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*Review Article*

# Topologizing Homeomorphism Groups

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This paper surveys topologies, called admissible group topologies, of the full group of self-homeomorphisms $\mathscr{H}(X)$ of a Tychonoff space $X$, which yield continuity of both the group operations and at the same time provide continuity of the evaluation function or, in other words, make the evaluation function a group action of $\mathscr{H}(X)$ on $X$. By means of a compact extension procedure, beyond local compactness and in two essentially different cases of rim-compactness, we show that the complete upper-semilattice $\mathscr{L}_H(X)$ of all admissible group topologies on $\mathscr{H}(X)$ admits a least element, that can be described simply as a set-open topology and contemporaneously as a uniform topology. But, then, carrying on another efficient way to produce admissible group topologies in substitution of, or in parallel with, the compact extension procedure, we show that rim-compactness is not a necessary condition for the existence of the least admissible group topology. Finally, we give necessary and sufficient conditions for the topology of uniform convergence on the bounded sets of a local proximity space to be an admissible group topology. Also, we cite that local compactness of $X$ is not a necessary condition for the compact-open topology to be an admissible group topology of $\mathscr{H}(X)$.

## 1. Introduction

The "incipit" of the homeomorphism group theory resides in the early seminal work of Birkoff [1]. With an apparent simplicity joined with an impressive bright proof strategy, Birkoff positively answered to the query: *there exists a topology on the full self-homeomorphism group $\mathscr{H}(X)$ of a compact metric space X which makes it into a topological group and a subspace of the Hilbert cube?* The area, originating from [1], has initially evolved relaxing the compactness condition by passing from the class of compact metric spaces, as in Birkoff, to the class of $T_2$ locally compact spaces, as in Arens [2]. In [2] Arens focused on those topologies which yield continuity of both the group operations, product and inverse function, and also, at the same time, provide continuity of the evaluation function $e : (f, x) \in \mathscr{H}(X) \times X \to f(x) \in X$ and posed the problem of the existence for noncompact spaces $X$ of the least element in the upper-semilattice (ordered by the usual inclusion) $\mathscr{L}_H(X)$ of all topologies with these two features, that he called *admissible group topologies*. Of course, there are many different ways to topologize $\mathscr{H}(X)$. For instance, it can be endowed with the subspace topology induced by any of all known function space topologies. Nevertheless, following Birkoff and Arens, we also focused our investigation on topologies which make $\mathscr{H}(X)$ a topological group and the evaluation function a group action of $\mathscr{H}(X)$ on $X$ and, rather obviously, looked at uniform topologies. In fact, uniform topologies make continuous the evaluation function. Furthermore, they make continuous both product and inverse function at $(i, i)$ and at $i$, respectively, where $i$ is the identity function of $X$. Being well aware that if $X$ is compact $T_2$, then the compact-open topology on $\mathscr{H}(X)$, which is also the uniform topology derived from the unique totally bounded uniformity on $X$, is an admissible group topology, we searched the admissible group topologies on $\mathscr{H}(X)$ by means of a compact extension procedure. Whenever $X$ is Tychonoff, since any self-homeomorphism of $X$ continuously extends to $\beta(X)$, the Stone-Čech compactification of $X$, then $\mathscr{H}(X)$ embeds as a subgroup in $\mathscr{H}(\beta X)$. Analogously, whenever $X$ is locally compact $T_2$, $\mathscr{H}(X)$ embeds as a subgroup in $\mathscr{H}(X_\infty)$, where $X_\infty$ is the one-point compactification of $X$. Thereby, the relativization to $\mathscr{H}(X)$ of the compact-open topology on $\mathscr{H}(\beta X)$ and that on $\mathscr{H}(X_\infty)$ are both admissible group topologies. Accordingly, the previous significant examples strongly suggest investigating those uniform topologies on $\mathscr{H}(X)$ derived from totally bounded uniformities on $X$ whose uniform completion is a $T_2$-compactification of $X$ to which any self-homeomorphism of $X$ continuously extends. We say that a $T_2$-compactification $\gamma(X)$ of $X$ *has the lifting property* if every self-homeomorphism of $X$ continuously extends to $\gamma(X)$. Whenever $\gamma(X)$ is a $T_2$-compactification of $X$ with the lifting property, the homeomorphism group $\mathscr{H}(X)$ embeds as subgroup in $\mathscr{H}(\gamma(X))$ equipped with the compact-open topology. Thus, the induced topology, that is, the topology of uniform convergence determined by the unique totally bounded uniformity associated with $\gamma(X)$, is an admissible group topology. Furthermore, the compact extension procedure appears as a powerful method to prove the existence of a least admissible group topology. The problem of the existence of a least element in $\mathscr{L}_H(X)$ for non-compact space $X$ goes back to Arens [2], who proved that, if $X$ is locally compact $T_2$, then the $g$-topology, which is generated by the collection of all sets of the type:

$$[C, W] = \{f \in \mathscr{H}(X) : f(C) \subseteq W\}, \qquad (1)$$

where $C$ is closed, $W$ is open in $X$ and $C$ or $X - W$ is compact, is the least admissible group topology. He also proved that, with the additional property of local connectedness for $X$, the $g$-topology agrees with the compact-open topology. In the direction of extending Arens' result beyond the class of locally compact spaces, it comes as very natural idea to *weaken local compactness into rim-compactness*, since, to a rim-compact $T_2$ space $X$ is attached the Freudenthal compactification $F(X)$ [3–5], to which any self-homeomorphism continuously extends [6]. A space $X$ is *rim-compact* if and only if any of its points admits arbitrarily small neighborhoods with compact boundaries. The group topology $\tau_F$ induced by $F(X)$ on $\mathscr{H}(X)$ has a simple description as the set-open topology admitting like subbasic open sets all sets $[C, W]$, as in (1), but where now $C$ is a closed set with compact boundary in $X$ and again $W$ is open in $X$. However, rim-compactness by itself is not enough to assure the admissible group topology $\tau_F$ determined by the Freudenthal compactification to be the least element in $\mathscr{L}_H(X)$. As for the space of natural numbers $\mathbb{N}$, for instance, the Freudenthal compactification $F(\mathbb{N})$ induces on $\mathscr{H}(\mathbb{N})$ the closed-open topology which differs from the compact-open topology which in the case is the $g$-topology. Nevertheless, we performed the result in two substantially different cases of rim-compactness: the former one, where $X$ is rim-compact, $T_2$, and locally connected, [7]; the latter one, in the first step, where $X$ is the rational number space $\mathbb{Q}$ equipped with the Euclidean topology and, next, where $X$ is a product of $T_2$ zero-dimensional spaces each satisfying the property: *any two nonempty clopen subspaces are homeomorphic*, [8]. In the former, whenever $X$ is a locally connected, rim-compact $T_2$ space, we construct in two steps a $T_2$-compactification of $X$, $\gamma(X)$, in which $\gamma(X)-X$ zero-dimensionally embeds and to which any self-homeomorphism of $X$ continuously extends. In the first step, $X$ comes densely embedded into the disjoint union of the Freudenthal compactifications of its components, $c(X)$, which is a locally compact $T_2$ space to which any self-homeomorphism of $X$ continuously extends. In the second step, in turn $c(X)$ comes embedded in its one-point compactification $\gamma(X)$, and, as a matter of fact, $\tau_{\gamma(X)}$ is the least element of $\mathscr{L}_H(X)$, that can be described as the set-open topology determined by all closed sets with compact boundaries contained in some component of $X$. The latter, the rational one, is very singular indeed. First, since any two non-empty open subspaces in $\mathbb{Q}$ are homeomorphic, $\mathscr{L}_H(\mathbb{Q})$ is a very big object. Next, Arens proved *"given an admissible topology for the group of homeomorphisms $\mathscr{H}$ of the rational number system, one can construct another admissible topology for $\mathscr{H}$ which is not weaker* (but now not stronger) *than the first."* And more, the minimal convergence structure on $\mathscr{H}(\mathbb{Q})$ which provides continuity of the evaluation function and both the group operations, denoted by $g$-convergence and assigned by requiring

$$\{f_\lambda\}_{\lambda\in\Lambda} \longrightarrow f \quad \text{iff } \{f_\lambda\}_{\lambda\in\Lambda} \xrightarrow{\text{c-c}} f, \qquad \{f_\lambda^{-1}\}_{\lambda\in\Lambda} \xrightarrow{\text{c-c}} f^{-1}, \qquad (2)$$

where $\xrightarrow{\text{c-c}}$ stands for continuous convergence, unfortunately is not topological [9, 10]. Therefore, in the beginning one has no clear indication and fluctuates between arguments promoting existence or nonexistence in $\mathscr{L}_H(\mathbb{Q})$ of a least element. What Arens wrote seems to contain a subliminal message of nonexistence. On the contrary, checking in details his construction or completing in their minimal group topologies the uniform topologies induced by non-Archimedean metric compactifications of $\mathbb{Q}$ anytime one runs into the closed-open topology which is induced by the Stone-Čech compactification which in the rational case is also the Freudenthal compactification [11]. Two arguments seem to promote the existence. On one side, the fine or Whitney topology on $\mathscr{H}(\mathbb{Q})$ determines an admissible group topology on $\mathscr{H}(\mathbb{Q})$ strictly finer than the closed-open topology: so, the closed-open topology is not too fine. On the other side, the Stone-Čech compactification is the only one $T_2$-compactification of $\mathbb{Q}$ with the lifting property: so, the closed-open topology seems enough fine. In conclusion, $\mathscr{H}(\mathbb{Q})$, even though it admits no least admissible topology [2], it still supports the clopen-open topology as the least admissible group topology. This issue is essentially achieved by the property: any two non-empty clopen subspaces of $\mathbb{Q}$ are homeomorphic, as it is derived from the topological characterization of $\mathbb{Q}$. Therefore, following the rational trace, we focus just on the class of zero-dimensional spaces satisfying the property: *any two non-empty clopen subspaces are homeomorphic* and their products. All zero-dimensional spaces of diversity one [12] and all compact zero-dimensional spaces of diversity two [13] are of this kind. Among them we recognize as leaders the rationals, the irrationals, the Baire spaces, and the Cantor discontinuum. In all previous results the least element in $\mathscr{L}_H(X)$ is achieved as a uniform topology that can be viewed also as a set-open topology. Accordingly, in the approach to the zero-dimensional case we explored the class of bases of clopen sets in $X$ to select the ones that determine a clopen-open topology that is an admissible group topology induced by a $T_2$-compactification of $X$ with the lifting property. The bases of clopen sets of $X$ closed under complements and invariant under homeomorphisms of $X$ emerge as the right tool: they make the match. We show that *if $X = \prod_{i\in I} X_i$ is* *a product of zero-dimensional spaces each of which satisfies the property: any two non-empty clopen subspaces are homeomorphic, then $\mathscr{L}_H(X)$ is a complete lattice*. Besides, its least element is a clopen-open topology with the left, the right, and the two-sided uniformities all non-Archimedean, thus zero-dimensional [8, 11, 14].

