8th IFAC Symposium on Nonlinear Control Systems
University of Bologna, Italy, September 1-3, 2010

Numerical issues in integrating holonomic kinematic inversion algorithms for redundant manipulators

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Abstract: Redundant robotic manipulators under kinematic control may exhibit unpredictable behaviours at joint level, since closed loop trajectories in Cartesian space do not in general map in closed loop trajectories in joint space. Holonomic kinematic inversion algorithms avoid this problem. In this paper, we point out some numerical problems that may arise when discretizing a holonomic kinematic inversion algorithm. It is shown that the knowledge of the additional set of holonomic constraints is needed in order to overcome such drawbacks. A simple case study that shows the potential effects of discretization is discussed.

Keywords: Robot kinematics, redundant manipulators, inverse kinematic problem, numerical methods

1. INTRODUCTION

Robotic manipulators are said to be kinematically redundant when they have more degrees of freedom than those strictly necessary to perform a given task. Since a general task consists in following an end-effector trajectory with a specified orientation, and thus requires six degrees of freedom, it follows that a manipulator with seven or more joints is redundant. More in general, let m be the number of required degrees of freedom of the task and n be the number of joints of the robot: the robot is thus redundant if $n > m$.

A redundant manipulator is able to perform a prescribed end-effector motion in infinite ways, which implies that the inverse kinematic problem has infinite solutions. This fact can be used in order to optimize some additional criteria such as singularity (Seraji and Colbaugh, 1990) or obstacle avoidance (Colbaugh et al., 1989), torque minimization, (Hollerbach and Suh, 1987b) and (Hollerbach and Suh, 1987a), and others.

Side effects of the adoption of kinematic redundancy are that the motion of the robot can be to some extent unpredictable. During a positioning task, the final configuration of the robot may depend on the planned end-effector trajectory even when the motion of the robot starts from the same initial joint configuration. Moreover, under a kinematic control strategy a closed end-effector trajectory can be mapped into an open trajectory on the configuration space. These facts are highly undesirable and may represent a limitation in the use of redundant manipulators.

Fortunately, there exists a class of kinematic inversion algorithms that avoid these problems. Such inversion algorithms are called holonomic, see (Roberts and Maciejewski, 1992), (De Luca et al., 1992), (Mussa-Ivaldi and Hogan, 1991), (Michellod et al., 2008) and (Rocco and Zanchettin, 2010).

In this paper, we point out some numerical problems that may arise when such algorithms are implemented. In particular, we show that holonomy can be lost once the algorithm is discretized, regardless of the discretization time. More precisely, it will be shown that the non-controllable part of the resulting discrete time system may turn to be unstable. It is well known, see (Murray et al., 1994), that the holonomy of a solution is equivalent to the existence of an additional set of holonomic constraints. The knowledge of such constraints can be used to overcome the numerical problems. The proposed solution is formally equivalent to the so called augmented Jacobian method (Siciliano and Khatib, 2008) and to the post-stabilization method (Ascher and Petzold, 1998).

The remaining of this paper is organized as follows. In Section 2 some mathematical background is reviewed. In Section 3 the problem of the numerical integration of a holonomic local control strategy is presented and discussed. Finally, in Section 4 such numerical aspects are discussed on a simple case study.

⋆ The research leading to these results has received funding from the European Community's Seventh Framework Programme FP7/2007-2013 - Challenge 2 - Cognitive Systems, Interaction, Robotics - under grant agreement No 230902 - ROSETTA.

2. PRELIMINARIES

Consider a robotic manipulator with n joints. If q_i $(i = 1,\dots,n)$ denotes the variable characterizing the position of the i-th joint, the configuration of the robot is given by the vector $q = [q_1\ q_2\ \dots\ q_n]^T$.

