need only three spaces X_0, X_{-1}, X_{-2} and the complex

$$X_0 \xrightarrow{L_1} X_{-1} \xrightarrow{L_1} X_{-2} \xrightarrow{L_1} 0 \quad (3.3)$$

The degree assignment is as follows: $\varepsilon \in X_0$, $h^\mu \in X_{-1}$ and $\mathcal{F}_\mu \in X_{-2}$.

The properties of the mappings L_i under permutation are defined in [28]. For instance

$$L_2(x_1, x_2) = -(-1)^{x_1 x_2} L_2(x_2, x_1) \quad (3.4)$$

In general

$$L_n(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = (-1)^\sigma \epsilon(\sigma; x) L_n(x_1, x_2, \ldots, x_n) \quad (3.5)$$

It is worth noting that if all the x_i's are odd $(-1)^\sigma \epsilon(\sigma; x) = 1$.

The product L_i are defined as follows. We first define the maps ℓ_i

$$\delta_\varepsilon h = \ell_1(\varepsilon) + \ell_2(\varepsilon, h) - \frac{1}{2}\ell_3(\varepsilon, h, h) - \frac{1}{3!}\ell_4(\varepsilon, h, h, h) + \ldots \quad (3.6)$$

Therefore, in our case,

$$\begin{aligned} \ell_1(\varepsilon)^\mu &= \partial_x^\mu \varepsilon(x, p) \\ \ell_2(\varepsilon, h)^\mu &= -i[h^\mu(x, p) \overset{*}{,} \varepsilon(x, p)] = -\ell_2(h, \varepsilon)^\mu \\ \ell_j(\varepsilon, h, \ldots, h)^\mu &= 0, \qquad j \geq 3 \end{aligned} \quad (3.7)$$

For these entries, i.e. $\varepsilon, (\varepsilon, h), (\varepsilon, h, h), \ldots$ we set $L_i = \ell_i$.

---

[4]Open string field theory is instead characterized by an A_∞ structure, see [28] and references therein.

From the above we can extract $L_2(\varepsilon, \varepsilon) \equiv \ell_2(\varepsilon, \varepsilon)$. We have

$$
\begin{aligned}
(\delta_{\varepsilon_1}\delta_{\varepsilon_2} - \delta_{\varepsilon_2}\delta_{\varepsilon_1}) h^\mu &= \delta_{\varepsilon_1}(\ell_1(\varepsilon_2) + \ell_2(\varepsilon_2, h)) - \delta_{\varepsilon_2}(\ell_1(\varepsilon_1) + \ell_2(\varepsilon_1, h)) \\
&= \delta_{\varepsilon_1}(\ell_2(\varepsilon_2, h)) - \delta_{\varepsilon_2}(\ell_2(\varepsilon_1, h)) \\
&= \ell_2(\varepsilon_2, \delta_{\varepsilon_1} h) - \ell_2(\varepsilon_1, \delta_{\varepsilon_2} h) = \ell_2(\varepsilon_2, \ell_1(\varepsilon_1)) - \ell_2(\varepsilon_1, \ell_1(\varepsilon_2)) + \mathcal{O}(h)
\end{aligned}
\tag{3.8}
$$

Now, the L_∞ relation (3.1) involving L_1 and L_2 is

$$
L_1(L_2(x_1, x_2)) = L_2(L_1(x_1), x_2) - (-1)^{\mathrm{x}_1 \mathrm{x}_2} L_2(L_1(x_2), x_1)
\tag{3.9}
$$

for two generic elements of x_1, x_2 of degree $\mathrm{x}_1, \mathrm{x}_2$, respectively. If we wish to satisfy it we have to identify

$$
(\delta_{\varepsilon_1}\delta_{\varepsilon_2} - \delta_{\varepsilon_2}\delta_{\varepsilon_1}) h = -\ell_1(\ell_2(\varepsilon_1, \varepsilon_2)) + \mathcal{O}(h)
\tag{3.10}
$$

By comparing this with (2.29) we obtain

$$
\ell_2(\varepsilon_1, \varepsilon_2) = i\,[\varepsilon_1 \overset{*}{,} \varepsilon_2]
\tag{3.11}
$$

The next step is to determine L_3. It must satisfy, in particular, the L_∞ relation

$$
\begin{aligned}
0 =\ & L_1(L_3(x_1, x_2, x_3)) \\
&+ L_3(L_1(x_1), x_2, x_3) + (-1)^{\mathrm{x}_1} L_3(x_1, L_1(x_2), x_3) + (-1)^{\mathrm{x}_1 + \mathrm{x}_2} L_3(x_1, x_2, L_1(x_3)) \\
&+ L_2(L_2(x_1, x_2), x_3) + (-1)^{(\mathrm{x}_1 + \mathrm{x}_2)\mathrm{x}_3} L_2(L_2(x_3, x_1), x_2) + (-1)^{(\mathrm{x}_2 + \mathrm{x}_3)\mathrm{x}_1} L_2(L_2(x_2, x_3), x_1)
\end{aligned}
\tag{3.12}
$$

We define first the ℓ_i with only h entries. They are given by the generalized EoM:

$$
\mathcal{F} = \ell_1(h) - \frac{1}{2}\ell_2(h, h) - \frac{1}{3!}\ell_3(h, h, h) + \ldots
\tag{3.13}
$$

Let us write $\mathcal{F}_\mu$, (2.54) in compact form as

$$
\mathcal{F}_\mu = \sum_{n=1}^{\infty} \frac{1}{n!} \langle\!\langle \mathcal{W}_\mu^{(n+1)}, h^{\otimes n} \rangle\!\rangle
\tag{3.14}
$$

then

$$
\begin{aligned}
\ell_n(h, \ldots, h) &= (-1)^{\frac{n(n-1)}{2}} \langle\!\langle \mathcal{W}_\mu^{(n+1)}, h^{\otimes n} \rangle\!\rangle \\
&= (-1)^{\frac{n(n-1)}{2}} \int \prod_{i=1}^{n} d^d x_i \frac{d^d p_i}{(2\pi)^d} \mathcal{W}^{(n+1)}_{\mu,\mu_1\ldots,\mu_n}(x, p, x_1, p_1, \ldots, x_n, p_n) \\
&\quad \times h^{\mu_1}(x_1, p_1) \ldots h^{\mu_n}(x_n, p_n)
\end{aligned}
\tag{3.15}
$$

in particular,

$$
\ell_1(h) = \langle\!\langle \mathcal{W}_\mu^{(2)}, h \rangle\!\rangle = \int d^d x_i \frac{d^d p_i}{(2\pi)^d} \mathcal{W}^{(2)}_{\mu,\mu_1}(x, p, x_1, p_1) h^{\mu_1}(x_1, p_1)
\tag{3.16}
$$

