Ideal Limit Theorems and Their Equivalence in (ℓ)-Group Setting

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Abstract

We prove some equivalence results between limit theorems for sequences of (ℓ)-group-valued measures, with respect to order ideal convergence. A fundamental role is played by the tool of uniform ideal exhaustiveness of a measure sequence already introduced for the real case or more generally for the Banach space case in our recent papers, to get some results on uniform strong boundedness and uniform countable additivity. We consider both the case in which strong boundedness, countable additivity and the related concepts are formulated with respect to a common order sequence and the context in which these notions are given in a classical like setting, that is not necessarily with respect to a same (O)-sequence. We show that, in general, uniform ideal exhaustiveness cannot be omitted. Finally we pose some open problems.

1. Introduction

Ideal convergence was introduced by Kostyrko, Šalát, and Wilczyński (2000/2001) and independently by Nuray and Ruckle (2000) with the name of "cofilter convergence", while filter convergence was introduced by Katětov (1968). Ideal convergence includes as a particular case the statistical convergence, introduced by Fast (1951) and Steinhaus (1951) (see also Connor, 1992; Fridy & Miller, 1991; Kolk, 1993; Šalát, 1980). Several fundamental properties of ideal convergence have been recently investigated. Among the literature, we quote for instance Boccuto, Das, and Dimitriou (2012); Boccuto, Dimitriou, and Papanastassiou (2013); Dems (2004/2005); Komisarski (2008); Laczkovich and Recław (2009); Lahiri and Das (2003); Letavaj (2011); Šalát, Tripathy, and Ziman (2004). This concept has been studied even in topological spaces (see e.g. Das, 2012; Das & Ghosal, 2011; Lahiri & Das, 2005; Lahiri & Das, 2007/2008) and in (ℓ)-groups (see also Boccuto & Dimitriou, 2011a, 2011b, 2011c, 2013b; Boccuto, Dimitriou, & Papanastassiou, 2010b, 2011b, 2011c, 2012a, 2012b, 2012d; Boccuto, Dimitriou, Papanastassiou, & Wilczyński, 2012, 2013). This notion has been several applications in the very recent literature. Among them we recall weak ideal compactness in measure spaces (see also Boccuto, Das, & Dimitriou, 2012; Boccuto, Das, Dimitriou, & Papanastassiou, 2012; Boccuto, Dimitriou, & Papanastassiou, 2010b; Dimitriou, 2011), Approximation Theory, signal sampling and reconstruction of images (see also Bardaro, Boccuto, Dimitriou, & Mantellini, 2012, 2013; Boccuto & Dimitriou, 2012, 2013c; Duman, 2007; Higgins & Stens, 1999), and in particular Brooks-Jewett, Dieudonné, Nikodým convergence and Vitali-Hahn-Saks-type theorems, with which we deal in this paper.

In the classical context, among the extensions of the classical convergence theorems (see Brooks & Jewett, 1970; Choksi, 2001; Dieudonné, 1951; Hahn, 1922; Nikodým, 1933a, 1933b; Saks, 1932, 1937; Vitali, 1907), we quote Candeloro (1985a, 1985b, 1985c), de Lucia & Morales (1986) and Drewnowski (1972a, 1972b) for topological group-valued measures and Boccuto (1996a, 1996b); Boccuto and Candeloro (2001/2002, 2002, 2004, 2010, 2011); Boccuto, Dimitriou, and Papanastassiou (2012c) for Riesz space- and/or lattice group-valued measures (see also de Lucia & Pap, 2002; Dimitriou, 2007, 2011 and their bibliographies). In Boccuto, Dimitriou, and Papanastassiou (2011b) some ideal limit theorems are proved for positive (ℓ)-group-valued measures, extending some results of Boccuto (1996b). In Boccuto, Dimitriou, and Papanastassiou (2011c, 2012d) some limit theorems are given in this setting for not necessarily positive measures and for a suitable class of filters, extending earlier results of Aviles Lopez, Cascales Salinas, Kadets, and Leonov (2007). Similar results have been recently proved in topological group context in Boccuto and Dimitriou (2013a). In Boccuto, Das, Dimitriou, and Papanastassiou (2012) and Boccuto and Dimitriou (2011a, 2011b) some other versions of ideal limit theorems for real-valued and lattice group-valued measures are given. In this framework, a fundamental role is played by uniform ideal exhaustiveness, which in general, when $\mathcal{I} \neq \mathcal{I}_{\mathrm{fin}}$, cannot be dropped (see also Boccuto & Dimitriou, 2011b). These theorems are given when σ-additivity and related concepts are formulated not necessarily with respect to a same (O)-sequence. Moreover, in Boccuto, Dimitriou, and Papanastassiou (2010b, 2012a) some versions of basic matrix theorems are given, extending earlier results of Aizpuru and Nicasio-Llach (2008); Aizpuru, Nicasio-Llach, and Rambla-Barreno (2010); Antosik and Swartz (1985). Note that in general these kinds of theorems, in their ideal version, do not produce immediate results for measure convergence like in the classical case, since in lattice groups the nature of order convergence is in general not topological, and because filter convergence is not inherited by subsequences. Furthermore note that, in the ideal setting, in general one cannot have results completely analogous to the classical limit theorems, even in the case of positive real-valued measures (see also Boccuto, Dimitriou, & Papanastassiou, 2012a).

In this paper we continue the investigation initiated in Boccuto, Das, Dimitriou, and Papanastassiou (2012); Boccuto and Dimitriou (2011a, 2011b), in the context of (ℓ)-group-valued measures and in connection with uniform ideal exhaustiveness. We consider both cases when countable additivity and strong boundedness are intended relatively to a same order sequence, and when these notions are formulated like in the classical approach, that is not necessarily with respect to a common (O)-sequence. We give some equivalence results between limit theorems in the setting of order ideal convergence. Here, absolute continuity is intended with respect to a general Fréchet-Nikodým topology. Similar equivalence results are given in Drewnowski (1972a) in topological groups. In particular, when it is proved that the Nikodým convergence theorem implies the Brooks-Jewett theorem, countably additive restrictions of finitely additive (s)-bounded topological group-valued measures, defined on suitable σ-algebras, are considered (see also Boccuto, Dimitriou, & Papanastassiou, 2010a, 2011a for a lattice group version). However in our setting, in order to relate finitely and countably additive measures, it is not advisable to use an approach of this kind. Indeed, in topological groups, the involved convergences fulfil some suitable properties, which are not always satisfied by order convergence in (ℓ)-groups, because in general it does not have a topological nature. So, to prove our results, we use the Stone Isomorphism technique (see also Sikorski, 1964), by means of which it is possible to construct a σ-additive extension of a finitely additive (s)-bounded measure, and to study the properties of the starting measures in relation with the corresponding ones of the considered extensions. For lattice group-valued measures, the Stone extension is examined in Boccuto (1995) when σ-additivity and (s)-boundedness are intended in a classical like sense, that is not necessarily with respect to a same order sequence, and in Boccuto and Candeloro (2002, 2004) when these notions are formulated with respect to a common (O)-sequence or regulator. Note that, in topological groups, it is possible to use not only the Drewnowski Lemma (see Drewnowski, 1972a), but also the Stone Isomorphism Technique, to construct σ-additive measures by starting from finitely additive (s)-bounded measures (see also Candeloro, 1985c; Sion, 1969, 1973). Moreover, to prove that the Brooks-Jewett theorem implies the Nikodým theorem, when we link uniform (s)-boundedness and σ-additivity, when these concepts are intended not necessarily with respect to a same order sequence, in general for technical reasons it is not advisable to consider a direct approach, and we use the Maeda-Ogasawara-Vulikh representation theorem for Dedekind complete (ℓ)-groups, studying the related properties of the corresponding real-valued measures. When we deal with a common (O)-sequence, it is possible to give direct proofs, and it is not always advisable to use the tool of the Maeda-Ogasawara-Vulikh representation theorem, because it yields informations in general only about convergence of suitable (ℓ)-group-valued sequences by means of convergence of suitable real-valued sequences, and not necessarily about whether they can be obtained with respect to a single (O)-sequence in the (ℓ)-group involved.

