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Research Article

Signal Processing for Nondifferentiable Data Defined on Cantor Sets: A Local Fractional Fourier Series Approach

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From the signal processing point of view, the nondifferentiable data defined on the Cantor sets are investigated in this paper. The local fractional Fourier series is used to process the signals, which are the local fractional continuous functions. Our results can be observed as significant extensions of the previously known results for the Fourier series in the framework of the local fractional calculus. Some examples are given to illustrate the efficiency and implementation of the present method.

1. Introduction

Fractional derivatives [1, 2], like the Caputo derivative, the Riemann-Liouville derivative, and the Grünwald-Letnikov derivative, were applied to model some anomalous phenomena, such as the anomalous diffusion [3, 4], Brownian motion [5], relaxation in dielectrics [6], transport of particles [7], and reaction kinetics [8]. From the signal processing point of view, the fractional-order signal processing is anomalous behavior of nature from practice activity. In literature [9–16], many researchers employed the fractional calculus theory to handle signals, which are continuous characteristics (having a similar behavior). Some applications of the fractional-order signal processing to electrochemical noises were presented [15]. In [16], Tao and coauthors suggested the signal processing by using the fractional Fourier transform. The short time fractional Fourier transform was used to handle the robotic manipulators with vibrations in [17]. The fractional wavelet transform for processing the composite signals of the active compounds was considered in [18].

There is also a class of signals of the random sequences, which have a fractal behavior, and some methods were suggested in [19–21]. Some signals are defined on the Cantor sets, such as the Cantor function and Cantor-like functions, which are nondifferentiable data.

Figure 1 shows an example of a signal for a Cantor function object while in Figures 2 and 3 there are some examples of signals on the Cantor-like functions defined on the Cantor sets.

With these signals being defined on Cantor sets, classical methods of signal analysis are not efficient to deal with them. To overcome these drawbacks, suitable methods for the signals on the Cantor sets are developed, such as the local fractional Fourier series [22–27], wavelet transform [28], and its discrete version [29].

In view of the special characteristics of the local fractional Fourier series [22, 23], as alternative method for Fourier series based upon the local fractional calculus, the aim of this paper is to investigate the signal processing of nondifferentiable data defined on the Cantor sets, which is a special case of local fractional continuous function [30]. The paper has been organized as follows. Section 2 gives the fundamental concepts of local fractional Fourier series. In Section 3, the nondifferentiable data defined on the Cantor sets are processed. Conclusions are given in Section 4.

2. Mathematical Tools

In this section, we present the fundamental concepts for the local fractional Fourier series and some results for the local

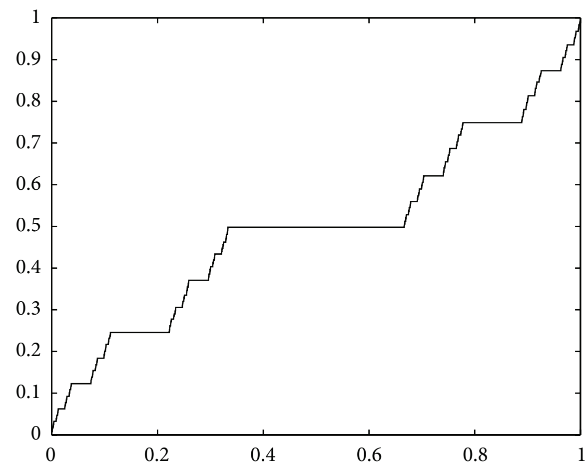

Figure 1: The Cantor function.

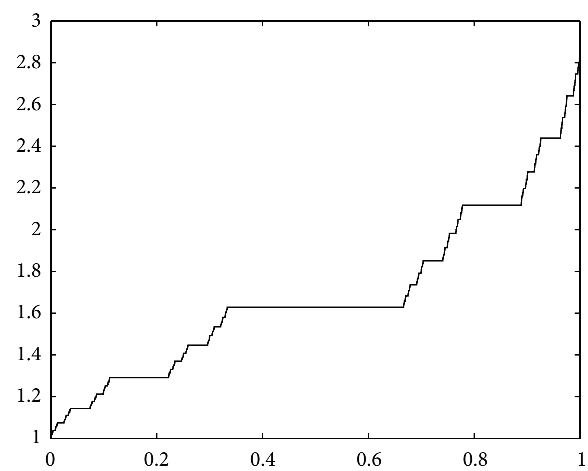

Figure 2: The Cantor-like function defined on the Cantor sets.

fractional integral operator [22–27], which are used in this paper.