Notice that $\mathcal{W}^{(n+1)}_{\mu,\mu_1\ldots,\mu_n}$ is not symmetric in the exchange of its indices. In fact it has only a cyclic symmetry. But in order to verify the L_∞ relations we have to know these products for different entries. Following [28] we define, for instance,

$$
2L_2(h_1, h_2) = \ell_2(h_1 + h_2, h_1 + h_2) - \ell_2(h_1, h_1) - \ell_2(h_2, h_2)
\tag{3.17}
$$

which is equivalent to

$$L_2(h_1, h_2) = \frac{1}{2} \left(\ell_2(h_1, h_2) + \ell_2(h_2, h_1) \right) \tag{3.18}$$

Similarly

$$L_3(h_1, h_2, h_3) = \frac{1}{6} \left(\ell_3(h_1, h_2, h_3) + \text{perm}(h_1, h_2, h_3) \right) \tag{3.19}$$

In general, when we have a non-symmetric n-linear function f_n of the variable h we can generate a symmetric function F_n linearly dependent on each of n variables $h_1, \ldots, h_n$ through the following process

$$\begin{aligned}
F_n&(h_1, \ldots, h_n)\\
&= \frac{1}{n!} \Big(f_n(h_1 + \ldots + h_n) - \Big[f_n(h_1 + \ldots + h_{n-1}) + f_n(h_1 + \ldots + h_{n-2} + h_n)\\
&\quad + \ldots + f_n(h_2 + \ldots + h_n) \Big] + \Big[f_n(h_1 + \ldots + h_{n-2}) + \cdots + f_n(h_3 + \ldots + h_n) \Big]\\
&\quad + \ldots\\
&\quad + (-1)^{n-k} \Big[f_n(h_1 + \ldots + h_k) + \cdots + f_n(h_{n-k+1} + \ldots + h_n) \Big]\\
&\quad + \ldots\\
&\quad + (-1)^{n-1} \Big[f_n(h_1) + \ldots + f_n(h_n) \Big] \tag{3.20}
\end{aligned}$$

We shall define $L_n(h_1, \ldots, h_n)$ by using this formula: replace F_n with L_n and f_n with ℓ_n, the latter being given by (3.15).

We shall see that beside $L_n(h_1, \ldots, h_n)$, (3.7) and (3.11) the only nonvanishing objects defining the L_∞ algebra of the HS effective action are

$$L_2(\varepsilon, E) = i[\varepsilon \overset{*}{,} E] \tag{3.21}$$

where E represents $\mathcal{F}_\mu$ or any of its homogeneous pieces.

In the rest of this section we shall prove that L_n defined in this way generate an L_∞ algebra.

3.2 Proof of the L_∞ relations

3.2.1 Relation $L_1^2 = 0$, degree -2

Now let us verify the remaining L_∞ relations. The first is $L_1^2 \equiv \ell_1^2 = 0$.[5]

Let us start from $\ell_1(\ell_1(\varepsilon))$. We recall that $\ell_1(\varepsilon) = \partial_x \varepsilon(x, p)$ and belongs to X_{-1}. Now

$$\ell_1(h) = \langle\!\langle \mathcal{W}_\mu^{(2)}, h \rangle\!\rangle \tag{3.22}$$

Replacing h with $\partial_x \varepsilon(x, p)$ corresponds to taking the variation of the lowest order in h of $\mathcal{F}_\mu$ with respect to h, i.e. with respect to (2.18). On the other hand the variation of $\mathcal{F}_\mu$ is given by (2.58) and is linear in $\mathcal{F}_\mu$. Therefore, since $\ell_1(\partial_x \varepsilon(x, p))$ is order 0 in h it must vanish. In fact it does, which corresponds to the gauge invariance of the EoM to the lowest order in h.

Next let us consider $\ell_1(\ell_1(h))$. It has degree -3, so it is necessarily 0 since $X_{-3} = 0$.

---

[5]We remark that if the generalized cosmological constant term (see end of section 2.4 and beginning of section 3) is non-vanishing, then $\ell_1^2 \neq 0$. In this case an enlarged version of L_∞, called *curved* L_∞, is necessary. We thank J. Stasheff for this piece of information. We will not explore this possibility here.

3.2.2 Relation $L_1L_2 = L_2L_1$, degree -1

Next, we know $\ell_2(\varepsilon_1, \varepsilon_2), \ell_2(\varepsilon, h)$ and $\ell_2(h_1, h_2)$, and we have to verify $L_1L_2 = L_2L_1$. The latter is written explicitly in (3.9) and takes the form

$$\ell_1(\ell_2(\varepsilon, h)) = L_2(\ell_1(\varepsilon), h) + L_2(\varepsilon, \ell_1(h)) \tag{3.23}$$
$$= \frac{1}{2}\Big(\ell_2(\ell_1(\varepsilon), h) + \ell_2(h, \ell_1(\varepsilon))\Big) + L_2(\varepsilon, \ell_1(h))$$

where we used (3.18). More explicitly (3.23) writes

$$-i\ell_1([h \overset{*}{,} \varepsilon])_\mu = \frac{1}{2}\Big(\ell_2(\partial^x \varepsilon, h) + \ell_2(h, \partial^x \varepsilon)\Big)_\mu + L_2(\varepsilon, \langle\!\langle \mathcal{W}^{(2)}_\mu, h \rangle\!\rangle) \tag{3.24}$$

i.e.

$$i\langle\!\langle \mathcal{W}^{(2)}_{\mu\nu}, [h^\nu \overset{*}{,} \varepsilon]) \rangle\!\rangle = \frac{1}{2}\Big(\langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}, \partial^\nu_x \varepsilon h^\lambda \rangle\!\rangle + \langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}, h^\nu \partial^\lambda_x \varepsilon \rangle\!\rangle\Big) - L_2(\varepsilon, \langle\!\langle \mathcal{W}^{(2)}_\mu, h \rangle\!\rangle) \tag{3.25}$$

