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Scaling and fractality in fatigue crack growth: Implications to Paris' law and Wöhler's curve

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Abstract

Size effects on crack growth are of paramount importance on the fatigue problem. Paris' and Wöhler's fatigue curves exhibit scale effects, which can be explained in the framework of incomplete self-similarity and fractal geometry concepts. In particular, scaling laws are found for the main fatigue parameters, that is, the fatigue threshold ΔK_{th} and the fatigue limit $\Delta\sigma_{fl}$. The fatigue threshold increases with the crack length, whereas the fatigue limit decreases with the specimen size. Eventually, the proposed models are positively compared to experimental data available in the literature.

Keywords: Fatigue crack growth; Paris' law; Wöhler's curve; Incomplete self-similarity; Fractal geometry; Crack-size effects; Specimen-size effects; Fatigue threshold; Fatigue limit

Nomenclature

a	crack length (L)
N	number of cycles to failure $(-)$
$\mathrm{d}a/\mathrm{d}N$	crack growth rate (L)
ΔK_I	stress-intensity factor range $(\mathrm{FL}^{-3/2})$
R	loading ratio $(-)$
K_{IC}	fracture toughness $(\mathrm{FL}^{-3/2})$

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σ_u	ultimate tensile strength (FL^{-2})
ΔK_{th}	fatigue threshold ($FL^{-3/2}$)
$\Delta\sigma$	stress range (FL^{-2})
$\Delta\sigma_{fl}$	fatigue limit (FL^{-2})
b	characteristic structural (or specimen) size (L)

1. Introduction

The prediction of fatigue life can be performed through two different methods: the first one, based on Paris' law (Paris and Erdogan 1963), relates sub-critical crack growth rate to the stress-intensity factor range, whereas in the second, based on Wöhler's curve (Wöhler 1870), the applied stress range is a function of the number of cycles to failure. These two different approaches can be intimately connected through the use of incomplete self-similarity and fractal modeling, so that anomalous crack-size and specimen-size effects are considered. In the first part of the paper, generalized Paris' and Wöhler's laws are derived in accordance with dimensional analysis and incomplete self-similarity concepts, which are able to provide an interpretation to the various empirical power-laws used.
Subsequently, through the use of a different approach, based on the application of fractal geometry concepts, similar scaling laws are found. In other words, for Paris' law, the assumption of the invasive fractal roughness of crack profile implies the incomplete self-similarity in the problem. Subsequently, on the basis of the scaling laws previously defined, it is possible to obtain the crack-size dependence of fatigue threshold, so that the so-called anomalous behaviour of short cracks with respect to their longer counterparts can be explained.
On the other hand, for Wöhler's curve, the material ligament is considered as a lacunar fractal set which, taking into account a cross-sectional weakening, provides the incomplete self-similarity in the problem, so that the specimen-size dependence of fatigue limit can be put forward.
The hypothesis of the invasive fractal roughness of crack profile provides an explanation for the increment in the fatigue threshold with the crack length, whereas the assumption of the lacunar fractal ligament is able to explain the decrement in the fatigue limit which occurs as the specimen size increases. Eventually, the proposed models are positively compared to experimental data available in the literature.

2. Incomplete self-similarity in the analysis of Paris' law and Wöhler's curve

Let us analyse the phenomenon of fatigue crack growth, according to Paris' law, where the crack growth rate, $\mathrm{d}a/\mathrm{d}N$, is the parameter to be determined (Barenblatt and Botvina 1980). This quantity depends on three different categories of variables, which take into account testing conditions, material properties and a geometric parameter, i.e. the crack length. Thus, we can write the following functional dependence, where the time dependence is neglected:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \Phi(\Delta K_I, 1-R; K_{IC}, \sigma_u, \Delta K_{th}; a) \tag{1}$$

Assuming K_{IC} and σ_u as dimensionally independent quantities, we reduce the number of parameters involved in the problem by applying Buckingham's Π Theorem (Buckingham 1915):

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \left(\frac{K_{IC}}{\sigma_u}\right)^2 \tilde{\Phi}\left(\frac{\Delta K_I}{K_{IC}}, 1-R; \frac{\Delta K_{th}}{K_{IC}}; \frac{{\sigma_u}^2}{{K_{IC}}^2} a\right) \tag{2}$$