If, for $x \in (a, b)$, a function $f(x)$ fulfills the condition

$$\left|f(x) - f(x_0)\right| < \varepsilon^{\alpha}, \tag{1}$$

with $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in R$, then $f(x) \in C_{\alpha}(a, b)$; namely, it is the so-called local fractional continuous on the interval (a, b). If the fractal dimension α is equal to 1, this definition reduces to the classical one.

Let $f(x) \in C_{\alpha}(-\infty, \infty)$. Local fractional trigonometric Fourier series of $f(x)$ is given by [22–27]

$$\begin{aligned} f(x) = a_0 &+ \sum_{k=1}^{\infty} a_k \sin_{\alpha}\left(k^{\alpha}\omega_0^{\alpha}x^{\alpha}\right) \\ &+ \sum_{k=1}^{\infty} b_k \cos_{\alpha}\left(k^{\alpha}\omega_0^{\alpha}x^{\alpha}\right). \end{aligned} \tag{2}$$

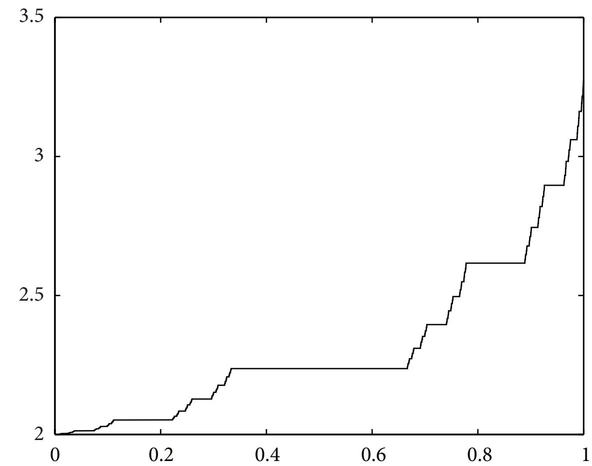

Figure 3: Another Cantor-like function defined on the same Cantor sets.

The local fractional Fourier coefficients read as follows [22–27]:

$$a_0 = \frac{\Gamma(1+\alpha)}{T^{\alpha}} \, {}_0I_T^{(\alpha)} f(x), \tag{3}$$

$$a_k = \frac{2^{\alpha}\Gamma(1+\alpha)}{T^{\alpha}} \, {}_0I_T^{(\alpha)} f(x) \sin_{\alpha}\left(k^{\alpha}\omega_0^{\alpha}x^{\alpha}\right), \tag{4}$$

$$b_k = \frac{2^{\alpha}\Gamma(1+\alpha)}{T^{\alpha}} \, {}_0I_T^{(\alpha)} f(x) \cos_{\alpha}\left(k^{\alpha}\omega_0^{\alpha}x^{\alpha}\right), \tag{5}$$

where the local fractional integral of $f(x)$ of order α in the interval $[a, b]$ is defined as [22–30]

$$\begin{aligned} {}_aI_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^{\alpha} \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f\left(t_j\right)\left(\Delta t_j\right)^{\alpha}, \end{aligned} \tag{6}$$

where the partition of the interval $[a, b]$ is denoted as $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_0, \Delta t_1, \ldots, \Delta t_j, \ldots\}$, and $j = 0, \ldots, N-1$, $t_0 = a$, and $t_N = b$. For more details on the local fractional Fourier series, see [22–27].

The Lebesgue-Cantor staircase function is defined by [30]

$$H_{\alpha}(F \cap (0, x)) = \Gamma(1+\alpha) \, {}_0I_x^{(\alpha)} 1, \tag{7}$$

where F is a Cantor set, $H_{\alpha}(\cdot)$ is the α dimensional Hausdorff measure, ${}_0I_x^{(\alpha)}(\cdot)$ is the local fractional integral operator [24–30], and $\Gamma(\cdot)$ is the Gamma function.

Following (7), we obtain [24, 30]

$$H_{\alpha}(F \cap (0, x)) = x^{\alpha}, \tag{8}$$

which is the Lebesgue-Cantor staircase function.

