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Research Article

Polynomials in Control Theory Parametrized by Their Roots

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The aim of this paper is to introduce the space of roots to study the topological properties of the spaces of polynomials. Instead of identifying a monic complex polynomial with the vector of its coefficients, we identify it with the set of its roots. Viète's map gives a homeomorphism between the space of roots and the space of coefficients and it gives an explicit formula to relate both spaces. Using this viewpoint we establish that the space of monic (Schur or Hurwitz) aperiodic polynomials is contractible. Additionally we obtain a Boundary Theorem.

1. Introduction

It is well known that for the stability of a linear system $\dot{x} = Ax$ it is required that all the roots of the corresponding characteristic polynomial $p(t)$ have negative real part in other words, $p(t)$ is a Hurwitz (stable) polynomial. There are various approaches to decide if a given polynomial is Hurwitz. Maybe the most popular of such methods is the Routh-Hurwitz criterion. Other important approaches are Lienard-Chipart conditions and the Hermite-Biehler Theorem (see Gantmacher [1], Lancaster and Tismenetsky, [2] and Bhattacharyya et al. [3]). On the other hand, to have the stability of a discrete time linear system $x_{n+1} = Ax_n$

it is necessary that all of the roots of the characteristic polynomial are strictly within the unit disc. A polynomial with this property is named Schur polynomial. Maybe Jury's test is the most studied criterion for checking if a given polynomial is a Schur polynomial [4], but also there exists the corresponding Hermite-Biehler Theorem for Schur polynomials [5] or we can mention as well the Schur stability test [3]. In addition to the stability of polynomials another important problem is the aperiodicity condition, which consists in obtaining from a (continuous or discrete) stable system a response that has no oscillations or has only a finite number of oscillations. Mathematically this requires that all the roots of the characteristic polynomial $p(t)$ are distinct and on the negative real axis, for the case of continuous systems; and distinct and in the real interval $(0,1)$ for the discrete case. Criteria to decide if a system is aperiodic are given for instance in [6–12]. An important reference where Hurwitz and Schur stability and aperiodicity are studied is the book of Jury [13].

However, if a continuous or discrete system is modeling a physical phenomenon then it is affected by disturbances. Consequently it is convenient to think that there are uncertainties in the elements of the matrix A and then there are uncertainties in the coefficients of the polynomial $p(t)$; that is, we have a family of polynomials and we like to know if all of the polynomials are Schur or Hurwitz polynomials (if the family is stable). The study of the stability of various families of polynomials has attracted the attention of a lot of researchers. The most famous result about families of polynomials is without doubt the Kharitonov Theorem which gives conditions for Hurwitz stability of interval polynomials [14]. There exists an analogous result to Kharitonov's Theorem for Schur polynomials [15] and also results about segments of Schur polynomials (see [16] or [3]). Results on balls of Schur and Hurwitz polynomials can be found in [17]. In the case of Hurwitz polynomials, the stability of segments of polynomials has been studied in [18–21]. The stability of rays and cones of polynomials has been studied in [22, 23]. Other studied families are polytopes of polynomials and here the most important result is the Edge Theorem [24] which says that the stability of a polytope is determined by the stability of its edges. Good references about families of Hurwitz and Schur polynomials are the books of Ackermann [25], Barmish [26], and Bhattacharyya [3]. The problem of robustness for the aperiodicity condition has been worked in [27–30]. About the specific case of intervals of aperiodic polynomials we can mentioned the works of Foo and Soh [31] and Mori et al. [32].

To define and study these families, one uses the fact that a real polynomial $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ can be identified with the vector of its coefficients $(a_0, a_1, \ldots, a_n)$. Then the set of real polynomials up to degree n can be identified with $\mathbb{R}^{n+1}$ and the sets of Schur and Hurwitz polynomials can be seen as sets contained in $\mathbb{R}^{n+1}$. Using this approach, several topological and geometric properties of the spaces of Schur and Hurwitz polynomials have been studied. For instance, it is known that the space of Schur polynomials is an open set [3, Theorem 1.3]; it is not a convex set [16] and it is a contractible set [33]. On the other hand, it is known that the space $\mathscr{H}_n$ of Hurwitz polynomials of degree n is an open set (see [3]) and it is not connected, since the coefficients of a Hurwitz polynomial have the same sign ([1]). However, the set of Hurwitz polynomials with positive coefficients, $\mathscr{H}_n^+$, is connected ([3, 34]) and it is a contractible space [35]. Furthermore it is known that $\mathscr{H}_n$ is not convex ([18, 19, 21, 23]).

The aim of this paper is to present a The aim of this paper is to present a different viewpoint in the study of topological properties of the spaces of Schur and Hurwitz (aperiodic) polynomials which is more natural since the definition of such polynomials is in terms of their roots. Instead of identifying a monic complex polynomial $p(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} + t^n$ with the vector of its coefficients, we identify it with the set of its roots

$\{z_1, \ldots, z_n\}$, where $z_i \in \mathbb{C}$ and $p(z_i) = 0$ for $i = 1, \ldots, n$. It is well known that the space of roots is homeomorphic to the space of the coefficients and Viète's map gives an explicit formula to relate both spaces.

Using this viewpoint we give simple proofs of some known results about the topology of the spaces of Schur and Hurwitz polynomials; for instance, we give a direct proof (see Section 6) that the space of Hurwitz polynomials of degree n with positive (resp. negative) coefficients is contractible. Additionally we prove that the space of monic (Schur or Hurwitz) *aperiodic* polynomials is contractible and we establish a Boundary Theorem; these results have not been reported in control literature. We would like to emphasize that despite Viète's map and the space of roots are well known in Mathematics, they have not been used explicitly (see Remark 7.2) to prove the aforementioned results and our contribution is to give simpler proofs using them. Compare Theorem 4.3 with [33, Lemma 1] and Theorem 4.4 with [35, Corrolary 4.1.28] and see Section 6 of the present paper.

One could also study a broad variety of families of polynomials directly in the space of roots and via Viète's map to get the corresponding results in the space of coefficients. In this way we use a topological approach to study the spaces of polynomials. Other works where topological and geometric ideas have been applied in control theory are the papers [36–44]. The rest of the paper is organized as follow: in Section 2 we introduce the concept of Hurwitz or Schur (aperiodic) vector and we give some notation which will be useful in the proofs; in Section 3 we write some known results; in Section 4 we define the space of roots, which is the approach that we suggest in this paper; in Section 5 we establish the main results; in Section 6 some observations about the relation between Schur and Hurwitz polynomials are given; in Section 7 two remarks are included to explain the relation of our paper with other previous works; finally, some final conclusions are established in Section 8.