The paper is structured as follows. In Section 2 we present some basic notions and results on (ℓ)-groups, order ideal convergence, submeasures, Fréchet-Nikodým topologies and (ℓ)-group-valued measures. In Section 3, using the Stone extension of measures in connection with uniform ideal exhaustiveness, we present our main results about ideal limit theorems and their equivalence, and we show that, in general, the condition of uniform ideal exhaustiveness cannot be dropped in our context, and it is impossible to obtain versions of limit theorems analogous to the classical ones, when pointwise convergence of the measures involved is replaced by ideal pointwise convergence. Finally we pose some open problems.

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2. Preliminaries

2.1 Lattice Groups

We recall some basic properties of lattice groups (see also Birkhoff, 1940; Boccuto, Riečan, & Vrábelová, 2009; Luxemburg & Zaanen, 1971; Riečan & Neubrunn, 1997; Vulikh, 1967).

An (ℓ)-group is said to be *Dedekind complete* iff every nonempty subset $A \subset R$, bounded from above, has lattice supremum in R, denoted by $\vee A$.

A Dedekind complete (ℓ)-group is *super Dedekind complete* iff every nonempty set $A \subset R$, bounded from above, has a countable subset $A' \subset R$, with $\vee A' = \vee A$.

A sequence $(\sigma_p)_p$ in R is an (O)-*sequence* iff it is decreasing and $\wedge_p \sigma_p = 0$, where the symbol $\wedge$ denotes the lattice infimum.

A bounded double sequence $(a_{t,l})_{t,l}$ in R is called a (D)-*sequence* or *regulator* iff $(a_{t,l})_l$ is an (O)-sequence for every $t \in \mathbb{N}$.

An (ℓ)-group R is said to be *weakly σ-distributive* iff

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \Big) = 0$$

for every (D)-sequence $(a_{t,l})_{t,l}$.

Note that weak σ-distributivity is a necessary and sufficient condition in order that, for any abstract nonempty set G and any algebra $\mathcal{A}$ of subsets of G, every σ-additive R-valued measure defined on $\mathcal{A}$ admits a σ-additive extension, defined on the σ-algebra $\Sigma(\mathcal{A})$ generated by $\mathcal{A}$ (see Wright, 1971).

A sequence $(x_n)_n$ in R is said to be *order-convergent* (or (O)-*convergent*) to $x \in R$ iff there is an (O)-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ there is $n_p \in \mathbb{N}$ with $|x_n - x| \le \sigma_p$ for every $n \ge n_p$. In this case we write $(O) \lim_n x_n = x$ (with respect to $(\sigma_p)_p$).

We now recall the Maeda-Ogasawara-Vulikh representation theorem in its version for Dedekind complete (ℓ)-groups (see also Bernau, 1965, Theorem 6; Boccuto & Dimitriou, 2011b, Theorem 2.3).

Theorem 2.1 *Given a Dedekind complete (ℓ)-group R, there exists a compact Hausdorff extremely disconnected topological space Ω, such that R can be lattice isomorphically embedded as a subgroup of $C_{\infty}(\Omega) = \{f \in \widetilde{\mathbb{R}}^{\Omega} : f$ is continuous, and $\{\omega \in \Omega : |f(\omega)| = +\infty\}$ is nowhere dense in $\Omega\}$. Moreover, if $(a_{\lambda})_{\lambda \in \Lambda}$ is any family such that $a_{\lambda} \in R$ for all λ, and $a = \vee_{\lambda} a_{\lambda} \in R$ (where the supremum is taken with respect to R), then $a = \vee_{\lambda} a_{\lambda}$ with respect to $C_{\infty}(\Omega)$, and the set $\{\omega \in \Omega : (\vee_{\lambda} a_{\lambda})(\omega) \ne \sup_{\lambda} [a_{\lambda}(\omega)]\}$ is meager in Ω.*

In this paper we deal with the order convergence for sequences in the (ℓ)-group setting. Another kind of convergence is widely studied in this context, the (D)-convergence (see also Boccuto, 2003; Boccuto, Riečan, & Vrábelová, 2009; Riečan & Neubrunn, 1997). Note that, in any Dedekind complete (ℓ)-group R, every (O)-convergent sequence (D)-converges to the same limit, while the converse is true if and only if R is weakly σ-distributive.

For technical reasons, there are some situations in which (O)-convergence is easier to handle than (D)-convergence, and other contexts in which it is preferable to consider (D)-convergence. In particular, we will often use the tool of replacing a *series* of (D)-sequences with a single regulator (Fremlin Lemma), and in this setting it is advisable to deal with regulators.

Lemma 2.2 (Fremlin Lemma, see also Fremlin, 1975, Lemma 1C; Riečan & Neubrunn, 1997, Theorem 3.2.3) *Let R be any Dedekind complete (ℓ)-group and $(a_{t,l}^{(n)})_{t,l}$, $n \in \mathbb{N}$, be a sequence of regulators in R. Then for every $u \in R$, $u \geq 0$ there is a (D)-sequence $(a_{t,l})_{t,l}$ in R with*

$$u \wedge \Big(\sum_{n=1}^{q} \big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)} \big) \Big) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \quad \text{for every } q \in \mathbb{N} \text{ and } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

The following result links order and (D)-sequences and will be useful to study some properties of lattice group-valued measures.

Theorem 2.3 (see also Boccuto, 2003, Theorems 3.1 and 3.4) *Given any Dedekind complete (ℓ)-group R and any (O)-sequence $(\sigma_l)_l$ in R, the double sequence defined by $a_{t,l} := \sigma_l$, $t, l \in \mathbb{N}$, is a regulator, with the property that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, if $l = \varphi(1)$, then*

$$\sigma_l \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}.$$

Conversely, if R is super Dedekind complete and weakly σ-distributive, then for every (D)-sequence $(a_{t,l})_{t,l}$ in R there is an (O)-sequence $(\beta_p)_p$ such that for each $p \in \mathbb{N}$ there exists $\varphi_p \in \mathbb{N}^{\mathbb{N}}$ with

$$\bigvee_{t=1}^{\infty} a_{t,\varphi_p(t)} \leq \beta_p.$$

2.2 Ideal Convergence

We now recall the main properties of ideal convergence in the (ℓ)-group setting.

A class of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is called an *ideal* of $\mathbb{N}$ iff $A \cup B \in \mathcal{I}$ whenever $A, B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and $B \subset A$ we get $B \in \mathcal{I}$. An ideal of $\mathbb{N}$ is said to be *admissible* iff $\mathbb{N} \notin \mathcal{I}$ and $\mathcal{I}$ contains all singletons.

Given an ideal $\mathcal{I}$ of $\mathbb{N}$, we call *dual filter* of $\mathcal{I}$ the family of sets $\mathcal{F} = \mathcal{F}(\mathcal{I}) := \{ \mathbb{N} \setminus A : A \in \mathcal{I} \}$.

