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*Research Article*

# The Distance to $L^\infty$ in the Grand Orlicz Spaces

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We establish a formula for the distance to $L^\infty$ from the grand Orlicz space $L^{\Phi)}(\Omega)$ introduced in Capone et al. (2008). A new formula for the distance to $L^\infty$ from the grand Lebesgue space $L^{n)}(\Omega)$ introduced in Iwaniec and Sbordone (1992) is also provided.

## 1. Introduction

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and let $1 < p < \infty$. The *grand* $L^p$*-space*, denoted by $L^{p)} = L^{p)}(\Omega)$, consists of functions $h \in \bigcap_{0<\varepsilon\leq p-1} L^{p-\varepsilon}(\Omega)$ such that

$$\|h\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \fint_\Omega |h(x)|^{p-\varepsilon} dx\right)^{1/(p-\varepsilon)} < \infty, \qquad (1)$$

where $\fint_\Omega = (1/|\Omega|)\int_\Omega$ denotes the average over $\Omega$. Note that $\|\cdot\|_{p)}$ is a norm and $L^{p)}(\Omega)$ is a Banach space. This space was introduced by Iwaniec and Sbordone in connection with the integrability of the Jacobian [1], and it comes into play in a various number of problems (see, e.g., [2–15]).

It is worth pointing out that $L^\infty(\Omega)$ is not a dense subspace of $L^{p)}(\Omega)$ (see [9]); it is proved in [16] that the distance to $L^\infty$ in $L^{p)}$ is given by

$$\operatorname{dist}_{L^{p)}}(h, L^\infty) = \limsup_{\varepsilon\to 0^+} \left(\varepsilon \fint_\Omega |h(x)|^{p-\varepsilon} dx\right)^{1/(p-\varepsilon)}. \qquad (2)$$

A generalization of the grand Lebesgue space is the *grand Orlicz space* $L^{\Phi)}(\Omega)$, introduced by Capone et al. in [17]. Let us recall that $\Phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is continuous, strictly increasing, and satisfies $\Phi(0) = 0$ and $\lim_{t\to\infty}\Phi(t) = \infty$. The *Orlicz space* $L^\Phi(\Omega)$ associated with $\Phi$ consists of all measurable functions $u : \Omega \to \mathbb{R}$ for which there exists $\lambda > 0$ such that

$$\int_\Omega \Phi\left(\frac{|u(x)|}{\lambda}\right) dx < \infty. \qquad (3)$$

Let us introduce the Luxemburg functional defined as

$$\|u\|_{L^\Phi(\Omega)} = \inf\left\{\lambda > 0 : \fint_\Omega \Phi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1\right\}. \qquad (4)$$

Because of the monotonicity of $\Phi$ we have

$$\Phi((1-\alpha)s + \alpha t) \leq [\Phi(s) + \Phi(t)] \quad \text{for every } s,t \in (0,\infty),\ \alpha \in (0,1), \qquad (5)$$

and among Orlicz functions we will consider the ones satisfying the following condition:

$$\Phi(\alpha t) \leq C(\alpha)\Phi(t) \quad \text{for every } t \in (0,\infty),\ \alpha \in (0,1), \qquad (6)$$

for some constant $C(\alpha)$ such that $C(\alpha) \to 0$ as $\alpha \to 0$. This will be done in order to ensure that the functional in (4) is a quasinorm. In what follows, we will lose no generality in assuming that

$$\Phi(1) = 1. \qquad (7)$$

Suppose that

$$\int_1^\infty \frac{\Phi(t)}{t^{n+1}}\,dt = \infty, \tag{8}$$

and let $N_\Phi : [0,1] \to [0,\infty)$ be the increasing weight defined as

$$N_\Phi(\sigma) = \frac{1}{\int_1^\infty \left([\Phi(t)]^{1/(1+\sigma)}/t^{n+1}\right)dt}. \tag{9}$$

Following a definition given in [18], we suppose that $N_\Phi$ is *tempered*; that is,

$$\begin{gathered} c_1 N_\Phi(\sigma) \le N_\Phi(2\sigma) \le c_2 N_\Phi(\sigma) \\ \text{for every } \sigma \in (0,\sigma_0),\ \sigma_0 \in \left(0,\frac{1}{2}\right) \end{gathered} \tag{10}$$

for some $c_1, c_2 > 0$.

An example of function $\Phi$ satisfying (6)–(10) is $\Phi(t) = t^n(1+\log(1+t))^{-\alpha}$ for $0 \le \alpha \le 1$, and in this case $N_\Phi(\sigma) \approx \sigma^{1-\alpha}$ as $\sigma \to 0^+$ when $0 \le \alpha < 1$ and $N_\Phi(\sigma) \approx |\log\sigma|^{-1}$ as $\sigma \to 0^+$ when $\alpha = 1$ (see Section 5 for details).