The Barenblatt-Botvina's approach assumes an incomplete self-similarity with the following power-law dependencies (Carpinteri and Paggi 2007):

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \frac{(K_{IC})^{2-\alpha_1}}{{\sigma_u}^2} \Delta {K_I}^{\alpha_1} (1-R)^{\alpha_2} \left(\frac{\Delta K_{th}}{K_{IC}}\right)^{\alpha_3} \left(\frac{{\sigma_u}^2}{{K_{IC}}^2} a\right)^{\alpha_4} \quad (3)$$

Eq.(3) can be considered as a generalized Paris' law, where all the main functional dependencies of the parameter C have been considered, thus permitting to capture specific anomalous deviations from the original formulation of Paris' law. It is interesting to note that, for $\alpha_1 = 2$, we obtain the *complete self-similarity*.

If we apply the incomplete self-similarity approach to Wöhlers's functional dependence (Carpinteri and Paggi 2009), we obtain:

$$N = \Psi\left(\Delta\sigma, 1-R; \sigma_u, K_{IC}, \Delta\sigma_{fl}; b\right) \quad (4)$$

where the cycles to failure, N, are the parameter to be determined, and the geometric parameter is the characteristic structural size, b. Similarly to what we have done for Paris' law, Buckingham's Π Theorem permits us to reduce the number of independent parameters, so that Eq.(4) becomes:

$$N = \widetilde{\Psi}\left(\frac{\Delta\sigma}{\sigma_u}, 1-R; \frac{\Delta\sigma_{fl}}{\sigma_u}; \frac{{\sigma_u}^2}{{K_{IC}}^2} b\right) \quad (5)$$

Subsequently, assuming an incomplete self-similarity, we obtain:

$$N = \left(\frac{\Delta\sigma}{\sigma_u}\right)^{\beta_1} (1-R)^{\beta_2} \left(\frac{\Delta\sigma_{fl}}{\sigma_u}\right)^{\beta_3} \left(\frac{{\sigma_u}^2}{{K_{IC}}^2} b\right)^{\beta_4} \quad (6)$$

Eq.(6), representing a generalized Wöhler's relationship, can be compared to Basquin's law $N = (\Delta\sigma_0/\Delta\sigma)^n$ (Basquin 1910).

3. Fractal approach to Paris' law and fatigue threshold

Let us consider the crack-size effect on Paris' law, which can be explained through the concepts of fractal geometry. By modelling the crack profile as an invasive fractal set with a fractal measure $a^* \simeq a^{1+d_\mathcal{G}}$, being $1 + d_\mathcal{G}$ the dimension of the fractal crack profile, the following relationships can be written (Carpinteri 1994):

$$\Delta K_I \simeq \Delta K_I^* a^{\frac{d_\mathcal{G}}{2}} \quad (7a)$$

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \frac{a^{-d_\mathcal{G}}}{1+d_\mathcal{G}} \frac{\mathrm{d}a^*}{\mathrm{d}N} \quad (7b)$$

where ΔK_I^* and $\mathrm{d}a^*/\mathrm{d}N$ are the renormalized stress-intensity factor range and the renormalized crack growth rate, respectively. A scaling law for Paris' parameter, C, can be obtained by rewriting Paris' law in terms of the fractal stress-intensity factor range and the fractal crack growth rate (Carpinteri An. and Spagnoli 2004):

$$C(a) = \frac{C^*}{1+d_\mathcal{G}} a^{-d_\mathcal{G}\left(1+\frac{m}{2}\right)} \quad (8)$$

where C^* is the fractal Paris' parameter. Inserting Eq.(8) into Paris' law, a crack-size dependent fatigue law is obtained:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \frac{C^*}{1+d_\mathcal{G}} a^{-d_\mathcal{G}\left(1+\frac{m}{2}\right)}\, \Delta K_I^m \tag{9}$$