An admissible ideal $\mathcal{I}$ of $\mathbb{N}$ is called a *P-ideal* iff for any sequence $(A_j)_j$ in $\mathcal{I}$ there are a sequence $(B_j)_j$ in $\mathcal{P}(\mathbb{N})$, such that the symmetric difference $A_j \Delta B_j$ is finite for all $j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

An admissible ideal $\mathcal{I}$ of $\mathbb{N}$ is said to be *maximal* iff, for every subset $A \subset \mathbb{N}$, we get that either $A \in \mathcal{I}$ or $\mathbb{N} \setminus A \in \mathcal{I}$.

Some examples of *P*-ideals of $\mathbb{N}$ are the ideal $\mathcal{I}_{\text{fin}}$ of all finite subsets of $\mathbb{N}$ and the ideal $\mathcal{I}_\delta$ of all subsets of $\mathbb{N}$ having null asymptotic density (see also Kostyrko, Šalát, & Wilczyński, 2000/2001; Farah, 2000). Observe that $\mathcal{I}_\delta$ is not maximal. Indeed, if $\mathbb{E}$ is the set of all even integers, then we get $\mathbb{E} \notin \mathcal{I}_\delta$ and $\mathbb{N} \setminus \mathbb{E} \notin \mathcal{I}_\delta$. However it is known that, if we assume the continuum hypothesis, then there are several maximal *P*-ideals of $\mathbb{N}$ (see also Henriksen, 1959, (1,7)).

Some other examples of *P*-ideals are the *Erdős-Ulam ideals* associated with a function $f : \mathbb{N} \to \mathbb{R}^+$, consisting on all subsets $A \subset \mathbb{N}$ for which

$$\lim_n \frac{\sum_{i \in A \cap [1,n]} f(i)}{\sum_{i=1}^{n} f(i)} = 0,$$

whose $\mathcal{I}_\delta$ is a particular case, obtained by taking $f(n) = 1$ for each $n \in \mathbb{N}$ (see also Farah, 2000, Example 1.2.3 (d)).

Let $\mathbb{N} = \bigcup_{k=1}^{\infty} \Delta_k$ be a partition of $\mathbb{N}$ into infinite sets, and $\mathcal{I}_0 = \{ A \subset \mathbb{N} : A \text{ intersects at most a finite number of } \Delta_k\text{'s} \}$.

The ideal $\mathcal{I}_0$ is not a *P*-ideal (see also Kostyrko, Šalát, & Wilczyński, 2000/2001, Example 3.1 (g)).

For further properties of ideals, see also Farah (2000) and the bibliography therein.

We now recall the concept of order ideal convergence (see also Boccuto, Dimitriou, & Papanastassiou, 2012a).

Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. A sequence $(x_n)_n$ in R $(O\mathcal{I})$-*converges to* $x \in R$ iff there exists an (O)-sequence $(\sigma_p)_p$ in R with $\{ n \in \mathbb{N} : |x_n - x| \not\leq \sigma_p \} \in \mathcal{I}$ for all $p \in \mathbb{N}$, and in this case we write $(O\mathcal{I}) \lim_n x_n = x$. If $R = \mathbb{R}$, we write simply $(\mathcal{I}) \lim_n x_n = x$.

The following result relates $(O\mathcal{I})$-convergence with the classical (O)-convergence (see also Boccuto, Dimitriou, & Papanastassiou, 2012a, Proposition 2.11).

Proposition 2.4 *Let R be any Dedekind complete (ℓ)-group and $\mathcal{I}$ be any admissible ideal of $\mathbb{N}$. If there is $B \in \mathcal{I}$ with $(O) \lim_{n \in \mathbb{N} \setminus B} x_n = x$ with respect to an (O)-sequence $(\sigma_p)_p$ in R, then $(O\mathcal{I}) \lim_n x_n = x$ with respect to $(\sigma_p)_p$.*

Proof. By hypothesis there is $B \in \mathcal{I}$ such that, if $M := \mathbb{N} \setminus B$, $M = \{m_1 < \ldots < m_h < \ldots\}$, then

$$(O) \lim_h x_{m_h} = x \tag{1}$$

with respect to a suitable (O)-sequence $(\sigma_p)_p$ in R.

Choose arbitrarily $p \in \mathbb{N}$. By (1) there exists $h_0 \in \mathbb{N}$ with $|x_{m_h} - x| \leq \sigma_p$ whenever $h \geq h_0$. Thus the set $A_p := \{n \in \mathbb{N} : |x_n - x| \not\leq \sigma_p\} \subset B \cup \{m_1, \ldots, m_{h_0-1}\}$ belongs to $\mathcal{I}$, since $\mathcal{I}$ is admissible. This ends the proof. $\square$

The converse of Proposition is in general not true, and holds if and only if $\mathcal{I}$ is a P-ideal (see also Boccuto & Dimitriou, 2011b; Boccuto, Dimitriou, Papanastassiou, & Wilczyński, 2013).

The following property of P-ideals will be useful in the sequel.

Proposition 2.5 (Boccuto & Dimitriou, 2011b, Proposition 3.2) *Let $(x_{n,j})_{n,j}$ be a double sequence in R, $\mathcal{I}$ be a P-ideal of $\mathbb{N}$, and suppose that $(O\mathcal{I}) \lim_n x_{n,j} = x_j$ for every $j \in \mathbb{N}$ with respect to a common (O)-sequence $(\sigma_p)_p$. Then there is a set $B_0 \in \mathcal{F}$ with $(O) \lim_{n \in B_0} x_{n,j} = x_j$ for all $j \in \mathbb{N}$, with respect to the same (O)-sequence $(\sigma_p)_p$.*

2.3 Set Functions and FN-Topologies

We now recall some notions and properties of submeasures, (ℓ)-group-valued measures and Fréchet-Nikodým topologies. From now on, let Σ be a σ-algebra of subsets of an abstract infinite set G.

A *submeasure* $\eta : \Sigma \to [0, +\infty]$ is a set function with $\eta(\emptyset) = 0$, $\eta(A) \leq \eta(B)$ whenever $A, B \in \Sigma$, $A \subset B$, and $\eta(A \cup B) \leq \eta(A) + \eta(B)$ whenever $A, B \in \Sigma$ and $A \cap B = \emptyset$. Note that, if η is a submeasure, then $\eta\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \eta(A_i)$ for any $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \Sigma$ (see also Drewnowski, 1972b, §2).

A submeasure η is *order continuous* iff $\lim_k \eta(H_k) = 0$ for every decreasing sequence $(H_k)_k$ in Σ with $\bigcap_{k=1}^{\infty} H_k = \emptyset$.

A topology τ on Σ is called a *Fréchet-Nikodým topology* iff the functions $(A, B) \mapsto A \Delta B$ and $(A, B) \mapsto A \cap B$ from $\Sigma \times \Sigma$ (endowed with the product topology) to Σ are continuous, and for each τ-neighborhood V of $\emptyset$ in Σ there is a τ-neighborhood U of $\emptyset$ in Σ with the property that, if $E \in \Sigma$ is contained in some suitable element of U, then $E \in V$ (see also Drewnowski, 1972b, §1).

Observe that a topology τ on Σ is a Fréchet-Nikodým topology if and only if there exists a family of submeasures $\mathcal{Z} := \{\eta_i : i \in \Lambda\}$, with the property that a base of τ-neighborhoods of $\emptyset$ in Σ is given by

$$\mathcal{D} := \{U_{\varepsilon, J} := \{A \in \Sigma : \eta_i(A) < \varepsilon \text{ for all } i \in J\} : \varepsilon \in \mathbb{R}^+, J \subset \Lambda \text{ is finite}\}$$

(see also Boccuto & Dimitriou, 2011b; Drewnowski, 1972a, 1972b; Weber, 2002).