The position of the end-effector is usually characterized by the vector $x = [x_1\ x_2\ \dots\ x_m]^T$ which describes its position and/or orientation. The direct kinematic mapping associated to a manipulator is thus a nonlinear function $f : \mathbb{R}^n \to \mathbb{R}^m$:

$$x = f(q) \tag{1}$$

A manipulator is said to be redundant if $n > m$. Solving the kinematic inversion problem means finding q for a given x such that the previous equation holds. Usually, this problem is addressed at velocity level. In other words, the first time derivative of (1) is taken into account:

$$\dot{x} = \frac{\partial f}{\partial q}\dot{q} \equiv J(q)\,\dot{q} \tag{2}$$

where J is a $m \times n$ matrix called task-Jacobian, or simply Jacobian [1].

Local control strategy Let $\mathbb{S}$ be a simply-connected open subset of $\mathbb{R}^n$ where the Jacobian matrix J is full rank and let G be a $n \times m$ matrix such that $JG = I_m$. Then G is called a local control strategy.

Considering the local control strategy G, a solution of the inverse kinematic problem is the trajectory of the following nonlinear input-affine system:

$$\dot{q} = G\dot{x} \tag{3}$$

Differently from non-redundant manipulators, the motion of redundant manipulators under a kinematic local control strategy can be unpredictable (Klein and Huang, 1983). More precisely, during a task the final configuration in the joint space of the robot may depend on the end-effector trajectory even when the motion of the robot starts from the same initial configuration (Schaufler et al., 2000). In this case the local control strategy is called non-holonomic (LaValle, 2006). On the other hand, a local control strategy that maps closed trajectories in the Cartesian space into closed trajectories in the joint space is called holonomic. The solution of (3) is holonomic if and only if there exists a function $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n-m}$ such that

$$\frac{\partial g}{\partial q}\dot{q} = 0 \tag{4}$$

where $\partial g/\partial q$ is full rank and equations (2) and (4) are linearly independent. Equation (4) states that there exists a generally nonlinear integrable invariant manifold such that the solution of (3) is subject to the additional constraint $g(q) = constant$.
Frobenius' theorem can be used to check the holonomy of a given local control strategy G. Such theorem is based on the definition of distribution (associated to a local control strategy) and involutivity.

Distribution The distribution associated to the local control strategy $G = [G_1 \ \dots \ G_m]$ (where G_i denotes the i-th column of G) is $span(G_1, \dots, G_m) = range(G)$.

Involutivity The distribution associated to the local control strategy G is said to be involutive if and only if

$$\forall (i, j) : [G_i, G_j] \in range(G) \tag{5}$$

where

$$[A, B] = \frac{\partial B}{\partial q}A - \frac{\partial A}{\partial q}B \tag{6}$$

denotes the Lie bracket operation (Isidori, 1995).

Theorem 2.1. (Frobenius). A local control strategy G generates a holonomic behaviour if and only if the underlying distribution is involutive.

The involutivity of G, and thus the holonomy of the method, can be checked by computing its Lie brackets. Moreover, in (Schaufler et al., 2000), a simplified criterion is proposed. In particular, the following criterion has been proven.

Theorem 2.2. A local control strategy G generates a holonomic behaviour if and only if

$$\forall (i, j) : [G_i, G_j] = 0 \tag{7}$$

In the literature, many different approaches to design holonomic local control strategies G for a given manipulator can be found, see e.g. (Klein and Ahmed, 1995), (Schaufler et al., 2000). The design of the local control strategy G can be done using minimization (Roberts and Maciejewski, 1992), with a proper selection of the null-space velocity (De Luca and Oriolo, 1997). Finally a holonomic control strategy can be designed using a weighted pseudo-inverse (Rocco and Zanchettin, 2010). Using this method the robotic programmer, instead of the robot manufacturer, might directly select the redundancy resolution criterion, namely the weight matrix, according to the requirements of his/her application.
From the control theory point of view, the issue to check whether a local control strategy G is holonomic or not is strictly related to the notion of controllability (Sastry, 1999), (Isidori, 1995). Let us consider the nonlinear input-affine drift-less dynamical system:

$$\dot{q} = Gu \tag{8}$$

The system (8) is small-time locally controllable if and only if $rank(C) = n$, where

$$C = \left[G_1, \dots, G_m, [G_i, G_j], \dots, \left(ad^k_{G_i}, G_j\right), \dots\right] \tag{9}$$

is called the controllability distribution, $k = 2, \dots, n-1$ and $\left(ad^k_{G_i}, G_j\right)$ denotes the repeated Lie bracket operation defined as follows:

$$\left(ad^1_A, B\right) = [A, B] \quad \left(ad^k_A, B\right) = \left[A, \left(ad^{k-1}_A, B\right)\right] \tag{10}$$