Eq.(9) can be regarded as a modified Paris' law, since C is no longer a material constant. Furthermore, evaluating Eqs.(7a) and (7b) in correspondence of the coordinates of the limit-points of Paris' regime, the following scaling laws can be introduced:

$$v_{th} \simeq \frac{v_{th}^*\ a^{-d_\mathcal{G}}}{1+d_\mathcal{G}} \tag{10a}$$

$$\Delta K_{th} \simeq \Delta K_{th}^*\, a^{\frac{d_\mathcal{G}}{2}} \tag{10b}$$

$$v_{cr} \simeq \frac{v_{cr}^*\ a^{-d_\mathcal{G}}}{1+d_\mathcal{G}} \tag{10c}$$

$$\Delta K_{cr} \simeq (1-R) K_{IC}^*\, a^{\frac{d_\mathcal{G}}{2}} \tag{10d}$$

According to Eqs.(10b) and (10d), the fatigue threshold and the fracture toughness increase with the crack length. Notice that this increment in ΔK_{th} and ΔK_{IC} is consistent with the fractal roughness of crack profile (Carpinteri and Paggi 2011). On the other hand, Eqs.(10a) and (10c) predict a decrease of v_{th} and v_{cr} with the crack length. Thus, the scaling laws previously introduced yield a simultaneous rightward and downward translation of Paris' curve increasing the crack length. Substituting the nominal crack growth rate and the nominal SIF range with the corresponding renormalized parameters, a fractal Cartesian coordinates system is obtained, so that the coordinates of the limit-points of Paris' regime correspond to the fractal quantities entering Eqs.(10a-d). Consequently, through the introduction of fractal coordinates, the set of Paris' curves, obtained varying the crack length, collapse onto a single crack-size independent Paris' curve.

4. Multi-Fractal approach to Paris' law and fatigue threshold

Although the general trend can be captured considering just the fractal approach, a transition occurs from a fractal regime for small cracks to a Euclidian regime for long cracks. Thus, by exploiting the concept of self-affinity, a multi-fractal scaling law should be defined for the Paris parameter C (Paggi and Carpinteri 2009):

$$C_{MF} = C_{MF}^{\infty} \left(1 + \frac{l_{ch}}{a}\right)^{d_\mathcal{G}\left(1+\frac{m}{2}\right)} \tag{11}$$

Remarkably, Eq.(11) predicts that, for very long cracks, crack-size effects disappear and a horizontal asymptote is found. On the other hand, for shorter cracks, the maximum possible disorder is reached and an oblique asymptote, with slope $d_\mathcal{G}\left(1+\frac{m}{2}\right)$, is obtained.

Similarly, a transition of fatigue threshold occurs from the long cracks regime, where the fatigue threshold is a material property, to the short cracks regime. Hence, with the aim to reproduce the experimental data, a multi-fractal scaling law should be considered in order to link the two extreme behaviours:

$$\Delta K_{th} = \Delta K_{th}^{\infty} \left(1 + \frac{l_{ch}}{a}\right)^{-\frac{d_\mathcal{G}}{2}} \tag{12}$$

Whereas ΔK_{th}^{∞} is the upper limit for fatigue threshold, which is reached for very long cracks, for very short cracks ($a \to 0$), the influence of disorder becomes progressively more important and the fatigue threshold tends to vanish. Eventually, notice that l_{ch} is function of the heterogeneity of the material microstructure. Validation of the fatigue threshold scaling law is performed fitting Eq.(12) with available experimental data (Kitagawa and Takahashi 1976,1979). The obtained values for the best-fitting parameters, ΔK_{th}^{∞} , l_{ch}, and d_G, are reported in Fig.1.