We recall the basic properties of measures with values in a Dedekind complete (ℓ)-group R.

Given a finitely additive measure $m : \Sigma \to R$, we denote by *positive part, negative part and semivariation* of m the quantities

$$m^+(A) = \bigvee_{B \in \Sigma, B \subset A} m(B), \quad m^-(A) = \bigvee_{B \in \Sigma, B \subset A} (-m(B)), \quad v(m)(A) = \bigvee_{B \in \Sigma, B \subset A} |m(B)|$$

respectively, where $A \in \Sigma$.

We say that the finitely additive measures $m_n : \Sigma \to R$, $n \in \mathbb{N}$, are *equibounded* on Σ iff there is a $w \in R$ with $|m_n(A)| \leq w$ for all $n \in \mathbb{N}$ and $A \in \Sigma$.

A finitely additive measure $m : \Sigma \to R$ is said to be (s)-*bounded* on Σ iff $(O) \lim_k v(m)(C_k) = 0$ for every disjoint sequence $(C_k)_k$ in Σ.

The finitely additive measures $m_n \colon \Sigma \to R$, $n \in \mathbb{N}$, are *uniformly (s)-bounded* on Σ iff

$$(O)\lim_k [\vee_n v(m_n)(C_k)] = 0$$

whenever $(C_k)_k$ is a sequence of pairwise disjoint elements of Σ.

A finitely additive measure $m \colon \Sigma \to R$ is said to be *σ-additive* on Σ iff for every disjoint sequence $(C_k)_k$ in Σ we get $(O)\lim_k v(m)\bigl(\bigcup_{l=k}^{\infty} C_l\bigr) = 0$. The measures $m_n \colon \Sigma \to R$, $n \in \mathbb{N}$, are *uniformly σ-additive* on Σ iff for each disjoint sequence $(C_k)_k$ in Σ we get $(O)\lim_k \bigl(\bigvee_n v(m_n)\bigl(\bigcup_{l=k}^{\infty} C_l\bigr)\bigr) = 0$.

Let τ be a Fréchet-Nikodým topology on Σ. A finitely additive measure $m \colon \Sigma \to R$ is said to be *τ-continuous* on Σ, iff for each decreasing sequence $(H_k)_k$ in Σ, with $\tau\text{-}\lim_k H_k = \emptyset$, we get

$$(O)\lim_k v(m)(H_k) = 0.$$

The finitely additive measures $m_n \colon \Sigma \to R$, $n \in \mathbb{N}$, are *uniformly τ-continuous* on Σ, iff

$$(O)\lim_k \bigl(\bigvee_n v(m_n)(H_k)\bigr) = 0$$

whenever $(H_k)_k$ is a decreasing sequence in Σ such that $\tau\text{-}\lim_k H_k = \emptyset$.

Let $\mathcal{G}, \mathcal{H} \subset \Sigma$ denote two lattices, satisfying the following property:

(R0) the complement with respect to G of every element of $\mathcal{H}$ belongs to $\mathcal{G}$ and $\mathcal{G}$ is closed under countable unions.

A finitely additive measure $m \colon \Sigma \to R$ is *regular* on Σ iff

(R1) for each $E \in \Sigma$ and $r \in \mathbb{N}$ there exist $F_r \in \mathcal{H}$, $G_r \in \mathcal{G}$ with $F_r \subset F_{r+1} \subset E \subset G_{r+1} \subset G_r$ for any r, and $(O)\lim_r v(m)(G_r \setminus F_r) = 0$.

The finitely additive measures $m_n \colon \Sigma \to R$, $n \in \mathbb{N}$, are *uniformly regular* on Σ iff for every $E \in \Sigma$ and $r \in \mathbb{N}$ there exist $F_r \in \mathcal{H}$, $G_r \in \mathcal{G}$ with $F_r \subset F_{r+1} \subset E \subset G_{r+1} \subset G_r$ for all r, and $(O)\lim_r \bigl(\bigvee_n v(m_n)(G_r \setminus F_r)\bigr) = 0$.

Analogously as above it is possible to formulate the notions of *global* and *global uniform* (s)-boundedness, σ-additivity, τ-continuity and regularity, by requiring that the involved (O)-limits exist with respect to a common (O)-sequence. Note that, in general, these concepts are not identical. Indeed there exist bounded finitely additive lattice group-valued measures, which are not globally (s)-bounded (see Boccuto & Candeloro, 2002, Example 2.17), while every bounded finitely additive (ℓ)-group-valued measure is (s)-bounded too (see Boccuto, Dimitriou, & Papanastassiou, 2010a, Theorem 3.1).

The following technical lemma will be useful in the sequel to link global uniform (s)-boundedness of (ℓ)-group-valued measures with the other related global properties. A similar result (see Boccuto & Dimitriou, 2011b, Lemma 3.3) holds also for (s)-bounded measures, not necessarily with respect to a same (O)-sequence, but for technical reasons in this case we use the Maeda-Ogasawara-Vulikh theorem and assume uniform (s)-boundedness of the measures $m_n(\cdot)(\omega)$, $n \in \mathbb{N}$, for ω belonging to a complement of a meager subset of Ω, where Ω is as in Theorem 2.1.

Lemma 2.6 *Let $\mathcal{G} \subset \Sigma$ be a lattice, closed under countable unions, $(\sigma_p)_p$ be an (O)-sequence in R and $m_n \colon \Sigma \to R$, $n \in \mathbb{N}$, be a sequence of finitely additive measures, globally uniformly (s)-bounded on Σ. Fix $W \in \Sigma$ and a decreasing sequence $(H_k)_k$ in $\mathcal{G}$, with $W \subset H_k$ for each $k \in \mathbb{N}$. If*

$$(O)\lim_k v(m_n)(H_k) = 0 \quad \text{for all } n \in \mathbb{N} \tag{2}$$

with respect to the same (O)-sequence $(\sigma_p)_p$, then

$$(O)\lim_k \bigl(\bigvee_n v(m_n)(H_k)\bigr) = 0$$

with respect to $(\sigma_p)_p$.

Proof. Put $\mathcal{W} := \{A \in \Sigma : A \cap W = \emptyset\}$. For every $A \in \mathcal{W}$ and $n, q \in \mathbb{N}$, we have

$$m_n(A) - m_n(A \setminus H_q) = m_n(A \cap H_q). \tag{3}$$

As $A \cap H_q \subset H_{q-1} \setminus W$ for any $q \in \mathbb{N}$, from (2) and (3), for each $n \in \mathbb{N}$ we get

$$m_n(A) = (O) \lim_q m_n(A \setminus H_q) \quad \text{for all } A \in \mathcal{W}. \tag{4}$$

If the thesis of the lemma is not true, then there exists $p \in \mathbb{N}$ such that for every $r \in \mathbb{N}$ there are $n, k \in \mathbb{N}$ with $k > r$ and $A \in \Sigma$ with $A \subset H_k \setminus W$, $|m_n(A)| \not\leq \sigma_p$, and thus, thanks to (4),

$$|m_n(A \setminus H_q)| \not\leq \sigma_p$$

for q large enough.

At the first step, we find a set $A_1 \in \Sigma$ and three integers $k_1 > 1$, $n_1 \in \mathbb{N}$ and $q_1 > \max\{k_1, n_1\}$, with $A_1 \subset H_{k_1} \setminus W$, $|m_{n_1}(A_1)| \not\leq \sigma_p$ and $|m_{n_1}(A_1 \setminus H_{q_1})| \not\leq \sigma_p$. From (2), in correspondence with $n = 1, 2, \ldots, n_1$ there exists $h_1 > q_1$ with

$$|m_n(A)| \leq \sigma_p \tag{5}$$

whenever $k \geq h_1$ and $A \subset H_k \setminus W$.