As rim-compactness is a weak and peripherical compactness property, one might think any further relaxation as impossible. But, we show that rim-compactness for $X$ is not a necessary condition for the existence of the least admissible group topology on $\mathscr{H}(X)$. More precisely, we show that the full group of self-homeomorphisms of the product space $\mathbb{R} \times \mathbb{Q}$, where $\mathbb{R}$ and $\mathbb{Q}$ are the sets of the real and rational numbers, respectively, both carrying the Euclidean topology, admits a least admissible group topology even though notoriously $\mathbb{R}\times\mathbb{Q}$ is not rim-compact, [15]. To achieve this result we carry on another efficient way to produce admissible group topologies in substitution of, or in parallel with, the compact extension procedure. By exploring the literature on the different ways to control efficiently closeness between self-homeomorphisms of a Tychonoff space, we arrive at several different remarkable ideas: drawn by covers yielding the open-cover topology [16]; by uniformities yielding uniform topologies [16–18]; or, in the metric case, also by the compatible metrics yielding the limitation topology [1, 19] and by the continuous functions to the positive real numbers yielding the fine or Whitney topology [20]. Namely, as for the metric setting, three of the examined methods collapse in just one. As a matter of fact, in the metric setting there are only two substantially different options to control closeness in $\mathscr{H}(X)$. An effective control of closeness can be managed, in one way, via the metrics compatible with $X$ and, in the other way, via the continuous functions from $X$ to the positive real numbers. The idea of how to discriminate comes from the rationals. The clopen-open topology on $\mathscr{H}(\mathbb{Q})$ is the uniform topology induced by the Čech uniformity of $\mathbb{Q}$, which in turn is the finest totally bounded uniformity compatible with $\mathbb{Q}$. Consequently, being $\mathbb{Q}$ metrisable and separable, thus admitting compatible totally bounded metrics, the clopen-open topology on $\mathscr{H}(\mathbb{Q})$ can be reformulated as the supremum of all uniform topologies induced by totally bounded metrics compatible with $\mathbb{Q}$. On the other hand, the fine or Whitney topology on $\mathscr{H}(\mathbb{Q})$ is a group topology [15]. Hence, we demonstrate that it collapses on just the fine uniform topology [16], which in the case is the supremum of all uniform topologies deriving from metrics compatible with $\mathbb{Q}$. These results pointout the suprema of uniform topologies deriving from metrics compatible with $X$, running in a given class, as the right tool. The presentation in [19] of the fine uniform topology is a compelling motivation to generalise it in order to produce new admissible group topologies on $\mathscr{H}(X)$ and its subgroups. Given a class $\mathscr{D}(X)$ of metrics compatible with $X$ and a group $\mathscr{G}(X)$ of self-homeomorphisms of $X$, we refer to the uniform topology induced on $\mathscr{G}(X)$ by the supremum of the uniformities on $X$ associated with the metrics in $\mathscr{D}(X)$ as the *fine uniform topology on $\mathscr{G}(X)$ associated with, or generated by, $\mathscr{D}(X)$*. Obviously, in this way the fine uniform topology is generated by the full homeomorphism group $\mathscr{H}(X)$ and by the class of all metrics compatible with $X$. Blending in a group of self-homeomorphisms $\mathscr{G}(X)$ with a class $\mathscr{D}(X)$ of metrics compatible with $X$ originates a new class of metrics compatible with $X$, which reveals interesting and useful features. A class $\mathscr{D}(X)$ is invariant under the group $\mathscr{G}(X)$ if, whenever the distance between every two points of $X$ is measured by a metric in $\mathscr{D}(X)$ applied to the pair of their images under a homeomorphic deformation of $X$ belonging to $\mathscr{G}(X)$, the new produced metric in this way belongs once again to $\mathscr{D}(X)$. We show that if $\mathscr{D}(X)$ is $\mathscr{G}(X)$-invariant, then the fine uniform topology induced by $\mathscr{D}(X)$ on $\mathscr{G}(X)$ is a group topology. Justified by this result, we refer to the fine uniform topology on $\mathscr{G}(X)$ generated by the minimal $\mathscr{G}(X)$-invariant enlargement of $\mathscr{D}(X)$ as the *fine group topology on $\mathscr{G}(X)$ generated by $\mathscr{D}(X)$*. A same group blended in with different classes of metrics gives rise to different fine group topologies. As for the rational case, for instance, the fine group topology generated on $\mathscr{H}(\mathbb{Q})$ by all totally bounded metrics compatible with $\mathbb{Q}$ and the fine group topology generated on $\mathscr{H}(\mathbb{Q})$ by all metrics compatible with $\mathbb{Q}$ are distinct from each other. Namely, the former one coincides with the clopen-open topology of $\mathscr{H}(\mathbb{Q})$ [7] and the latter one with the fine or Whitney topology on $\mathscr{H}(\mathbb{Q})$. And the clopen-open topology and the fine or Whitney topology on $\mathscr{H}(\mathbb{Q})$ do not agree, being the fine or Whitney topology strictly stronger than the clopen-open topology [7]. Finally, we show that *any admissible group topology on $\mathscr{H}(\mathbb{R} \times \mathbb{Q})$ is stronger than the fine group topology determined from the class of metrics on $\mathbb{R}\times\mathbb{Q}$ of the type $d_1 \times d_2$ as $d_1$ is the stereographic metric on $\mathbb{R}$ and $d_2$ runs over all totally bounded metrics on $\mathbb{Q}$* [15].

The issues so far discussed lead us to show: *a uniform topology on $\mathscr{H}(X)$ derived from a totally bounded uniformity on $X$ is a group topology (hence an admissible group topology) if and only if it is derived from a totally bounded uniformity of $X$ associated with a $T_2$-compactification of $X$ with the lifting property* [21].

On the other hand, if $X$ is locally compact $T_2$, then the compact-open topology on $\mathscr{H}(X)$, which is also the topology of uniform convergence on compacta derived from any uniformity on $X$, is admissible and yields continuity of the product function. Unfortunately in general, the compact-open topology does not provide continuity of the inverse function. But, with the following additional property: $(*)$ *any point of $X$ has a compact connected neighborhood*, due to Dijkstra [22], the compact-open topology becomes a group topology and, as a consequence, the least admissible group topology of $\mathscr{H}(X)$. According to this issue the compact-open topology on $\mathscr{H}(X)$ is quoted as the most eligible one if $X$ is a manifold of finite dimension or $X$ is an infinite dimensional manifold modelled on the Hilbert cube [23]. In looking for topologies of uniform convergence on members of a given family, containing all compact sets, which are admissible group topologies, we focus beyond local compactness. In order to do so, we follow as suggestive example that of bounded sets of an infinite dimensional normed vector space carrying as proximity the metric proximity associated with the norm. We emphasise first that local compactness of $X$ is equivalent to the family of compact sets of $X$ being a *boundedness* of

$X$ [24], which, jointly with any EF-proximity of $X$, gives a *local proximity space* [25]. As a consequence, we make this particular case fall within the more general one in which compact sets are substituted with bounded sets in a local proximity space, while the property $(*)$ is replaced by the following one: $(**)$ *for each nonempty bounded set $B$ there exist a finite number of connected bounded sets $B_1, \ldots, B_n$ such that* $B \ll_\delta \operatorname{int}(B_1) \cup \cdots \cup \operatorname{int}(B_n)$. So doing, we achieve the following issue: *if $(X, \mathscr{B}, \delta)$ is a local proximity space with the property $(**)$ and any homeomorphism of $X$ preserves both boundedness and proximity, then the topology of uniform convergence on bounded sets derived from the unique totally bounded uniformity associated with $\delta$ is an admissible group topology on $\mathscr{H}(X)$*.

The uniform topologies so far considered are totally bounded, and the concept of totally bounded uniformity can be dually recast as EF-proximity and then as strong inclusion, [26]. As a consequence, it is worthwhile to reformulate uniform topologies derived from totally bounded uniformities as proximal set-open topologies. Taking up the common proximity nature of set-open topologies as the compact-open topology, the bounded-open topology and the topology of convergence in proximity, Naimpally, jointly with the author, introduced as unifying tool the notion of proximal set-open topology, simply replacing the usual inclusion with a strong one [27]. The *proximal set-open topology relative to a network $\alpha$ and an EF-proximity $\delta$*, designed by the acronym $\mathrm{PSOT}_{\alpha,\delta}$ or, simply, $\mathrm{PSOT}_\delta$, when $\alpha$ is the set $\mathrm{CL}(X)$ of all non empty closed subsets of $X$, is that having as subbasic open sets the ones of the following form:

$$[C, A]_\delta := \{f \in \mathscr{H}(X) : f(C) \ll_\delta A\}, \tag{3}$$

where $C$ runs through $\alpha$, $A$ runs through all open subsets in $Y$, and $\ll_\delta$ is the strong inclusion naturally associated with $\delta$. Whenever $\alpha$ is a closed and hereditarily closed network of $X$, then $\mathrm{PSOT}_{\alpha,\delta}$ agrees with the topology of uniform convergence relative to $\alpha$ derived from the unique totally bounded uniformity naturally associated with $\delta$. Consequently, $\mathrm{PSOT}_\delta$ agrees with the uniform topology on $\mathscr{H}(X)$ derived from the unique totally bounded uniformity compatible with $\delta$. By endowing $\mathscr{H}(X)$ with a PSOT, our two previous results can be reformulated as follows. The former, when $\alpha$ is $\mathrm{CL}(X)$, is: *a $\mathrm{PSOT}_\delta$ is a group topology on $\mathscr{H}(X)$ if and only if it is $\mathrm{PSOT}_{\delta'}$ relative to a proximity $\delta'$ whose Smirnov compactification has the lifting property*. After recalling that the concepts of local proximity on a Tychonoff space $X$ and $T_2$ local compactification of $X$ are dual [25] and a $T_2$ local compactification of $X$ has *the lifting property* if and only if any self-homeomorphism of $X$ continuously extends to it, then the latter result, when $\alpha$ is a boundedness of $X$ which jointly with $\delta$ gives a local proximity space [25], can be recasted as: *if $(X, \mathscr{B}, \delta)$ is a local proximity space with the property $(**)$ and the $T_2$ local compactification of $X$ naturally associated with it has the lifting property, then $PSOT_{\mathscr{B},\delta}$ is an admissible group topology on $\mathscr{H}(X)$*[21].