2. The Spaces of Schur, Hurwitz, and Aperiodic Polynomials

Let $\mathbb{F}$ be either the real or the complex numbers. Consider the set

$$\mathcal{P}_{\leq n}^{\mathbb{F}} = \{a_0 + a_1 t + \cdots + a_n t^n \mid a_i \in \mathbb{F}\} \tag{2.1}$$

of polynomials in one variable, of degree less than or equal to n, with coefficients in $\mathbb{F}$. The set $\mathcal{P}_{\leq n}^{\mathbb{F}}$ is a vector space and choosing the monomials $\{1, t, \ldots, t^{n-1}, t^n\}$ as a basis, we can give explicitly an isomorphism between $\mathcal{P}_{\leq n}^{\mathbb{F}}$ and $\mathbb{F}^{n+1}$ identifying the polynomial $a_0 + a_1 t + \cdots + a_n t^n$ with the vector $(a_0, a_1, \ldots, a_n)$ in $\mathbb{F}^{n+1}$. Clearly $\mathcal{P}_{\leq n-1}^{\mathbb{F}} \subset \mathcal{P}_{\leq n}^{\mathbb{F}}$ and under the isomorphism $\mathcal{P}_{\leq n-1}^{\mathbb{F}}$ corresponds to the hyperplane defined by the equation $a_n = 0$. Thus the set of polynomials of degree n with coefficients in $\mathbb{F}$, denoted by $\mathcal{P}_{n}^{\mathbb{F}}$, corresponds to the set $\{(a_0, a_1, \ldots, a_n) \in \mathbb{F}^{n+1} \mid a_n \neq 0\}$. If we denote by $\mathcal{M}\mathcal{P}_{n}^{\mathbb{F}}$ the set of *monic* polynomials of degree n, then it corresponds to the hyperplane in $\mathbb{F}^{n+1}$ defined by the equation $a_n = 1$, that is, vectors in $\mathbb{F}^{n+1}$ of the form $(a_0, a_1, \ldots, a_{n-1}, 1)$. Usually we will identify $\mathcal{M}\mathcal{P}_{n}^{\mathbb{F}}$ directly with $\mathbb{F}^n$, identifying the monic polynomial $a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} + t^n$ with the vector $[a_0, a_1, \ldots, a_{n-1}] \in \mathbb{F}^n$, we shall use square brackets to avoid confusion with its corresponding vector

$$(a_0, a_1, \ldots, a_{n-1}, 1) \in \mathbb{F}^{n+1}. \tag{2.2}$$

Using this isomorphism we can endow $\mathcal{P}_{\leq n}^{\mathbb{C}}$ (respectively $\mathcal{P}_{\leq n}^{\mathbb{R}}$) with the Hermitian (respectively Euclidean) inner product and its induced topology. Also we can think of the inclusion of the set of real polynomials $\mathcal{P}_{\leq n}^{\mathbb{R}}$ in the set of complex polynomials $\mathcal{P}_{\leq n}^{\mathbb{C}}$ as the inclusion of $\mathbb{R}^{n+1}$ in $\mathbb{C}^{n+1}$.

A (real or complex) polynomial $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ is called a *Schur polynomial* if all its roots are in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid \|z\| < 1\}$. The polynomial $p(t)$ is called a *Schur aperiodic polynomial* if all its roots are distinct, real, and in the interval $(0,1)$.

A (real or complex) polynomial $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ is called a *Hurwitz polynomial* if all its roots have negative real part. The polynomial $p(t)$ is called a *Hurwitz aperiodic polynomial* if all its roots are distinct, real, and negative.

Following [34], we call a vector $(a_0, a_1, \ldots, a_n) \in \mathbb{F}^{n+1}$ *Schur*, *Schur aperiodic*, *Hurwitz*, or *Hurwitz aperiodic* if it corresponds, respectively, to a Schur, Schur aperiodic, Hurwitz, or Hurwitz aperiodic polynomial under the aforementioned isomorphism.

Let $\mathcal{S}_{\leq n}^{\mathbb{F}}$ be the sets of Schur vectors in $\mathbb{F}^{n+1}$ and let $\mathcal{S}_n^{\mathbb{F}}$ denote the set of Schur vectors which correspond to Schur polynomials of degree n. Then we have that

$$\mathcal{S}_{\leq n}^{\mathbb{F}} = \mathcal{S}_n^{\mathbb{F}} \cup \mathcal{S}_{\leq n-1}^{\mathbb{F}}. \tag{2.3}$$

Analogously, let $\mathcal{H}_{\leq n}^{\mathbb{F}}$ be the sets of Hurwitz vectors in $\mathbb{F}^{n+1}$ and let $\mathcal{H}_n^{\mathbb{F}}$ denote the set of Hurwitz vectors which correspond to Hurwitz polynomials of degree n. Then we have that

$$\mathcal{H}_{\leq n}^{\mathbb{F}} = \mathcal{H}_n^{\mathbb{F}} \cup \mathcal{H}_{\leq n-1}^{\mathbb{F}}. \tag{2.4}$$

In the same way, let $\mathcal{SA}_{\leq n}^{\mathbb{F}}$ and $\mathcal{HA}_{\leq n}^{\mathbb{F}}$ be, respectively, the sets of Schur aperiodic and Hurwitz aperiodic polynomials. Then the previous decompositions restrict to the subsets of aperiodic polynomials

$$\begin{gathered} \mathcal{SA}_{\leq n}^{\mathbb{F}} = \mathcal{SA}_n^{\mathbb{F}} \cup \mathcal{SA}_{\leq n-1}^{\mathbb{F}}, \\ \mathcal{HA}_{\leq n}^{\mathbb{F}} = \mathcal{HA}_n^{\mathbb{F}} \cup \mathcal{HA}_{\leq n-1}^{\mathbb{F}}. \end{gathered} \tag{2.5}$$

From the fact that if $a \in \mathbb{F}^{n+1}$ the polynomials corresponding to a and λa have the same roots for any $\lambda \in \mathbb{F}$ with $\lambda \neq 0$, we have that if $a \in \mathcal{S}_n^{\mathbb{F}}$ (resp., $a \in \mathcal{H}_n^{\mathbb{F}}$, $a \in \mathcal{SA}_n^{\mathbb{F}}$, $a \in \mathcal{HA}_n^{\mathbb{F}}$), then $\lambda a \in \mathcal{S}_n^{\mathbb{F}}$ (resp. $\lambda a \in \mathcal{H}_n^{\mathbb{F}}$, $\lambda a \in \mathcal{SA}_n^{\mathbb{F}}$, $\lambda a \in \mathcal{HA}_n^{\mathbb{F}}$) for any $\lambda \neq 0$. Therefore, to study the space $\mathcal{S}_n^{\mathbb{F}}$ (resp. $\mathcal{H}_n^{\mathbb{F}}$, $\mathcal{SA}_n^{\mathbb{F}}$, $\mathcal{HA}_n^{\mathbb{F}}$) it is enough to study the space of monic Schur (resp. Hurwitz, Schur aperiodic, Hurwitz aperiodic) polynomials of degree n with coefficients in $\mathbb{F}$, which we will denote by $\mathcal{MS}_n^{\mathbb{F}}$ (resp. $\mathcal{MH}_n^{\mathbb{F}}$, $\mathcal{MSA}_n$, $\mathcal{MHA}_n$). In the case of $\mathcal{MSA}_n$ and $\mathcal{MHA}_n$ we drop the superscript $\mathbb{F}$ since the coefficients of a monic (Schur or Hurwitz) aperiodic polynomial are real, because all of its roots are real. If $\mathcal{Q}_n^{\mathbb{F}}$ is any of $\mathcal{S}_n^{\mathbb{F}}$, $\mathcal{H}_n^{\mathbb{F}}$, $\mathcal{SA}_n^{\mathbb{F}}$, or $\mathcal{HA}_n^{\mathbb{F}}$, and $\mathcal{MQ}_n^{\mathbb{F}}$ is the corresponding set of monic polynomials, we have that

$$\mathcal{Q}_n^{\mathbb{F}} \quad \text{is homeomorphic to } \mathcal{MQ}_n^{\mathbb{F}} \times \mathbb{F}^*, \tag{2.6}$$

where $\mathbb{F}^* = \mathbb{F} - \{0\}$.