To understand this relation one must unfold (2.58). On one side we have

$$\delta_\varepsilon \mathcal{F}_\mu = \sum_{n=1}^\infty \frac{1}{n!} \Big(\sum_{i=1}^n \langle\!\langle \mathcal{W}^{(n+1)}_{\mu\mu_1\ldots\mu_i\ldots\mu_n}, h^{\mu_1} \ldots \partial_x^{\mu_i} \varepsilon \ldots h^{\mu_n} \rangle\!\rangle \tag{3.26}$$
$$- i \sum_{i=1}^n \langle\!\langle \mathcal{W}^{(n+1)}_{\mu\mu_1\ldots\mu_i\ldots\mu_n}, h^{\mu_1} \ldots [h^{\mu_i} \overset{*}{,} \varepsilon] \ldots h^{\mu_n} \rangle\!\rangle \Big)$$

On the other side

$$i[\varepsilon \overset{*}{,} \mathcal{F}_\mu] = i \sum_{n=1}^\infty \frac{1}{n!} [\varepsilon \overset{*}{,} \langle\!\langle \mathcal{W}^{(n+1)}_\mu, h^{\otimes n} \rangle\!\rangle] \tag{3.27}$$

The two must be equal order by order in h. Thus we have

$$i[\varepsilon \overset{*}{,} \langle\!\langle \mathcal{W}^{(n+1)}_\mu, h^{\otimes n} \rangle\!\rangle] = \frac{1}{n+1} \sum_{i=1}^{n+1} \langle\!\langle \mathcal{W}^{(n+2)}_{\mu\mu_1\ldots\mu_i\ldots\mu_{n+1}}, h^{\mu_1} \ldots \partial_x^{\mu_i} \varepsilon \ldots h^{\mu_{n+1}} \rangle\!\rangle \tag{3.28}$$
$$- i \sum_{i=1}^n \langle\!\langle \mathcal{W}^{(n+1)}_{\mu\mu_1\ldots\mu_i\ldots\mu_n}, h^{\mu_1} \ldots [h^{\mu_i} \overset{*}{,} \varepsilon] \ldots h^{\mu_n} \rangle\!\rangle$$

This is a not too disguised form of the Ward identity for the symmetry (2.18). Setting $n = 1$ gives precisely (3.25) provided

$$L_2(\varepsilon, \langle\!\langle \mathcal{W}^{(2)}_\mu, h \rangle\!\rangle) = i[\varepsilon \overset{*}{,} \langle\!\langle \mathcal{W}^{(2)}_\mu, h \rangle\!\rangle] \tag{3.29}$$

The quantity $\mathcal{F}^{(1)} = \langle\!\langle \mathcal{W}^{(2)}_\mu, h \rangle\!\rangle$ is the lowest order piece of the EoM (of degree -2), see (3.14). So we can say

$$L_2(\varepsilon, \mathcal{F}^{(1)}) \equiv \ell_2(\varepsilon, \mathcal{F}^{(1)}) = i[\varepsilon \overset{*}{,} \mathcal{F}^{(1)}] \tag{3.30}$$

In general,

$$\ell_2(\varepsilon, \mathcal{F}) = i[\varepsilon \overset{*}{,} \mathcal{F}] \tag{3.31}$$

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The next relation to be verified is

$$L_1(L_2(h_1, h_2)) = L_2(L_1(h_1), h_2) - L_2(h_1, L_1(h_2)) \qquad (3.32)$$

The entries of L_2 on the r.h.s. have degree -3, so they must vanish. On the other hand $L_2(h_1, h_2)$ on the l.h.s. has degree -2, and is mapped to degree -3 by L_1. So it is consistent to equate both sides to 0. In particular we can set $L_2(\mathcal{F}^{(1)}, h) = 0$ (and, more generally, $L_2(X_{-2}, h) = 0$).

3.2.3 Relation $L_3 L_1 + L_2 L_2 + L_1 L_3 = 0$, degree 0

First we should evaluate $L_3(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Its degree is 1, therefore it exits the complex. Is it consistent to set it to 0? The relevant L_∞ relation is

$$
\begin{aligned}
0 = {} & \ell_1(L_3(x_1, x_2, x_3)) \\
& + L_3(\ell_1(x_1), x_2, x_3) + (-1)^{x_1} L_3(x_1, \ell_1(x_2), x_3) + (-1)^{x_1+x_2} L_3(x_1, x_2, \ell_1(x_3)) \\
& + L_2(L_2(x_1, x_2), x_3) + (-1)^{(x_1+x_2)x_3} L_2(L_2(x_3, x_1), x_2) + (-1)^{(x_2+x_3)x_1} L_2(L_2(x_2, x_3), x_1)
\end{aligned}
\qquad (3.33)
$$

In our case the second line equals $\partial_x L_3(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Thus if we set $L_3(\varepsilon_1, \varepsilon_2, \varepsilon_3) = 0$, the first two lines vanish. Using (3.11), we see that the third line is nothing but the $*$-Jacobi identity.

Arguing the same way and using the next L_∞ relation, which involves L_4, one can show that $L_4(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = 0$, etc.

From (3.7) we also know that $L_3(\varepsilon, h_1, h_2) \equiv \ell_3(\varepsilon, h_1, h_2) = 0$. Following [28] we will set also $L_3(\varepsilon_1, \varepsilon_2, h) = 0$, $L_3(\varepsilon_1, \varepsilon_2, \mathcal{F}^{(1)}) = 0$. Therefore

$$L_3(\varepsilon_1, \varepsilon_2, \varepsilon_3) = 0, \quad L_3(\varepsilon, h_1, h_2) = 0, \quad L_3(\varepsilon_1, \varepsilon_2, h) = 0, \quad L_3(\varepsilon_1, \varepsilon_2, \mathcal{F}^{(1)}) = 0 \quad (3.34)$$

Let us consider next the entries $\varepsilon_1, \varepsilon_2, h$. The terms of the first two lines in (3.12) vanish due to (3.34). The last line is

$$
\begin{aligned}
& \ell_2(\ell_2(\varepsilon_1, \varepsilon_2), h) + \ell_2(\ell_2(h, \varepsilon_1), \varepsilon_2) + \ell_2(\ell_2(\varepsilon_2, h), \varepsilon_1) \\
& = [h^\mu \overset{*}{,} [\varepsilon_1 \overset{*}{,} \varepsilon_2]] - [[h^\mu \overset{*}{,} \varepsilon_1] \overset{*}{,} \varepsilon_2] + [[h^\mu \overset{*}{,} \varepsilon_2] \overset{*}{,} \varepsilon_1]
\end{aligned}
\qquad (3.35)
$$

which vanishes due to $*$-Jacobi identity.