At the second step, there are $A_2 \in \Sigma$, $k_2 > h_1$, $n_2 \in \mathbb{N}$ and $q_2 > \max\{k_2, n_2\}$, with $A_2 \subset H_{k_2} \setminus W$ and

$$|m_{n_2}(A_2)| \not\leq \sigma_p; \quad |m_{n_2}(A_2 \setminus H_{q_2})| \not\leq \sigma_p. \tag{6}$$

From (5) and (6) it follows that $k_2 > k_1$.

By induction, we find a sequence $(A_k)_k$ in Σ and three strictly increasing sequences in $\mathbb{N}$, $(k_r)_r$, $(n_r)_r$, $(q_r)_r$, with $q_r > k_r > q_{r-1}$ for all $r \geq 2$; $q_r > n_r$, $A_r \subset H_{k_r} \setminus W$, $|m_{n_r}(A_r \setminus H_{q_r})| \not\leq \sigma_p$ for all $r \in \mathbb{N}$. But this is impossible, because the sets $A_r \setminus H_{q_r}$, $r \in \mathbb{N}$, are pairwise disjoint, and the measures m_n, $n \in \mathbb{N}$, are globally uniformly (s)-bounded on Σ with respect to $(\sigma_p)_p$. $\qquad\square$

The proof of the following result is similar to that of Lemma 2.6 (see also Boccuto & Candeloro, 2010, Lemma 3.2 and Corollary 3.3; Boccuto & Dimitriou, 2011b, Lemma 3.3).

Lemma 2.7 *Let Ω be as in Theorem 2.1, and assume that there exist a meager set $N_1 \subset \Omega$ such that the real-valued measures $m_n(\cdot)(\omega)$, $n \in \mathbb{N}$, are uniformly (s)-bounded on Σ for all $\omega \in \Omega \setminus N_1$. Let $\mathcal{G}$ be as in Lemma 2.6, and $(H_k)_k$ be a fixed decreasing sequence in $\mathcal{G}$, with $W \subset H_k$ for each $k \in \mathbb{N}$. If*

$$\lim_k \left(\sup_{A \in \Sigma, A \subset H_k} |m_n(A)(\omega)| \right) = 0$$

for every $n \in \mathbb{N}$ and ω belonging to the complement of a meager set $N_2 \subset \Omega$, then

$$\lim_k \left(\sup_n \left(\sup_{A \in \Sigma, A \subset H_k} |m_n(A)(\omega)| \right) \right) = 0$$

for all $\omega \in \Omega \setminus (N_1 \cup N_2)$.

3. The Main Results

We begin with recalling the concept of uniform ideal exhaustiveness for measures (see also Athanassiadou, Dimitriou, Papachristodoulos, & Papanastassiou, 2012; Boccuto, Das, Dimitriou, & Papanastassiou, 2012; Boccuto & Dimitriou, 2011a, 2011b; Boccuto, Dimitriou, Papanastassiou, & Wilczyński, 2011, 2012), which plays a very important role in the versions of limit theorems with respect to ideal order convergence, and we deal with some properties of the Stone extension of a finitely additive measure, in connection with uniform ideal exhaustiveness and ideal pointwise convergence with respect to a common (O)-sequence.

In what follows, we suppose that R is a Dedekind complete (ℓ)-group, $\mathcal{I}$ is a P-ideal of $\mathbb{N}$ and $\lambda: \Sigma \to [0, +\infty]$ is a finitely additive measure, such that Σ is separable with respect to the Fréchet-Nikodým topology generated by λ (shortly, *λ-separable*). Let $\mathcal{B} := \{F_j : j \in \mathbb{N}\}$ be a countable λ-dense subset of Σ.

A sequence of finitely additive measures $m_n : \Sigma \to R$, $n \in \mathbb{N}$, is *λ-uniformly $\mathcal{I}$-exhaustive on Σ* iff there is an (O)-sequence $(\sigma_p)_p$ in R such that for every $p \in \mathbb{N}$ there are a positive real number δ and a set $D \in \mathcal{I}$, with $|m_n(E) - m_n(F)| \le \sigma_p$ whenever $E, F \in \Sigma$ with

$$|\lambda(E) - \lambda(F)| \le \delta \tag{7}$$

and for each $n \in \mathbb{N} \setminus D$.

A sequence $m_n \colon \Sigma \to R$, $n \ge 0$, of finitely additive measures, together with λ, *satisfies property* $(*)$ *with respect to R and $\mathcal{I}$* iff it is λ-uniformly $\mathcal{I}$-exhaustive on Σ and $(O\mathcal{I}) \lim_n m_n(E) = m_0(E)$ for every $E \in \Sigma$ with respect to a common (O)-sequence.

The following lemma will be useful to prove our results about equivalence of limit theorems.

Lemma 3.1 *Let $m_n \colon \Sigma \to R$, $n \in \mathbb{N}$, be a λ-uniformly $\mathcal{I}$-exhaustive sequence of finitely additive measures, and assume that*

$$(O\mathcal{I}) \lim_n m_n(F_j) =: m(F_j) \quad \text{for any } j \in \mathbb{N}$$

with respect to a common (O)-sequence. Then,

α) there exist a set $M_0 \in \mathcal{F} = \mathcal{F}(\mathcal{I})$ and a finitely additive measure $m_0 \colon \Sigma \to R$, which extends m and such that $(O) \lim_{n \in M_0} m_n(E) = m_0(E)$ for each $E \in \Sigma$ and with respect to a same (O)-sequence $(b_p)_p$;

$\alpha\alpha$) the measures m_n, $n \in M_0$, and m_0 satisfy together with λ property $(*)$ with respect to R and $\mathcal{I}_{fin}$;

$\alpha\alpha\alpha$) if Ω is as in Theorem 2.1, then there is a meager set $N_0 \subset \Omega$ such that for each $\omega \in \Omega \setminus N_0$ the real-valued measures $m_n(\cdot)(\omega)$, $n \in M_0$, and $m_0(\cdot)(\omega)$, satisfy together with λ property $(*)$ with respect to $\mathbb{R}$ and $\mathcal{I}_{fin}$.

Proof. α) It is a consequence of Boccuto and Dimitriou (2011b) Lemma 3.9 and Proposition 2.5.

$\alpha\alpha$) By λ-uniform $\mathcal{I}$-exhaustiveness of the m_n's there is an (O)-sequence $(\sigma_p)_p$ such that for every $p \in \mathbb{N}$ there are a positive real number δ and a set $A_p \in \mathcal{I}$ with $|m_n(E) - m_n(F)| \le \sigma_p$ for any $E, F \in \Sigma$ with $|\lambda(E) - \lambda(F)| \le \delta$ and $n \notin A_p$. For each $p \in \mathbb{N}$, set $M'_p := \mathbb{N} \setminus A_p$. Since $\mathcal{I}$ is a P-ideal, there exists a sequence $(M_p)_p$ of subsets of $\mathbb{N}$ such that $M_p \Delta M'_p$ is finite for each $p \in \mathbb{N}$ and $M := \bigcap_{p=1}^{\infty} M_p \in \mathcal{F}$. For every $p \in \mathbb{N}$, set $Z_p := M \setminus M'_p$. Note that Z_p is finite for every $p \in \mathbb{N}$, and so $|m_n(E) - m_n(F)| \le \sigma_p$ whenever $E, F \in \Sigma$ with $|\lambda(E) - \lambda(F)| \le \delta$ and $n \in M \setminus Z_p$. This proves $\alpha\alpha$).