In view of Theorem 2.2, it is straightforward to notice that a holonomic local control strategy does not define a controllable system. In fact, if G is holonomic, there exists a consistent change of variables $x = f(q)$ and $y = g(q)$ such that the system (8) is equivalent to the following one:

$$\dot{x} = u \quad \dot{y} = 0 \tag{11}$$

Equation (11) points out that there exists a part of the system, related to the variable y, that does not depend on the input u.
Notice that, since the system is marginally stable, a closed loop inversion technique is needed. In the remaining of this work, we consider the following closed-loop inverse kinematic (CLIK) system:

$$\dot{q} = G(\dot{x} + Ke) \quad e = x - f(q) \tag{12}$$

where $-K$ is any $m \times m$ Hurwitz matrix.
Since the system is not controllable, the proposed feedback stabilization (12) acts only on the controllable part of the system. In particular, applying to (12) the change of variables $e = x - f(q)$ and $y = g(q)$, the system (12) is equivalent to the following one:

$$\dot{e} = -Ke \quad \dot{y} = 0 \tag{13}$$

3. NUMERICAL ASPECTS

3.1 Position of the problem

In order to be implemented on a robotic controller, the CLIK system (12) has to be discretized. Due to the hard-real time fashion of this application, numerical methods that require a high computational demand are not considered in this work. For this reason, we will analyze single-step Runge-Kutta methods that require only one evaluation of the matrix G. Moreover, since explicit methods are not able to preserve the asymptotic stability of stable linear systems, implicit methods will be analyzed only.
Let $\phi_{\Delta t}(\cdot)$ be a single step discretizing method (e.g. backward Euler, midpoint rule, etc.) where Δt denotes the discretizing

[1] From now on, the dependence on q will be omitted.

time interval. Then the discrete-time system obtained from (12) takes the following expression[2]:

$$q^k = \phi_{\Delta t}\left(q^{k-1}\right) \tag{14}$$

Although the system to be discretized is linear with respect to a proper change of variables, depending on $\phi_{\Delta t}(\cdot)$ (i.e. depending on the chosen discretizing method), the exact solution of the equation $\dot{y} = 0$ is not guaranteed. In particular, $(\Delta y)^k = y^k - y^{k-1} = g\left(q^k\right) - g\left(q^{k-1}\right)$ might converge to zero, but also diverge. In general, once the system is discretized, the holonomy, which implies that $(\Delta y)^k = y^k - y^{k-1} = 0$, might be lost or achieved only asymptotically. If the asymptotic convergence might be acceptable, the divergence of $(\Delta y)^k$ is obviously undesirable, since it implies that the discretized inverse kinematics algorithm no longer satisfies the holonomic constraint $g(q) = const$.

3.2 Using different discretizing methods

In the following, the problem is addressed computing the Taylor expansion of Δy at time instant k.
Assume that G is holonomic and consider the implicit (or backward) Euler method[3] applied to the system (8):

$$q^k = q^{k-1} + \Delta t G\left(q^k\right) u\left(t^k\right) \tag{16}$$

Applying the Taylor series expansion to $(\Delta y)^k$ centered in q^k, one obtains:

$$(\Delta y)^k = g\left(q^k\right) - g\left(q^k\right) + \left.\frac{\partial g}{\partial q}\right|_{q^k} \gamma + \frac{1}{2}\gamma^T \left.\frac{\partial^2 g}{\partial q^2}\right|_{q^k} \gamma + \cdots \tag{17}$$

where $\gamma = q^k - q^{k-1} = \Delta t G\left(q^k\right) u\left(t^k\right)$. Therefore

$$(\Delta y)^k = \frac{1}{2}\gamma^T \left.\frac{\partial^2 g}{\partial q^2}\right|_{q^k} \gamma + \cdots \tag{18}$$