The *grand Orlicz space* $L^{\Phi)}(\Omega)$ consists of all measurable functions $u : \Omega \to \mathbb{R}$ for which there exists $\lambda > 0$ such that

$$\sup_{0<\sigma<\sigma_0} N_\Phi(\sigma) \fint_0^{|\Omega|} t^\sigma \Phi\left(\frac{u^*(t)}{\lambda}\right)dt < \infty, \tag{11}$$

where $u^*$ is the decreasing rearrangement of $u$

$$u^*(t) = \inf\{h \ge 0 : \mu_u(h) \le t \text{ for every } t \in [0,|\Omega|]\} \tag{12}$$

and $\mu_u$ is the distribution function of $u$

$$\mu_u(h) = |\{x \in \Omega : |u(x)| > h\}| \quad \text{for every } h \ge 0. \tag{13}$$

The quasinorm denoted by $\|\cdot\|_{L^{\Phi)}(\Omega)}$ is defined as follows:

$$\|u\|_{L^{\Phi)}(\Omega)} = \inf\left\{\lambda > 0 : \sup_{0<\sigma<\sigma_0} N_\Phi(\sigma) \fint_0^{|\Omega|} t^\sigma \Phi\left(\frac{u^*(t)}{\lambda}\right)dt \le 1\right\}. \tag{14}$$

We address that if we take $\Phi(t) = t^n$ also the grand Orlicz space $L^{\Phi)}(\Omega)$ reduces to the grand Lebesgue space $L^{n)}(\Omega)$ (see [17, Proposition 2.6], [6]).

Our main result provides a formula for the distance of a function $u \in L^{\Phi)}(\Omega)$ to $L^\infty(\Omega)$, defined by

$$\operatorname{dist}_{L^{\Phi)}(\Omega)}(u, L^\infty(\Omega)) = \inf_{\varphi \in L^\infty(\Omega)} \|u-\varphi\|_{L^{\Phi)}(\Omega)}. \tag{15}$$

**Theorem 1.** *Let $\Omega$ be a bounded open set of $\mathbb{R}^n$. Assume that $\Phi : [0,\infty) \to [0,\infty)$ is an Orlicz function verifying (6)–(10). For every function $u \in L^{\Phi)}(\Omega)$, one has*

$$\begin{aligned} &\operatorname{dist}_{L^{\Phi)}(\Omega)}(u, L^\infty(\Omega)) \\ &\quad = \inf\left\{\lambda > 0 : \limsup_{\sigma\to 0^+} N_\Phi(\sigma) \fint_0^{|\Omega|} t^\sigma \Phi\left(\frac{u^*(t)}{\lambda}\right)dt \le 1\right\}. \end{aligned} \tag{16}$$

Our theorem is in the framework of the results of paper [19], which cannot be directly applied to our context, without a preliminary check that the grand Orlicz spaces $L^{\Phi)}$ can be characterized as interpolation or extrapolation spaces. We also refer to [5, 16, 20–24] for the problem of finding formulae for the distance to a subspace in a given function space.

Theorem 1 gives, as byproduct, a characterization of the closure of $L^\infty(\Omega)$ in $L^{\Phi)}(\Omega)$ with respect to the norm $\|\cdot\|_{L^{\Phi)}(\Omega)}$, which will be denoted by $L_b^{\Phi)}(\Omega)$.

**Theorem 2.** *A function $u$ belongs to $L_b^{\Phi)}(\Omega)$ if and only if*

$$\lim_{\sigma\to 0^+} N_\Phi(\sigma) \fint_0^{|\Omega|} t^\sigma \Phi(\beta u^*(t))\,dt = 0 \quad \forall \beta > 0. \tag{17}$$

For the special choice $\Phi(t) = t^n$, Theorem 1 also provides new formula for the distance to $L^\infty$ in $L^{n)}$ (see Theorem 5).

## 2. The Main Result

We start this section recalling few basic properties of the decreasing rearrangement $u^*$ of a measurable function $u : \Omega \to \mathbb{R}$ defined in a bounded open set $\Omega$ of $\mathbb{R}^n$. We refer the reader to [25, Propositions 1.7 and 1.8] for details.

**Lemma 3.** *Let $u, v : \Omega \to \mathbb{R}$ be measurable functions defined in a bounded open set $\Omega$ of $\mathbb{R}^n$:*

$$\begin{gathered} (u+v)^*(t_1+t_2) \le u^*(t_1) + v^*(t_2) \\ \text{for every } t_1, t_2 \ge 0 \text{ with } t_1+t_2 \le |\Omega|, \end{gathered} \tag{18}$$

$$(cv)^* = cv^* \quad \text{for every } c > 0, \tag{19}$$

$$v^*(0) = \|v\|_{L^\infty(\Omega)} \quad \text{if } v \in L^\infty(\Omega). \tag{20}$$

We need a technical result providing a useful property of the quantity

$$(u)_{L^{\Phi)}(\Omega)} = \inf\left\{\lambda > 0 : \limsup_{\sigma\to 0^+} N_\Phi(\sigma) \fint_0^{|\Omega|} t^\sigma \Phi\left(\frac{u^*(t)}{\lambda}\right)dt \le 1\right\}. \tag{21}$$

We recall that the goal of Theorem 1 consists in proving that $(u)_{L^{\Phi)}(\Omega)}$ is equal to $\mathrm{dist}_{L^{\Phi)}(\Omega)}(u, L^\infty(\Omega))$. We notice that if $\varphi \in L^\infty(\Omega)$, then $(\varphi)_{L^{\Phi)}(\Omega)} = 0$, because from (8) and (10) it is $N_\Phi(\sigma) \to 0$ as $\sigma \to 0^+$ and the average remains bounded for every $\sigma > 0$.