Again in local compactness, in the paper [28], unpublished as per my knowledge, Wicks gave necessary and sufficient conditions for the compact-open topology being a group topology by using nonstandard methods on one side and action on hyperspace on the other side, which is so inspiring [29]. But, under local compactness is Dijkstra's property a necessary condition for the compact-open topology being a group topology? And is local compactness a necessary condition for the compact-open topology being a group topology that more makes the evaluation map jointly continuous? In both cases we give a negative answer by using as counterexample first a model of locally compact topologist's comb, a typical space that is not locally connected, and then a nonlocally compact one. Wicks proved that $\mathscr{H}(X)$ equipped with the compact-open topology $\tau_{\mathrm{c.o}}$ being a topological group is equivalent to joint continuity of the evaluation map $E : (f, C) \in \mathscr{H}(X) \times \mathrm{CL}X \to f(C) \in \mathrm{CL}X$ with respect to $\tau_{\mathrm{c.o}}$ and the Fell hypertopology $\tau_F$. Since for the compact-open topology three different formulations as set-open topology, as the topology of uniform convergence on compacta, and also as proximal set-open topology can be displayed, three possible generalizations in topology, proximity, and uniformity arise from those. After analyzing the compact case, we improve and contemporaneously generalize the compact case in the topological, uniform, and proximal frameworks by replacing the compact-open topology with a set-open topology based on a Urysohn family, with a topology of uniform convergence on a uniformly Urysohn family, with a proximal set-open topology relative to a proximity and a boundedness giving a local proximity space, respectively. Finally, we show that *the topologicality of $\mathscr{H}(X)$ is equivalent to topologicality of the evaluation map $E : (f, C) \in \mathscr{H}(X) \times CLX \to f(C) \in CLX$, as in the Wicks case, in each generalized case*. We limit only to cite this final result since the paper containing it and others has to be published [29].

## 2. Background and Works

Firstly, we give some useful background and summarise a number of already stated basic facts. Definitions and terminology quoted below are drawn by [26, 30–33].

*2.1. Topologies on $\mathscr{H}(X)$.* Let $X$ be a Tychonoff space, $\mathscr{H}(X)$ the group of all self-homeomorphisms of $X$, and $e : (f, x) \in \mathscr{H}(X) \times X \to f(x) \in X$ the evaluation map. We start by recalling some necessary background about continuous convergence and related topics. Remember that if $(\Lambda, \leq)$, $(M, \leq')$ are directed sets, then $\Lambda \times M$ admits as a direction $(\leq)$ defined by

$$(\lambda, \mu)(\leq)(\overline{\lambda}, \overline{\mu}) \Longleftrightarrow \lambda \leq \overline{\lambda},\ \mu \leq' \overline{\mu}. \tag{4}$$

Whenever $\{f_\lambda\}_{\lambda\in\Lambda}$ is a net in $\mathscr{H}(X)$ and $\{x_\mu\}_{\mu\in M}$ is a net in $X$, then $\{f_\lambda(x_\mu)\}_{(\lambda,\mu)\in\Lambda\times M}$ stands for the net in $X$ determined by $\Lambda \times M$ with direction $(\leq)$.

A net $\{f_\lambda\}_{\lambda\in\Lambda}$ in $\mathscr{H}(X)$ *continuously converges* to $f \in \mathscr{H}(X)$, in short $\{f_\lambda\}_{\lambda\in\Lambda} \xrightarrow{\mathrm{c.c}} f$, if and only if whenever a net $\{x_\mu\}_{\mu\in M} \to x$ in $X$, then $\{f_\lambda(x_\mu)\}_{(\lambda,\mu)\in\Lambda\times M} \to f(x)$ in $X$.

Topologies on $\mathscr{H}(X)$ providing continuity of the evaluation function $e : (f, x) \in \mathscr{H}(X) \times X \rightarrow f(x) \in X$ are called *admissible*.

(i) Any admissible topology on $\mathscr{H}(X)$ induces a convergence that implies continuous convergence [34].

(ii) Let $(X, \mathscr{U})$ be a Weil uniform space [17, 31, 33]. Then the topology of uniform convergence induced by $\mathscr{U}$ on $\mathscr{H}(X)$ is admissible [34].

Topologies on $\mathscr{H}(X)$ compatible with the group operations are called *group topologies*.

(i) Let $(X, \mathscr{U})$ be a Weil uniform space. Then the topology of uniform convergence induced by $\mathscr{U}$ on $\mathscr{H}(X)$ provides continuity of the product at $(i, i)$ and of the inverse function at $i$, where $i$ is the identity function of $X$.

(ii) Let $X$ be a compact $T_2$ space. Then the compact-open topology on $\mathscr{H}(X)$ is an admissible group topology on $\mathscr{H}(X)$. Furthermore, it is exactly the topology of continuous convergence [1, 2].

(iii) Let $(X, d)$ be a compact metric space and $\widehat{d}$ the *supremum metric* determined from $d$ on $\mathscr{H}(X)$ by the usual formula

$$\widehat{d}(f, g) := \sup \{d(f(x), g(x)) : x \in X\}. \tag{5}$$

Then the metric $d^*$ defined by the formula

$$d^*(f, g) = \max\{\widehat{d}(f, g), \widehat{d}(f^{-1}, g^{-1})\}, \quad f, g \in \mathscr{H}(X) \tag{6}$$

induces, as $\widehat{d}$ does, the compact-open topology on $\mathscr{H}(X)$ and metrizes the two-sided uniformity so making $\mathscr{H}(X)$ into a Polish space [1].

(iv) Of course, every admissible group topology makes the evaluation function as a group action.

(v) There is always on $\mathscr{H}(X)$ a minimal convergence structure which provides continuity of the evaluation function and both the group operations. It is assigned by the formula

$$\{f_\lambda\}_{\lambda\in\Lambda} \longrightarrow f \quad \text{iff } \{f_\lambda\}_{\lambda\in\Lambda} \xrightarrow{\text{c·c}} f, \qquad \{f_\lambda^{-1}\}_{\lambda\in\Lambda} \xrightarrow{\text{c·c}} f^{-1}. \tag{7}$$

The natural notation for it is as $g$-convergence. The $g$-convergence is not topological in general [9, 10].

(vi) Of course, every admissible group topology on $\mathscr{H}(X)$ induces a convergence which implies the $g$-convergence.

Let $\mathscr{L}_H(X)$ stand for the set of all admissible group topologies on $\mathscr{H}(X)$ ordered by the usual inclusion. Since any topology finer than an admissible one is in its turn admissible and the join of subsets of group topologies is again a group topology, $\mathscr{L}_H(X)$ is a complete upper semilattice. Obviously, the discrete topology is in $\mathscr{L}_H(X)$ and is, indeed, the maximum. The existence in $\mathscr{L}_H(X)$ of the minimum is equivalent to $\mathscr{L}_H(X)$ being a complete lattice. The problem of the existence of a least element in $\mathscr{L}_H(X)$ for noncompact space $X$ goes back to Arens [2], who proved that:

(i) if $X$ is locally compact $T_2$, then the $g$-topology, which is generated by the collection of all sets:

$$[C, W] = \{f \in \mathscr{H}(X) : f(C) \subseteq W\}, \tag{1*}$$

where $C$ is closed, $W$ is open in $X$ and $C$ or $X - W$ is compact, is the least admissible group topology. He also proved that, with the additional property of local connectedness for $X$, the $g$-topology agrees with the compact-open topology.

Secondly, we differentiate the topologies on $\mathscr{H}(X)$ according to their derivation from the following: uniformities yielding uniform topologies, covers yielding the open-cover topology, the compatible metrics yielding the limitation topology, and the continuous functions to the positive reals yielding the fine or Whitney topology.

*2.2. How Uniformities on X Yield a Uniform Control on $\mathscr{H}(X)$.* Let $X$ stand for a Tychonoff space. Every Weil uniformity $\mathscr{U}$ compatible with $X$ induces on $\mathscr{H}(X)$ the *uniformity of the uniform convergence with respect to* $\mathscr{U}$, which admits as basic diagonal neighborhoods the sets

$$\widehat{U} := \{(f, g) \in \mathscr{H}(X) \times \mathscr{H}(X) : (f(x), g(x)) \in U, \forall x \in X\} \tag{8}$$

as $U$ runs over all diagonal neighborhoods in $\mathscr{U}$. The uniformity of the uniform convergence w.r.t. $\mathscr{U}$ on $\mathscr{H}(X)$ generates in its turn the *uniform topology* or the *topology of the uniform convergence w.r.t.* $\mathscr{U}$, that we will denote by $\tau_U$. Whenever the uniformity $\mathscr{U}$ is metrisable and $d$ is a bounded metric compatible with it, then the uniform topology $\tau_U$ is just the topology of the supremum metric $\widehat{d}$. The uniform topology induced on $\mathscr{H}(X)$ by the finest uniformity compatible with $X$ is usually referred to as the *fine uniform topology on* $\mathscr{H}(X)$. Following [16], we will denote it by $\tau_f$. Moreover, the supremum of uniform topologies on $\mathscr{H}(X)$ relative to Weil uniformities on $X$, running in a given class, agrees with the uniform topology relative to the supremum uniformity in that class. Finally, if $X$ is a metrisable separable space, which thus admits compatible totally bounded metrics, then the uniform topology on $\mathscr{H}(X)$ induced by the Čech uniformity of $X$, which is also the finest totally bounded uniformity compatible with $X$, is the supremum of all uniform topologies deriving from totally bounded metrics compatible with $X$.

*2.3. Closeness by Covers: The Open-Cover Topology.* Let $\mathscr{A}$ be an open cover of $X$ and $f, g \in \mathscr{H}(X)$. Then $f$ is said to be *$\mathscr{A}$-close* to $g$ if for each $x$ in $X$ there exists some $U \in \mathscr{A}$ such that both $f(x), g(x)$ belong to $U$. At any $f \in \mathscr{H}(X)$ the *open-cover topology* admits as arbitrarily small neighborhoods the sets of the form:

$$\mathscr{A}(f) := \{g \in \mathscr{H}(X) : g \text{ is } \mathscr{A}\text{—close to } f\}, \tag{9}$$

with $\mathscr{A}$ being an open cover of $X$ [16].

*2.4. Closeness by Real Functions in the Metric Case: The Fine or Whitney Topology.* Let $(X, d)$ stand for a metric space. At any $f \in \mathscr{H}(X)$ the *fine* or *Whitney topology* on $\mathscr{H}(X)$, that we will denote by $\tau_W$, admits as arbitrarily small neighborhoods the following sets, also called *tubes*:

$$T(f, \varepsilon) := \{g \in \mathscr{H}(X) : d(f(x), g(x)) < \varepsilon(x), \forall x \in X\}, \tag{10}$$

$\varepsilon$ being a continuous function from $X$ to the positive real numbers.

It is known that, having been given a topological characterisation, the fine topology $\tau_W$ is independent of the metric $d$ [20].

*2.5. Closeness by Metrics: The Limitation Topology.* Let $(X, d)$ stand for a metric space again. At any $f \in \mathscr{H}(X)$ the *limitation topology* on $\mathscr{H}(X)$ admits as arbitrarily small open neighborhoods sets as the following:

$$B(f, d) := \left\{g \in \mathscr{H}(X) : \sup\{d(f(x), g(x)) : x \in X\} < 1\right\} \tag{11}$$

as $d$ runs over all metrics compatible with $X$ [1, 19].