$\alpha\alpha\alpha$) Let m_0, M_0, $(b_p)_p$ be as in α) and $(\sigma_p)_p$ be as in $\alpha\alpha$). By α) and Theorem there is a meager set $N_0 \subset \Omega$ such that the sequences $(b_p(\omega))_p$ and $(\sigma_p(\omega))_p$ are (O)-sequences in $\mathbb{R}$ for each $\omega \in \Omega \setminus N_0$, and with the property that for every $p \in \mathbb{N}$ and $E \in \Sigma$ there is an integer $\bar{n} \in M_0$ with

$$|m_n(E)(\omega) - m_0(E)(\omega)| \le b_p(\omega) \quad \text{for all } n \ge \bar{n}, n \in M_0, \text{ and } \omega \in \Omega \setminus N_0. \tag{8}$$

Furthermore, from $\alpha\alpha$) and λ-uniform $\mathcal{I}_{\mathrm{fin}}$-exhaustiveness of the m_n's, $n \in M_0$, it follows that for each $p \in \mathbb{N}$ there are $\delta > 0$ and $Z_p \in \mathcal{I}_{\mathrm{fin}}$ with

$$|m_n(E)(\omega) - m_n(F)(\omega)| \le \sigma_p(\omega) \tag{9}$$

whenever $E, F \in \Sigma$, $|\lambda(E) - \lambda(F)| \le \delta$, $n \in M \setminus Z_p$ and $\omega \in \Omega \setminus N_0$. From (8) and (9) it follows that the measures $m_n(\omega)$, $n \in M_0$, are λ-uniformly $\mathcal{I}_{\mathrm{fin}}$-exhaustive and $\lim_{n \in M_0} m_n(E)(\omega) = m(E)(\omega)$ for every $\omega \in \Omega \setminus N_0$ and $E \in \Sigma$. This proves $\alpha\alpha\alpha$). $\qquad\square$

In order to prove the equivalence between our ideal limit theorems, we will relate (globally) (s)-bounded and (globally) σ-additive measures. Indeed, in general, many problems involving finitely additive measures can be solved by finding suitable σ-additive measures, related to them, and then studying their properties. One of the main tools in this setting is the *Stone extension*, by means of which it is possible to construct a globally σ-additive measure, defined on a larger σ-algebra than the original one (see also Boccuto & Candeloro, 2002; Boccuto & Candeloro, 2004; Sikorski, 1964).

Let R be a super Dedekind complete and weakly σ-distributive (ℓ)-group, $\lambda \colon \Sigma \to [0, +\infty]$, $m_n \colon \Sigma \to R$, $n \in \mathbb{N}$, be finitely additive measures, Q be the Stone space associated with Σ, that is a totally disconnected Hausdorff

compact space, such that the algebra Q of its clopen subsets is algebraically isomorphic to Σ. If we denote by $\psi : \Sigma \to Q$ such isomorphism, then it is possible to "transfer" the measures λ and m_n, $n \geq 0$, to Q, by putting $(\lambda \circ \psi^{-1})(E) = \lambda(\psi^{-1}(E))$, $(m_n \circ \psi^{-1})(E) = m_n(\psi^{-1}(E))$, $E \in Q$. Observe that, by the particular structure of Q, every monotone sequence $(H_k)_k$ of sets of Q is eventually constant. This implies that the measures $\lambda \circ \psi^{-1}$ and $m_n \circ \psi^{-1}$ are globally σ-additive. By Boccuto and Candeloro (2004, Theorem 4.4), these measures admit (unique) globally σ-additive extensions $\widetilde{\lambda}$, $\widetilde{m_n}$ respectively, to the σ-algebra $\Sigma(Q)$ generated by Q. These extensions are called the *Stone extensions* of λ and m_n respectively. We now prove that the Stone extensions "inherit" property $(*)$.

Theorem 3.2 *Let $\lambda: \Sigma \to \mathbb{R}$, $m_n: \Sigma \to R$, $n \geq 0$, be finitely additive measures, which together with λ satisfy property $(*)$ with respect to $\mathcal{I}$ and R.*

Then the σ-algebra $\Sigma(Q)$ is $\widetilde{\lambda}$-separable, and the measures $\widetilde{\lambda}: \Sigma(Q) \to \mathbb{R}$, $\widetilde{m_n}: \Sigma(Q) \to R$ satisfy together with $\widetilde{\lambda}$ property $(*)$ with respect to $\mathcal{I}$ and R.

Proof. We first claim that $\Sigma(Q)$ is $\widetilde{\lambda}$-separable. Fix arbitrarily $\varepsilon > 0$. By Boccuto and Candeloro (2002, Theorem 4.4), for each $A \in \Sigma(Q)$ there is a set $E \in Q$ with $|\widetilde{\lambda}(A) - \widetilde{\lambda}(E)| \leq \frac{\varepsilon}{2}$. Since Σ is λ-separable and $\{F_j : j \in \mathbb{N}\}$ is a countable λ-dense subset of Σ, the set $\mathcal{E} := \{\psi(F_j) : j \in \mathbb{N}\}$ is a countable $\widetilde{\lambda}$-dense subset of Q. Hence there is $j \in \mathbb{N}$ with $|\widetilde{\lambda}(E) - \widetilde{\lambda}(\psi(F_j))| \leq \frac{\varepsilon}{2}$, from which it follows that $|\widetilde{\lambda}(A) - \widetilde{\lambda}(\psi(F_j))| \leq \varepsilon$. This proves the claim, and we get also that $\mathcal{E}$ is a countable $\widetilde{\lambda}$-dense subset of $\Sigma(Q)$.

We now prove that the $\widetilde{m_n}$'s are $\widetilde{\lambda}$-uniformly $\mathcal{I}$-exhaustive. According to λ-uniform $\mathcal{I}$-exhaustiveness of the m_n's on Σ, let $(\sigma_p)_p$ be an (O)-sequence related with it, choose arbitrarily $p \in \mathbb{N}$ and pick $\delta > 0$, $D \in \mathcal{I}$ in correspondence with p.

By proceeding analogously as in Boccuto and Candeloro (2004, Theorems 4.4 and 4.5), it is possible to see that for every $n \in \mathbb{N}$ there is a (D)-sequence $(a_{t,l}^{(n)})_{t,l}$ in R, such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $A_1, A_2 \in \Sigma(Q)$ there are $E_1^{(n)}$, $E_2^{(n)} \in Q$ with

$$|\widetilde{\lambda}(E_s^{(n)}) - \widetilde{\lambda}(A_s)| \leq \frac{\delta}{3}, \quad |\widetilde{m_n}(E_s^{(n)}) - \widetilde{m_n}(A_s)| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)}, \quad s = 1, 2.$$

Since, by construction, the $\widetilde{m_n}$'s are equibounded (see also Boccuto & Candeloro, 2004), then by Lemma 2.2, in correspondence with $u := \bigvee_{A \in \Sigma(Q), n \in \mathbb{N}} |\widetilde{m_n}(A)|$ and the (D)-sequences $(a_{t,l}^{(n)})_{t,l}$, $n \in \mathbb{N}$, there is a (D)-sequence $(a_{t,l})_{t,l}$ in R with

$$(2\,u) \wedge \Big(\sum_{n=1}^{q} \big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)}\big)\Big) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

for every $q \in \mathbb{N}$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$, and hence

$$|\widetilde{m_n}(E_s^{(n)}) - \widetilde{m_n}(A_s)| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}, \quad s = 1, 2.$$