**Lemma 4.** *Let $\Phi : [0,\infty) \to [0,\infty)$ be an Orlicz function satisfying the assumptions of Theorem 1. Assume that $u \in L^{\Phi)}(\Omega)$ and $\varphi \in L^\infty(\Omega)$. Then*

$$(u)_{L^{\Phi)}(\Omega)} = (u-\varphi)_{L^{\Phi)}(\Omega)}. \tag{22}$$

*Proof.* Let $\lambda > (u)_{L^{\Phi)}(\Omega)}$ and let $\varepsilon \in (0,\lambda)$ so that

$$\lambda > \lambda - \varepsilon > (u)_{L^{\Phi)}(\Omega)}. \tag{23}$$

We use (18) with $t_1 = t \in [0,|\Omega|]$ and $t_2 = 0$ and (20), and we get

$$(u-\varphi)^*(t) \le u^*(t) + \|\varphi\|_{L^\infty(\Omega)}. \tag{24}$$

We use (5) to get

$$\begin{aligned} \Phi\left(\frac{(u-\varphi)^*(t)}{\lambda}\right) &\le \Phi\left(\frac{u^*(t)}{\lambda} + \frac{\|\varphi\|_{L^\infty(\Omega)}}{\lambda}\right) \\ &= \Phi\left(\frac{\lambda-\varepsilon}{\lambda}\frac{u^*(t)}{\lambda-\varepsilon} + \frac{\varepsilon}{\lambda}\frac{\|\varphi\|_{L^\infty(\Omega)}}{\varepsilon}\right) \\ &\le \Phi\left(\frac{u^*(t)}{\lambda-\varepsilon}\right) + \Phi\left(\frac{\|\varphi\|_{L^\infty(\Omega)}}{\varepsilon}\right). \end{aligned} \tag{25}$$

We multiply by $t^\sigma$ and we integrate over $[0,|\Omega|]$ to get

$$\begin{aligned} ⨍_0^{|\Omega|} t^\sigma \Phi\left(\frac{(u-\varphi)^*(t)}{\lambda}\right) dt &\le ⨍_0^{|\Omega|} t^\sigma \Phi\left(\frac{u^*(t)}{\lambda-\varepsilon}\right) dt \\ &\quad + \Phi\left(\frac{\|\varphi\|_{L^\infty(\Omega)}}{\varepsilon}\right) ⨍_0^{|\Omega|} t^\sigma dt \\ &= ⨍_0^{|\Omega|} t^\sigma \Phi\left(\frac{u^*(t)}{\lambda-\varepsilon}\right) dt \\ &\quad + \Phi\left(\frac{\|\varphi\|_{L^\infty(\Omega)}}{\varepsilon}\right) \frac{|\Omega|^\sigma}{1+\sigma}. \end{aligned} \tag{26}$$

We multiply by $N_\Phi(\sigma)$, and since $N_\Phi(\sigma) \to 0$ as $\sigma \to 0^+$, we have

$$\begin{aligned} &\limsup_{\sigma\to 0^+} N_\Phi(\sigma) ⨍_0^{|\Omega|} t^\sigma \Phi\left(\frac{(u-\varphi)^*(t)}{\lambda}\right) dt \\ &\quad \le \limsup_{\sigma\to 0^+} N_\Phi(\sigma) ⨍_0^{|\Omega|} t^\sigma \Phi\left(\frac{u^*(t)}{\lambda-\varepsilon}\right) dt. \end{aligned} \tag{27}$$

From (23) we get

$$\limsup_{\sigma\to 0^+} N_\Phi(\sigma) \Phi ⨍_0^{|\Omega|} t^\sigma \left(\frac{(u-\varphi)^*(t)}{\lambda}\right) dt \le 1. \tag{28}$$

We apply the definition of $(u-\varphi)_{L^{\Omega)}(\Phi)}$, and we have

$$(u-\varphi)_{L^{\Phi)}(\Omega)} \le \lambda, \tag{29}$$

and then, passing to the limit as $\lambda \to (u)_{L^{\Phi)}(\Omega)}$, we have

$$(u-\varphi)_{L^{\Phi)}(\Omega)} \le (u)_{L^{\Phi)}(\Omega)}. \tag{30}$$

By replacing $u$ with $u-\varphi$ and $\varphi$ with $-\varphi$ in (30), we obtain the converse inequality

$$(u)_{L^{\Phi)}(\Omega)} \le (u-\varphi)_{L^{\Phi)}(\Omega)}. \tag{31}$$

Equality (22) is finally proved. □

Now, we are in a position to prove Theorem 1.