Some useful formulas, which are used in this paper, are presented as follows [30]:

$$
\begin{aligned}
&{}_aI_b^{(\alpha)}\left[f(x)\pm g(x)\right]={}_aI_b^{(\alpha)}f(x)\pm{}_aI_b^{(\alpha)}g(x),\\
&{}_aI_b^{(\alpha)}\left[Cf(x)\right]=C\,{}_aI_b^{(\alpha)}f(x),\\
&{}_aI_b^{(\alpha)}f(x)=g(b)-g(a),\\
&{}_aI_x^{(\alpha)}f(x)\,g^{(\alpha)}(x)\\
&\quad=\left[f(x)\,g(x)\right]\big|_a^x-{}_aI_x^{(\alpha)}f^{(\alpha)}(x)\,g(x),\\
&{}_0I_x^{(\alpha)}E_\alpha(x^\alpha)=E_\alpha(x^\alpha)-1,\\
&{}_0I_x^{(\alpha)}\sin_\alpha(a^\alpha x^\alpha)=\frac{1}{a^\alpha}\left[\cos_\alpha(a^\alpha x^\alpha)-1\right],\\
&{}_0I_x^{(\alpha)}\cos_\alpha(a^\alpha x^\alpha)=\frac{1}{a^\alpha}\sin_\alpha(a^\alpha x^\alpha),\\
&{}_0I_x^{(\alpha)}\frac{x^\alpha}{\Gamma(1+\alpha)}\sin_\alpha(a^\alpha x^\alpha)\\
&\quad=-\frac{1}{a^\alpha}\left[\frac{x^\alpha}{\Gamma(1+\alpha)}\cos_\alpha(a^\alpha x^\alpha)-\frac{1}{a^\alpha}\sin_\alpha(a^\alpha x^\alpha)\right],\\
&{}_0I_x^{(\alpha)}\frac{x^\alpha}{\Gamma(1+\alpha)}\cos_\alpha(a^\alpha x^\alpha)\\
&\quad=\frac{1}{a^\alpha}\left\{\frac{x^\alpha}{\Gamma(1+\alpha)}\sin_\alpha(a^\alpha x^\alpha)\right.\\
&\qquad\left.-\frac{1}{a^\alpha}\left[\cos_\alpha(a^\alpha x^\alpha)-1\right]\right\},
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&{}_0I_x^{(\alpha)}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sin_\alpha(a^\alpha x^\alpha)\\
&\quad=-\frac{1}{a^\alpha}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\cos_\alpha(a^\alpha x^\alpha)\\
&\qquad+\frac{1}{a^{2\alpha}}\left\{\frac{x^\alpha}{\Gamma(1+\alpha)}\sin_\alpha(a^\alpha x^\alpha)\right.\\
&\qquad\left.-\frac{1}{a^\alpha}\left[\cos_\alpha(a^\alpha x^\alpha)-1\right]\right\},
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&{}_0I_x^{(\alpha)}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\cos_\alpha(a^\alpha x^\alpha)\\
&\quad=\frac{1}{a^\alpha}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sin_\alpha(a^\alpha x^\alpha)\\
&\qquad+\frac{1}{a^{2\alpha}}\left[\frac{x^\alpha}{\Gamma(1+\alpha)}\cos_\alpha(a^\alpha x^\alpha)-\frac{1}{a^\alpha}\sin_\alpha(a^\alpha x^\alpha)\right],
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&{}_0I_x^{(\alpha)}\frac{x^{3\alpha}}{\Gamma(1+2\alpha)}\sin_\alpha(a^\alpha x^\alpha)\\
&\quad=-\frac{1}{a^\alpha}\frac{x^{3\alpha}}{\Gamma(1+2\alpha)}\cos_\alpha(a^\alpha x^\alpha)\\
&\qquad+\frac{1}{a^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sin_\alpha(a^\alpha x^\alpha)\\
&\qquad+\frac{1}{a^{3\alpha}}\left[\frac{x^\alpha}{\Gamma(1+\alpha)}\cos_\alpha(a^\alpha x^\alpha)\right.\\
&\qquad\left.-\frac{1}{a^\alpha}\sin_\alpha(a^\alpha x^\alpha)\right],
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&{}_0I_x^{(\alpha)}\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}\cos_\alpha(a^\alpha x^\alpha)\\
&\quad=\frac{1}{a^\alpha}\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}\sin_\alpha(a^\alpha x^\alpha)\\
&\qquad-\frac{1}{a^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\cos_\alpha(a^\alpha x^\alpha)\\
&\qquad+\frac{1}{a^{3\alpha}}\left\{\frac{x^\alpha}{\Gamma(1+\alpha)}\sin_\alpha(a^\alpha x^\alpha)\right.\\
&\qquad\left.-\frac{1}{a^\alpha}\left[\cos_\alpha(a^\alpha x^\alpha)-1\right]\right\},
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&{}_0I_x^{(\alpha)}\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}\sin_\alpha(a^\alpha x^\alpha)\\
&\quad=-\frac{1}{a^\alpha}\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}\cos_\alpha(a^\alpha x^\alpha)\\
&\qquad+\frac{1}{a^{2\alpha}}\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}\sin_\alpha(a^\alpha x^\alpha)\\
&\qquad-\frac{1}{a^{3\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\cos_\alpha(a^\alpha x^\alpha)\\
&\qquad+\frac{1}{a^{4\alpha}}\left\{\frac{x^\alpha}{\Gamma(1+\alpha)}\sin_\alpha(a^\alpha x^\alpha)\right.\\
&\qquad\left.-\frac{1}{a^\alpha}\left[\cos_\alpha(a^\alpha x^\alpha)-1\right]\right\},
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&{}_0I_x^{(\alpha)}\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}\cos_\alpha(a^\alpha x^\alpha)\\
&\quad=\frac{1}{a^\alpha}\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}\sin_\alpha(a^\alpha x^\alpha)\\
&\qquad-\frac{1}{a^{2\alpha}}\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}\cos_\alpha(a^\alpha x^\alpha)\\
&\qquad+\frac{1}{a^{3\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sin_\alpha(a^\alpha x^\alpha)
\end{aligned}
$$