In [19] it has been proven that the limitation topology on $\mathscr{H}(X)$ is an admissible group topology.

*2.6. Comparison.* In the metric setting three of the examined methods collapse in just one because of the two following circumstances. The former one is why the open-cover topology and the limitation topology agree: any open cover in a metric space $X$ can be refined by the cover of balls of radius 1, $\{B_d(x, 1) : x \in X\}$, relative to a suitable metric $d$ compatible with $X$. The latter one is why the fine uniform topology and the limitation topology agree: the fine uniformity of a metric space $X$ is the supremum of all metrisable uniformities compatible with $X$. Accordingly, as for the metric setting, closeness in $\mathscr{H}(X)$ can be substantially controlled in two ways: via the metrics compatible with $X$ or via the continuous functions from $X$ to the positive real numbers. Usually, the fine or Whitney topology $\tau_W$ is finer than the fine uniform topology $\tau_f$.

**Theorem 1.** *If $\tau_W$ is a group topology, then $\tau_W = \tau_f$, [15].*

## 3. Compact Extension Procedure

Implicitly due to Birkhoff, a natural way to get admissible group topologies works efficiently. Whenever $X$ is Tychonoff, since any self-homeomorphism of $X$ continuously extends to $\beta(X)$, the Stone-Čech compactification of $X$, then $\mathscr{H}(X)$ embeds as a subgroup in $\mathscr{H}(\beta X)$. Analogously, whenever $X$ is locally compact $T_2$, $\mathscr{H}(X)$ embeds as a subgroup in $\mathscr{H}(X_\infty)$, where $X_\infty$ is the one-point compactification of $X$. Thereby, the relativization to $\mathscr{H}(X)$ of the compact-open topology on $\mathscr{H}(\beta X)$ and that on $\mathscr{H}(X_\infty)$ are both admissible group topologies. Accordingly, the previous significant examples strongly suggest investigating those uniform topologies on $\mathscr{H}(X)$ derived from totally bounded uniformities on $X$ whose uniform completion is a $T_2$-compactification of $X$ to which any self-homeomorphism of $X$ continuously extends. We say that a $T_2$-compactification $\gamma(X)$ of $X$ *has the lifting property* if every self-homeomorphism of $X$ continuously extends to $\gamma(X)$. Remember that whenever $X$ is a Tychonoff, locally compact $T_2$, and rim-compact $T_2$ space, any self-homeomorphism extends to a self-homeomorphism of its Stone-Čech compactification $\beta X$, its one-point compactification $X_\infty$, its Freudenthal compactification $F(X)$, respectively. In other words $\beta X, X_\infty,$ and $F(X)$, when they make sense, are all compactifications of $X$ with the lifting property.

**Theorem 2.** *Let $\gamma(X)$ be a $T_2$-compactification of $X$ with the lifting property. Then the relativization $\tau_{\gamma(X)}$ to $\mathscr{H}(X)$ of the compact-open topology on $\mathscr{H}(\gamma(X))$ is an admissible group topology on $\mathscr{H}(X)$, [7].*

Starting with a totally bounded uniformity we construct a $T_2$-compactification with the lifting property as follows.

Let $\mathscr{U}$ be a collection of subsets of $X \times X$. For any $U \in \mathscr{U}$ and any $h \in \mathscr{H}(X)$ put

$$U_h := \{(x, y) \in X \times X : (h(x), h(y)) \in U\}. \tag{12}$$

Furthermore, set

$$\mathscr{S}_{\mathscr{H}} := \{U_h : U \in \mathscr{U}, h \in \mathscr{H}(X)\}. \tag{13}$$

**Theorem 3.** *Let $\mathscr{U}$ be a uniformity on $X$. Then the following hold:*

(a) *The family $\mathscr{S}_{\mathscr{H}}$ is a subbase for a uniformity $\mathscr{U}_{\mathscr{H}}$ on $X$, which is separated whenever $\mathscr{U}$ is so.*

(b) *The uniformity $\mathscr{U}_{\mathscr{H}}$ is totally bounded whenever $\mathscr{U}$ is so.*

(c) *Any self-homeomorphism of $X$ is a uniformly continuous function with respect to $\mathscr{U}_{\mathscr{H}}$ or equivalently $\mathscr{U}_{\mathscr{H}}$ has the lifting property.*

(d) *The uniformity $\mathscr{U}_{\mathscr{H}}$ is the least uniformity with the lifting property finer than $\mathscr{U}$.*

For every uniformity $\mathscr{U}$ the property (d) motivates us to refer to $\mathscr{U}_{\mathscr{H}}$ as the *minimal $\mathscr{H}(X)$-enlargement* of $\mathscr{U}$. Minimal $\mathscr{H}(X)$-enlargements have interesting properties.

**Proposition 4.** *Let $\mathscr{U}$ be a totally bounded uniformity on $X$. Then the uniform topology $\tau_{\mathscr{U}_{\mathscr{H}}}$ on $\mathscr{H}(X)$ derived from $\mathscr{U}_{\mathscr{H}}$ is a group topology; hence it is an admissible group topology.*

In the case $\mathscr{U}$ is totally bounded the previous result induces us to refer to the uniform topology $\tau_{\mathscr{U}_{\mathscr{H}}}$ as the *fine group topology associated with $\mathscr{U}$*.

**Proposition 5.** *Let $\mathscr{U}$ be a totally bounded uniformity on $X$. Then the uniform topology on $\mathscr{H}(X), \tau_{\mathscr{U}}$, derived from $\mathscr{U}$ is* *a group topology if and only if it agrees with the uniform topology $\tau_{\mathscr{U}_{\mathscr{H}}}$ derived from $\mathscr{U}_{\mathscr{H}}$.*

The previous result can be summarised as follows.

**Theorem 6.** *A uniform topology on $\mathscr{H}(X)$ derived from a totally bounded uniformity on X is a group topology (hence an admissible group topology) if and only if it is derived from a totally bounded uniformity of X associated with a $T_2$-compactification of X with the lifting property, [21, 29].*

## 4. Completeness of $\mathscr{L}_H(X)$ in Rim-Compactness

In the direction of extending Arens' result beyond the class of locally compact spaces, it comes as very natural idea to *weaken local compactness into rim-compactness*, since to a rim-compact $T_2$ space $X$ is attached the Freudenthal compactification $F(X)$ [3–5], to which any self-homeomorphism continuously extends [6]. So, we focus our attention on rim-compact $T_2$ spaces and in particular on their Freudenthal compactification. The Freudenthal compactification in rim-compactness plays a key role as the one-point compactification does in local compactness. A space $X$ is *rim-compact*, *peripherically compact*, or *semicompact* if any point has arbitrarily small neighborhoods whose boundaries are compact. For example, removing from a compact metric space a totally disconnected $F_\sigma$-set is a way to produce rim-compact $T_2$ spaces [35]. Of course, 0-dimensional spaces are rim-compact. We briefly summarize the characters of the Freudenthal compactification which we will refer to. Any rim-compact $T_2$ space $X$ admits $T_2$-compactifications $\gamma(X)$ whose growth $\gamma(X) - X$ is zero-dimensionally embedded in $\gamma(X)$ that is, every point in the growth $\gamma(X) - X$ has arbitrarily small neighborhoods whose boundaries lie in $X$. The Freudenthal compactification $F(X)$ is the maximal $T_2$-compactification of $X$ whose growth $F(X) - X$ is zero-dimensionally embedded in $F(X)$. The Freudenthal compactification can be also described as the completion of the totally bounded uniformity determined by the covering uniformity generated from all binary coverings $\{X - A, X - B\}$, where $A$ and $B$ are open sets with compact boundaries. The Freudenthal compactification is the unique perfect $T_2$-compactification in which the growth zero-dimensionally embeds. A compactification $\gamma(X)$ of a space $X$ is called *perfect* if, for each point $x \in \gamma(X) - X$ and each open neighborhood $U$ of $x$ in $\gamma(X)$, the set $U \cap X$ is not a disjoint union of two open sets $V$ and $W$ such that $x \in \mathrm{CL}_{\gamma(X)}(V) \cap \mathrm{CL}_{\gamma(X)}(W)$. Any homeomorphism between two rim-compact $T_2$-spaces extends to a homeomorphism between their Freudenthal compactifications. Hence, the Freudenthal compactification has the lifting property. Finally, the Freudenthal compactification is the Smirnov compactification associated to the Freudenthal proximity: *two closed sets are far if and only if they can be separated by a compact set.* If $X$ is rim-compact $T_2$ connected and locally connected, then its Freudenthal compactification is locally connected.

We are now able to give a very simple description as set-open topologies for $\tau_{\beta X}$, whenever $X$ is normal, and for $\tau_F$. We recall that a *set-open topology* on $\mathscr{H}(X)$ admits as subbasic open sets those sets of the type [18]

$$[C, W] = \{f \in \mathscr{H}(X) : f(C) \subseteq W\}, \tag{14}$$

where $C$ runs in a fixed collection of closed sets of $X$ and $W$ is open in $X$. When $C$ runs over all closed sets in $X$, then we get *the closed-open topology.*

**Theorem 7.** *When $X$ is $T_4$, the relativization $\tau_{\beta X}$ of the compact-open topology on $\mathscr{H}(\beta X)$ is the closed-open topology.*

**Theorem 8.** *Let $X$ be a rim-compact $T_2$ space. Then the relativization $\tau_F$ to $\mathscr{H}(X)$ of the compact-open topology on $\mathscr{H}(F(X))$ is a set-open topology. It admits as subbasic open sets those ones of the type*

$$[C, W] = \{f \in \mathscr{H}(X) : f(C) \subseteq W\}, \tag{15}$$

*where $C$ runs in the family of all closed sets whose boundaries are compact and $W$ runs in the topology of $X$.*

Unfortunately, we have no hope for minimality of $\tau_F$ without adding some more condition. In fact, there are rim-compact $T_2$ spaces whose Freudenthal compactification does not determine a least admissible group topology, as for the space $\mathbb{N}$ of natural numbers, for instance. Since $\mathbb{N}$ is locally compact and locally connected, $\mathscr{H}(\mathbb{N})$ admits a least group topology which is just the compact-open topology [2], while that one induced by the Freudenthal compactification is just the closed-open topology. The closed-open topology is in this case strictly finer than the compact-open topology on $\mathscr{H}(\mathbb{N})$. The neighborhood of the identity function $f$ in the closed-open topology $[P, P]$, where $P$ is the set of all even integers, cannot contain any neighborhood of $f$ of the type $[K_1, W_1] \cap \cdots \cap [K_n, W_n]$, with $K_1, \ldots, K_n$ compact, hence finite, and $W_1, \ldots, W_n$ bounded open. Suppose $\bigcup_{i=1\cdots n} W_i \subset [0, m]$ for some odd integer $m$. Put

$$g(n) = n, \quad n \le m; \qquad g(m+h) = m+h-1, \quad h \ge 2, h \text{ is even}; \tag{16}$$
$$g(m+h) = m+h+1, \quad h \ge 1, h \text{ is odd}.$$

Then $g$ is in $\mathscr{H}(\mathbb{N})$ and in $[K_1, W_1] \cap \cdots \cap [K_n, W_n]$ but does not belong to $[P, P]$. If $n > m, n$ is even, and $n = m + h$, then $h$ has to be odd and $g(n) = n + 1$ is odd.