*Proof of Theorem 1.* From Lemma 4 we know that

$$(u)_{L^{\Phi)}(\Omega)} = (u-\varphi)_{L^{\Phi)}(\Omega)}, \tag{32}$$

for every $\varphi \in L^\infty(\Omega)$. This clearly proves that

$$\mathrm{dist}_{L^{\Phi)}(\Omega)}(u, L^\infty(\Omega)) \ge (u)_{L^{\Phi)}(\Omega)} \tag{33}$$

since $(u)_{L^{\Phi)}(\Omega)} = (u-\varphi)_{L^{\Phi)}(\Omega)} \le \|u-\varphi\|_{L^{\Phi)}(\Omega)}$ for every $\varphi \in L^\infty(\Omega)$.

Now, we want to show that

$$\mathrm{dist}_{L^{\Phi)}(\Omega)}(u, L^\infty(\Omega)) \le (u)_{L^{\Phi)}(\Omega)}. \tag{34}$$

In order to achieve the claimed inequality, we prove that if

$$\lambda_0 > (u)_{L^{\Phi)}(\Omega)}, \tag{35}$$

then

$$\mathrm{dist}_{L^{\Phi)}(\Omega)}(u, L^\infty(\Omega)) < \lambda_0. \tag{36}$$

Without loss of generality we may assume that $u \notin L^\infty(\Omega)$. From (35) we find $\lambda \in (0,\lambda_0)$ such that

$$\limsup_{\sigma\to 0^+} N_\Phi(\sigma) ⨍_0^{|\Omega|} t^\sigma \Phi\left(\frac{u^*(t)}{\lambda}\right) dt \le 1. \tag{37}$$

For each $\varepsilon > 0$ there exists $\sigma_\varepsilon \in (0,\sigma_0)$ such that

$$N_\Phi(\sigma) ⨍_0^{|\Omega|} t^\sigma \Phi\left(\frac{u^*(t)}{\lambda}\right) dt \le 1+\varepsilon \quad \text{for every } \sigma \in (0,\sigma_\varepsilon]. \tag{38}$$

Let $h^* > 0$ be such that

$$\mu_u(h^*) < 1, \tag{39}$$

and let $\sigma \in (\sigma_\varepsilon, \sigma_0)$. From (38), we find some constant $h_\lambda$ (depending on $\lambda$), with $h_\lambda > h^*$, such that

$$\frac{N_\Phi(1)}{|\Omega|} \int_0^{\mu_u(h_\lambda)} t^{\sigma_\varepsilon} \Phi\left(\frac{u^*(t)}{\lambda}\right) dt \le 1. \tag{40}$$

Using the monotonicity of weight $N_\Phi$, the fact that $h_\lambda > h^*$, and (39), we deduce from (40) that

$$\frac{N_\Phi(\sigma)}{|\Omega|}\int_0^{\mu_u(h_\lambda)} t^\sigma \Phi\left(\frac{u^*(t)}{\lambda}\right)dt \le 1 \quad \text{for every } \sigma \in (\sigma_\varepsilon, \sigma_0). \tag{41}$$

We set $u_\lambda(x) = u(x)$ if $|u(x)| \le h_\lambda$ and, $u_\lambda(x) = 0$ if $|u(x)| > h_\lambda$, and we show that

$$\begin{aligned}(u-u_\lambda)^*(t) &= u^*(t) \quad \text{for } t \in [0, \mu(h_\lambda)),\\ (u-u_\lambda)^*(t) &= 0 \quad \text{for } t \in [\mu(h_\lambda), |\Omega|].\end{aligned} \tag{42}$$

Let us observe that

$$\mu_{u-u_\lambda}(h) = \mu_u(h_\lambda) \quad \text{if } 0 < h \le h_\lambda, \tag{43}$$

while

$$\mu_{u-u_\lambda}(h) = \mu_u(h) \quad \text{if } h \ge h_\lambda. \tag{44}$$

Using the fact that the distribution function is decreasing, we easily see that

$$\mu_{u-u_\lambda}(h) \le \mu_u(h_\lambda) \quad \text{for every } h \ge 0. \tag{45}$$

Therefore, if we let $t \in [\mu(h_\lambda), |\Omega|]$, we see that condition

$$\mu_{u-u_\lambda}(h) \le t \tag{46}$$

is verified for all $h \ge 0$. Thus $(u-u_\lambda)^*(t) = 0$ for $t \in [\mu(h_\lambda), |\Omega|]$. On the other hand, if we let $t \in [0, \mu(h_\lambda))$, we see that condition (46) is the same as requiring

$$\mu_u(h) \le t. \tag{47}$$

Thus $(u-u_\lambda)^*(t) = u^*(t)$ holds if $t \in [0, \mu(h_\lambda))$, and (42) is proved.