$$+\frac{1}{a^{4\alpha}}\left[\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cos_{\alpha}\left(a^{\alpha}x^{\alpha}\right)-\frac{1}{a^{\alpha}}\sin_{\alpha}\left(a^{\alpha}x^{\alpha}\right)\right], \tag{15}$$

$${}_0I_x^{(\alpha)}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}=\frac{x^{(k+1)\alpha}}{\Gamma[1+(k+1)\alpha]}. \tag{16}$$

3. Signal Processing for Data Defined on the Cantor Sets

This section deals with the nondifferentiable data defined on the Cantor sets. Some examples of nondifferentiable functions defined on the Cantor sets are given and the corresponding local fractional Fourier series are explicitly computed.

Example 1. We present the nondifferentiable data defined on the Cantor sets in the following form:

$$f(t)=\frac{t^{\alpha}}{\Gamma(1+\alpha)},\quad (0\le t\le 2\pi), \tag{17}$$

where the fractal dimension α is equal to $\ln 2/\ln 3$.

Using (3), (4), and (5), we have the local fractional Fourier coefficients as follows:

$$\begin{aligned}
a_0&=\frac{\Gamma(1+\alpha)}{(2\pi)^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}f(x)=\frac{\Gamma(1+\alpha)}{(2\pi)^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\
&=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(2\pi)^{\alpha},\\
a_k&=\frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}f(x)\sin_{\alpha}\left(k^{\alpha}x^{\alpha}\right)\\
&=\frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}\frac{t^{\alpha}}{\Gamma(1+\alpha)}\sin_{\alpha}\left(k^{\alpha}x^{\alpha}\right)\\
&=-\left[\frac{1}{k^{\alpha}}\left[\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cos_{\alpha}\left(k^{\alpha}x^{\alpha}\right)-\frac{1}{k^{\alpha}}\sin_{\alpha}\left(k^{\alpha}x^{\alpha}\right)\right]\right]\Bigg|_0^{2\pi}\\
&=-\left(\frac{2}{k}\right)^{\alpha},\\
b_k&=\frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}f(x)\cos_{\alpha}\left(k^{\alpha}x^{\alpha}\right)\\
&=\frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}\frac{t^{\alpha}}{\Gamma(1+\alpha)}\cos_{\alpha}\left(k^{\alpha}x^{\alpha}\right)\\
&=\left[\frac{1}{k^{\alpha}}\left\{\frac{x^{\alpha}}{\Gamma(1+\alpha)}\sin_{\alpha}\left(k^{\alpha}x^{\alpha}\right)-\frac{1}{k^{\alpha}}\left[\cos_{\alpha}\left(k^{\alpha}x^{\alpha}\right)-1\right]\right\}\right]\Bigg|_0^{2\pi}=0.
\end{aligned} \tag{18}$$