Now we consider the entries ε, h_1, h_2. Plugging them into (3.12), the first line vanishes because of (3.34). The rest is

$$
\begin{aligned}
0 = {} & \frac{1}{6}\Big(\ell_3(\ell_1(\varepsilon), h_1, h_2) + \text{perm}_3 \Big) \\
& + L_3(\varepsilon, \ell_1(h_1), h_2) - L_3(\varepsilon, h_1, \ell_1(h_2)) \\
& + \frac{1}{2}\Big(\ell_2(\ell_2(\varepsilon, h_1), h_2) + \ell_2(h_2, \ell_2(\varepsilon, h_1)) - \ell_2(\ell_2(h_2, \varepsilon), h_1) \\
& \qquad - \ell_2(h_1, \ell(h_2, \varepsilon)) + \ell_2(\ell_2(h_1, h_2), \varepsilon) + \ell_2(\ell_2(h_2, h_1), \varepsilon) \Big)
\end{aligned}
\qquad (3.36)
$$

where perm_3 means the permutation of the three entries of ℓ_3. Writing down explicitly the first line, it takes the form

$$\frac{1}{6}\left(\ell_3(\ell_1(\varepsilon), h_1, h_2) + \text{perm}_3 \right) = -\frac{1}{6}\left(\langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}, \partial_x^\nu \varepsilon \, h_1^\lambda \, h_2^\rho \rangle\!\rangle + \text{perm}_3 \right) \qquad (3.37)$$

The last two lines of (3.36) give

$$\ell_2(\ell_2(\varepsilon, h_1), h_2) + \ell_2(h_2, \ell_2(\varepsilon, h_1)) - \ell_2(\ell_2(h_2, \varepsilon), h_1) - \ell_2(h_1, \ell_2(h_2, \varepsilon)) + \ell_2(\ell_2(h_1, h_2), \varepsilon)$$

$$+\ell_2(\ell_2(h_2, h_1), \varepsilon) = +i\Big(\langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}, [h_1^\nu \overset{*}{,} \varepsilon] h_2^\lambda \rangle\!\rangle + \langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}, h_2^\lambda [h_1^\nu \overset{*}{,} \varepsilon] \rangle\!\rangle + \langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}, [h_2^\nu \overset{*}{,} \varepsilon] h_1^\lambda \rangle\!\rangle$$

$$+\langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}, h_1^\lambda [h_2^\nu \overset{*}{,} \varepsilon] \rangle\!\rangle + [\varepsilon \overset{*}{,} \langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}, h_1^\nu h_2^\lambda \rangle\!\rangle] + [\varepsilon \overset{*}{,} \langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}, h_2^\lambda h_1^\nu \rangle\!\rangle] \Big) \quad (3.38)$$

Summing the rhs's of (3.37) and (3.38) one gets, apart from the second line, (3.36) expressed in terms of the expressions appearing in the r.h.s. of (3.28) with entries h_1, h_2, instead of one single h.

Now let us consider (3.28) for $n = 2$, i.e.

$$i[\varepsilon \overset{*}{,} \langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}, h^\nu h^\lambda \rangle\!\rangle] = \frac{1}{3} \langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}, \partial_x^\nu \varepsilon h^\lambda h^\rho + h^\nu \partial_x^\lambda \varepsilon h^\rho + h^\nu h^\lambda \partial_x^\rho \varepsilon \rangle\!\rangle \quad (3.39)$$

$$-i \langle\!\langle \mathcal{W}^{(3)}_{\mu\nu\lambda}, [h^\nu \overset{*}{,} \varepsilon] h^\lambda + h^\nu [h^\lambda \overset{*}{,} \varepsilon] \rangle\!\rangle.$$

This can be read as

$$-i[\varepsilon \overset{*}{,} \ell_2(h, h)] = -\frac{1}{3} \Big(\ell_3(\partial_x \varepsilon, h, h) + \ell_3(h, \partial_x \varepsilon, h) + \ell_3(h, h, \partial_x \varepsilon) \Big)$$

$$+i\ell_2(h, [h \overset{*}{,} \varepsilon]) + i\ell_2([h \overset{*}{,} \varepsilon], h) \quad (3.40)$$

Now we consider the same equation obtained by replacing h with $h_1 + h_2$ according to the symmetrization procedure in (3.17). We get in this way the symmetrized equation

$$-i[\varepsilon \overset{*}{,} \ell_2(h_1, h_2)] - i[\varepsilon \overset{*}{,} \ell_2(h_2, h_1)] \quad (3.41)$$

$$= -\frac{1}{3} \Big(\ell_3(\partial_x \varepsilon, h_1, h_2) + \ell_3(\partial_x \varepsilon, h_2, h_1) + \ell_3(h_1, \partial_x \varepsilon, h_2)$$

$$+\ell_3(h_2, \partial_x \varepsilon, h_1) + \ell_3(h_1, h_2, \partial_x \varepsilon) + \ell_3(h_2, h_1, \partial_x \varepsilon) \Big)$$

$$+i\ell_2(h_1, [h_2 \overset{*}{,} \varepsilon]) + i\ell_2(h_2, [h_1 \overset{*}{,} \varepsilon]) + i\ell_2([h_1 \overset{*}{,} \varepsilon], h_2) + i\ell_2([h_2 \overset{*}{,} \varepsilon], h_1)$$

This is the same as the sum of the first, third and fourth lines of (3.36), or, alternatively, the sum of the rhs's of (3.37) and (3.38).

Thus (3.36) is satisfied if the two remaining terms in the second line vanish. They are all of the type $L_3(\varepsilon, h, \mathcal{F}^{(1)})$ and we can assume that such types of terms vanish. So, beside (3.34) we have

$$L_3(\varepsilon, h, E) = -L_3(\varepsilon, E, h) = 0 \quad (3.42)$$

where E represent $\mathcal{F}_\mu$ or anything in X_{-2}.