For that, we focus our attention on the class of rim-compact $T_2$ spaces whose Freudenthal compactification is locally connected at any ideal point. Naturally there exist rim-compact but not locally compact $T_2$ spaces having their Freudenthal compactification locally connected at any ideal point. We can give as an example the subspace $X$ obtained from $I \times I$, the unit square in the plane, by removing from it the points whose coordinates are both irrational. The space $X$ is rim-compact $T_2$ but not locally compact. Moreover, its Freudenthal compactification is just $I \times I$.

Trying to capture minimality in local connectedness we get a previous basic result. Let $X$ be a Tychonoff space and

$\gamma(X)$ a $T_2$-compactification of $X$. The space $X$ is *locally connected in* $\gamma(X)$ provided that any point in $\gamma(X) - X$ admits arbitrarily small open neighborhoods $U$ such that $U \cap X$ is connected [36]. Whenever a space $X$ is connected, locally connected, locally compact, and second-countable $T_2$, then $X$ is locally connected in $F(X)$ (Freudenthal's original construction). Naturally if $X$ is locally connected in $\gamma(X)$, then $\gamma(X)$ is locally connected at any ideal point.

**Theorem 9.** *If $X$ is a locally compact $T_2$ space, then $F(X)$ is locally connected at any ideal point if and only if $X$ is locally connected in it.*

A result about local compactness involving as particular case the real line and more generally connected non compact Lie groups is the following.

**Theorem 10.** *Let $X$ be a rim-compact $T_2$ and locally connected space. If $F(X)$ is an $n$-point compactification, then $F(X)$ is locally connected at any ideal point.*

**Theorem 11.** *Whenever $X$ is a rim-compact $T_2$ space and its Freudenthal compactification $F(X)$ is locally connected at any ideal point, then the group topology $\tau_F$ induced on $\mathscr{H}(X)$ from $F(X)$ is the least in the upper-semilattice $\mathscr{L}_{\mathscr{H}}(X)$ of all admissible group topologies on $\mathscr{H}(X)$.*

A relationship with local compactness resides in the following.

**Corollary 12.** *If $X$ is a locally connected space and its Freudenthal compactification has only a finite number of ideal points, then the group topology induced by the one-point compactification and the Freudenthal compactification agree.*

In a more general context in which unfortunately the group topologies do not have a simple description and a convergence strategy, even though rather technical, has to be managed we have the following.

**Theorem 13.** *If $X$ is rim-compact $T_2$ and admits a $T_2$-compactification $\gamma(X)$ with the lifting property, locally connected at any ideal point, in which $\gamma(X) - X$ zero-dimensionally embeds, then the group topology $\tau_{\gamma(X)}$ induced by $\gamma(X)$ on $\mathscr{H}(X)$ is the least of all admissible group topologies on $\mathscr{H}(X)$.*

By essentially using the previous basic result, then we construct in two steps a $T_2$-compactification of $X, \gamma(X)$, in which $\gamma(X) - X$ zero-dimensionally embeds and to which any self-homeomorphism of $X$ continuously extends. At the first step, $X$ comes densely embedded in the disjoint union of the Freudenthal compactifications of its components, $c(X)$, which is a locally compact $T_2$ space to which any self-homeomorphism of $X$ continuously extends. At the second step, in turn $c(X)$ becomes embedded in its one-point compactification $\gamma(X)$, and, as a matter of fact, $\tau_{\gamma(X)}$ is the least element of $\mathscr{L}_H(X)$.

**Theorem 14.** *Suppose $X$ is a rim-compact $T_2$ and locally connected space. Then:*

(i) *$X$ embeds in a $T_2$-compactification $\gamma(X)$ which induces on $\mathscr{H}(X)$ the least admissible group topology $\tau_{\gamma(X)}$,*

(ii) *$\tau_{\gamma(X)}$ is the set-open topology determined by all closed sets with compact boundaries contained in some component of $X$ [7].*

Whenever $X$ is finite union of disjoint connected subspaces, as in particular if $X$ is connected, the compactification $\gamma(X)$ agrees with the Freudenthal compactification, but it is generally different as in the natural case. Under local compactness the previous construction works but is evidently redundant.

*Example 15.* Let $R_n, n \in N^+$, be obtained from the rectangle $[0,1] \times [r_n, 1/n]$, where $1/(n+1) < r_n < 1/n$ after removing inside points whose coordinates are both rational. Put $R = \cup\{R_n : n \in N^+\}$. Add to $R$ the segment $I = \{(x,0) : 0 \le x \le 1\}$. Consider $X = R \cup I$ as subspace in the Euclidean plane. The space $X$ is a rim-compact $T_2$ space not locally compact, not connected, and not locally connected. Its Freudenthal compactification $F(X)$ agrees with the closure of the space $R$; then it is metrisable and locally connected at any ideal point. So $\mathscr{H}(X)$ admits a least admissible group topology which is induced by the supmetric deriving from the Euclidean metric on $X$.

## 5. The Rational Case

The rational case apparently is singular. First, since any two nonempty open subspaces in $\mathbb{Q}$ are homeomorphic, $\mathscr{H}(\mathbb{Q})$ is a very big object. Anyway, $\mathscr{H}(\mathbb{Q})$, even though it admits no least admissible topology [2], still supports the clopen-open topology as the least admissible group topology.

**Theorem 16.** *Let $\gamma(\mathbb{Q})$ be an arbitrary $T_2$-compactification of $\mathbb{Q}$ but distinct from $\beta(\mathbb{Q})$. Then there always exists a self-homeomorphism of $\mathbb{Q}$ which does not continuously extend to $\gamma(\mathbb{Q})$.*

Remember that $\mathbb{Q}$ is strongly zero-dimensional, hence rim-compact. Its Stone-Čech compactification is zero-dimensional and perfect, so is its Freudenthal compactification [37]. The relativization to $\mathscr{H}(\mathbb{Q})$ of the compact-open topology on $\mathscr{H}(\beta\mathbb{Q})$ is the closed-open topology.

Of course, the main issue in the rational case is the following one.

**Theorem 17.** *Any admissible group topology $\tau$ on $\mathscr{H}(\mathbb{Q})$ is finer than the closed-open topology [7].*

We now investigate whether the *fine*, *strong*, or *Whitney topology* on $\mathscr{H}(\mathbb{Q})$ [20] induces naturally an admissible group topology on $\mathscr{H}(\mathbb{Q})$ strictly finer than the closed-open topology. To make easier the relationship between the Whitney topology and the group operations, preliminarly we need to acquire the following two Lemmas.

Let $C(\mathbb{Q}, \mathbb{R}^+)$ denote the set of all real-valued positive continuous functions on $\mathbb{Q}$.

**Lemma 18.** *Let $\varepsilon \in C(\mathbb{Q}, \mathbb{R}^+)$. Then there exists a locally constant function $\eta \in C(\mathbb{Q}, \mathbb{R}^+)$ such that $\eta < \varepsilon$.*

**Lemma 19.** *For each $f \in C(\mathbb{Q}, \mathbb{R}^+)$ and each locally constant $\varepsilon \in C(\mathbb{Q}, \mathbb{R}^+)$, there exists $\varphi_{f,\varepsilon} \in C(\mathbb{Q}, \mathbb{R}^+)$ such that anytime $x, y \in \mathbb{Q}$ and $|x - y| < \varphi_{f,\varepsilon}(x)$ then $|f(x) - f(y)| < \varepsilon(x)$.*

Let $\varepsilon \in C(\mathbb{Q}, \mathbb{R}^+)$, and let $d$ be the Euclidean metric on $\mathbb{Q}$. Denote

$$U_1(\varepsilon) = \{(f, g) \in \mathscr{H}(\mathbb{Q}) \times \mathscr{H}(\mathbb{Q}) : d(f(x), g(x)) < \varepsilon(x), \forall x \in \mathbb{Q}\}. \quad (17)$$

It is well known that $\{U_1(\varepsilon) : \varepsilon \in C(\mathbb{Q}, \mathbb{R}^+)\}$ is a base for a uniformity $\mathscr{U}_1$ on $\mathscr{H}(\mathbb{Q})$ which induces the fine, strong, or Whitney topology which is independent of the metric $d$ since $\mathbb{Q}$ is paracompact [20]. The fine or Whitney topology admits as typical basic neighborhoods for any $f$

$$U_1(f, \varepsilon) = \left\{g \in \mathscr{H}(\mathbb{Q}) : d(f(x), g(x)) < \varepsilon(x), \forall x \in \mathbb{Q}\right\}, \quad (18)$$

where $\varepsilon \in C(\mathbb{Q}, \mathbb{R}^+)$.

**Theorem 20.** *The Whitney topology on $\mathscr{H}(\mathbb{Q})$ provides continuity of the usual product $(f, g) \in \mathscr{H}(\mathbb{Q}) \times \mathscr{H}(\mathbb{Q}) \to gof \in \mathscr{H}(\mathbb{Q})$.*

Now, denote $U_2(\varepsilon) = \{(f, g) \in \mathscr{H}(\mathbb{Q}) \times \mathscr{H}(\mathbb{Q}) : d(f^{-1}(x), g^{-1}(x)) < \varepsilon(x), \text{ for all } x \in \mathbb{Q}\}$.

It is easily verified that $\{U_2(\varepsilon) : \varepsilon \in C(\mathbb{Q}, \mathbb{R}^+)\}$ is a base for a uniformity $\mathscr{U}_2$ on $\mathscr{H}(\mathbb{Q})$, which induces a topology providing, in analogy with the previous result, continuity of the usual product. Jointly $\mathscr{U}_1, \mathscr{U}_2$ generate a new uniformity $\mathscr{U}$ on $\mathscr{H}(\mathbb{Q})$ having as basic diagonal neighborhoods $U(\varepsilon) = U_1(\varepsilon) \cap U_2(\varepsilon)$, $\varepsilon \in C(\mathbb{Q}, \mathbb{R}^+)$.

The uniformity $\mathscr{U}$ induces a topology on $\mathscr{H}(\mathbb{Q})$ whose typical basic neighborhoods for any $f$ are

$$U(f, \varepsilon) = \left\{g \in \mathscr{H}(\mathbb{Q}) : \max\left[d(f(x), g(x)), d\left(f^{-1}(x), g^{-1}(x)\right)\right] < \varepsilon(x), \forall x \in \mathbb{Q}\right\}, \quad (19)$$

where $\varepsilon \in C(\mathbb{Q}, \mathbb{R}^+)$. We, justified from the following result, call it the *fine group topology* on $\mathscr{H}(\mathbb{Q})$.