It follows directly from (42) that

$$\begin{aligned}N_\Phi(\sigma)\fint_0^{|\Omega|} t^\sigma \Phi\left(\frac{(u-u_\lambda)^*(t)}{\lambda}\right)dt\\ = \frac{N_\Phi(\sigma)}{|\Omega|}\int_0^{\mu_u(h_\lambda)} t^\sigma \Phi\left(\frac{u^*(t)}{\lambda}\right)dt,\end{aligned} \tag{48}$$

for every $\sigma \in (0, \sigma_0)$. Hence, we make use of (38) if $\sigma \in (0, \sigma_\varepsilon]$ and of (41) if $\sigma \in (\sigma_\varepsilon, \sigma_0)$ to conclude that

$$N_\Phi(\sigma)\fint_0^{|\Omega|} t^\sigma \Phi\left(\frac{(u-u_\lambda)^*(t)}{\lambda}\right)dt \le 1+\varepsilon \quad \text{for every } \sigma \in (0, \sigma_0). \tag{49}$$

In particular,

$$\sup_{0<\sigma<\sigma_0} N_\Phi(\sigma)\fint_0^{|\Omega|} t^\sigma \Phi\left(\frac{(u-u_\lambda)^*(t)}{\lambda}\right)dt \le 1+\varepsilon. \tag{50}$$

Since (50) holds for every $\varepsilon > 0$, we obtain that its left-hand side is smaller than 1, and therefore $\|u-u_\lambda\|_{L^{\Phi)}(\Omega)} \le \lambda$. We get

$$\operatorname{dist}_{L^{\Phi)}(\Omega)}(u, L^\infty(\Omega)) \le \|u-u_\lambda\|_{L^{\Phi)}(\Omega)} \le \lambda < \lambda_0. \tag{51}$$

Hence (36) is established. Since $\lambda_0$ is any arbitrary number for which (35) holds, we may pass to the limit as $\lambda_0$ approaches $(u)_{L^{\Phi)}(\Omega)}$ in (36) to get

$$\operatorname{dist}_{L^{\Phi)}(\Omega)}(u, L^\infty(\Omega)) \le (u)_{L^{\Phi)}(\Omega)}. \tag{52}$$

Combining (52) with (33) we obtain (16) as desired. □

*Proof of Theorem 2.* As a consequence of Theorem 1, it is clear that $u \in L_b^{\Phi)}(\Omega)$ if and only if

$$\limsup_{\sigma\to 0^+} N_\Phi(\sigma)\fint_0^{|\Omega|} t^\sigma \Phi\left(\frac{u^*(t)}{\lambda}\right)dt \le 1 \quad \forall\lambda > 0. \tag{53}$$

We fix an arbitrary $\alpha \in (0,1)$ and we set $\lambda = \alpha/\beta$. Using (6) we have

$$\begin{aligned}&N_\Phi(\sigma)\fint_0^{|\Omega|} t^\sigma \Phi(\beta u^*(t))\,dt\\ &\quad= N_\Phi(\sigma)\fint_0^{|\Omega|} t^\sigma \Phi\left(\alpha\frac{u^*(t)}{\lambda}\right)dt\\ &\quad\le C(\alpha)\,N_\Phi(\sigma)\fint_0^{|\Omega|} t^\sigma \Phi\left(\frac{u^*(t)}{\lambda}\right)dt.\end{aligned} \tag{54}$$

Hence, using (53) we have

$$\limsup_{\sigma\to 0^+} N_\Phi(\sigma)\fint_0^{|\Omega|} t^\sigma \Phi(\beta u^*(t))\,dt \le C(\alpha), \tag{55}$$

and (17) follows since $C(\alpha) \to 0$ as $\alpha \to 0$. □

## 3. The Case of the Grand Lebesgue Space $L^{n)}$

We denote by $(u)_{L^{n)}(\Omega)}$ the functional $(u)_{L^{\Phi)}(\Omega)}$ as in (21) when $\Phi(t) = t^n$. In this case, $(u)_{L^{n)}(\Omega)}$ takes the form

$$(u)_{L^{n)}(\Omega)} = \limsup_{\sigma\to 0^+}\left(\frac{n\sigma}{1+\sigma}\fint_0^{|\Omega|} t^\sigma |u^*(t)|^n dt\right)^{1/n}. \tag{56}$$

Our next result proves that the distance given by formula (2) reduces to $(u)_{L^{n)}(\Omega)}$.

**Theorem 5.** *Let $\Omega$ be a bounded open set of $\mathbb{R}^n$. For every function $u \in L^{n)}(\Omega)$, one has*

$$\begin{aligned}&\operatorname{dist}_{L^{\Phi)}(\Omega)}(u, L^\infty(\Omega))\\ &\quad= \limsup_{\sigma\to 0^+}\left(\frac{n\sigma}{1+\sigma}\fint_0^{|\Omega|} t^\sigma |u^*(t)|^n\, dt\right)^{1/n}.\end{aligned} \tag{57}$$

*Proof.* First we prove that

$$\limsup_{\varepsilon\to 0^+}\left(\varepsilon\fint_\Omega |u(x)|^{n-\varepsilon}dx\right)^{1/(n-\varepsilon)} \le \limsup_{\sigma\to 0^+}\left(\frac{n\sigma}{1+\sigma}\fint_0^{|\Omega|} t^\sigma\left|u^*(t)\right|^n dt\right)^{1/n}. \tag{58}$$