The relation with entries $\varepsilon_1, \varepsilon_2$ and E is nontrivial and has to be verified. Consider again (3.12) with entries $\varepsilon_1, \varepsilon_2$ and E. Due to (3.34), (3.42) the relation (3.12) reduces to the last line:

$$\ell_2(\ell_2(\varepsilon_1, \varepsilon_2), E) + \ell_2(\ell_2(E, \varepsilon_1), \varepsilon_2) + \ell_2(\ell_2(\varepsilon_2, E), \varepsilon_1) \quad (3.43)$$

$$= i\ell_2([\varepsilon_1 \overset{*}{,} \varepsilon_2], E) + i\ell_2([E \overset{*}{,} \varepsilon_1], \varepsilon_2) + i\ell_2([\varepsilon_2 \overset{*}{,} E], \varepsilon_1)$$

$$= +[E \overset{*}{,} [\varepsilon_1 \overset{*}{,} \varepsilon_2]] - [[E \overset{*}{,} \varepsilon_1] \overset{*}{,} \varepsilon_2] - [[\varepsilon_2 \overset{*}{,} E] \overset{*}{,} \varepsilon_1]$$

which vanishes because of the $*$-Jacobi identity.

3.2.4   Relation $L_1L_4 - L_2L_3 + L_3L_2 - L_4L_1 = 0$, degree 1

The L_∞ relation to be proved at degree 1 is

$$L_1(L_4(x_1, x_2, x_3, x_4)) \tag{3.44}$$
$$-L_2(L_3(x_1, x_2, x_3), x_4) + (-1)^{x_3 x_4} L_2(L_3(x_1, x_2, x_4), x_3)$$
$$+(-1)^{(1+x_1)x_2} L_2(x_2, L_3(x_1, x_3, x_4)) - (-1)^{x_1} L_2(x_1, L_3(x_2, x_3, x_4))$$
$$+L_3(L_2(x_1, x_2), x_3, x_4) + (-1)^{1+x_2 x_3} L_3(L_2(x_1, x_3), x_2, x_4)$$
$$+(-1)^{x_4(x_2+x_3)} L_3(L_2(x_1, x_4), x_2, x_3)$$
$$-L_3(x_1, L_2(x_2, x_3), x_4) + (-1)^{x_3 x_4} L_3(x_1, L_2(x_2, x_4), x_3) + L_3(x_1, x_2, L_2(x_3, x_4))$$
$$-L_4(L_1(x_1), x_2, x_3, x_4) - (-1)^{x_1} L_4(x_1, L_1(x_2), x_3, x_4)$$
$$-(-1)^{x_1+x_2} L_4(x_1, x_2, L_1(x_3), x_4) - (-1)^{x_1+x_2+x_4} L_4(x_1, x_2, x_3, L_1(x_4)) = 0$$

We have

$$L_4(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = 0, \quad L_4(\varepsilon_1, \varepsilon_2, \varepsilon_3, h) = 0, \quad L_4(\varepsilon_1, \varepsilon_2, h_1, h_2) = 0, \quad L_4(\varepsilon, h_1, h_2, h_3) = 0 \tag{3.45}$$

The first and second equality have positive degree, so they must vanish. The fourth has been proven above, see (3.7). The other is an ansatz to be checked by consistency.

The relation (3.44) with four ε entries has already been commented. The same relation with three ε entries and one h is also trivial as a consequence of (3.34) and (3.45). The same happens in the case of two ε entries and two h, as a consequence again of (3.34) and (3.45).

Now let us consider the case of one ε and three h's. Plugging them into (3.44) here is what we get in terms of ℓ_i's (only the nonzero terms are written down)

$$0 = -\frac{1}{6}\Big(\ell_2(\varepsilon, \ell_3(h_1, h_2, h_3)) + \mathrm{perm}_3\Big) \tag{3.46}$$
$$+\frac{1}{6}\Big(\ell_3(\ell_2(\varepsilon, h_1), h_2, h_3) + \ell_3(\ell_2(\varepsilon, h_2), h_1, h_3) + \ell_3(\ell_2(\varepsilon, h_3), h_1, h_2) + \mathrm{perm}_3\Big)$$
$$-\frac{1}{4!}\Big(\ell_4(\ell_1(\varepsilon), h_1, h_2, h_3) + \mathrm{perm}_4\Big)$$
$$-L_4(\varepsilon, \ell_1(h_1), h_2, h_3) + L_4(\varepsilon, h_1, \ell_1(h_2), h_3) - L_4(\varepsilon, h_1, h_2, \ell_1(h_3))$$

where $\mathrm{perm}_3, \mathrm{perm}_4$ refer to the permutations of the ℓ_3, ℓ_4 entries, respectively. Disregarding for the moment the last line, which is of type $L_4(\varepsilon, E, h, h)$, this equation becomes

$$0 = \frac{i}{6}\Big([\varepsilon \overset{*}{,} \langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}, h_1^\nu h_2^\lambda h_3^\rho \rangle\!\rangle] + \mathrm{perm}(h_1, h_2, h_3) \tag{3.47}$$
$$+\langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}, [h_1^\nu \overset{*}{,} \varepsilon] h_2^\lambda h_3^\rho \rangle\!\rangle + \mathrm{perm}([h_1 \overset{*}{,} \varepsilon], h_2, h_3)$$
$$+\langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}, [h_2^\nu \overset{*}{,} \varepsilon] h_1^\lambda h_3^\rho \rangle\!\rangle + \mathrm{perm}([h_2 \overset{*}{,} \varepsilon], h_1, h_3)$$
$$+\langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}, [h_3^\nu \overset{*}{,} \varepsilon] h_1^\lambda h_2^\rho \rangle\!\rangle + \mathrm{perm}([h_3 \overset{*}{,} \varepsilon], h_1, h_2)\Big)$$
$$-\frac{1}{4!}\Big(\langle\!\langle \mathcal{W}^{(5)}_{\mu\nu\lambda\rho\sigma}, \partial_x^\nu \varepsilon\, h_1^\lambda h_2^\rho h_3^\sigma \rangle\!\rangle + \mathrm{perm}(\partial_x\varepsilon, h_1, h_2, h_3)\Big)$$