In the case of real polynomials we can say more. Since we are mainly interested in real polynomials we shall denote $\mathcal{H}_n^{\mathbb{R}}$ simply by $\mathcal{H}_n$, $\mathcal{S}_n^{\mathbb{R}}$ simply by $\mathcal{S}_n$, and so forth. As before, if $\mathcal{Q}_n = \mathcal{Q}_n^{\mathbb{R}}$ is any of $\mathcal{S}_n$, $\mathcal{H}_n$, $\mathcal{SA}_n$, or $\mathcal{HA}_n$, and $\mathcal{MQ}_n$ is the corresponding set of monic polynomials, we have that the space $\mathcal{Q}_n$ is homeomorphic to the disjoint union of two cylinders over the space of corresponding real monic polynomials $\mathcal{M}\mathcal{Q}_n$, that is

$$\mathcal{Q}_n \cong \mathcal{M}\mathcal{Q}_n \times (-\infty, 0) \cup \mathcal{M}\mathcal{Q}_n \times (0, \infty). \tag{2.7}$$

For the case of real Hurwitz polynomials $\mathcal{H}_n$ we can give an explicit description of each of such cylinders. The coefficients of a real Hurwitz polynomial have the same sign [1], therefore we can express it as the union of two sets

$$\mathcal{H}_n = \mathcal{H}_n^+ \cup \mathcal{H}_n^-, \tag{2.8}$$

where $\mathcal{H}_n^+$ and $\mathcal{H}_n^-$ are, respectively, the set of real Hurwitz vectors with positive and negative coefficients. If $a = (a_0, a_1, \ldots, a_n) \in \mathcal{H}_n^+$, then $-a \in \mathcal{H}_n^-$, that is, $\mathcal{H}_n^- = -\mathcal{H}_n^+$. Hence, to study the space $\mathcal{H}_n$ it is enough to study $\mathcal{H}_n^+$ (compare with [34, Proposition 2.1]). In fact, topologically the space $\mathcal{H}_n^+$ is homeomorphic to one of the aforementioned cylinders over the space $\mathcal{M}\mathcal{H}_n$, that is,

$$\mathcal{H}_n^+ \cong \mathcal{M}\mathcal{H}_n \times (0, \infty), \tag{2.9}$$

geometrically it corresponds to a cone in $\mathbb{R}^{n+1}$ over $\mathcal{M}\mathcal{H}_n$ with vertex at the origin (not including the vertex). This is expressed by the following map which maps $\mathcal{M}\mathcal{H}_n \times (0, \infty)$ diffeomorphically onto $\mathcal{H}_n^+$ in $\mathbb{R}^{n+1}$

$$\begin{gathered} \mathcal{M}\mathcal{H}_n \times (0, \infty) \longrightarrow \mathbb{R}^{n+1} \\ ([a_0, a_1, \ldots, a_{n-1}], \lambda) \longmapsto (\lambda a_0, \lambda a_1, \ldots, \lambda a_{n-1}, \lambda), \end{gathered} \tag{2.10}$$

where we identify a monic polynomial with a vector in $\mathbb{R}^n$ as before.

Hence we have that

$$\mathcal{H}_{\leq n} = \mathcal{H}_n^+ \cup \mathcal{H}_n^- \cup \mathcal{H}_{\leq n-1}. \tag{2.11}$$

Analogously,

$$\mathcal{H}\mathcal{A}_{\leq n} = \mathcal{H}\mathcal{A}_n^+ \cup \mathcal{H}\mathcal{A}_n^- \cup \mathcal{H}\mathcal{A}_{\leq n-1}, \tag{2.12}$$

and $\mathcal{H}\mathcal{A}_n^+ \cong \mathcal{M}\mathcal{H}\mathcal{A}_n \times (0, \infty)$.

3. Symmetric Products, Configuration Spaces, and Viète's Theorems

Let Σ_n be the *symmetric* or *permutation* group of the set $\{1, 2, \ldots, n\}$. Let X be a topological space and let $X^n = X \times \cdots \times X$ be its nth Cartesian product for $n \geq 1$. Consider the action of Σ_n on X^n given by permutation of coordinates

$$\begin{gathered}\Sigma_n \times X^n \longrightarrow X^n, \\ \tau \cdot (x_1, \ldots, x_n) = \left(x_{\tau(1)}, \ldots, x_{\tau(n)}\right).\end{gathered} \tag{3.1}$$

The orbit space of this action

$$\operatorname{Sym}^n(X) = \frac{X^n}{\Sigma_n}, \tag{3.2}$$

endowed with the quotient topology, is called the nth *symmetric product* of X. The equivalence class of the n-tuple $(x_1, \ldots, x_n)$ will be denoted by $\{x_1, \ldots, x_n\}$. Notice that an element $\{x_1, \ldots, x_n\} \in \operatorname{Sym}^n(X)$ is a set of n elements of X without order.

Denote by $F(X, n)$ the set of n-tuples of distinct points in X, that is,

$$F(X, n) = \left\{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\right\}. \tag{3.3}$$

By definition $F(X, n) \subset X^n$ and clearly $F(X, n)$ are invariant under the action of Σ_n, hence; the orbit space

$$B(X, n) = \frac{F(X, n)}{\Sigma_n}, \tag{3.4}$$

called the nth *configuration space* of X, consists of the sets of n distinct elements of X without order. Also by definition we have that $B(X, n) \subset \operatorname{Sym}^n(X)$.

Consider the 2-sphere S^2 as the *Riemann sphere* consisting of the complex numbers together with the point at infinity, denoted by ∞. Also consider the *complex projective space* of dimension n as the quotient space

$$\mathbb{CP}^n = \frac{\mathbb{C}^{n+1} - \{0\}}{\sim}, \tag{3.5}$$

identifying $(a_0, \ldots, a_n)$ with $(\lambda a_0, \ldots, \lambda a_n)$ for nonzero $\lambda \in \mathbb{C}$. We denote the equivalence class of $(a_0, \ldots, a_n)$ using homogeneous coordinates $(a_0 : \cdots : a_n)$. If as before we identify $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$ with a polynomial of degree less than or equal to n, then the class $(a_0 : \cdots : a_n)$ consists of all the polynomials which have the same roots as $(a_0, \ldots, a_n)$. Since in every class $(a_0 : \cdots : a_n)$ there is a vector which represents a monic polynomial, we can think of $\mathbb{CP}^n$ as the space of complex monic polynomials of degree less than or equal to n and we shall denote it by $\mathcal{MP}^{\mathbb{C}}_{\leq n}$.

A point in $\operatorname{Sym}^n(S^2)$ is an unordered n-tuple $\{z_1, \ldots, z_n\}$ of complex numbers or ∞. There exists a nonzero polynomial, unique up to a nonzero complex factor, of degree less than or equal to n whose roots are precisely $\{z_1, \ldots, z_n\}$, where we consider ∞ to be a root of the polynomial if its degree is less than n. Considering the coefficients of this polynomial as homogeneous coordinates on $\mathbb{CP}^n$ we get a map from $\mathrm{Sym}^n(S^2)$ to $\mathbb{CP}^n$ called *Viète's projective map*. This map can be written explicitly as follows.