**Theorem 21.** *The topology generated by the base $\{U(f, \varepsilon) : f \in \mathscr{H}(\mathbb{Q}), \varepsilon \in C(\mathbb{Q}, \mathbb{R}^+)\}$ is an admissible group topology on $\mathscr{H}(\mathbb{Q})$, strictly finer than the closed-open topology [7].*

## 6. Group Action on 0-Dimensional Spaces and Completeness

The full homeomorphism group $\mathscr{H}(\mathbb{Q})$ of the rational numbers space $\mathbb{Q}$ equipped with the Euclidean topology admits as least admissible group topology the closed-open topology induced by the Stone-Čech compactification of $\mathbb{Q}$, which, in the case, agrees with the Freudenthal compactification of $\mathbb{Q}$. In trying to extend a similar result to a larger class of zero-dimensional spaces we briefly review properties of some of their $T_2$-compactifications and in particular of their Freudenthal compactifications. A Tychonoff space $X$ is *zero-dimensional* if it admits a base of clopen sets. A clopen set in $X$ is a subset of $X$ that is at the same time closed and open. A zero-dimensional space is rim-compact.

In the rational case, the proof strategy is based on the property $(\star)$ *any two non-empty clopen subspaces are homeomorphic*. So we focus our attention on the class of spaces with this property and their products. This class includes all zero-dimensional spaces of diversity one (or divine) and all compact zero-dimensional spaces of diversity two (or semidivine), as introduced and investigated by Rajagopalan and others [12, 13]. An infinite Tychonoff space $X$ is *of diversity one* if any two non-empty open subspaces are homeomorphic and is *of diversity two* if there exist two classes of homeomorphism for the open non-empty subspaces of $X$. The rationals, the irrationals, and the Baire spaces are of diversity one by their topological characterizations. The Cantor discontinuum is of diversity two. In a compact space of diversity two any two non-empty clopen subspaces are homeomorphic. No space of diversity one can be compact or locally compact, connected or locally connected. Diversity one or two is not preserved under products. Every space of diversity one is rich of homeomorphisms that move any point, since it can be expressed as countable disjoint union of homeomorphic copies of itself. For further details see [12, 13].

**Theorem 22.** *If $X$ is a zero-dimensional space, then the topology on $\mathscr{H}(X)$, $\tau_{F(X)}$, induced by the Freudenthal compactification $F(X)$ is the clopen-open topology.*

**Theorem 23.** *If $X$ has the property $(\star)$, then so does $F(X)$.*

**Theorem 24.** *If $X$ is a zero-dimensional, nonlocally compact space that satisfies the property $(\star)$, then its Freudenthal compactification $F(X)$ is the unique $T_2$-compactification of $X$ with the lifting property and zero-dimensional growth.*

Recall that a Tychonoff space $X$ is *strongly zero-dimensional* if any two non-empty disjoint zero sets can be separated by the empty set.

**Theorem 25.** *If $X$ is a strongly zero-dimensional, non-locally compact space that satisfies the property $(\star)$, then its Stone-Čech compactification $\beta(X)$ is the unique perfect $T_2$-compactification of $X$ and also the unique $T_2$-compactification of $X$ with the lifting property [8].*

Supposing $X$ is a zero-dimensional space, we call *nice* any base of clopen sets in $X$ that is closed under complements and invariant under homeomorphisms of $X$. Any base $\mathscr{B}$ of clopen sets embeds in the nice base $\{h(E) : E \in \mathscr{B}$ or $X - E \in \mathscr{B}, h \in \mathscr{H}(X)\}$, that is also the minimal nice base containing $\mathscr{B}$. If $\mathscr{B}$ is a base of clopen sets, the minimal nice base containing $\mathscr{B}$ is referred to as the *nice closure* of $\mathscr{B}$.

Recall that a Weil uniformity is *non-Archimedean* when it admits a base of diagonal neighborhoods that are equivalence relations in $X$. For further details see [11].

**Theorem 26.** *Let $X$ be a zero-dimensional space, $\mathscr{B}$ a nice base of $X$, and $\tau_{\mathscr{B}}$ the set-open topology determined by $\mathscr{B}$. Then the following holds:*

(i) *$\tau_{\mathscr{B}}$ is an admissible group topology, that is, $\tau_{\mathscr{B}} \in \mathscr{L}_H(X)$.*

(ii) *The left, the right, and the two-sided uniformities associated with $\tau_{\mathscr{B}}$ are all non-Archimedean.*

(iii) *$\tau_{\mathscr{B}}$ is the topology of uniform convergence induced by a $T_2$-compactification of $X$ with the lifting property [8].*

**Corollary 27.** *Let $X$ be a zero-dimensional space and $\mathscr{B}$ a nice base of $X$. Then the set-open topology $\tau_{\mathscr{B}}$ determined from $\mathscr{B}$ is zero-dimensional.*

Let $\{X_i : i \in I\}$ be a family of zero-dimensional spaces in each of which any two non-empty clopen subspaces are homeomorphic. Let $X = \prod_{i\in I} X_i$ be equipped with the product topology. We call *standard nice base for $X$* the nice closure of the standard clopen base generated by the subbasic clopen sets of the type $E_j \times \prod_{i\neq j} X_i$, where $E_j$ runs over all clopen sets in $X_j$ and $j$ in $I$. We refer to the clopen-open topology generated by the standard base as the *standard clopen-open topology*.

**Theorem 28.** *Let $\{X_i : i \in I\}$ be a family of zero-dimensional spaces in each of which any two non-empty clopen subspaces are homeomorphic. Let $X = \prod_{i\in I} X_i$ be equipped with the product topology. Then $\mathscr{L}_H(X)$ is a complete lattice. The standard clopen-open topology is the minimum of $\mathscr{L}_H(X)$ [8].*

We conclude with the following.

**Theorem 29.** *If $X$ is a zero-dimensional space in which any two non-empty clopen subspaces are homeomorphic, then $\mathscr{L}_H(X)$ is a complete lattice. The minimum is the clopen-open topology that is induced by the Freudenthal compactification.*

**Corollary 30.** *If $X$ is a zero-dimensional metrisable space of diversity one, then $\mathscr{L}_H(X)$ is a complete lattice. The minimum of $\mathscr{L}_H(X)$ is the closed-open topology that is induced by the Stone-Čech compactification.*

## 7. Fine Group Topologies

Now, we carry on another efficient way to produce admissible group topologies in substitution of, or in parallel with, the compact extension procedure. Let $X$ stand for a metrisable space. The presentation in [19] of the fine uniform topology is a compelling motivation to generalise it in order to produce admissible group topologies on $\mathscr{H}(X)$ and its subgroups. But first, we introduce *a way to produce new metrics from old ones* [15]. We suitably combine a self-homeomorphism $h$ of $X$ with a metric $d$ compatible with $X$ and so generate a new metric $d_h$, which is once again compatible with $X$. Namely, if the space $X$ is subject to a homeomorphic deformation $h$ and we measure the distance between two points in $X$ as the $d$—distance of their $h$—images, we construct a new metric $d_h$ defined more precisely by the following formula:

$$d_h(x, y) := d(h(x), h(y)) \quad \forall x, y \in X, \tag{$\div$}$$

compatible with $X$ and further totally bounded when so is $d$.

Let $\mathscr{D}(X)$ be a class of metrics compatible with $X$ and $\mathscr{G}(X)$ a subgroup of $\mathscr{H}(X)$. We will refer to the uniform topology induced on $\mathscr{G}(X)$ by the supremum of the uniformities on $X$ associated with the metrics in $\mathscr{D}(X)$ as the *fine uniform topology on $\mathscr{G}(X)$ associated with (or generated by) $\mathscr{D}(X)$*, and we will denote it by $\tau_{\mathscr{D},\mathscr{G}}$. Of course, the fine uniform topology $\tau_f$ is then generated by the full homeomorphism group $\mathscr{H}(X)$ and by the class of all metrics compatible with $X$.

Whenever $\mathscr{D}(X)$ is closed under the scalar multiplication, it is easy to show that at any $f \in \mathscr{G}(X)$ the topology $\tau_{\mathscr{D},\mathscr{G}}$ admits as subbasic open neighborhoods the sets of the kind

$$\begin{aligned} B(f, d) \\ := \{g \in \mathscr{G}(X) : \sup\{d(f(x), g(x)) : x \in X\} < 1\} \end{aligned} \tag{20}$$

as $d$ runs over $\mathscr{D}(X)$. Blending in a group of self-homeomorphisms $\mathscr{G}(X)$ with a class $\mathscr{D}(X)$ of metrics compatible with $X$ gives rise to a new class of metrics compatible with $X$, which reveals useful features.

We say that a class $\mathscr{D}(X)$ is *invariant under the group $\mathscr{G}(X)$* or *$\mathscr{G}(X)$-invariant* if, whenever we submit the space $X$ to any homeomorphic deformation $h$ in $\mathscr{G}(X)$ and we measure the distance between two points of $X$ as the $d$—distance of the pair of their $h$—images, where $d$ is a metric in $\mathscr{D}(X)$, the new produced metric $d_h$, defined in ($\div$), belongs once again to $\mathscr{D}(X)$.

If $\mathscr{D}(X)$ is $\mathscr{G}(X)$-invariant, then the fine uniform topology $\tau_{\mathscr{D},\mathscr{G}}$ is a group topology on $\mathscr{G}(X)$.

Every class of metrics $\mathscr{D}(X)$ admits as $\mathscr{G}(X)$-invariant enlargement the wider class $\{d_h : d \in \mathscr{D}(X), h \in \mathscr{G}(X)\}$, which is also the minimal $\mathscr{G}(X)$-invariant enlargement of $\mathscr{D}(X)$. The previous result enables us to define the fine uniform topology on $\mathscr{G}(X)$ generated by the minimal $\mathscr{G}(X)$-invariant enlargement of $\mathscr{D}(X)$ as the *fine group topology on $\mathscr{G}(X)$ generated by $\mathscr{D}(X)$*.

*7.1. A Same Group Blended in with Different Classes of Metrics Gives Rise to Different Fine Group Topologies.* If $X$ is metrisable and separable, thus admitting totally bounded compatible metrics, then the fine group topology generated on $\mathscr{H}(X)$ by all totally bounded metrics compatible with $X$ is, in general, distinct from the fine uniform topology generated on $\mathscr{H}(X)$ by all metrics compatible with $X$. The rational numbers provide the right counterexample that follows. The fine group topology generated on $\mathscr{H}(\mathbb{Q})$ by all totally bounded metrics compatible with $\mathbb{Q}$ and the fine group topology generated on $\mathscr{H}(\mathbb{Q})$ by all metrics compatible with $\mathbb{Q}$ are distinct from each other. Namely, the former one coincides with the clopen-open topology of $\mathscr{H}(\mathbb{Q})$ [7]. The latter one has to coincide with the fine or Whitney topology on $\mathscr{H}(\mathbb{Q})$, since this one, in the case, is a group topology. And, as proven in [7], the clopen-open topology and the fine or Whitney topology on $\mathscr{H}(\mathbb{Q})$ do not agree, being the fine or Whitney topology strictly stronger than the clopen-open topology.