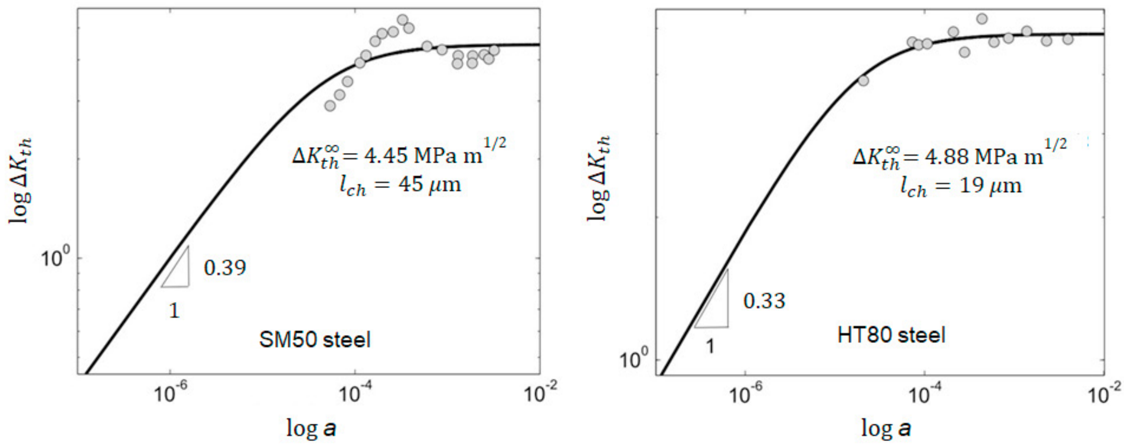

Figure 1. Experimental assessment of fatigue threshold scaling (Kitagawa 1976, 1979): ΔK_{th} measured in MPa m$^{1/2}$; a measured in m.

5. Fractal approach to Wöhler's curve

Let us consider size effects on Wöhler's curve. Introducing fractal concepts to model the lacunarity of cross-section, we can write the following relationship for the stress range (Carpinteri 1994, Carpinteri An. and Spagnoli 2009):

$$\Delta\sigma \simeq \Delta\sigma^* \, b^{-d_\sigma} \tag{13}$$

where $\Delta\sigma^*$ and d_σ are the fractal stress range and the fractal dimension decrement, respectively. Notice that Eq.(13) predicts a decrease of fatigue strength with the specimen size, N being the same. Similarly to what was done for Paris' curve, it is possible to evaluate Eq.(13) in correspondence of the limit-points of Wöhler's curve, so that the following scaling laws for $\Delta\sigma_u$ and $\Delta\sigma_{fl}$ are obtained (Carpinteri An. et al. 2002):

$$\Delta\sigma_u \simeq (1-R)\sigma_u^* \; b^{-d_\sigma} \tag{14a}$$

$$\Delta\sigma_{fl} \simeq \Delta\sigma_{fl}^* \; b^{-d_\sigma} \tag{14b}$$

Consistently with the concept of lacunar fractality, we obtain a negative trend for the ultimate tensile strength and the fatigue limit by increasing the specimen size. Thus, Eqs.(14a) and (14b) yield a downward translation of Wöhler's curve increasing the structural size. In other words, only a vertical translation is expected since N_{cr} and N_{fl} are dimensionless parameters. Substituting the nominal stress range with the corresponding renormalized parameter, we obtain a fractal Cartesian coordinates system, so that the set of Wöhler's curves, obtained varying the structural size, collapse onto a single specimen-size independent Wöhler's curve.
Validation of the fatigue limit scaling law is performed fitting Eq.(14b) with available experimental data obtained by Hatanaka et al. (1983). Considering two different materials and dog-bone specimens of 8, 20, 30 and 40 mm of diameter, tests were carried out through a rotating bending machine. The obtained values for the best-fitting

parameters, $\Delta\sigma_{fl}^{*}$ and d_{σ}, are reported in Fig.2. Notice that, in accordance to LEFM, the dimensional decrement, d_{σ}, is always lower than 1/2.