For comparison let us go back to (3.28) with $n = 3$. It writes

$$i[\varepsilon \overset{*}{,} \langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}, h^\nu h^\lambda h^\rho \rangle\!\rangle]$$
$$= \frac{1}{4} \langle\!\langle \mathcal{W}^{(5)}_{\mu\nu\lambda\rho\sigma}, \partial^\nu_x \varepsilon h^\lambda h^\rho h^\sigma + h^\nu \partial^\lambda_x \varepsilon h^\rho h^\sigma + h^\nu h^\lambda \partial^\rho_x \varepsilon h^\sigma + h^\nu h^\lambda h^\rho \partial^\sigma_x \varepsilon \rangle\!\rangle$$
$$- i \langle\!\langle \mathcal{W}^{(4)}_{\mu\nu\lambda\rho}, [h^\nu \overset{*}{,} \varepsilon] h^\lambda h^\rho + h^\nu [h^\lambda \overset{*}{,} \varepsilon] h^\rho + h^\nu h^\lambda [h^\rho \overset{*}{,} \varepsilon] \rangle\!\rangle \tag{3.48}$$

If now we transform the l.h.s. of this equation to a trilinear function of h_1, h_2, h_3 according to the recipe (3.20), we obtain precisely eq.(3.47). As a consequence we are forced to set

$$L_4(\varepsilon, E, h, h) = L_4(\varepsilon, h, E, h) = L_4(\varepsilon, h, h, E) = 0 \tag{3.49}$$

Considering the entries $\varepsilon, \varepsilon, E, h$ in (3.44) one can show that

$$L_4(\varepsilon, \varepsilon, E, h) = 0 \tag{3.50}$$

for consistency. Using this and evaluating (3.44) with entries $\varepsilon, \varepsilon, h, h$, one can see that the third ansatz in (3.45) is justified.

3.2.5 Relation $L_1 L_n + \ldots \pm L_n L_1 = 0$, degree $n - 3$

The general L_∞ relation is (3.1). As the $n = 4$ example shows, for $n \geq 4$ it is consistent to set the values of L_n to zero except when all the entries have degree -1. Schematically, out of (3.1), the only nontrivial relation is

$$-L_2(\varepsilon, L_{n-1}(h, \ldots, h)) + L_{n-1}(L_2(\varepsilon, h), h, \ldots, h) + (-1)^{n-1} L_n(L_1(\varepsilon), h, \ldots, h) = 0 \tag{3.51}$$

Written in explicit form in terms of ℓ_n, it is

$$-\frac{1}{(n-1)!} \Big(\ell_2(\varepsilon, \ell_{n-1}(h_1, \ldots, h_{n-1})) + \mathrm{perm}_{n-1} \Big) \tag{3.52}$$
$$+\frac{1}{(n-1)!} \Big(\ell_{n-1}(\ell_2(\varepsilon, h_1), h_2, \ldots, h_{n-1}) + \ell_{n-1}(\ell_2(\varepsilon, h_2), h_1, \ldots, h_{n-1}) + \ldots$$
$$+\ell_{n-1}(\ell_2(\varepsilon, h_{n-1}), h_1, \ldots, h_{n-2}) + \mathrm{perm}_{n-1} \Big)$$
$$+\frac{(-1)^{n-1}}{n!} \Big(\ell_n(\ell_1(\varepsilon), h_1, \ldots, h_{n-1}) + \mathrm{perm}_n \Big) = 0$$

In order to obtain this it is essential to remark that, for entries of degree -1, the factor $(-1)^\sigma \epsilon(\sigma; x)$ in (3.1) is 1.

Using now the definition (3.15) and simplifying, (3.52) becomes

$$-i\Big([\varepsilon \overset{*}{,} \langle\!\langle \mathcal{W}^{(n)}_{\mu\nu_1\ldots\nu_{n-1}}, h_1^{\nu_1} \ldots h_{n-1}^{\nu_{n-1}} \rangle\!\rangle] + \mathrm{perm}_{n-1} \Big) \tag{3.53}$$
$$+i\Big(\langle\!\langle \mathcal{W}^{(n)}_{\mu\nu_1\ldots\nu_{n-1}}, [\varepsilon \overset{*}{,} h_1^{\nu_1}] h_2^{\nu_2} \ldots h_{n-1}^{\nu_{n-1}} \rangle\!\rangle + \mathcal{W}^{(n)}_{\mu\nu_1\ldots\nu_{n-1}}, [\varepsilon \overset{*}{,} h_2^{\nu_1}] h_1^{\nu_2} \ldots h_{n-1}^{\nu_{n-1}} \rangle\!\rangle$$
$$+ \ldots + \mathcal{W}^{(n)}_{\mu\nu_1\ldots\nu_{n-1}}, [\varepsilon \overset{*}{,} h_{n-1}^{\nu_1}] h_1^{\nu_2} \ldots h_{n-2}^{\nu_{n-1}} \rangle\!\rangle + \mathrm{perm}_{n-1} \Big)$$
$$+\frac{1}{n} \Big(\langle\!\langle \mathcal{W}^{(n+1)}_{\mu\nu_1\ldots\nu_n}, \partial^{\nu_1}_x \varepsilon h_1^{\nu_2} h_2^{\nu_3} \ldots h_{n-1}^{\nu_n} \rangle\!\rangle + \mathrm{perm}_n \Big) = 0$$

where perm_{n-1} means the permutations of $h_1, \ldots, h_{n-1}$, and perm_n means the permutations of $h_1, \ldots, h_{n-1}$ and $\partial_x \varepsilon$.

Now, from (3.28) we get

$$i[\varepsilon \overset{*}{,} \langle\!\langle \mathcal{W}^{(n)}_{\mu\nu_1\ldots\nu_{n-1}}, h^{\mu_1}\ldots h^{\mu_{n-1}} \rangle\!\rangle] - i\sum_{i=1}^{n-1} \langle\!\langle \mathcal{W}^{(n)}_{\mu\mu_1\ldots\mu_i\ldots\mu_{n-1}}, h^{\mu_1}\ldots[\varepsilon \overset{*}{,} h^{\mu_i}]\ldots h^{\mu_{n-1}} \rangle\!\rangle$$

$$-\frac{1}{n}\sum_{i=1}^{n} \langle\!\langle \mathcal{W}^{(n+1)}_{\mu\mu_1\ldots\mu_i\ldots\mu_n}, h^{\mu_1}\ldots\partial_x^{\mu_i}\varepsilon\ldots h^{\mu_n} \rangle\!\rangle = 0 \tag{3.54}$$

If now we transform the l.h.s. of this equation to a multilinear function of $h_1, \ldots, h_{n-1}$ according to the recipe (3.20), we obtain precisely (3.53). This completes the proof of the n-th L_∞ relation.