which means that only linear constraints are exactly preserved.
Consider now the implicit midpoint method (Ascher and Petzold, 1998):

$$q^k = q^{k-1} + \Delta t G\left(q^{k-1/2}\right) u\left(t^{k-1/2}\right) \tag{19}$$

where $q^{k-1/2} = \left(q^k + q^{k-1}\right)/2$ and $t^{k-1/2} = \left(t^k + t^{k-1}\right)/2$. Applying the Taylor expansion to $(\Delta y)^k$ centered in $q^{k-1/2}$ one obtains the expression (15), where $\delta = \left(q^k - q^{k-1}\right)/2$. When computing $(\Delta y)^k$ the first two terms and the quadratic terms vanish. Moreover, since $\partial g/\partial q G = 0$ and

$$\delta = \frac{\Delta t}{2} G\left(q^{k-1/2}\right) u\left(t^{k-1/2}\right) \tag{20}$$

the two linear terms in the Taylor series vanish, as well. However, nothing can be said for the third order terms. This fact implies that the implicit midpoint rule preserves, in general, quadratic constraints only.

3.3 Possible remedies

The use of holonomic local control strategies G that enforce quadratic constraints only is very restrictive. In practical applications, the adoption of simple discretizing methods (e.g. the backward Euler or the midpoint rule) is often the only way to implement a kinematic inversion algorithm. Therefore, the drawbacks mentioned before have to be taken into account explicitly.
In the literature on numerical integration of dynamical systems, many approaches to solve this problem (which is usually called *constraint drift*) exist, see (Ascher, 1997) for an overview. In particular:

- integration methods for constrained mechanical systems (Baumgarte, 1972), (Ascher and Petzold, 1998);
- stabilization of ordinary differential equations on invariant manifolds (Hairer, 2001);
- post-stabilization methods (Ascher and Petzold, 1998).

Unfortunately, to the best of the authors' knowledge, there is no integration method that avoid the constraint drift which can be applied without the knowledge of the constraint itself. Notice that a holonomic local control strategy G does not provide any information on the underlying holonomic constraint except the fact that $range(G) = null(\partial g/\partial q)$, see (Rocco and Zanchettin, 2010).
In (Baillieul, 1985) and (Rocco and Zanchettin, 2010) it has been shown that any holonomic local control strategy G can be written in terms of an augmented Jacobian (Siciliano and Khatib, 2008). In particular, $\dot{q} = G\dot{x}$ is the solution of the following problem:

$$\begin{bmatrix} J \\ \frac{\partial g}{\partial q} \end{bmatrix} \dot{q} = J_A \dot{q} = \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} \tag{21}$$

where the augmented Jacobian J_A is non singular and $\partial g/\partial q$ is such that $\partial g/\partial q G = 0$. If the function $g(\cdot)$ is known, the dynamical system (21) can be stabilized, leading to the following CLIK algorithm:

$$\dot{q} = J_A^{-1}\left(\begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} + K_A \begin{bmatrix} x - f(q) \\ g(q_0) - g(q) \end{bmatrix}\right) \tag{22}$$

where $-K_A$ is any $n \times n$ Hurwitz matrix and $q_0 = q(0)$ is the initial configuration of the robot. This solution is well known and has been proposed originally in (Egeland, 1987). Differently from (12), the joint variables are generated through an asymptotically stable system. In fact, applying to (22) the following change of variables $e_x = x - f(q)$, $e_y = g(q_0) - g(q)$ which is consistent away from singularities of J_A, it can be easily proven, computing their time derivative, that both e_x and e_y tend to zero. Therefore, (22) can be discretized without running into the numerical problems mentioned before.
Finally, it can be proven that, under simple assumptions, the augmented Jacobian method is a particular application of the post-stabilization approach. In fact

$$J_A^{-1} = \begin{bmatrix} G & N\left(\frac{\partial g}{\partial q} N\right)^{-1} \end{bmatrix} \tag{23}$$

where N is any null-space base of the Jacobian matrix J. Therefore, under the assumption that K_A is block diagonal, i.e. $K_A = diag(K, \bar{K})$, (22) can be rewritten as follows:

$$\begin{aligned} \dot{q} &= J_A^{-1}\left(\begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} + K_A \begin{bmatrix} x - f(q) \\ g(q_0) - g(q) \end{bmatrix}\right) = \\ &= G(\dot{x} + Ke) + N\left(\frac{\partial g}{\partial q} N\right)^{-1} \bar{K}\left(g(q_0) - g(q)\right) = \\ &= G(\dot{x} + Ke) + F\left(g(q_0) - g(q)\right) \end{aligned} \tag{24}$$