Let σ_k^n, $k = 1, \ldots, n$ be the *elementary symmetric polynomials* in n variables $\sigma_k^n : \mathbb{C}^n \to \mathbb{C}$ which are defined as follows:

$$\begin{gathered}\sigma_0^n(z_1, \ldots, z_n) = 1, \\ \sigma_k^n(z_1, \ldots, z_n) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} z_{j_1} z_{j_2} \cdots z_{j_k}, \quad 1 \le k \le n.\end{gathered} \tag{3.6}$$

Let $\tau \in \Sigma_n$, then we have that

$$\sigma_k^n(\tau \cdot (z_1, \ldots, z_n)) = \sigma_k^n\left(z_{\tau(1)}, \ldots, z_{\tau(n)}\right) = \sigma_k^n(z_1, \ldots, z_n), \tag{3.7}$$

so the polynomials σ_k^n descend to the nth symmetric product $S^n(\mathbb{C})$ giving continuous functions

$$\sigma_k^n : \mathrm{Sym}^n(\mathbb{C}) \longrightarrow \mathbb{C}. \tag{3.8}$$

Viète's projective map assigns to $\{z_1, \ldots, z_n\} \in \mathrm{Sym}^n(S^2)$ the class in $\mathbb{CP}^n$ corresponding to all the complex polynomials with roots $\{z_1, \ldots, z_n\}$. To simplify notation we omit the argument of the polynomials σ_k^l since the superindex l means to evaluate in the l finite roots:

$$\begin{gathered}\nu : \mathrm{Sym}^n\left(S^2\right) \longrightarrow \mathbb{CP}^n, \\ \{z_1, \ldots, z_n\} \longmapsto \left((-1)^n \sigma_n^n : \cdots : -\sigma_1^n : \sigma_0^n\right), \\ \{z_1, \ldots, z_{n-1}, \infty\} \longmapsto \left((-1)^{n-1} \sigma_{n-1}^{n-1} : \cdots : \sigma_0^{n-1} : 0\right), \\ \{z_1, \ldots, z_{n-2}, \infty, \infty\} \longmapsto \left((-1)^{n-2} \sigma_{n-2}^{n-2} : \cdots : \sigma_0^{n-2} : 0 : 0\right), \\ \vdots \\ \{\infty, \ldots, \infty\} \longmapsto (1 : 0 : 0 : \cdots : 0).\end{gathered} \tag{3.9}$$

We have the following known theorem; see, for instance, [45, Bei. 3.2], [46, Exa. 5.2.4] or [47, Section 10.2 and App. A].

Theorem 3.1. *Viète's projective map is a homeomorphism between* $\mathrm{Sym}^n(S^2)$ *and* $\mathbb{CP}^n$.

Proof (Sketch). Viète's projective map ν is a continuous bijection from a compact space to a Hausdorff space and therefore a homeomorphism. □

Clearly $\mathrm{Sym}^n(\mathbb{C})$ is contained in $\mathrm{Sym}^n(S^2)$ as the n-tuples $\{z_1, \ldots, z_n\}$ which consist only of complex numbers, without ∞'s. We have that under the Viète's projective map the image of $\mathrm{Sym}^n(\mathbb{C})$ consists of the classes in $\mathbb{CP}^n$ which corresponds to polynomials of degree exactly n, that is, classes of the form $(a_0 : \cdots : a_{n-1} : 1)$. Since each class has a monic polynomial as a representative, we can identify the image of $\mathrm{Sym}^n(\mathbb{C})$ with the space $\mathcal{MP}_n^{\mathbb{C}}$ of monic polynomials of degree n, which we know is homeomorphic to $\mathbb{C}^n$.

The restriction $\nu : \mathrm{Sym}^n(\mathbb{C}) \to \mathbb{C}^n$ of Viète's projective map to $\mathrm{Sym}^n(\mathbb{C})$ is given by

$$\begin{gathered} \nu : \mathrm{Sym}^n(\mathbb{C}) \longrightarrow \mathbb{C}^n, \\ \{z_1, \ldots, z_n\} \longmapsto \left[(-1)^n \sigma_n^n, \ldots, \sigma_2^n, -\sigma_1^n\right], \end{gathered} \tag{3.10}$$

where the vector in $\mathbb{C}^n$ corresponds to the monic complex polynomial with roots $\{z_1, \ldots, z_n\}$. We denote it also by ν and we call it *Viète's map*.

Hence, as a corollary of Theorem 3.1 we get the classical Viète's Theorem

Theorem 3.2. *Viète's map is a homeomorphism between* $\mathrm{Sym}^n(\mathbb{C})$ *and* $\mathbb{C}^n$.

There are results related to Viète's Theorems which are consequence of a classical theorem by Maxwell [48]; see [49] or [47, Section 10.2 and App. A].

4. The Space of Roots

Let $\mathcal{R}_n$ denote the nth symmetric product $\mathrm{Sym}^n(\mathbb{C})$. We call $\mathcal{R}_n$ the space of *roots* of complex polynomials of degree n. Notice as before, that an element of $\mathcal{R}_n$ is a set of n complex numbers $\{z_1, \ldots, z_n\}$ without order.

What Theorem 3.2 says is that to study the topological properties of a subspace of monic complex polynomials, it is equivalent to parametrize them in terms of their coefficients or in terms of their roots.

We define the following subspaces of $\mathcal{R}_n$.

(i) The space of *roots of real polynomials*:

$$\mathcal{R}_n^{\mathbb{R}} = \left\{ \{z_1, \ldots, z_n\} \in \mathcal{R}_n \;\middle|\; \begin{array}{ll} z_{2j} = \overline{z}_{2j-1}, & j = 1, \ldots, k \text{ with } 2k \le n, \\ z_l \in \mathbb{R}, & 2k+1 \le l \le n. \end{array} \right\}. \tag{4.1}$$

Clearly its image under Viète's map is $\mathbb{R}^n$ which is homeomorphic to the space $\mathcal{MP}_n^{\mathbb{R}}$ of monic real polynomials.

(ii) The space of *roots of complex Schur polynomials*:

$$\mathcal{SR}_n^{\mathbb{C}} = \{\{z_1, \ldots, z_n\} \in \mathcal{R}_n \mid z_i \in \mathbb{D},\ i = 1, \ldots, n\}. \tag{4.2}$$

Its image under Viète's map is the set of monic complex Schur polynomials $\mathcal{MS}_n^{\mathbb{C}}$.

(iii) The space of *roots of real Schur polynomials*:

$$\mathcal{SR}_n = \mathcal{SR}_n^{\mathbb{R}} = \mathcal{R}_n^{\mathbb{R}} \cap \mathcal{SR}_n^{\mathbb{C}}. \tag{4.3}$$

Its image under Viète's map is the set of monic real Schur polynomials $\mathcal{MS}_n$.

(iv) The space of *roots of Schur aperiodic polynomials*: let $J = (0,1)$,

$$\mathcal{SAR}_n = B(J,n). \tag{4.4}$$

Its image under Viète's map is the set of monic Schur aperiodic polynomials $\mathcal{MSA}_n$.

(v) The space of *roots of complex Hurwitz polynomials*:

$$\mathcal{HR}_n^{\mathbb{C}} = \{\{z_1,\ldots,z_n\} \in \mathcal{R}_n \mid \Re(z_i) < 0,\ i = 1,\ldots,n\}. \tag{4.5}$$

Its image under Viète's map is the set of monic complex Hurwitz polynomials $\mathcal{MH}_n^{\mathbb{C}}$.

(vi) The space of *roots of real Hurwitz polynomials*:

$$\mathcal{HR}_n = \mathcal{HR}_n^{\mathbb{R}} = \mathcal{R}_n^{\mathbb{R}} \cap \mathcal{HR}_n^{\mathbb{C}}. \tag{4.6}$$

Its image under Viète's map is the set of monic real Hurwitz polynomials $\mathcal{MH}_n$.

(vii) The space of *roots of Hurwitz aperiodic polynomials*: denote by $\mathbb{R}^-$ the negative real axis,

$$\mathcal{HAR}_n = B\left(\mathbb{R}^-,n\right). \tag{4.7}$$

Its image under Viète's map is the set of monic Hurwitz aperiodic polynomials $\mathcal{MHA}_n$.