To this aim, we consider $\varepsilon,\sigma>0$ and $k>1$ such that

$$n-\varepsilon=\frac{n}{1+k\sigma}. \tag{59}$$

Using Hölder's inequality we have

$$\begin{aligned}\varepsilon\fint_\Omega |u(x)|^{n-\varepsilon}dx &= \frac{nk\sigma}{1+k\sigma}\fint_0^{|\Omega|} t^{\sigma/(1+k\sigma)}\left|u^*(t)\right|^{n/(1+k\sigma)}t^{-\sigma/(1+k\sigma)}dt\\ &\le \frac{nk\sigma}{1+k\sigma}\left(\fint_0^{|\Omega|} t^\sigma\left|u^*(t)\right|^n dt\right)^{1/(1+k\sigma)}\left(\fint_0^{|\Omega|} t^{-1/k}dt\right)^{k\sigma/(1+k\sigma)}\end{aligned} \tag{60}$$

which in turn implies that

$$\begin{aligned}\left(\varepsilon\fint_\Omega |u(x)|^{n-\varepsilon}dx\right)^{1/(n-\varepsilon)} &\le \left(\frac{k}{(k-1)\,|\Omega|^{1/k}}\right)^{k\sigma/n}\left(\frac{nk\sigma}{1+k\sigma}\right)^{(1+k\sigma)/n}\times\left(\fint_0^{|\Omega|} t^\sigma\left|u^*(t)\right|^n dt\right)^{1/n}\\ &= \left(\frac{k}{(k-1)\,|\Omega|^{1/k}}\right)^{k\sigma/n}\times\left(\frac{nk\sigma}{1+k\sigma}\right)^{(1+k\sigma)/n}\left(\frac{1+\sigma}{n\sigma}\right)^{1/n}\times\left(\frac{n\sigma}{1+\sigma}\fint_0^{|\Omega|} t^\sigma\left|u^*(t)\right|^n dt\right)^{1/n}\\ &= \left(\frac{k}{(k-1)\,|\Omega|^{1/k}}\right)^{k\sigma/n}\times\left(\frac{k}{1+k\sigma}\right)^{(1+k\sigma)/n}(1+\sigma)^{1/n}(n\sigma)^{k\sigma/n}\times\left(\frac{n\sigma}{1+\sigma}\fint_0^{|\Omega|} t^\sigma\left|u^*(t)\right|^n dt\right)^{1/n}.\end{aligned} \tag{61}$$

Since $(n\sigma)^{k\sigma/n}\to 1$ as $\sigma\to 0^+$, we deduce from (61) that

$$\limsup_{\varepsilon\to 0^+}\left(\varepsilon\fint_\Omega |u(x)|^{n-\varepsilon}\,dx\right)^{1/(n-\varepsilon)} \le k^{1/n}\limsup_{\sigma\to 0^+}\left(\frac{n\sigma}{1+\sigma}\fint_0^{|\Omega|} t^\sigma\left|u^*(t)\right|^n dt\right)^{1/n}. \tag{62}$$

Since $k$ is any number strictly greater than 1, (62) immediately implies (58).

We wish to prove the converse inequality

$$\limsup_{\varepsilon\to 0^+}\left(\varepsilon\fint_\Omega |u(x)|^{n-\varepsilon}dx\right)^{1/(n-\varepsilon)} \ge \limsup_{\sigma\to 0^+}\left(\frac{n\sigma}{1+\sigma}\fint_0^{|\Omega|} t^\sigma\left|u^*(t)\right|^n dt\right)^{1/n}. \tag{63}$$

For each $t\in[0,|\Omega|]$, we have

$$\begin{aligned}\fint_\Omega |u(x)|^{n/(1+\sigma)}dx &= \fint_0^{|\Omega|}\left|u^*(s)\right|^{n/(1+\sigma)}ds\\ &\ge \frac{1}{|\Omega|}\int_0^t\left|u^*(s)\right|^{n/(1+\sigma)}ds\\ &\ge \frac{1}{|\Omega|}t\left|u^*(t)\right|^{n/(1+\sigma)}.\end{aligned} \tag{64}$$

Thus

$$\begin{aligned}\fint_0^{|\Omega|} t^\sigma\left|u^*(t)\right|^n dt &= \fint_0^{|\Omega|} t^\sigma\left|u^*(t)\right|^{n\sigma/(1+\sigma)}\left|u^*(t)\right|^{n/(1+\sigma)}dt\\ &\le |\Omega|^\sigma\left(\fint_0^{|\Omega|}\left|u^*(t)\right|^{n/(1+\sigma)}dt\right)^{1+\sigma}.\end{aligned} \tag{65}$$