Hence, from (18)-(19), $f(t)$ is expressed as follows:

$$f(x)=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(2\pi)^{\alpha}-\sum_{k=1}^{\infty}\left(\frac{2}{k}\right)^{\alpha}\sin_{\alpha}\left(k^{\alpha}x^{\alpha}\right). \tag{19}$$

Example 2. Let us consider the nondifferentiable data defined on the Cantor sets in the following form:

$$f(t)=\frac{t^{2\alpha}}{\Gamma(1+2\alpha)},\quad (0\le t\le 2\pi), \tag{20}$$

where the fractal dimension α is equal to $\ln 2/\ln 3$.

According to (3) and (16), we obtain the local fractional Fourier series as follows:

$$\begin{aligned}
a_0&=\frac{\Gamma(1+\alpha)}{(2\pi)^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}f(x)=\frac{\Gamma(1+\alpha)}{(2\pi)^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\
&=\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}(2\pi)^{2\alpha}.
\end{aligned} \tag{21}$$

From (4) and (10), we have the following local fractional Fourier series coefficient:

$$\begin{aligned}
a_k&=\frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}f(x)\sin_{\alpha}\left(k^{\alpha}x^{\alpha}\right)\\
&=\frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\sin_{\alpha}\left(k^{\alpha}x^{\alpha}\right)\\
&=\frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\left[-\frac{1}{k^{\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\cos_{\alpha}\left(k^{\alpha}x^{\alpha}\right)+\frac{1}{k^{2\alpha}}\times\left\{\frac{x^{\alpha}}{\Gamma(1+\alpha)}\sin_{\alpha}\left(k^{\alpha}x^{\alpha}\right)-\frac{1}{k^{\alpha}}\left[\cos_{\alpha}\left(k^{\alpha}x^{\alpha}\right)-1\right]\right\}\right]\Bigg|_0^{2\pi}\\
&=-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{4\pi}{k}\right)^{\alpha}.
\end{aligned} \tag{22}$$

Using (5) and (11), we give

$$\begin{aligned}
b_k&=\frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}f(x)\cos_{\alpha}\left(k^{\alpha}x^{\alpha}\right)\\
&=\frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\cos_{\alpha}\left(k^{\alpha}x^{\alpha}\right)\\
&=\frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\left[\frac{1}{k^{\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sin_{\alpha}\left(k^{\alpha}x^{\alpha}\right)+\frac{1}{k^{2\alpha}}\times\left[\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cos_{\alpha}\left(k^{\alpha}x^{\alpha}\right)-\frac{1}{k^{\alpha}}\sin_{\alpha}\left(k^{\alpha}x^{\alpha}\right)\right]\right]\Bigg|_0^{2\pi}\\
&=\left(\frac{2}{k^{2}}\right)^{\alpha}.
\end{aligned} \tag{23}$$

Therefore, $f(t)$ is expressed as follows:

$$f(x) = \frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}(2\pi)^{2\alpha} - \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{k=1}^{\infty}\left(\frac{4\pi}{k}\right)^{\alpha}\sin_{\alpha}(k^{\alpha}x^{\alpha}) + \sum_{i=1}^{\infty}\left(\frac{2}{k^2}\right)^{\alpha}\cos_{\alpha}(k^{\alpha}x^{\alpha}). \quad (24)$$

Example 3. The nondifferentiable data defined on the Cantor sets in the following form is given by the local fractional Fourier series as follows:

$$f(t) = \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \quad (0 \le t \le 2\pi), \quad (25)$$

where the fractal dimension α is equal to $\ln 2/\ln 3$.