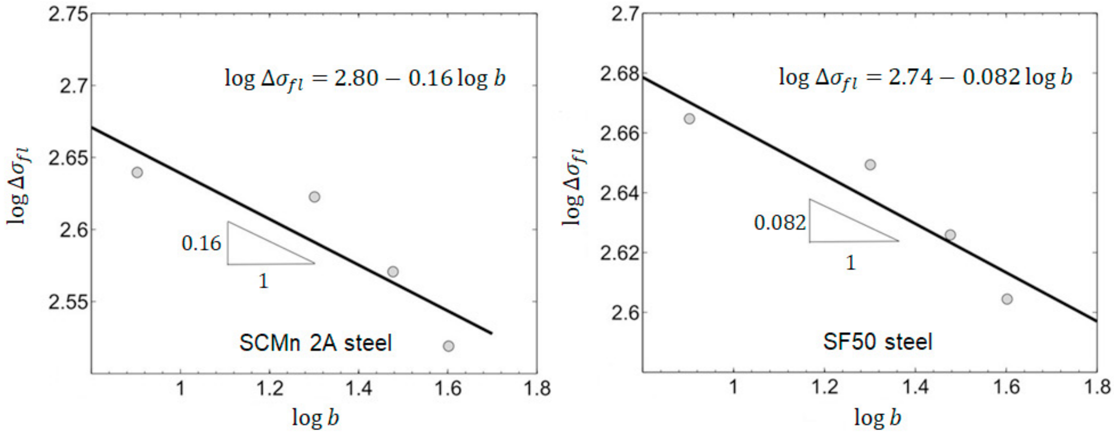

Figure 2. Experimental assessment of fatigue limit scaling [16]: $\Delta\sigma_{fl}$ measured in MPa; b measured in mm.

Conclusions

We have proposed the application of incomplete self-similarity and fractal approaches to capture the scale effects on Paris' law and Wöhler's curve. In particular, by modelling the crack profile as an invasive fractal set, it is possible to obtain crack-size effects on fatigue threshold, whereas the hypothesis of lacunar fractal ligament permits us to determine specimen-size effects on fatigue limit. Eventually, a positive comparison with experimental data available in the literature is shown.

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References

Basquin O.H. "The exponential law of endurance tests", *ASTM Proceedings* 10 (1910): 625-630.

Barenblatt G.I., and L. Botvina "Incomplete self-similarity of fatigue in the linear range of crack growth", *Fatigue & Fracture of Engineering Materials & Structures* 3 (1980): 193-202.

Buckingham E. "Model experiments and the forms of empirical equations", *ASME Transaction* 37 (1915): 263-296.

Carpinteri A. "Scaling laws and renormalization groups for strength and toughness of disordered materials", *International Journal of Solids and Structures* 31 (1994): 291-302.

Carpinteri A., and M. Paggi "Self-similarity and crack growth instability in the correlation between the Paris' constants", *Engineering Fracture Mechanics* 74 (2007): 1041-1053.

Carpinteri A., and M. Paggi "A unified interpretation of the power laws in fatigue and the analytical correlations between cyclic properties of engineering materials", *International Journal of Fatigue* 31 (2009): 1524-1531.

Carpinteri A., and M. Paggi "Dimensional analysis and fractal modeling of fatigue crack growth", *Journal of ASTM International* 8 (2011): 1-13.

Carpinteri An., A. Spagnoli, and S. Vantadori "An approach to size effect in fatigue of metals using fractal theories", *Fatigue & Fracture Engineering Materials & Structures* 25 (2002): 619-627.

Carpinteri An., and A. Spagnoli "A fractal analysis of size effect on fatigue crack growth", *International Journal of Fatigue* 26 (2004): 125-133.

Carpinteri An., and A. Spagnoli "Size effect in S-N curves: A fractal approach to finite-life fatigue strength", *International Journal of Fatigue* 31 (2009): 927-933.

Hatanaka K., S. Shimizu and A. Nagae "Size effect on rotating bending fatigue in steels", *Bulletin of JSME* 26 (1983): 1288-1295.

Kitagawa H., and S. Takahashi "Applicability of fracture mechanics to very small cracks or the cracks in the early stage", *Proceedings of the Second International Conference on Mechanical Behavior of Materials*. American Society for Metals 1976.

Kitagawa H., and S. Takahashi "Fracture mechanical approach to very small fatigue cracks and to the threshold", *Transaction of the Japan Society of Mechanical Engineers* 45 (1979): 1289-1303.

Paggi M., and A. Carpinteri "Fractal and multifractal approaches for the analysis of crack-size dependent scaling laws in fatigue", *Chaos, Solitons and Fractals* 40 (2009): 1136-1145.

Paris P., and F. Erdogan "A critical analysis of crack propagation laws", *Journal of Basic Engineering Transaction ASME* 85 (1963): 528-534.

Wöhler A. "Über die Festigkeitsversuche mit Eisen und Stahl", *Zeitschrift für Bauwesen* 20 (1870): 73-106.