[2] This formula denotes the solution of the implicit equation in the unknown q^k that can be obtained using the Newton-Raphson method.
[3] Here and in the following, the symbol u will denote the feedforward term $\dot{x}$ or the mixed feedback/feedforward term $\dot{x} + Ke$, without distinction.

$$(\Delta y)^k = g\left(q^{k-1/2}\right) - g\left(q^{k-1/2}\right) + \left.\frac{\partial g}{\partial q}\right|_{q^{k-1/2}} \delta + \left.\frac{\partial g}{\partial q}\right|_{q^{k-1/2}} \delta + \frac{1}{2}\delta^T \left.\frac{\partial^2 g}{\partial q^2}\right|_{q^{k-1/2}} \delta - \frac{1}{2}(-\delta)^T \left.\frac{\partial^2 g}{\partial q^2}\right|_{q^{k-1/2}} (-\delta) + \cdots \tag{15}$$

where $F = N\left(\partial g/\partial qN\right)^{-1} \bar{K}$ is the so called projection matrix. Notice that, following the general formulation of the projection method, the stabilizing factor $F\left(g\left(q_0\right) - q\left(q\right)\right)$ might modify the end-effector tracking performance. Using the proposed formulation, i.e. the augmented Jacobian method, the two components of the right hand side of (24) are orthogonal. In other words, the second term does not affect the end-effector motion.

4. A CASE STUDY

Consider the planar PPR manipulator sketched in Fig. 1. The robot is redundant for the task of positioning the end-effector (point p, in Fig. 1) with unspecified orientation ($n = 3 > m = 2$). The Jacobian matrix of this manipulator (the length of the third link is unitary) is expressed as:

$$J = \begin{bmatrix} 1 & 0 & -s_3 \\ 0 & 1 & c_3 \end{bmatrix} \tag{25}$$

where $c_3 = \cos\left(q_3\right)$, $s_3 = \sin\left(q_3\right)$. Consider the following

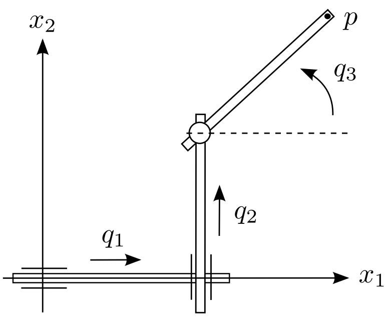

Fig. 1. A planar PPR manipulator

holonomic local control strategy:

$$G = \begin{bmatrix} 1 - s_3c_3 - s_3q_1 & \left(q_2 + s_3\right)s_3 \\ \left(q_1 + c_3\right)c_3 & 1 - s_3c_3 - c_3q_2 \\ -q_1 - c_3 & q_2 + s_3 \end{bmatrix} \tag{26}$$

Letting $g\left(q\right) = 1/2\left(q_1 + c_3\right)^2 - 1/2\left(q_2 + s_3\right)^2 + q_3$, it can be shown that $\partial g/\partial qG = 0$.

The CLIK system of equation (12) with $K = diag\left(1000, 1000\right)$ has been discretized using the backward Euler method, with $\Delta t = 1\ ms$. Time histories of the norm of the kinematic error, $\|x - f\left(q\right)\|_2$, and of the constraint, $g\left(q\right)$, are shown in Figs. 2 and 3, respectively. Despite the norm of the kinematic error, $\|x - f\left(q\right)\|_2$, is bounded, Fig. 3 shows that the discretized system is no longer stable and the resulting motion is no longer holonomic.