Making use of (3) and (16), the local fractional Fourier series of $f(t)$ reads as follows:

$$a_0 = \frac{\Gamma(1+\alpha)}{(2\pi)^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)} f(x) = \frac{\Gamma(1+\alpha)}{(2\pi)^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+4\alpha)}(2\pi)^{3\alpha}. \quad (26)$$

By applying (4) and (12), we get

$$\begin{aligned} a_k &= \frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)} f(x)\sin_{\alpha}(k^{\alpha}x^{\alpha}) \\ &= \frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\sin_{\alpha}(k^{\alpha}x^{\alpha}) \\ &= \frac{\Gamma(1+\alpha)}{\pi^{\alpha}} \\ &\quad\times\left[-\frac{1}{k^{\alpha}}\frac{x^{3\alpha}}{\Gamma(1+2\alpha)}\cos_{\alpha}(k^{\alpha}x^{\alpha}) + \frac{1}{k^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sin_{\alpha}(k^{\alpha}x^{\alpha}) + \frac{1}{k^{3\alpha}}\left(\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cos_{\alpha}(k^{\alpha}x^{\alpha}) - \frac{1}{k^{\alpha}}\sin_{\alpha}(k^{\alpha}x^{\alpha})\right)\right]\Bigg|_0^{2\pi} \\ &= \frac{1}{\pi^{\alpha}}\left(\left(\frac{2\pi}{k^3}\right)^{\alpha} - \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{(2\pi)^3}{k}\right)^{\alpha}\right). \end{aligned} \quad (27)$$

By using (5) and (13), we have

$$\begin{aligned} b_k &= \frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)} f(x)\cos_{\alpha}(k^{\alpha}x^{\alpha}) \\ &= \frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\cos_{\alpha}(k^{\alpha}x^{\alpha}) \\ &= \frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\left[\frac{1}{k^{\alpha}}\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}\sin_{\alpha}(k^{\alpha}x^{\alpha}) - \frac{1}{k^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\cos_{\alpha}(k^{\alpha}x^{\alpha}) + \frac{1}{k^{3\alpha}}\left\{\frac{x^{\alpha}}{\Gamma(1+\alpha)}\sin_{\alpha}(k^{\alpha}x^{\alpha}) - \frac{1}{k^{\alpha}}\left[\cos_{\alpha}(k^{\alpha}x^{\alpha}) - 1\right]\right\}\right]\Bigg|_0^{2\pi} \\ &= -\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{4\pi}{k^2}\right)^{\alpha}. \end{aligned} \quad (28)$$

Hence, from (18)–(28), the nondifferentiable signal can be expressed as follows:

$$\begin{aligned} f(x) &= \frac{\Gamma(1+\alpha)}{\Gamma(1+4\alpha)}(2\pi)^{3\alpha} + \sum_{k=1}^{\infty}\frac{1}{\pi^{\alpha}}\left(\left(\frac{2\pi}{k^3}\right)^{\alpha} - \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{(2\pi)^3}{k}\right)^{\alpha}\right) \\ &\quad\times\sin_{\alpha}(k^{\alpha}x^{\alpha}) - \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=1}^{\infty}\left(\frac{4\pi}{k^2}\right)^{\alpha}\cos_{\alpha}(k^{\alpha}x^{\alpha}). \end{aligned} \quad (29)$$

Example 4. Let us consider the data defined on Cantor sets

$$f(t) = \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}, \quad (0 \le t \le 2\pi) \quad (30)$$

to be expressed in local fractional Fourier series.

According to (3) and (16), we have

$$a_0 = \frac{\Gamma(1+\alpha)}{(2\pi)^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)} f(x) = \frac{\Gamma(1+\alpha)}{(2\pi)^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+5\alpha)}(2\pi)^{4\alpha}. \quad (31)$$