As an example of the use of the space of roots $\mathcal{R}_n$ we get simple proofs of some known results about the topology of the spaces $\mathcal{MS}_n^{\mathbb{F}}$ and $\mathcal{MH}_n^{\mathbb{F}}$.

Proposition 4.1. *The spaces $\mathcal{MS}_n^{\mathbb{F}}$ and $\mathcal{MH}_n^{\mathbb{F}}$ are open in $\mathbb{F}^n$.*

Proof. It is clear from the definition of the spaces $\mathcal{SR}_n^{\mathbb{C}}$ and $\mathcal{HR}_n^{\mathbb{C}}$ that they are open subset of $\mathcal{R}_n$; therefore under Viète's homeomorphism $\mathcal{MS}_n^{\mathbb{C}}$ and $\mathcal{MH}_n^{\mathbb{C}}$ are open in $\mathbb{C}^n$. On the other hand, $\mathcal{SR}_n$ and $\mathcal{HR}_n$ are open in $\mathcal{R}_n^{\mathbb{R}}$ and therefore $\mathcal{MS}_n$ and $\mathcal{MH}_n$ are open in $\mathbb{R}^n$. □

Proposition 4.2. *The boundary $\partial\mathcal{MS}_n^{\mathbb{F}}$ consists of all coefficient vectors in $\mathbb{F}^n$ which correspond to polynomials with all their roots in $\overline{\mathbb{D}}$ and which have at least one root on $\partial\mathbb{D}$.*

The boundary $\partial\mathcal{MH}_n^{\mathbb{F}}$ consists of all coefficient vectors in $\mathbb{F}^n$ which correspond to polynomials with all their roots in $\overline{\mathbb{C}}_-$ and which have at least one root on $\partial\mathbb{C}_- = i\mathbb{R}$.

Proof. By the definition of the spaces $\mathcal{SR}_n^{\mathbb{C}}$, $\mathcal{SR}_n$, $\mathcal{HR}_n^{\mathbb{C}}$, and $\mathcal{HR}_n$, their boundaries are given by

$$\begin{gathered}\partial\mathcal{SR}_n^{\mathbb{C}} = \left\{\{z_1,\dots,z_n\} \in \mathcal{R}_n \mid z_i \in \overline{\mathbb{D}}, i = 1,\dots,n, \text{and at least one is in } \partial\mathbb{D}\right\},\\ \partial\mathcal{SR}_n = \partial\mathcal{SR}_n^{\mathbb{C}} \cap \mathcal{R}_n^{\mathbb{R}},\\ \partial\mathcal{HR}_n^{\mathbb{C}} = \left\{\{z_1,\dots,z_n\} \in \mathcal{R}_n \mid z_i \in \overline{\mathbb{C}}_-, i = 1,\dots,n, \text{and at least one is in } i\mathbb{R}\right\},\\ \partial\mathcal{HR}_n = \partial\mathcal{HR}_n^{\mathbb{C}} \cap \mathcal{R}_n^{\mathbb{R}}.\end{gathered} \tag{4.8}$$

Using the homeomorphism given by *Viète's map* we get the proposition. □

Let $I = [0,1]$, a topological space X is *contractible* if there exists a homotopy $F : X \times I \to X$ that starts with the identity and ends with the constant map $c(x) = x_0$, for some $x_0 \in X$. Such a homotopy is called a *contraction*.

The following theorem is proved in [33, Lemma 1] (see Remark 7.2).

Theorem 4.3. *The spaces $\mathcal{SR}_n^{\mathbb{C}}$ and $\mathcal{SR}_n$ are contractible. Therefore the spaces $\mathcal{MS}_n^{\mathbb{C}}$ and $\mathcal{MS}_n$ are contractible.*

Proof. The following homotopy gives a contraction of $\mathcal{R}_n$ to the point $\{0,\dots,0\}$

$$\begin{gathered}G : \mathcal{R}_n \times I \longrightarrow \mathcal{R}_n,\\ G(\{z_1,\dots,z_n\}, r) = \{(1-r)z_1,\dots,(1-r)z_n\}.\end{gathered} \tag{4.9}$$

If $\{z_1,\dots,z_n\} \in \mathcal{SR}_n^{\mathbb{C}}$, by definition we have that $\|z_i\| < 1$, $i = 1,\dots,n$ and since $(1-r) < 1$, we have that $\|(1-r)z_i\| < 1$ for all $r \in I$ and therefore the contraction G restricts to a contraction $G : \mathcal{SR}_n^{\mathbb{C}} \times I \to \mathcal{SR}_n^{\mathbb{C}}$ proving that $\mathcal{SR}_n^{\mathbb{C}}$ is contractible.

For the case $\{z_1,\dots,z_n\} \in \mathcal{SR}_n$ we just need to check that if we have a pair of conjugate roots, say $z_{2j} = \overline{z}_{2j-1}$, they stay a conjugate pair through all the homotopy, but clearly $(1-r)z_{2j} = \overline{(1-r)z_{2j-1}}$ for all $r \in I$. Therefore the contraction F restricts to a contraction $F : \mathcal{SR}_n \times I \to \mathcal{SR}_n$ proving that $\mathcal{SR}_n$ is contractible. □

The following Theorem is indicated in [35, Corrolary 1.4.28, Ex. 13] (see Section 6).

Theorem 4.4. *The spaces $\mathcal{HR}_n^{\mathbb{C}}$ and $\mathcal{HR}_n$ are contractible. Therefore the spaces $\mathcal{MH}_n^{\mathbb{C}}$ and $\mathcal{MH}_n$ are contractible.*

Proof. The following homotopy is a contraction of $\mathcal{R}_n$ to the point $\{-1,\dots,-1\}$:

$$\begin{gathered}F : \mathcal{R}_n \times I \longrightarrow \mathcal{R}_n,\\ F(\{z_1,\dots,z_n\}, r) = \{(z_1+1)(1-r)-1,\dots,(z_n+1)(1-r)-1\}.\end{gathered} \tag{4.10}$$

If $\{z_1,\ldots,z_n\} \in \mathcal{HR}_n^{\mathbb{C}}$, by definition we have that

$$z_k = -a_k + ib_k, \quad a_k, b_k \in \mathbb{R},\ a_k > 0,\ k = 1,\ldots,n. \tag{4.11}$$

Hence

$$(z_k + 1)(1-r) - 1 = -[a_k(1-r) + r] + ib_k(1-r), \quad k = 1,\ldots,n, \tag{4.12}$$

which has negative real part for all $r \in I$. Therefore the contraction F restricts to a contraction $F : \mathcal{HR}_n^{\mathbb{C}} \times I \to \mathcal{HR}_n^{\mathbb{C}}$ proving that $\mathcal{HR}_n^{\mathbb{C}}$ is contractible.

For the case when $\{z_1,\ldots,z_n\} \in \mathcal{HR}_n$ we just need to check that if we have a pair of conjugate roots, say $z_{2j-1} = -a + ib$ and $z_{2j} = -a - ib$, they stay a conjugate pair through all the homotopy. From (4.12) we have that

$$\begin{aligned} (z_{2j-1} + 1)(1-r) - 1 &= -[a(1-r) + r] + ib(1-r), \\ (z_{2j} + 1)(1-r) - 1 &= -[a(1-r) + r] - ib(1-r), \end{aligned} \tag{4.13}$$

which are conjugate for all $r \in I$. Therefore the contraction F restricts to a contraction $F : \mathcal{HR}_n \times I \to \mathcal{HR}_n$ proving that $\mathcal{HR}_n$ is contractible. □

Recall that a topological space X is said *simply connected* if it is path connected and for some base point $x_0 \in X$ the fundamental group $\pi_1(X, x_0)$ is trivial (see [46, Section 2.5] for the definition of fundamental group).