4 Conclusion

In this paper we have carried out the worldline quantization of a Dirac fermion field coupled to external sources. In particular, we have determined the formula for the effective action, by expanding it in a perturbative series, and determined the generalized equations of motion. This has allowed us, in the second part of the paper, to show that this set up of the theory accommodates an L_∞ algebra. We remark that this applies to the full effective action, i.e. not only to its local part, but also to its non-local part.

Although we do not give here an explicit proof, the same symmetry characterizes also the effective action obtained by integrating out a scalar field coupled to the same external sources. The proof in the scalar case is actually easier, because the corresponding $\mathcal{W}^{(n)}$'s come out automatically symmetric (for the basic formulas, see [17]).

An L_∞ symmetry is different from the familiar Lie algebra symmetry in that the equation of motion plays an essential role, in other words the symmetry is dynamical (for an early formulation in this sense, see [41]). The full implications of this (more general) symmetry are not yet clear. It characterizes a large class of *perturbative* field theories [28], but certainly not all. For instance, it is not present in the open string field theory à la Witten, where it is replaced by an A_∞ algebra. The classification of field and string theories on the basis of such homotopic-like algebra symmetries is under way. For the time being we intend to use it as a basic working tool in our attempt to generate higher spin theories by integrating out matter fields.

In what concerns us here, the L_∞ algebra symmetry is a symmetry of the equation of motion of the effective action resulting from integrating out the matter fields. The L_∞ symmetry descends from the Ward identities of the current correlators of the matter model. These Ward identities for current correlators imply the higher spin symmetry of effective action. Therefore one can say that the L_∞ symmetry is the source of the HS symmetry of the effective action. We recall again that the local part of this action is to be identified, in our approach, with the classical HS action. This has been proved so far only at the quadratic level. That it is true at the interacting levels is the bet of our program.

Another character of our paper is the worldline quantization. Let us repeat (see introduction) that it is not imperative to use the worldline formalism. As we have done in previous papers, one could use the traditional quantization and compute the effective

action by means of Feynman diagrams. The problem with this approach is that we do not have a way to fix a priori the form of the currents and the form of the symmetry transformations except by trial and error (a method that becomes rapidly unsustainable for increasing spin). The worldline quantization grants both at the same time. In this resides the importance of the worldline quantization.

The way we interpret the L_∞ relations among correlators is very similar to the usual Ward identities for an ordinary gauge symmetry (we have already pointed out above this parallelism): these relations must hold for both the classical and quantum theory, they are the relevant defining relations. Their possible breakdown is analogous to the appearances of anomalies in ordinary gauge Ward identities. It is interesting that possible obstructions to constructing higher spin theories in our scheme might be identified with such anomalies.[6]

Finally another consideration: while so far L_∞ algebras have been discussed mostly in relation to classical (first quantized, in the string field theory case) actions, as we have remarked above, our L_∞[7] structure characterizes the full effective field action (including its non-local part). This is perhaps in keeping with what was noticed in [7, 8]: the effective action for a single higher spin field, at least at the quadratic order, is characterized by a unique Fronsdal differential operator inflected in various non-local forms. In any case it is reassuring to find such a symmetry in the one-loop effective actions obtained by integrating out matter fields. Our idea of using this method to generate higher spin field theories is perhaps not groundless.

Acknowledgments

References

[1] X.O. Camanho, J.D. Edelstein, J. Maldacena and A. Zhiboedov, *Causality constraints on corrections to the graviton three-point coupling*, *JHEP* **02** (2016) 020 [arXiv:1407.5597] [INSPIRE].

[2] M.A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in* $(3+1)$-*dimensions*, *Phys. Lett.* **B 243** (1990) 378 [INSPIRE].

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[6]It is worth remarking that if such an anomaly occurs in the WI (2.52), i.e. $\delta_\varepsilon \mathcal{W}[h] = \mathcal{A}[\varepsilon, h] \neq 0$, it must satisfy a consistency condition, analogous to the WZ condition for the ordinary anomalies: $\delta_{\varepsilon_2}\mathcal{A}[\varepsilon_1, h] - \delta_{\varepsilon_1}\mathcal{A}[\varepsilon_2, h] = \mathcal{A}[[\varepsilon_1 \stackrel{*}{,} \varepsilon_2], h]$, as a consequence of (2.29).

[7]In our case we should perhaps call it L_∞^*, due to the essential role played in it by the Moyal product.

[3] M.A. Vasiliev, *Properties of equations of motion of interacting gauge fields of all spins in* $(3 + 1)$-*dimensions*, *Class. Quant. Grav.* **8** (1991) 1387 [INSPIRE].

[4] M.A. Vasiliev, *Algebraic aspects of the higher spin problem*, *Phys. Lett.* **B 257** (1991) 111 [INSPIRE].

[5] M.A. Vasiliev, *More on equations of motion for interacting massless fields of all spins in* $(3 + 1)$-*dimensions*, *Phys. Lett.* **B 285** (1992) 225 [INSPIRE].

[6] L. Bonora, M. Cvitan, P. Dominis Prester, B. Lima de Souza and I. Smolić, *Massive fermion model in* 3d *and higher spin currents*, *JHEP* **05** (2016) 072 [arXiv:1602.07178] [INSPIRE].

[7] L. Bonora, M. Cvitan, P. Dominis Prester, S. Giaccari, B. Lima de Souza and T. Štemberga, *One-loop effective actions and higher spins*, *JHEP* **12** (2016) 084 [arXiv:1609.02088] [INSPIRE].

[8] L. Bonora, M. Cvitan, P. Dominis Prester, S. Giaccari and T. Štemberga, *One-loop effective actions and higher spins. Part II*, *JHEP* **01** (2018) 080 [arXiv:1709.01738] [INSPIRE].

[9] A.D. Sakharov, *Vacuum quantum fluctuations in curved space and the theory of gravitation*, *Sov. Phys. Dokl.* **12** (1968) 1040 [*Dokl. Akad. Nauk Ser. Fiz.* **177** (1967) 70] [*Sov. Phys. Usp.* **34** (1991) 394] [*Gen. Rel. Grav.* **32** (2000) 365] [INSPIRE].

[10] C. Fronsdal, *Massless fields with integer spin*, *Phys. Rev.* **D 18** (1978) 3624 [INSPIRE].