We consider $\varepsilon,\sigma>0$ such that

$$n-\varepsilon=\frac{n}{1+\sigma}. \tag{66}$$

Then

$$\begin{aligned}\left(\frac{n\sigma}{1+\sigma}\fint_0^{|\Omega|} t^\sigma\left|u^*(t)\right|^n dt\right)^{1/n} &\le \left[\left(\frac{n\sigma}{1+\sigma}\right)^{-\sigma}|\Omega|^\sigma\left(\frac{n\sigma}{1+\sigma}\fint_0^{|\Omega|}\left|u^*(t)\right|^{n/(1+\sigma)}dt\right)^{1+\sigma}\right]^{1/n}\\ &= \left(\frac{n\sigma}{1+\sigma}\right)^{-\sigma/n}|\Omega|^{\sigma/n}\left(\frac{n\sigma}{1+\sigma}\fint_0^{|\Omega|}\left|u^*(t)\right|^{n/(1+\sigma)}dt\right)^{(1+\sigma)/n}\\ &= \left(\frac{n\sigma}{1+\sigma}\right)^{-\sigma/n}|\Omega|^{\sigma/n}\left(\varepsilon\fint_\Omega |u(x)|^{n-\varepsilon}dx\right)^{1/(n-\varepsilon)},\end{aligned} \tag{67}$$

which proves (63). □

## 4. Few Properties of the Distance

In this concluding section we provide certain properties of the functional $(\cdot)_{L^{\Phi)}(\Omega)}$.

**Lemma 6.** *Let $\Phi:[0,\infty)\to[0,\infty)$ be an Orlicz function satisfying the assumptions of Theorem 1 and let $v\in L^{\Phi)}(\Omega)$. Assume that*

$$\limsup_{\sigma\to 0^+} N_\Phi(\sigma)\fint_0^{|\Omega|} t^\sigma\Phi\left(\frac{v^*(t)}{\lambda}\right)dt\le L, \tag{68}$$

*for some constants positive $\lambda$ and $L$. Then, there exists a positive constant $C_0$ depending only on $L$ such that*

$$(v)_{L^{\Phi)}(\Omega)} \leq C_0\lambda. \quad (69)$$

*Proof.* Let $C(\alpha)$ be the constant appearing in (6). We may take $\alpha_0 \in (0,1)$ such that

$$C(\alpha_0) L \leq 1 \quad (70)$$

since $C(\alpha) \to 0$ as $\alpha \to 0$. We use (6) to get

$$\begin{aligned} N_\Phi(\sigma) \fint_0^{|\Omega|} t^\sigma \Phi\left(\alpha_0 \frac{v^*(t)}{\lambda}\right) dt \\ \leq C(\alpha_0) N_\Phi(\sigma) \fint_0^{|\Omega|} t^\sigma \Phi\left(\frac{v^*(t)}{\lambda}\right) dt. \end{aligned} \quad (71)$$

We take the lim sup as $\sigma \to 0^+$ and use (68) to get

$$\limsup_{\sigma\to 0^+} N_\Phi(\sigma) \fint_0^{|\Omega|} t^\sigma \Phi\left(\alpha_0 \frac{v^*(t)}{\lambda}\right) dt \leq C(\alpha_0) L. \quad (72)$$

Therefore, from (72) and (70) we have

$$\limsup_{\sigma\to 0^+} N_\Phi(\sigma) \fint_0^{|\Omega|} t^\sigma \Phi\left(\alpha_0 \frac{v^*(t)}{\lambda}\right) dt \leq 1. \quad (73)$$

The desired constant $C_0$ is obtained by setting $C_0 = 1/\alpha_0$. We address that $C_0$ is independent of $v$, and thus the proof is completed. □

*Remark 7.* It is clear from the definition of $(u)_{L^{\Phi)}(\Omega)}$ that we can pick $C_0 = 1$ if $L = 1$.

Our next lemma provides a sort of triangle inequality involving the functional $(\cdot)_{L^{\Phi)}(\Omega)}$.

**Lemma 8.** *Let $\Phi : [0,\infty) \to [0,\infty)$ be an Orlicz function satisfying the assumptions of Theorem 1 and let $u, v \in L^{\Phi)}(\Omega)$. Then, there exists a constant $C_1$ depending only on $\Phi$ such that*

$$(u+v)_{L^{\Phi)}(\Omega)} \leq C_1 \left[(u)_{L^{\Phi)}(\Omega)} + (v)_{L^{\Phi)}(\Omega)}\right]. \quad (74)$$

*Proof.* Take

$$\lambda_1 > (u)_{L^{\Phi)}(\Omega)}, \qquad \lambda_2 > (v)_{L^{\Phi)}(\Omega)}. \quad (75)$$

Let $t \in [0, |\Omega|]$. We use (18) with $t_1 = t_2 = t/2$, the monotonicity of $\Phi$, to obtain

$$\begin{aligned} \Phi\left(\frac{(u+v)^*(t)}{\lambda_1+\lambda_2}\right) &\leq \Phi\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{u^*(t/2)}{\lambda_1} + \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{v^*(t/2)}{\lambda_2}\right) \\ &\leq \left[\Phi\left(\frac{u^*(t/2)}{\lambda_1}\right) + \Phi\left(\frac{v^*(t/2)}{\lambda_2}\right)\right]. \end{aligned} \quad (76)$$