Corollary 4.5. *The spaces $\mathcal{SR}_n^{\mathbb{F}}$, $\mathcal{MS}_n^{\mathbb{F}}$, $\mathcal{HR}_n^{\mathbb{F}}$, and $\mathcal{MH}_n^{\mathbb{F}}$ are connected and simply connected.*

Proof. The space $\mathcal{SR}_n^{\mathbb{F}}$ is connected because the contraction G in the proof of Theorem 4.3 gives a path contained in $\mathcal{SR}_n^{\mathbb{F}}$ from any set of Schur roots $\{z_1,\ldots,z_n\}$ in $\mathcal{SR}_n^{\mathbb{F}}$ to the set $\{0,\ldots,0\}$.

All the homotopy groups of a contractible space are trivial [46, Theorem 3.5.8 (g)], in particular the fundamental group, therefore $\mathcal{SR}_n^{\mathbb{F}}$ is simply connected. Since $\mathcal{MS}_n^{\mathbb{F}}$ is homeomorphic to $\mathcal{SR}_n^{\mathbb{F}}$, it is also connected and simply connected.

The proof for the spaces $\mathcal{HR}_n^{\mathbb{F}}$ and $\mathcal{MH}_n^{\mathbb{F}}$ is analogous using the contraction F in the proof of Theorem 4.4 and the set $\{-1,\ldots,-1\}$. □

Corollary 4.5 for the space $\mathcal{MH}_n$ is proved in the article [34, Lemma A1, Theorem 2.1] but there is a mistake in the part of the simply connectedness (see Remark 7.1).

Corollary 4.6. *The spaces $\mathcal{S}_n^{\mathbb{C}}$ and $\mathcal{H}_n^{\mathbb{C}}$ are homotopically equivalent to a circle S^1.*

Proof. Let $Q_n^{\mathbb{C}}$ be $\mathcal{S}_n^{\mathbb{C}}$ or $\mathcal{H}_n^{\mathbb{C}}$. By (2.6) we have that $Q_n^{\mathbb{C}}$ is homeomorphic to $\mathcal{M}Q_n^{\mathbb{C}} \times \mathbb{C}^*$ which is homotopically equivalent to a circle S^1, since by Theorem 4.3, $\mathcal{M}Q_n^{\mathbb{C}}$ is contractible and $\mathbb{C}^* = \mathbb{C} - \{0\}$ is homotopically equivalent to a circle S^1. □

Corollary 4.7. *The spaces $\mathcal{S}_n$ and $\mathcal{H}_n$ consist of two contractible connected components. For $\mathcal{H}_n$ these contractible connected components are $\mathcal{H}_n^+$ and $\mathcal{H}_n^-$.*

Proof. Let Q_n be $\mathcal{S}_n$ or $\mathcal{H}_n$. By (2.7) we have that $Q_n \cong \mathcal{M}Q_n \times (-\infty, 0) \cup \mathcal{M}Q_n \times (0, \infty)$; by Corollary 4.5, $\mathcal{M}Q_n$ is connected and therefore $\mathcal{M}Q_n \times (-\infty, 0)$ and $\mathcal{M}Q_n \times (0, \infty)$ are the connected components of Q_n, with each of them being contractible, since by Theorems 4.3 and 4.4, $\mathcal{M}Q_n$ is contractible, and the product of two contractible spaces is contractible. By (2.8) and (2.9), $\mathcal{H}_n^{\pm}$ are the connected components of $\mathcal{H}_n$. □

5. The Topology of the Spaces of Aperiodic Polynomials

In this section we shall study the topology of the spaces of Schur and Hurwitz aperiodic polynomials $\mathcal{SA}_n^{\mathbb{F}}$ and $\mathcal{HA}_n^{\mathbb{F}}$. As we saw at the end of Section 3, it is enough to study the spaces of monic polynomials $\mathcal{MSA}_n$ and $\mathcal{MHA}_n$; these in turn, by Theorem 3.2 and Section 4 are, respectively, homeomorphic to the spaces of roots of Schur aperiodic polynomials $\mathcal{SAR}_n$ and roots of Hurwitz aperiodic polynomials $\mathcal{HAR}_n$. Recall that

$$\mathcal{SAR}_n = B(J, n), \qquad \mathcal{HAR}_n = B\left(\mathbb{R}^-, n\right), \tag{5.1}$$

where $B(X, n)$ is the nth *configuration space of* X, $J = (0, 1)$, and $\mathbb{R}^-$ is the negative real axis.

Theorem 5.1. *The spaces $\mathcal{SAR}_n$ and $\mathcal{HAR}_n$ are contractible. Therefore the spaces $\mathcal{MSA}_n$ and $\mathcal{MHA}_n$ are contractible.*

Proof. Since the spaces $(0, 1)$ and $\mathbb{R}^-$ are homeomorphic, and these in turn are homeomorphic to the real line $\mathbb{R}$, we have that

$$\mathcal{SAR}_n \cong \mathcal{HAR}_n \cong B(\mathbb{R}, n), \tag{5.2}$$

therefore, it is enough to prove that the space $B(\mathbb{R}, n)$ is contractible.

The space $F(\mathbb{R}, n)$ is precisely $\mathbb{R}^n$ minus all the hyperplanes of the form $x_i = x_j$ with $i \neq j$, which consists of $n!$ connected components all of them contractible, since they are convex subspaces of $\mathbb{R}^n$. When we take the quotient of $F(\mathbb{R}, n)$ by the action of the *symmetric group* Σ_n to get $B(\mathbb{R}, n)$, all the $n!$ connected components are identified in single contractible connected component. □

The fact that the space $B(\mathbb{R}, n)$ is contractible is a very well-known result in topology but its identification with the space of *monic aperiodic polynomials*, we believe, is new in control theory.

The proofs of the following corollaries are analogous to the proofs of Corollaries 4.5, 4.6, and 4.7, respectively.

Corollary 5.2. *The spaces $\mathcal{SAR}_n$, $\mathcal{HAR}_n$, $\mathcal{MSA}_n$, and $\mathcal{MHA}_n$ are simply connected.*

Corollary 5.3. *The spaces $\mathcal{SA}_n^{\mathbb{C}}$ and $\mathcal{HA}_n^{\mathbb{C}}$ are homotopically equivalent to a circle S^1.*

Corollary 5.4. *The spaces $\mathcal{SA}_n$ and $\mathcal{HA}_n$ consist of two contractible connected components. For $\mathcal{HA}_n$ these contractible connected components are $\mathcal{HA}_n^+$ and $\mathcal{HA}_n^-$.*

If Q_n is any of $\mathcal{S}_n$, $\mathcal{H}_n^{\pm}$, $\mathcal{SA}_n$, or $\mathcal{HA}_n^{\pm}$, as a consequence of Corollaries 4.7 and 5.4, all the homotopy groups of Q_n are trivial (see [46, Theorem 3.5.8 (g)]). In other words, we have the following theorem (see [46, Lemma 3.1.5]).