[11] J. Fang and C. Fronsdal, *Massless fields with half integral spin*, *Phys. Rev.* **D 18** (1978) 3630 [INSPIRE].

[12] D. Francia and A. Sagnotti, *On the geometry of higher spin gauge fields*, *Class. Quant. Grav.* **20** (2003) S473 [*Comment. Phys. Math. Soc. Sci. Fenn.* **166** (2004) 165] [PoS(JHW2003)005] [hep-th/0212185] [INSPIRE].

[13] D. Francia and A. Sagnotti, *Free geometric equations for higher spins*, *Phys. Lett.* **B 543** (2002) 303 [hep-th/0207002] [INSPIRE].

[14] M.J. Strassler, *Field theory without Feynman diagrams: one loop effective actions*, *Nucl. Phys.* **B 385** (1992) 145 [hep-ph/9205205] [INSPIRE].

[15] A.Y. Segal, *Conformal higher spin theory*, *Nucl. Phys.* **B 664** (2003) 59 [hep-th/0207212] [INSPIRE].

[16] A. Yu. Segal, *Point particle in general background fields versus gauge theories of traceless symmetric tensors*, *Int. J. Mod. Phys.* **A 18** (2003) 4999 [hep-th/0110056] [INSPIRE].

[17] X. Bekaert, E. Joung and J. Mourad, *Effective action in a higher-spin background*, *JHEP* **02** (2011) 048 [arXiv:1012.2103] [INSPIRE].

[18] M.G. Schmidt and C. Schubert, *The worldline path integral approach to Feynman graphs*, in 28th *International Symposium on Particle Theory*, Wendisch-Rietz Germany, 30 August–3 September 1994, pg. 0240 [hep-ph/9412358] [INSPIRE].

[19] M.G. Schmidt and C. Schubert, *Worldline Green functions for multiloop diagrams*, *Phys. Lett.* **B 331** (1994) 69 [hep-th/9403158] [INSPIRE].

[20] C. Schubert, *An introduction to the worldline technique for quantum field theory calculations*, *Acta Phys. Polon.* **B 27** (1996) 3965 [hep-th/9610108] [INSPIRE].

[21] E. D'Hoker and D.G. Gagne, *Worldline path integrals for fermions with general couplings*, *Nucl. Phys.* **B 467** (1996) 297 [hep-th/9512080] [INSPIRE].

[22] F. Bastianelli and A. Zirotti, *Worldline formalism in a gravitational background*, *Nucl. Phys.* **B 642** (2002) 372 [hep-th/0205182] [INSPIRE].

[23] F. Bastianelli, O. Corradini and A. Zirotti, *BRST treatment of zero modes for the worldline formalism in curved space*, *JHEP* **01** (2004) 023 [hep-th/0312064] [INSPIRE].

[24] F. Bastianelli, P. Benincasa and S. Giombi, *Worldline approach to vector and antisymmetric tensor fields*, *JHEP* **04** (2005) 010 [hep-th/0503155] [INSPIRE].

[25] F. Bastianelli, P. Benincasa and S. Giombi, *Worldline approach to vector and antisymmetric tensor fields*, *JHEP* **04** (2005) 010 [hep-th/0503155] [INSPIRE].

[26] P. Dai and W. Siegel, *Worldline Green functions for arbitrary Feynman diagrams*, *Nucl. Phys.* **B 770** (2007) 107 [hep-th/0608062] [INSPIRE].

[27] R. Bonezzi, *Induced action for conformal higher spins from worldline path integrals*, *Universe* **3** (2017) 64 [arXiv:1709.00850] [INSPIRE].

[28] O. Hohm and B. Zwiebach, L_∞ *algebras and field theory*, *Fortsch. Phys.* **65** (2017) 1700014 [arXiv:1701.08824] [INSPIRE].

[29] M.R. Gaberdiel and B. Zwiebach, *Tensor constructions of open string theories. 1: foundations*, *Nucl. Phys.* **B 505** (1997) 569 [hep-th/9705038] [INSPIRE].

[30] B. Zwiebach, *Oriented open-closed string theory revisited*, *Annals Phys.* **267** (1998) 193 [hep-th/9705241] [INSPIRE].

[31] H. Kajiura and J. Stasheff, *Homotopy algebras inspired by classical open-closed string field theory*, *Commun. Math. Phys.* **263** (2006) 553 [math/0410291] [INSPIRE].

[32] T. Lada and J. Stasheff, *Introduction to SH Lie algebras for physicists*, *Int. J. Theor. Phys.* **32** (1993) 1087 [hep-th/9209099] [INSPIRE].

[33] T. Lada and M. Markl, *Strongly homotopy Lie algebras*, hep-th/9406095 [INSPIRE].

[34] G. Barnich, R. Fulp, T. Lada and J. Stasheff, *The sh Lie structure of Poisson brackets in field theory*, *Commun. Math. Phys.* **191** (1998) 585 [hep-th/9702176] [INSPIRE].

[35] A.M. Zeitlin, *Homotopy Lie superalgebra in Yang-Mills theory*, *JHEP* **09** (2007) 068 [arXiv:0708.1773] [INSPIRE].

[36] A.M. Zeitlin, *String field theory-inspired algebraic structures in gauge theories*, *J. Math. Phys.* **50** (2009) 063501 [arXiv:0711.3843] [INSPIRE].

[37] A.M. Zeitlin, *Conformal field theory and algebraic structure of gauge theory*, *JHEP* **03** (2010) 056 [arXiv:0812.1840] [INSPIRE].

[38] D. Gaiotto, G.W. Moore and E. Witten, *Algebra of the infrared: string field theoretic structures in massive* $N = (2,2)$ *field theory in two dimensions*, arXiv:1506.04087 [INSPIRE].

[39] R. Blumenhagen, M. Fuchs and M. Traube, W *algebras are* L_∞ *algebras*, *JHEP* **07** (2017) 060 [arXiv:1705.00736] [INSPIRE].

[40] R. Blumenhagen, M. Fuchs and M. Traube, *On the structure of quantum* L_∞ *algebras*, *JHEP* **10** (2017) 163 [arXiv:1706.09034] [INSPIRE].

[41] F.A. Berends, G.J.H. Burgers and H. van Dam, *On the theoretical problems in constructing interactions involving higher spin massless particles*, *Nucl. Phys.* **B 260** (1985) 295 [INSPIRE].