Fix $\sigma \in (0,1)$. We multiply by $t^\sigma$ and we integrate over $[0, |\Omega|]$ to get

$$\begin{aligned} \int_0^{|\Omega|} t^\sigma \Phi\left(\frac{(u+v)^*(t)}{\lambda_1+\lambda_2}\right) dt \leq \left[\int_0^{|\Omega|} t^\sigma \Phi\left(\frac{u^*(t/2)}{\lambda_1}\right) dt \right. \\ \left. + \int_0^{|\Omega|} t^\sigma \Phi\left(\frac{v^*(t/2)}{\lambda_2}\right) dt\right]. \end{aligned} \quad (77)$$

With the aid of two changes of variables in the integrals appearing at the right-hand side of (77) we have

$$\begin{aligned} &\int_0^{|\Omega|} t^\sigma \Phi\left(\frac{(u+v)^*(t)}{\lambda_1+\lambda_2}\right) dt \\ &\quad \leq 2^{\sigma+1}\left[\int_0^{|\Omega|/2} s^\sigma \Phi\left(\frac{u^*(s)}{\lambda_1}\right) ds \right. \\ &\qquad \left. + \int_0^{|\Omega|/2} r^\sigma \Phi\left(\frac{v^*(r)}{\lambda_2}\right) dr\right], \end{aligned} \quad (78)$$

which in turn implies

$$\begin{aligned} &\fint_0^{|\Omega|} t^\sigma \Phi\left(\frac{(u+v)^*(t)}{\lambda_1+\lambda_2}\right) dt \\ &\quad \leq 4\left[\fint_0^{|\Omega|} s^\sigma \Phi\left(\frac{u^*(s)}{\lambda_1}\right) ds + \fint_0^{|\Omega|} r^\sigma \Phi\left(\frac{v^*(r)}{\lambda_2}\right) dr\right]. \end{aligned} \quad (79)$$

We multiply both sides of (79) by $N_\Phi(\sigma)$, and we take the lim sup as $\sigma \to 0^+$ and use (75) to get

$$\limsup_{\sigma\to 0^+} N_\Phi(\sigma) \fint_0^{|\Omega|} t^\sigma \Phi\left(\frac{(u+v)^*(t)}{\lambda_1+\lambda_2}\right) dt \leq 8. \quad (80)$$

We appeal to Lemma 6 to conclude that there exists a constant $C_1$ such that

$$(u+v)_{L^{\Phi)}(\Omega)} \leq C_1(\lambda_1+\lambda_2). \quad (81)$$

Finally, (74) follows letting $\lambda_1 \to (u)_{L^{\Phi)}(\Omega)}$ and $\lambda_2 \to (v)_{L^{\Phi)}(\Omega)}$, respectively. □

## 5. An Example

In this section we study the behaviour of weight $N_\Phi(\sigma)$ as $\sigma \to 0^+$ when $\Phi(t) = t^n(1+\log(1+t))^{-\alpha}$ with $0 \leq \alpha \leq 1$. We follow closely the lines of Example 3.6 in [17].

*Example 9.* Let $\Phi(t) = t^n(1+\log(1+t))^{-\alpha}$ and let $0 \leq \alpha \leq 1$. We start by proving that

$$N_\Phi(\sigma) \approx \sigma^{1-\alpha} \quad \text{as } \sigma \to 0^+ \quad \text{when } 0 \leq \alpha < 1. \quad (82)$$

To see this, let $\theta > 0$. Then

$$\begin{aligned}\int_1^{\infty} \frac{\Phi(t)^{1-\theta}}{t^{n+1}} dt &\approx \int_1^{\infty} t^{-n\theta-1}(1+\log t)^{-\alpha(1-\theta)} dt \\ &\approx \int_1^{\infty} e^{-n\theta u} u^{-\alpha(1-\theta)} du \\ &\approx \int_{\theta}^{\infty} e^{-nv} \left(\frac{v}{\theta}\right)^{-\alpha(1-\theta)} \frac{dv}{\theta} \\ &\approx \theta^{\alpha-1} \int_{\theta}^{\infty} e^{-nv} v^{-\alpha(1-\theta)} dv \\ &\approx \theta^{\alpha-1} \left(1 + \int_{\theta}^{1} v^{-\alpha(1-\theta)} dv\right).\end{aligned} \tag{83}$$

We pick $\theta = \sigma/(1+\sigma)$ in such a way that

$$N_{\Phi}(\sigma) = \frac{1}{\int_1^{\infty} \left([\Phi(t)]^{1/(1+\sigma)}/t^{n+1}\right) dt} \approx \sigma^{1-\alpha}. \tag{84}$$

A similar argument leads to

$$N_{\Phi}(\sigma) \approx |\log \sigma|^{-1} \quad \text{as } \sigma \to 0^+ \text{ when } \alpha = 1. \tag{85}$$

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