Since R is super Dedekind complete and weakly σ-distributive, by Theorem 2.3 we find an (O)-sequence $(v_p)_p$, such that for every $p \in \mathbb{N}$ there exists $\varphi_p \in \mathbb{N}^{\mathbb{N}}$, such that

$$\bigvee_{t=1}^{\infty} a_{t,\varphi_p(t)} \leq v_p.$$

Thus we obtain that for each $p, n \in \mathbb{N}$ and $A_1, A_2 \in \Sigma(Q)$ there exist $E_1^{(n)}, E_2^{(n)} \in Q$ with

$$|\widetilde{m_n}(E_s^{(n)}) - \widetilde{m_n}(A_s)| \leq v_p, \quad s = 1, 2. \tag{10}$$

Moreover we get:

$$|\widetilde{\lambda}(E_1^{(n)}) - \widetilde{\lambda}(E_2^{(n)})| \leq |\widetilde{\lambda}(E_1^{(n)}) - \widetilde{\lambda}(A_1)| + |\widetilde{\lambda}(A_1) - \widetilde{\lambda}(A_2)| + |\widetilde{\lambda}(A_2) - \widetilde{\lambda}(E_2^{(n)})| \leq \frac{3\delta}{3} = \delta,$$

namely $|\lambda(\psi^{-1}(E_1^{(n)})) - \lambda(\psi^{-1}(E_2^{(n)}))| \leq \delta$. Thus condition (7) is satisfied, and then we have

$$
\begin{aligned}
|\widetilde{m_n}(A_1) - \widetilde{m_n}(A_2)| &\leq \\
\leq\ & |\widetilde{m_n}(E_1^{(n)}) - \widetilde{m_n}(A_1)| + |\widetilde{m_n}(E_1^{(n)}) - \widetilde{m_n}(E_2^{(n)})| + |\widetilde{m_n}(A_2) - \widetilde{m_n}(E_2^{(n)})| = \\
=\ & |\widetilde{m_n}(E_1^{(n)}) - \widetilde{m_n}(A_1)| + |m_n(\psi^{-1}(E_1^{(n)})) - m_n(\psi^{-1}(E_2^{(n)}))| + \\
+\ & |\widetilde{m_n}(A_2) - \widetilde{m_n}(E_2^{(n)})| \leq \sigma_p + 2\,v_p.
\end{aligned}
$$

Let $w_p := \sigma_p + 2\,v_p$, $p \in \mathbb{N}$. Note that $(w_p)_p$ is an (O)-sequence, and we have obtained that for each $p \in \mathbb{N}$ there exist $\delta > 0$ and $D \in \mathcal{I}$ such that for every pair A_1, A_2 of elements of $\Sigma(Q)$ with $|\widetilde{\lambda}(A_1) - \widetilde{\lambda}(A_2)| \leq \dfrac{\delta}{3}$ and for each $n \in \mathbb{N} \setminus D$ we get $|\widetilde{m_n}(A_1) - \widetilde{m_n}(A_2)| \leq w_p$.

Thus we have proved that, if the measures $m_n : \Sigma \to R$ are λ-uniformly $\mathcal{I}$-exhaustive on Σ, then the measures $\widetilde{m_n}$: $\Sigma(Q) \to R$ are $\widetilde{\lambda}$-uniformly $\mathcal{I}$-exhaustive on $\Sigma(Q)$.

The last step is to prove that

$$(O\mathcal{I}) \lim_n \widetilde{m_n}(A) = \widetilde{m_0}(A) \quad \text{for all } A \in \Sigma(Q) \tag{11}$$

with respect to a common (O)-sequence. Since the $\psi(F_j)$'s, $j \in \mathbb{N}$, form a countable $\widetilde{\lambda}$-dense subset of $\Sigma(Q)$ and satisfy (11), then by Lemma 3.1, $\alpha)$ applied to the sequence $\widetilde{m_n} : \Sigma(Q) \to R$, $n \in \mathbb{N}$, there are an (O)-sequence $(\xi_p)_p$ in R and a set $M_0^* \in \mathcal{F}(\mathcal{I})$, with $(O) \lim_{n \in M_0^*} \widetilde{m_n}(A) = \widetilde{m_0}(A)$ for each $A \in \Sigma(Q)$ with respect to $(\xi_p)_p$. From this and Proposition 2.4 we obtain (11). This concludes the proof of the theorem. $\qquad\square$

We now recall the following Brooks-Jewett-type theorem in the (ℓ)-group context with (O)-convergence (with respect to a same (O)-sequence or not).

Theorem 3.3 (see Boccuto & Candeloro, 2010, Theorem 3.1; Boccuto & Dimitriou, 2011b, Theorem 3.4; Boccuto & Candeloro, 2004, Theorem 6.8) *Let R be a Dedekind complete (ℓ)-group, and assume that $m_n: \Sigma \to R$, $n \in \mathbb{N}$, is a sequence of equibounded finitely additive measures, (O)-convergent pointwise on Σ to a measure $m_0: \Sigma \to R$ with respect to a common (O)-sequence.*

Then the measures $m_n(\cdot)(\omega)$, $n \in \mathbb{N}$, are uniformly (s)-bounded on Σ for ω belonging to the complement of a meager set $N \subset \Omega$. Moreover $m_0(\cdot)(\omega)$, $\omega \in \Omega \setminus N$, is (s)-bounded, the m_n's are uniformly (s)-bounded and m_0 is (s)-bounded on Σ.

Furthermore, if the m_n's are globally (s)-bounded and R is super Dedekind complete and weakly σ-distributive, then they are also globally uniformly (s)-bounded.

We now are ready to prove the following limit theorems for (ℓ)-group-valued measures with respect to ideal convergence and their equivalence (see also Boccuto & Dimitriou, 2011b, Theorem 3.10).

Theorem 3.4 (Brooks-Jewett (BJ)) *Let R be a Dedekind complete (ℓ)-group, $\lambda: \Sigma \to [0, +\infty]$ be a finitely additive measure, such that Σ is λ-separable, $\mathcal{I}$ be a P-ideal of $\mathbb{N}$, $m_0: \Sigma \to R$, $m_n : \Sigma \to R$, $n \in \mathbb{N}$, be equibounded finitely additive measures, which together with λ satisfy property $(*)$ with respect to R and $\mathcal{I}$. Then,*

I) *there exists a set $M_0 \in \mathcal{F}(\mathcal{I})$, such that the measures m_n, $n \in M_0$, are uniformly (s)-bounded on Σ.*

II) *If the m_n's are globally (s)-bounded and R is super Dedekind complete and weakly σ-distributive, then M_0 can be chosen in such a way that the measures m_n, $n \in M_0$, are globally uniformly (s)-bounded on Σ.*

Theorem 3.5 (Vitali-Hahn-Saks (VHS)) *Let R, Σ, λ, $\mathcal{I}$, m_n be as in Theorem 3.4, and τ be a Fréchet-Nikodým topology on Σ.*

I) *If each m_n is τ-continuous, then there exists $M_0 \in \mathcal{F}(\mathcal{I})$, such that the measures m_n, $n \in M_0$, are uniformly τ-continuous on Σ.*

II) *If the m_n's are globally (s)-bounded and globally τ-continuous, and R is super Dedekind complete and weakly σ-distributive, then M_0 can be chosen to have global uniform τ-continuity of the m_n's, $n \in M_0$.*

Theorem 3.6 (Nikodým (N))

I) *If R, Σ, λ, $\mathcal{I}$ are as above and the m_n's, $n \in \mathbb{N}$, are σ-additive, then there is $M_0 \in \mathcal{F}(\mathcal{I})$, such that the measures m_n, $n \in M_0$, are uniformly σ-additive on Σ.*

II) *If each m_n is globally σ-additive and R is super Dedekind complete and weakly σ-distributive, then M_0 can be chosen in order that the measures m_n, $n \in M_0$, are globally uniformly σ-additive.*

Theorem 3.7 (Dieudonné (D)) *Let R, Σ, λ, $\mathcal{I}$, m_n be as in Theorem 3.4.*

I) *If each m_n is regular, then a set $M_0 \in \mathcal{F}(\mathcal{I})$ can be found, with the property that the measures m_n, $n \in M_0$, are uniformly (s)-bounded and uniformly regular on Σ.*

II) *If each m_n is globally (s)-bounded and globally regular, and R is super Dedekind complete and weakly σ-distributive, then M_0 can be chosen in such a way that the measures m_n, $n \in M_0$, are globally uniformly (s)-bounded and globally uniformly regular.*

To prove Theorem 3.4 (BJ), observe that there exist $M_0 \in \mathcal{F}(\mathcal{I})$ and $N_0 \subset \Omega$, satisfying the thesis of Lemma 3.1. The assertion of (BJ) follows by applying Theorem 3.3 to the sequence m_n, $n \in M_0$, and to N_0.