## 8. The Space $\mathbb{R} \times \mathbb{Q}$

As rim-compactness is a weak and peripherical compactness property, one might think of any further relaxation as impossible. But, we show that rim-compactness for $X$ is not a necessary condition for the existence of the least admissible group topology on $\mathscr{H}(X)$. More precisely, we show that the full group of self-homeomorphisms of the product space $\mathbb{R} \times \mathbb{Q}$, where $\mathbb{R}$ and $\mathbb{Q}$ are the sets of the real and rational numbers, respectively, both carrying the Euclidean topology, admits a least admissible group topology even though notoriously $\mathbb{R} \times \mathbb{Q}$ is not rim-compact, [15].

Since, if $C$ is closed and $A$ is open in $\mathbb{Q}$ and $C \subseteq A$, there exists a clopen set $E$ such that $C \subseteq E \subseteq A$, then the sets like the following:

$$[E, E] := \{f \in H(\mathbb{Q}) : f(E) \subseteq E\} \tag{21}$$

as $E$ runs over all clopen sets in $\mathbb{Q}$, give arbitrarily small neighborhoods at the identity function of $\mathbb{Q}$. This entails the coincidence of the closed-open topology with the clopen-open topology on $\mathscr{H}(\mathbb{Q})$. At the same time, the clopen-open topology on $\mathscr{H}(\mathbb{Q})$ is the uniform topology induced by the Čech uniformity of $\mathbb{Q}$, which is the finest totally bounded uniformity compatible with $\mathbb{Q}$. Consequently, the clopen-open topology on $\mathscr{H}(\mathbb{Q})$ can be reformulated as the supremum of all uniform topologies induced on $\mathscr{H}(\mathbb{Q})$ by totally bounded uniformities compatible with $\mathbb{Q}$. Then, being $\mathbb{Q}$ metrisable and separable, the same is the supremum of all uniform topologies induced by totally bounded metrics compatible with $\mathbb{Q}$.

Let us turn now our attention to $\mathbb{R}\times\mathbb{Q}$. Since the boundary of any non-empty bounded open subset of $\mathbb{R} \times \mathbb{Q}$ is not compact, the product $\mathbb{R} \times \mathbb{Q}$ is not rim-compact when both $\mathbb{R}$ and $\mathbb{Q}$ carry the Euclidean metric. The study of a complex object as $\mathscr{H}(\mathbb{R} \times \mathbb{Q})$ is certainly simplified by splitting any self-homeomorphism $F$ of $\mathbb{R} \times \mathbb{Q}$ into its two natural halves $p_1 \circ F$, $p_2 \circ F$, where $p_1, p_2$ are the usual projections of $\mathbb{R}\times\mathbb{Q}$ over $\mathbb{R}$ and $\mathbb{Q}$, respectively. The study of the two halves, separately, allows us to acquire their own features and their interplay.

Let us focus on the second half $p_2 \circ F$. The following two facts are to be considered. The components of $\mathbb{R} \times \mathbb{Q}$ are the subsets of the type $\mathbb{R} \times \{q\}$, as $q$ runs over $\mathbb{Q}$. Furthermore, every homeomorphism takes components to components. Consequently, for any given $q$ in $\mathbb{Q}$, the following occurs:

$$p_2 \circ F(x, q) = p_2 \circ F(x', q), \quad \forall x, x' \in \mathbb{R}. \tag{22}$$

This means that $p_2 \circ F$ is independent of the point $x$ in $\mathbb{R}$. This feature of $p_2 \circ F$ makes coherent its substitution with the map from $\mathbb{Q}$ to itself

$$f_2 : q \in \mathbb{Q} \longrightarrow p_2 \circ F(x, q) \in \mathbb{Q} \tag{$\bullet$}$$

whatever is the point $x$ in $\mathbb{R}$. Accordingly, it seems natural to identify the self-homeomorphism $F$ with the pair $(f_1, f_2)$, where $f_1 = p_1 \circ F : \mathbb{R} \times \mathbb{Q} \to \mathbb{R}$ and $f_2 : \mathbb{Q} \to \mathbb{Q}$ is determined from $p_2 \circ F$ as in ($\bullet$). Of course, both $f_1, f_2$ are continuous. The identity map of $\mathbb{R}\times\mathbb{Q}$ identifies with the pair $(p_1, i_{\mathbb{Q}})$, where $p_1$ is again the usual projection of $\mathbb{R} \times \mathbb{Q}$ on $\mathbb{R}$ and $i_{\mathbb{Q}}$ is the identity map of $\mathbb{Q}$. Next, if $F$ identifies with $(f_1, f_2)$ and $G$ with $(g_1, g_2)$, then their composition $G \circ F$ identifies with the pair $(h_1, h_2)$, where

$$\begin{gathered} h_1(x, q) = g_1(f_1(x, q), f_2(q)), \quad \forall (x, q) \in \mathbb{R} \times \mathbb{Q}, \\ h_2(q) = g_2(f_2(q)), \quad \forall q \in \mathbb{Q}. \end{gathered} \tag{23}$$

Hence, if the inverse homeomorphism $F^{-1}$ of $F$ identifies with $(g_1, g_2)$, then

$$\begin{gathered} g_1(f_1(x, q), f_2(q)) = x, \quad \forall (x, q) \in \mathbb{R} \times \mathbb{Q}, \\ g_2(f_2(q)) = q, \quad \forall q \in \mathbb{Q}. \end{gathered} \tag{24}$$

This implies $g_2 = f_2^{-1}$. Thus, $f_2$ is in turn a homeomorphism of $\mathbb{Q}$ to itself whenever $F$ is a homeomorphism of $\mathbb{R} \times \mathbb{Q}$ to itself.

The identification leads to a natural embedding of $\mathscr{H}(\mathbb{R} \times \mathbb{Q})$ in $C(\mathbb{R} \times \mathbb{Q}) \times \mathscr{H}(\mathbb{Q})$, where $C(\mathbb{R} \times \mathbb{Q})$ is the set of all continuous functions from $\mathbb{R} \times \mathbb{Q}$ to the reals.

We now recall the notion of product metric on a product space. Let $(X_1, d_1), (X_2, d_2)$ stand for two metric spaces. Then, their product $X_1 \times X_2$ can be metrised by the *product metric* $d_1 \times d_2$, which is defined by

$$\begin{aligned} & d_1 \times d_2((x_1, x_2), (y_1, y_2)) \\ & \quad := \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}. \end{aligned} \tag{25}$$

If we suppose $\mathscr{H}(\mathbb{R} \times \mathbb{Q})$ embedded via the canonical identification, as described above, in $C(\mathbb{R} \times \mathbb{Q}) \times \mathscr{H}(\mathbb{Q})$ and denote by $d_1$ the stereographic metric on $\mathbb{R}$, which measures the distance between two points in $\mathbb{R}$ as the geodesic distance of their images in the unit circle $S^1$ of the Euclidean plane by the inverse of the stereographic projection, then the following holds true.

**Theorem 31.** *Every admissible group topology on $\mathscr{H}(\mathbb{R}\times\mathbb{Q})$ is stronger than the fine group topology generated on $\mathscr{H}(\mathbb{R} \times \mathbb{Q})$ by the class of all metrics on $\mathbb{R} \times \mathbb{Q}$ of the type $d_1 \times d_2$, where $d_1$ is the stereographic metric on $\mathbb{R}$ and $d_2$ runs over all totally bounded metrics compatible with $\mathbb{Q}$ [15].*

## 9. Locally Compact Extension Procedure

In looking for topologies of uniform convergence on members of a given family, containing all compact sets, which are admissible group topologies, we focus beyond local compactness. In order to do so, we follow as suggestive example that of bounded sets of an infinite dimensional normed vector space carrying as proximity the metric proximity associated with the norm. We emphasise first that local compactness of $X$ is equivalent to the family of compact sets of $X$ being a *boundedness* of $X$ [24], which, jointly with any EF-proximity of $X$, gives a *local proximity space* [25]. As a consequence, we make this particular case fall within the more general one in which compact sets are substituted with bounded sets in a local proximity space, while the property $(*)$ *any point has a compact connected neighborhood* is replaced by the following one: $(**)$ *for each nonempty bounded set $B$ there exist a finite number of connected bounded sets $B_1,\ldots,B_n$ such that* $B \ll_\delta \operatorname{int}(B_1) \cup \cdots \cup \operatorname{int}(B_n)$.

*9.1. Uniformity, Proximity, and $T_2$-Compactifications.* Uniformities, proximities, and $T_2$-compactifications have an intensive reciprocal interaction. EF-proximity and totally bounded uniformity are dual concepts. Any uniformity $\mathcal{U}$ on $X$ naturally determines an EF-proximity on $X$ by setting for $A, B \subseteq X$, $A \not\delta_{\mathcal{U}} B$ if and only if there exists a diagonal neighborhood $U \in \mathcal{U}$ such that $U[A] \cap B \neq \emptyset$. The class of all uniformities on $X$ determining the same EF-proximity $\delta$ on $X$ contains a unique totally bounded uniformity, which is also the least element in the class. In the opposite, by the Smirnov compactification theorem [26], any EF-proximity $\delta$ on $X$ determines, up to homeomorphism, a $T_2$-compactification $\gamma(X)$ of $X$, whose unique compatible uniformity in turn induces on $X$ a totally bounded uniformity $\mathcal{U}^*$, whose naturally associated proximity is just the starting $\delta$. Both proximity and uniformity give rise to exhaustive procedures to generate all $T_2$-compactifications of a Tychonoff space.

Let $(X,\delta)$ be an EF-proximity space, $\tau_\delta$ the natural underlying topology, $\mathcal{U}^*$ the unique totally bounded uniformity compatible with $\delta$, and $\gamma(X)$ the uniform completion of $(X,\mathcal{U}^*)$. Given that $\gamma(X)$ is obviously the Smirnov compactification of $(X,\delta)$ up to homeomorphism, the following is easily acquired.

**Proposition 32.** *The following properties are equivalent:*

(a) *Any self-homeomorphism of the underlying topological space $(X,\tau_\delta)$ continuously extends to $\gamma(X)$.*

(b) *Any self-homeomorphism of $X$ is a proximity function w.r.t. $\delta$.*

(c) *Any self-homeomorphism of $X$ is a uniformly continuous function w.r.t. $\mathcal{U}^*$.*

It is to be reminded that a $T_2$-compactification $\gamma(X)$ of $X$ has the lifting property if and only if any self-homeomorphism of $X$ continuously extends to it. According to the previous Lemma we naturally say that *a proximity has the lifting property* if it satisfies property (b) and that *a uniformity has the lifting property* if it satisfies property (c).