Theorem 5.5 (Boundary Theorem). *If $S \subset Q_n$ is the image of an m-sphere under a continuous map $f : S^m \to Q_n$, for any $m \in \mathbb{N}$, then f can be extended to a continuous map $F : D^{m+1} \to Q_n$, where D^{m+1} is a closed ball of dimension $m+1$ and $S^m = \partial D^{m+1}$.*

6. The Relation between Schur and Hurwitz Polynomials

Theorem 4.3 was proved by Fam and Meditch in [33, Lemma 1] giving an explicit contraction in the space of *monic* Schur vectors. Their result is stated only for the space of *real monic Schur polynomials* $\mathcal{MS}_n$ although their contraction also works for the space of *complex monic Schur polynomials* $\mathcal{MS}_n^{\mathbb{C}}$. If we compose the contraction G of Theorem 4.3 with Viète's map we obtain the contraction G' given by Fam and Meditch (actually the reversed contraction, interchanging r by $1-r$; also compare with [35, Proposition 4.1.25]).

Theorem 4.4 is set as an exercise in [35, Corrolary 1.4.28, Ex. 13]; it is only stated for *real monic Hurwitz polynomials* but it can also be proved for *complex monic Hurwitz polynomials*. The proof indicated is an indirect one since it is based on the contractibility of the space $\mathcal{MS}_n^{\mathbb{F}}$ and the following transformation to relate Schur and Hurwitz polynomials. Consider the Möbius transformation

$$m : S^2 \longrightarrow S^2, \qquad m(z) = \frac{z+1}{z-1}. \tag{6.1}$$

It is a biholomorphism from the Riemann sphere onto itself which transforms the open left half plane $\mathbb{C}_-$ onto the open unit disk $\mathbb{D}$ and vice versa, since $m^{-1} = m$. In particular we have that $m(0) = -1$, $m(1) = \infty$. It is also important to notice that if $z \in \mathbb{R} \subset S^2$, with $z \neq 1$, then $m(z) \in \mathbb{R}$ and if $z_2 = \overline{z}_1$ then $m(z_2) = \overline{m(z_1)}$.

Let $p(t) = a_0 + a_1 t + \cdots + a_n t^n \in \mathcal{P}_{\leq n}^{\mathbb{C}}$ and suppose it has roots $\{z_1, \dots, z_n\}$. Define the *Möbius transform*

$$\widetilde{\ } : \mathcal{P}_{\leq n}^{\mathbb{C}} \longrightarrow \mathcal{P}_{\leq n}^{\mathbb{C}}, \qquad p \longmapsto \widetilde{p}. \tag{6.2}$$

By

$$\begin{aligned} \widetilde{p}(s) &= (s-1)^n p\left(\frac{s+1}{s-1}\right) \\ &= \sum_{i=0}^{n} a_i (z+1)^i (z-1)^{n-i} \\ &= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{i=j}^{k+j} a_i \binom{i}{j} \binom{n-i}{n-k-j} s^k. \end{aligned} \tag{6.3}$$

This is a linear isomorphism; it is involutive modulo a non-zero constant, that is, $\widetilde{\widetilde{p}} = 2^n p$, and one of its main properties is that if $p \in \mathcal{P}_n^{\mathbb{C}}$ of degree n with $p(1) \neq 0$, p is a Schur polynomial if and only if $\widetilde{p}$ is a Hurwitz polynomial of degree n; moreover $\widetilde{p}$ has roots $\{m(z_1), \ldots, m(z_n)\}$ (see [35, Lemma 3.4.81] for further properties).

From the fact that if $z \in \mathbb{R}$ then $m(z) \in \mathbb{R}$ and if $z_2 = \overline{z}_1$ then $m(z_2) = \overline{m(z_1)}$ we have that if p is a real polynomial then its Möbius transform $\widetilde{p}$ is again a real polynomial. However from (6.3) we can see that the Möbius transform of a monic polynomial p in general is not a monic polynomial since the leading coefficient of $\widetilde{p}$ in this case ($a_n = 1$) is given by $\widetilde{a}_n = 1 + \sum_{i=0}^{n-1} a_i$. Therefore we cannot use directly the Möbius transform to relate $\mathcal{MS}_n^{\mathbb{F}}$ with $\mathcal{MH}_n^{\mathbb{F}}$. To avoid this problem one defines the *normalized Möbius transform* $\breve{p}$ of p by

$$\breve{p}(s) = \left(1 + \sum_{l=0}^{n-1} a_l\right)^{-1} \widetilde{p}(s). \tag{6.4}$$

In this way we get a continuous map

$$\breve{\ }: \mathcal{MP}_{\leq n}^{\mathbb{C}} \longrightarrow \mathcal{MP}_{\leq n}^{\mathbb{C}}. \tag{6.5}$$

By the aforementioned property of $\widetilde{p}$ we have that restricting to $\mathcal{MH}_n^{\mathbb{F}}$ we get the homeomorphism

$$\begin{aligned} \breve{\ }: \mathcal{MH}_n^{\mathbb{F}} &\longrightarrow \mathcal{MS}_n^{\mathbb{F}}, \\ p &\longmapsto \breve{p}, \end{aligned} \tag{6.6}$$

which is its own inverse, that is, $\breve{\breve{p}} = p$.

Now we can use the *normalized Möbius transform* to prove that $\mathcal{MH}_n^{\mathbb{F}}$ is contractible. Let $G' : \mathcal{MS}_n^{\mathbb{F}} \times I \to \mathcal{MS}_n^{\mathbb{F}}$ be the contraction of $\mathcal{MS}_n^{\mathbb{F}}$ given by the image of the contraction G in the proof of Theorem 4.3 under Viète's map.

Define the contraction $H : \mathcal{MH}_n^{\mathbb{F}} \times I \to \mathcal{MH}_n^{\mathbb{F}}$ by $H(p, r) = \breve{G}'(\breve{p}, r)$, that is, following the diagram.

$$\begin{tikzcd} \mathcal{MH}_n^{\mathbb{F}} \times I \arrow[r, "\breve{\ }\times \mathrm{Id}"] \arrow[d, dashed, "H"'] & \mathcal{MS}_n^{\mathbb{F}} \times I \arrow[d, "G'"] \\ \mathcal{MH}_n^{\mathbb{F}} & \mathcal{MS}_n^{\mathbb{F}} \arrow[l, "\breve{\ }"'] \end{tikzcd} \tag{6.7}$$

The contraction G' contracts the space $\mathcal{MS}_n^{\mathbb{F}}$ to the Schur vector $[0, \ldots, 0] \in \mathbb{R}^n$, which corresponds to the polynomial t^n, then the contraction H contracts the space $\mathcal{MH}_n^{\mathbb{F}}$ to the Hurwitz vector $[\binom{n}{n}, \binom{n}{n-1}, \ldots, \binom{n}{1}]$ which corresponds to the polynomial $(s+1)^n$, since the *normalized Möbius transform* of t^n is $(s+1)^n$.

Using the space of roots we can give a simpler proof of the homeomorphism between the space of degree n monic Hurwitz polynomials $\mathcal{MH}_n^{\mathbb{F}}$ and the space of degree n monic Schur polynomials $\mathcal{MS}_n^{\mathbb{F}}$. The proof is simpler in the sense that one does not need any of the properties of the *Möbius transform* given in [35, Lemma 3.4.81] but only the fact that the Möbius transformation (6.1) is a homeomorphism which transforms $\mathbb{C}_-$ into $\mathbb{D}$. The Möbius transformation m given in (6.1) induces a homeomorphism from the nth Cartesian product

$S^2 \times \cdots S^2$ of the Riemann sphere onto itself. This homeomorphism is equivariant with respect to the action of Σ_n and therefore it induces a homeomorphism on the nth symmetric product of S^2

$$\begin{gathered}\tilde{m} : \operatorname{Sym}^n\left(S^2\right) \longrightarrow \operatorname{Sym}^n\left(S^2\right), \\ \{z_1, \dots, z_n\} \longmapsto \{m(z_1), \dots, m(z_n)\},\end{gathered} \tag{6.8}$$

such that $\tilde{m}^{-1} = \tilde{m}$. Since m maps homeomorphically the open left half plane $\mathbb{C}_-$ onto the open unit disk $\mathbb{D}$, it is clear that $\tilde{m}$ maps the space of Hurwitz roots $\mathcal{HR}_n^{\mathbb{F}}$ homeomorphically onto the space of Schur roots $\mathcal{SR}_n^{\mathbb{F}}$. The space $\mathcal{HR}_n^{\mathbb{F}}$ is homeomorphic to the space $\mathcal{MH}_n^{\mathbb{F}}$ and the space $\mathcal{SR}_n^{\mathbb{F}}$ is homeomorphic to the space $\mathcal{MS}_n^{\mathbb{F}}$ via Viète's map. Therefore $\mathcal{MH}_n^{\mathbb{F}}$ and $\mathcal{MS}_n^{\mathbb{F}}$ are homeomorphic.