From (4) and (14), we arrive at the following local fractional Fourier series coefficient:

$$\begin{aligned} a_k &= \frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)} f(x)\sin_{\alpha}(k^{\alpha}x^{\alpha}) \\ &= \frac{\Gamma(1+\alpha)}{\pi^{\alpha}}\,{}_0I_{2\pi}^{(\alpha)}\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}\sin_{\alpha}(k^{\alpha}x^{\alpha}) \end{aligned}$$

$$= \frac{\Gamma(1+\alpha)}{\pi^{\alpha}} \left[-\frac{1}{k^{\alpha}} \frac{x^{4\alpha}}{\Gamma(1+4\alpha)} \cos_{\alpha}(k^{\alpha}x^{\alpha}) + \frac{1}{k^{2\alpha}} \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} \sin_{\alpha}(k^{\alpha}x^{\alpha}) - \frac{1}{k^{3\alpha}} \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \cos_{\alpha}(k^{\alpha}x^{\alpha}) + \frac{1}{k^{4\alpha}} \left\{ \frac{x^{\alpha}}{\Gamma(1+\alpha)} \sin_{\alpha}(k^{\alpha}x^{\alpha}) - \frac{1}{k^{\alpha}} \left[\cos_{\alpha}(k^{\alpha}x^{\alpha}) - 1 \right] \right\} \right] \Bigg|_0^{2\pi}$$

$$= -\frac{\Gamma(1+\alpha)}{\pi^{\alpha}} \left(\frac{\left(\left((2\pi)^4/k\right)\right)^{\alpha}}{\Gamma(1+4\alpha)} + \frac{\left((2\pi)^2/k^3\right)^{\alpha}}{\Gamma(1+2\alpha)} \right). \tag{32}$$

Using (5) and (15), we obtain the following local fractional Fourier series coefficient:

$$b_k = \frac{\Gamma(1+\alpha)}{\pi^{\alpha}} {}_0I_{2\pi}^{(\alpha)} f(x) \cos_{\alpha}(k^{\alpha}x^{\alpha})$$
$$= \frac{\Gamma(1+\alpha)}{\pi^{\alpha}} {}_0I_{2\pi}^{(\alpha)} \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} \cos_{\alpha}(k^{\alpha}x^{\alpha})$$
$$= \frac{\Gamma(1+\alpha)}{\pi^{\alpha}} \left[\frac{1}{k^{\alpha}} \frac{x^{4\alpha}}{\Gamma(1+4\alpha)} \sin_{\alpha}(k^{\alpha}x^{\alpha}) - \frac{1}{k^{2\alpha}} \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} \cos_{\alpha}(k^{\alpha}x^{\alpha}) + \frac{1}{k^{3\alpha}} \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \sin_{\alpha}(k^{\alpha}x^{\alpha}) + \frac{1}{k^{4\alpha}} \left[\frac{x^{\alpha}}{\Gamma(1+\alpha)} \cos_{\alpha}(k^{\alpha}x^{\alpha}) - \frac{1}{k^{\alpha}} \sin_{\alpha}(k^{\alpha}x^{\alpha}) \right] \right] \Bigg|_0^{2\pi}$$
$$= \frac{\Gamma(1+\alpha)}{\pi^{\alpha}} \left(\frac{\left(2\pi/k^4\right)^{\alpha}}{\Gamma(1+\alpha)} - \frac{\left((2\pi)^3/k^2\right)^{\alpha}}{\Gamma(1+3\alpha)} \right). \tag{33}$$

Hence, we obtain the following local fractional Fourier coefficient:

$$f(x) = \frac{\Gamma(1+\alpha)}{\Gamma(1+5\alpha)} (2\pi)^{4\alpha} - \sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)}{\pi^{\alpha}} \left(\frac{\left(\left((2\pi)^4/k\right)\right)^{\alpha}}{\Gamma(1+4\alpha)} + \frac{\left((2\pi)^2/k^3\right)^{\alpha}}{\Gamma(1+2\alpha)} \right) \times \sin_{\alpha}(k^{\alpha}x^{\alpha}) + \sum_{i=1}^{\infty} \frac{\Gamma(1+\alpha)}{\pi^{\alpha}} \left(\frac{\left(2\pi/k^4\right)^{\alpha}}{\Gamma(1+\alpha)} - \frac{\left((2\pi)^3/k^2\right)^{\alpha}}{\Gamma(1+3\alpha)} \right) \times \cos_{\alpha}(k^{\alpha}x^{\alpha}). \tag{34}$$

4. Conclusions

Local fractional Fourier series are a generalization for Fourier series defined on the Cantor sets based upon the local fractional calculus. In this work, we use the local fractional Fourier series to deal with the nondifferentiable data defined on the Cantor sets. Some explicit computations for nondifferentiable data defined on the Cantor sets are also given to show the efficiency of the present method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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