It is remarkable that, for each positive integer $n$, any metric uniformity compatible with the space $\mathbb{R}^n$, equipped with the Euclidean topology, for which any homeomorphism is uniformly continuous, or, which is equivalent, with the lifting property, is totally bounded [38].

*9.2. Strong Inclusion.* The concept of EF-proximity can be recasted as *strong inclusion*, double containment, or nontangential inclusion. For any given EF-proximity $\delta$ on a space $X$ the relative dual strong inclusion is the binary relation over the power set $\operatorname{Exp}(X)$ of $X$ defined as follows:

$$A \ll_\delta B \quad \text{iff } A \not\delta X - B. \tag{26}$$

Conversely, for any given binary relation over $\operatorname{Exp}(X)$, $\ll$, which is a strong inclusion, the relative dual EF-proximity $\delta$ is the binary relation over $\operatorname{Exp}(X)$ defined by

$$A \not\delta B \quad \text{iff } A \ll_\delta X - B. \tag{27}$$

The relations $\delta$ and $\ll_\delta$ are interchangeable.

Furthermore, later on we essentially use the following *betweenness property*. Let $\delta$ be an EF-proximity. If $A \ll_\delta B$, then there exists a $\tau_\delta$-closed set $C$ such that $A \ll_\delta \operatorname{int}(C) \subseteq C \ll_\delta B$.

*9.3. Proximal Set-Open Topologies on $\mathcal{H}(X)$.* Let $\mathcal{U}$ be a uniformity compatible with $X$ and let $\alpha$ stand for a family of nonempty subsets of $X$. The *topology of uniform convergence on members of $\alpha$ derived from $\mathcal{U}$*, which we denote by $\tau_{\alpha,\mathcal{U}}$, is that admitting as subbasic open sets at any $f \in \mathcal{H}(X)$ the following ones:

$$(A,U,f) := \{h \in \mathcal{H}(X) : (f(x), h(x)) \in U,\ \forall x \in A\}, \tag{28}$$

where $A$ runs through $\alpha$ and $U$ varies in $\mathcal{U}$.

Since the uniform topologies so far considered are relative to totally bounded uniformities, it is worthwhile to reformulate them as proximal set-open topologies. To unify the concepts of compact-open topology, bounded-open topology, and topology of proximity convergence [18], Naimpally, jointly with the author, introduced the unifying tool of *proximal set-open topology relative to a network and a proximity* [27]. This recasting takes up the opportunity of reformulating topologies of uniform convergence on members of a network, when the range space carries a proximity. A collection $\alpha$ of subsets of a topological space $X$ is said to be a *network* on $X$ provided that for any point $x$ in $X$ and any open subset $A$ of $X$ containing $x$ there is a member $C$ in $\alpha$ such that $x \in C \subseteq A$. A network $\alpha$ is a *closed network* if any element in $\alpha$ is closed and is a *hereditarily closed network* if any closed subset of any element in $\alpha$ is again in $\alpha$.

Let $(X,\delta)$ be an EF-proximity space and $\alpha$ a network in $X$, then the *proximal set-open topology relative to $\alpha$ and $\delta$*, in short denoted by the acronym $\mathrm{PSOT}_{\alpha,\delta}$ or, simply, $\mathrm{PSOT}_\delta$ when $\alpha$ is the network $\mathrm{CL}(X)$ of all non empty closed subsets of $X$, is that admitting as subbasic open sets the following ones:

$$[A,W]_\delta := \{f \in \mathcal{H}(X) : f(A) \ll_\delta W\}, \tag{29}$$

where $A$ runs through $\alpha$ and $W$ is open in $X$. When $\alpha$ is the family of all compact subsets of $X$, for any proximity we get the compact-open topology, which is the prototype within the class of set-open topologies.

The proximal set-open topologies have remarkable properties [27].

**Theorem 33.** *Let $\alpha$ be a closed, hereditarily closed network in $X$ and $\delta$ an EF-proximity on $Y$. Then $PSOT_{\alpha,\delta}$ is the topology of uniform convergence on members of $\alpha$ derived from the unique totally bounded uniformity compatible with $\delta$.*

*9.4. Boundedness plus Proximity.* Blending proximity with boundedness gives local proximity. Local proximities play the same role in the construction of $T_2$ local compactifications of a Tychonoff space $X$ as that of EF-proximities in the construction of $T_2$-compactifications of $X$.

Let $X$ be a Tychonoff space. Any given $T_2$ local compactification $l(X)$ of $X$ takes up two features of $X$. Whereas the former one is the separated EF-proximity on $X$ induced by the one-point compactification of $l(X)$, the latter one is the boundedness made by all subsets of $X$ whose closures in $l(X)$ are compact. By joining proximity and boundedness in the unique concept of *local proximity*, Leader put this example in abstract [25].

A non empty collection $\mathscr{B}$ of subsets of a set $X$ is called a *boundedness* in $X$ if and only if

(a) $A \in \mathscr{B}$ and $B \subseteq A$ imply $B \in \mathscr{B}$ and (b) $A, B \in \mathscr{B}$ implies $A \cup B \in \mathscr{B}$.

The elements of $\mathscr{B}$ are called *bounded sets*. It is to be underlined that in [24] Hu proposed the notion of space with a boundedness as a natural generalisation of that of metric space.

We expressly remark that we look at a local proximity as localisation of an EF-proximity modulo of a free regular filter [25]. A *local proximity space* $(X, \mathscr{B}, \delta)$ consists of a set $X$, together with an EF-proximity $\delta$ on $X$ and a boundedness $\mathscr{B}$ in $X$ containing all singletons, which satisfies the following axiom: *if $A \in \mathscr{B}, C \subseteq X$, and $A \ll C$, then there exists some $B \in \mathscr{B}$ such that $A \ll B \ll C$*, where $\ll$ is the strong inclusion of $\delta$.

It is remarkable that the boundedness in a local proximity space $(X, \mathscr{B}, \delta)$ is also a uniformly Urysohn family w.r.t. the unique totally bounded uniformity naturally associated with $\delta$ [30]. In a local proximity space the closure of a bounded set is again bounded. Every compact subset of a local proximity space is bounded. Every local proximity space is also locally bounded. As a matter of fact, proximity spaces are just those ones where the underlying set $X$ is bounded. Besides, the following holds true [25].

**Theorem 34.** *For a Tychonoff space $X$ there exists a bijection between the set of all, up to equivalence, $T_2$ locally compact dense extensions of $X$ and the set of all separated local proximities on $X$ [27]. If $X$ is bounded, the $T_2$ local compactification associated with $(X, \mathscr{B}, \delta)$ is just the Smirnov compactification relative to $\delta$, while, if $X$ is unbounded, it can be obtained by removing from the Smirnov compactification relative to $\delta$ the point determined in that by the free regular filter $\mathscr{F} = \{X \setminus B : B \in \mathscr{B}\}$.*

*9.5. Proximity and Homeomorphism Groups.* Let $(X, \delta)$ be an EF-proximity space. It is easy to show the following.

**Proposition 35.** *Let $\mathscr{G}(X)$ be a subgroup of the full group $\mathscr{H}(X)$ of self-homeomorphisms of the underlying topological space $X$. Assuming that $\mathscr{G}(X)$ is equipped with $PSOT_\delta$, then the evaluation function $e : (f, x) \in \mathscr{G}(X) \times X \to f(x) \in X$ is continuous.*

Furthermore, given that a proximity-isomorphism or $\delta$-isomorphism is a self-homeomorphism of $X$ that preserves proximity in both ways, then the following holds.

**Proposition 36.** *If $(X, \delta)$ is an EF-proximity space, then $PSOT_\delta$ is a group topology on the full group of $\delta$-isomorphisms of $X$.*

We summarise the previous two results as follows.

**Theorem 37.** *If $(X, \delta)$ is an EF-proximity space, then the full group of $\delta$-isomorphisms of $X$, equipped with $PSOT_\delta$, is a topological group which continuously acts on $X$ by the evaluation function $e$.*

**Proposition 38.** *Whenever $X$ is a $T_2$ locally compact space, the PSOT associated with the Alexandroff proximity, known as the $g$-topology, is the least admissible group topology on $\mathscr{H}(X)$.*

**Proposition 39.** *Whenever $X$ is a $T_2$, rim-compact, and locally connected space, the PSOT associated with the Freudenthal proximity is the least admissible group topology on $\mathscr{H}(X)$.*

**Proposition 40.** *Whenever $X$ is the rational numbers space $\mathbb{Q}$, equipped with the Euclidean topology, the PSOT associated with the Čech proximity is the least admissible group topology on $\mathscr{H}(\mathbb{Q})$.*

Now, assume that a $T_2$ local compactification has the *lifting property* if and only if any homeomorphism preserves both boundedness and proximity; that is, any homeomorphic image of a bounded set is bounded, and if $B \ll_\delta W$, then $f(B) \ll_\delta f(W)$, where $f$ runs through $\mathscr{H}(X)$, $B$ is bounded, and $W$ is open.

It is to be recalled that a local proximity space $(X, \mathscr{B}, \delta)$ verifies the property $(**)$ if and only if *for each non empty bounded set $B$ there exist a finite number of connected bounded sets $B_1, \ldots, B_n$* such that $B \ll_\delta \operatorname{int}(B_1) \cup \cdots \cup \operatorname{int}(B_n)$.

Whenever $(X, \mathscr{B}, \delta)$ is a local proximity space, then the subcollection of $\mathscr{B}$ of all closed bounded subsets of $X$ is a closed, hereditarily closed network of $X$. Accordingly, $\mathrm{PSOT}_{\mathscr{B},\delta}$ is the topology of uniform convergence on members of $\mathscr{B}$ derived from the unique totally bounded uniformity associated with $\delta$. Unfortunately, $\mathrm{PSOT}_{\mathscr{B},\delta}$ is not in general an admissible group topology nor a group topology.

Nevertheless, what stated above is sufficient to draw the following final issue.

**Theorem 41.** *If $(X, \mathscr{B}, \delta)$ is an unbounded local proximity space with the property $(**)$ and any self-homeomorphism of $X$ preserves both boundedness and proximity, then the topology* *of uniform convergence on bounded sets derived from the unique totally bounded uniformity associated with $\delta$ is an admissible group topology on $\mathscr{H}(X)$, [21].*

This final result can be recasted as follows.

**Theorem 42.** *Whenever $(X, \mathscr{B}, \delta)$ is a local proximity space with the property $(**)$ and the $T_2$ local compactification associated with it has the lifting property, then $PSOT_{\mathscr{B},\delta}$ is an admissible group topology on $\mathscr{H}(X)$.*

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