The same system has been discretized with the implicit midpoint rule. Time histories of the norm of the kinematic error, $\|x - f\left(q\right)\|_2$, and of the constraint, $g\left(q\right)$, are shown in Figs. 4 and 5, respectively. It can be noticed that, even using a more accurate discretizing method, the exact integration of the constraint equation is not achieved, again. This fact results in a drift, as shown in Fig. 5.

Finally, Figs. 6 and 7 show the time history of the norm of the kinematic error and of the constraint, respectively, when the augmented Jacobian method of equation (22) with $K_A = diag\left(1000, 1000, 1000\right)$ is discretized with the backward Euler method with $\Delta t = 1\ ms$. As one can see, the augmented Jacobian

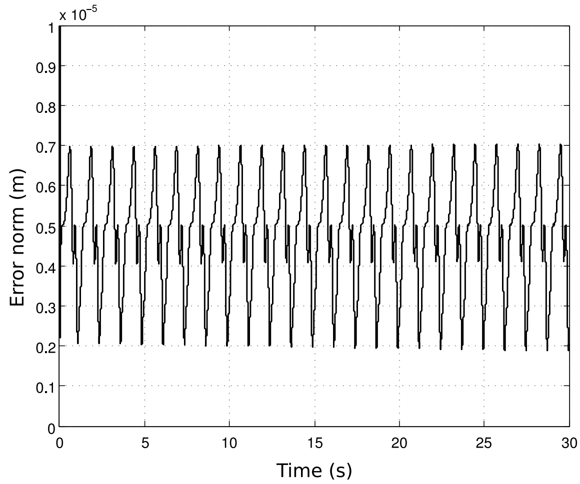

Fig. 2. Backward Euler - error norm $\|x - f\left(q\right)\|_2$

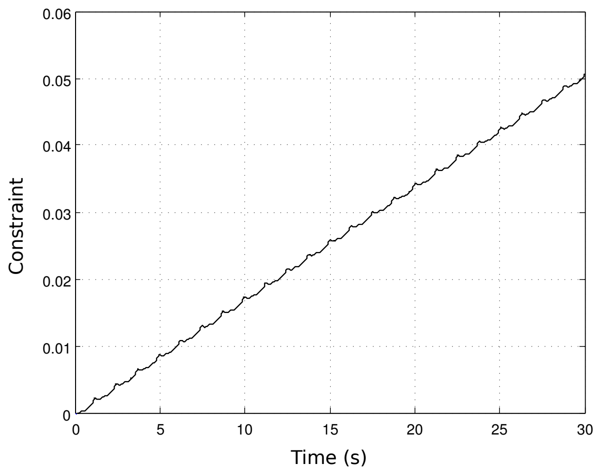

Fig. 3. Backward Euler - constraint $g\left(q\right) - g\left(q_0\right)$

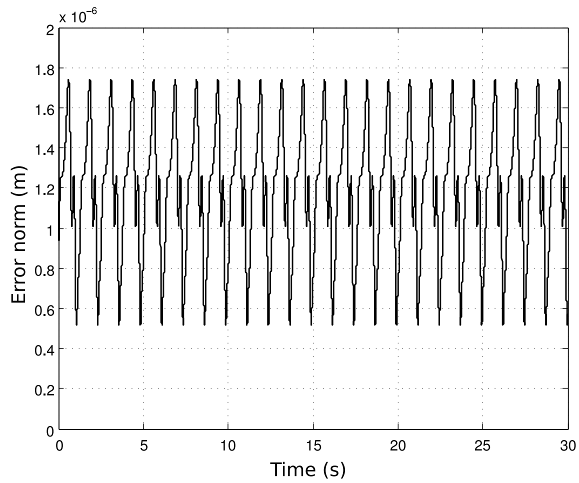

Fig. 4. Implicit midpoint - error norm $\|x - f\left(q\right)\|_2$

method is able to preserve the constraint even when discretized with the simplest implicit discretizing method. In addition, the efficiency of the kinematic inversion is comparable with the one