Remember that $\mathbb{CP}^n$ can be thought as the space of complex monic polynomials of degree less than or equal to n, denoted by $\mathcal{MP}_{\leq n}^{\mathbb{C}}$, and that *Viète's projective map* is a homeomorphism $\nu : \operatorname{Sym}^n(S^2) \to \mathcal{MP}_{\leq n}^{\mathbb{C}}$. Combining *Viète's projective map* with the homeomorphism $\tilde{m}$, we get a natural homeomorphism h from $\mathcal{MP}_{\leq n}^{\mathbb{C}}$ to itself given by the following diagram:

$$\begin{tikzcd} \mathcal{MP}_{\leq n}^{\mathbb{C}} \arrow[r, dashed, "h"] \arrow[d, "\nu^{-1}"'] & \mathcal{MP}_{\leq n}^{\mathbb{C}} \\ \operatorname{Sym}^n(S^2) \arrow[r, "\tilde{m}"] & \operatorname{Sym}^n(S^2) \arrow[u, "\nu"'] \end{tikzcd} \tag{6.9}$$

given explicitly as follows. Let $p \in \mathcal{MP}_{\leq n}^{\mathbb{C}}$ and suppose it has roots $\{z_1, \dots, z_n\}$, that is, $p(t) = \prod_{i=1}^{n}(t - z_i)$, then

$$h(p)(s) = \prod_{i=1}^{n}(s - m(z_i)), \tag{6.10}$$

the monic polynomial with roots $\{m(z_1), \dots, m(z_n)\}$. Remember that we consider ∞ to be a root of the polynomial if its degree is less than n. Therefore the homeomorphism h is precisely the *normalized Möbius transform* $\breve{\cdot}$; to see this, compare (6.10) with (70) in the proof of Lemma 3.4.81 in [35] when p is monic of degree n.

7. Two Remarks

We have the following two remarks about previous works.

Remark 7.1. The proof of Theorem 2.1 in [34] that the space $\mathcal{MH}_n$ (H_{1+}^n in notation of [34]) is simply connected is not correct. In Lemma A3 the space

$$\mathbb{R}_{1+}^{n+1} = \left\{(a_0, \dots, a_{n-1}, 1) \in \mathbb{R}^{n+1} \mid a_i > 0,\ i = 0, \dots, n-1\right\}, \tag{7.1}$$

which is homeomorphic to $\mathbb{R}^n$, is expressed as the union of two open connected subsets H^n_{1+} and U^n_{1+} with common boundary B^n_{1+}, that is,

$$\mathbb{R}^{n+1}_{1+} = H^n_{1+} \cup B^n_{1+} \cup U^n_{1+}. \tag{7.2}$$

Then the argument is that if H^n_{1+} is not simply connected, then U^n_{1+} is not connected and this contradicts the connectivity of U^n_{1+}. This argument is valid only in dimension 2 (i.e., in $\mathbb{R}^3_{1+} \cong \mathbb{R}^2$) but it is not true in higher dimensions. One counterexample is to consider the subset of $\mathbb{R}^3$ given by a closed solid torus

$$T = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \left(\sqrt{(x^2 + y^2)} - 2 \right)^2 + z^2 \leq 1 \right\}, \tag{7.3}$$

which is the solid obtained rotating the 2-disk D^2 in the plane xz with center in $(2, 0, 0)$, that is, $D^2 = \{(x, 0, z) \in \mathbb{R}^3 \mid (x-2)^2 + z^2 \leq 1\}$, around the z-axis, giving the shape of a "doughnut." The space T is homeomorphic to $D^2 \times S^1$. Denote by H the interior of the torus, by B its boundary, and by U the complement of T in $\mathbb{R}^3$. Then $\mathbb{R}^3 = H \cup B \cup U$, with H and U being open and connected with common boundary B. The space H is not simply connected because it is homotopically equivalent to a circle S^1. Also in $\mathbb{R}^3$, even if B was homeomorphic to a 2-sphere S^2, the unbounded component of $\mathbb{R}^3 - B$ is not necessarily simply connected; an example of this is the famous *Alexander Horned Sphere* (see [50, Example 2B.2]).

A counterexample in dimension $n > 2$ is similar taking $T = D^{n-1} \times S^1$. The same mistake is also made in [51, Theorem 3.2].

Now with the proof that $\mathcal{H}^+_n$ is contractible (Corollary 4.7), in particular simply connected, the *Edge and Boundary Theorems* of [34], which use as a main ingredient the simple connectivity of $\mathcal{H}^+_n$, remain valid.

Remark 7.2. The approach of using the roots space $\mathcal{R}_n$ has been used implicitly in previous studies of spaces of polynomials. For instance, in [33, Lemma 1] to prove the contractibility of the space of real monic Schur polynomials $\mathcal{MS}_n$, implicitly they proved the contractibility of the space of roots which are in the open disk in the complex plane. Also in [34, Lemma A1] to prove that H^n_{1+} is connected, they implicitly constructed a path in the space of roots.

8. Conclusions

Viète's map gives an explicit homeomorphism between the space of roots $\mathcal{R}_n$ and the space of monic complex polynomials $\mathcal{MP}^{\mathbb{C}}_n$. Restricting this homeomorphism to the spaces of Schur and Hurwitz (aperiodic) roots one can study in a more natural way some topological properties of the spaces of Schur and Hurwitz (aperiodic) polynomials. Using this viewpoint we give simple proofs of some topological properties of the spaces of Schur and Hurwitz polynomials; in particular we prove that the spaces $\mathcal{H}^+_n$ and $\mathcal{H}^-_n$ of Hurwitz polynomials of degree n with positive and negative coefficients, respectively, are contractible and therefore simply connected. As a new result we prove that the spaces of *monic Schur aperiodic polynomials* $\mathcal{MSA}_n$ and of *monic Hurwitz aperiodic polynomials* $\mathcal{MHA}_n$ are contractible. As a consequence of the contractibility of the spaces $\mathcal{S}_n$, $\mathcal{H}^\pm_n$, $\mathcal{SA}_n$, or $\mathcal{HA}^\pm_n$, we get the *Boundary Theorem* given in Theorem 5.5.

Also using the space of roots and Viète's projective map we see that the *normalized Möbius transform* is a natural transformation from the space $\mathcal{MP}_{\leq n}^{\mathbb{C}}$ of *complex monic polynomials* of degree less than or equal to n to itself, instead of just being seen as a "correction" to the *Möbius transform* to get monic polynomials. It gives a homeomorphism between the space of monic (complex or real) Schur polynomials and the space of (complex or real) Hurwitz polynomials.

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