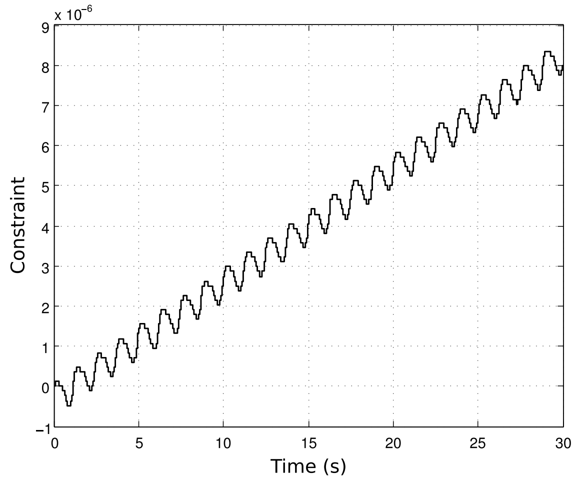

Fig. 5. Implicit midpoint - constraint $g(q) - g(q_0)$

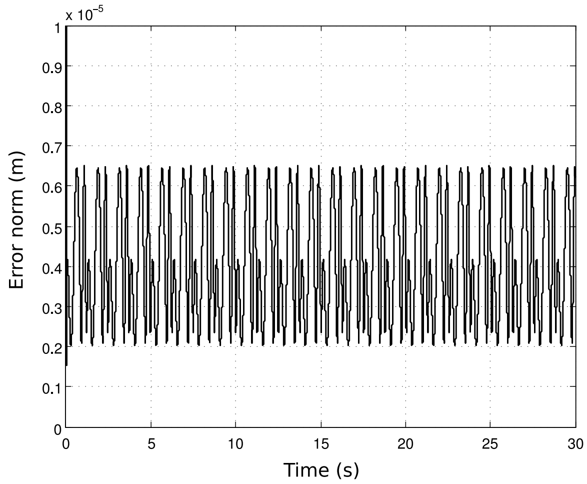

Fig. 6. Augmented Jacobian, backward Euler - error norm $\|x - f(q)\|_2$

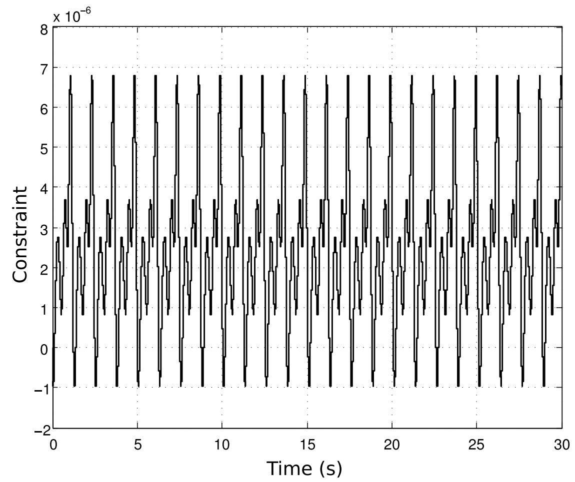

Fig. 7. Augmented Jacobian, backward Euler - constraint $g(q) - g(q_0)$

obtained using the local control strategy G in (26) discretized with the backward Euler method.

5. CONCLUSIONS AND FUTURE WORKS

In this paper, we have pointed out some numerical problems that might arise when implementing a holonomic kinematic inversion algorithm for redundant manipulators. In particular, we have shown that using simple discretizing methods, which is common for most practical applications, the holonomy of the discrete-time system is not guaranteed, in general. It has been shown that, even in a simple case study, the discretized system may easily turn to be unstable. This fact is clearly undesirable in a typical long-time integration problem as the kinematic inversion is.
An example has shown that the augmented Jacobian method preserves the holonomic constraint even when the discretized with a simple backward Euler algorithm.
However, according to the existing methodologies to enforce holonomy, the knowledge of the additional holonomic constraint is needed to avoid numerical problems. Since the stabilization method strictly depends on the redundancy resolution criterion, the existing methods are not suited for a user-oriented point of view. Future research is thus needed to address this problem.

ACKNOWLEDGEMENT

The authors wish to thank Arturo Locatelli and Nicola Schiavoni for fruitful discussions on the redundancy resolution issues.

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