We now prove equivalence between (BJ) II), (VHS) II), (N) II) and (D) II).

We begin with the implication (BJ) II) $\Longrightarrow$ (VHS) II). Let $m_n \colon \Sigma \to R$, $n \in \mathbb{N}$, be a sequence of globally (s)-bounded and globally τ-continuous finitely additive equibounded measures, satisfying together with λ property $(\ast)$ with respect to R and $\mathcal{I}$. By Lemma 3.1, $\alpha\alpha)$, there is $M_0 \in \mathcal{F}(\mathcal{I})$ such that the measures m_n, $n \in M_0$, and m_0 satisfy property $(\ast)$ with respect to R and $\mathcal{I}_{\text{fin}}$. By (BJ) II) used with $\mathcal{I} = \mathcal{I}_{\text{fin}}$, there is a set $M_0' \subset M_0$, such that $M_0 \setminus M_0'$ is finite and with the property that the measures m_n, $n \in M_0'$, are globally uniformly (s)-bounded, that is there is an (O)-sequence $(\sigma_p)_p$ with $(O) \lim_k [\sup_{n \in M_0'} v(m_n)(C_k)] = 0$ for any disjoint sequence $(C_k)_k$ in Σ and with respect to the same (O)-sequence $(\sigma_p)_p$. Note that $M_0' \in \mathcal{F}(\mathcal{I})$.

Fix arbitrarily any decreasing sequence $(H_k)_k$ in Σ, with τ-$\lim_k H_k = \emptyset$. By global τ-continuity of m_n, $n \in \mathbb{N}$, we get $(O) \lim_k v(m_n)(H_k) = 0$ for all $n \in \mathbb{N}$ and with respect to a same (O)-sequence $(\zeta_p)_p$. By Lemma 2.6, we obtain

$$(O) \lim_k \left(\bigvee_{n \in M_0'} v(m_n)(H_k) \right) = 0 \tag{12}$$

with respect to $(\zeta_p)_p$, and so we get global uniform τ-continuity of the m_n's, $n \in M_0'$. Thus, (BJ) II) implies (VHS) II).

The proof of (BJ) II) $\Longrightarrow$ (D) II) is analogous to that of (BJ) II) $\Longrightarrow$ (VHS) II).

We now prove (VHS) II) $\Longrightarrow$ (N) II). Let τ be the Fréchet-Nikodým topology generated by the family of *all* order continuous submeasures defined on Σ. If $(H_k)_k$ is any decreasing sequence in Σ with τ-$\lim_k H_k = \emptyset$ and $H = \bigcap_{k=1}^{\infty} H_k$, then we get $\eta(H) = 0$ for *every* order continuous submeasure η on Σ, and hence $H = \emptyset$. From this it follows that, if $m_n \colon \Sigma \to R$, $n \in \mathbb{N}$, is a sequence of globally σ-additive measures, then they are globally τ-continuous. Since the m_n's are also globally (s)-bounded, then by (VHS) II) they are globally uniformly τ-continuous, and hence also globally uniformly σ-additive. Thus, (VHS) II) implies (N) II).

We now prove (N) II) $\Longrightarrow$ (BJ) II). Let $m_n \colon \Sigma \to R$, $n \in \mathbb{N}$, be a sequence of equibounded finitely additive globally (s)-bounded measures, satisfying together with λ property $(\ast)$ with respect to $\mathcal{I}$ and R. By Lemma 3.1, $\alpha\alpha)$, a set $M_0 \in \mathcal{F}(\mathcal{I})$ can be found, with the property that the measures m_n, $n \in M_0$, and m_0 satisfy property $(\ast)$ with respect to $\mathcal{I}_{\text{fin}}$ and R. If $\widetilde{m_n} \colon \Sigma(Q) \to R$, $n \in \mathbb{N}$, and $\widetilde{\lambda} \colon \Sigma(Q) \to R$, are the Stone extensions of m_n and λ respectively, then, by Theorem 3.2, the σ-algebra $\Sigma(Q)$ is $\widetilde{\lambda}$-separable, and the $\widetilde{m_n}$'s, $n \in M_0$, are σ-additive measures, satisfying together with $\widetilde{\lambda}$ property $(\ast)$ with respect to $\mathcal{I}_{\text{fin}}$ and R. By (N) II) used with $\Sigma(Q)$ and $\mathcal{I}_{\text{fin}}$, we find a finite set $M_0'' \subset M_0$, such that the measures $\widetilde{m_n}$, $n \in M^* := M_0 \setminus M_0''$, are globally uniformly σ-additive, and hence also globally uniformly (s)-bounded, on $\Sigma(Q)$. "Coming back" to Σ, we get global uniform (s)-boundedness of the measures m_n, $n \in M^*$. Since $M^* \in \mathcal{F}(\mathcal{I})$, then it follows that (N) II) implies (BJ) II).

We now prove (D) II) $\Longrightarrow$ (BJ) II). Let $\mathcal{G}^*$, $\mathcal{H}^*$ be the lattices of all open and all closed subsets of the Stone space Q which belong to $\Sigma(Q)$ respectively. It is not difficult to see that $\mathcal{G}^*$ and $\mathcal{H}^*$ satisfy condition (R0).

Let $m_n \colon \Sigma \to R$, $n \geq 0$, be equibounded globally (s)-bounded finitely additive measures satisfying, together with λ, property $(\ast)$ with respect to $\mathcal{I}$ and R. Arguing as in the previous implication, let us consider the global σ-additive Stone extensions $\widetilde{m_n} \colon \Sigma(Q) \to R$ and $\widetilde{\lambda} \colon \Sigma(Q) \to [0, +\infty]$ of m_n and λ respectively.

We now prove that the $\widetilde{m_n}$'s are globally regular. Fix arbitrarily $n \in \mathbb{N}$. Obviously, condition (R1) is fulfilled for each set $E \in Q$. We now claim that the class of all sets satisfying (R1) is a σ-algebra. From this it will follow that every set $E \in \Sigma(Q)$ fulfils (R1), and hence that $\widetilde{m_n}$ is globally regular. Without loss of generality, assume that $\widetilde{m_n}$ is positive (indeed, in the general case, it will be enough to consider $\widetilde{m_n}^+$ and $\widetilde{m_n}^-$). It is readily seen that, if $E \in \Sigma(Q)$ fulfils (R1), then $Q \setminus E$ does. Let E_k, $k \in \mathbb{N}$, be a disjoint sequence in $\Sigma(Q)$, satisfying (R1) and $(\sigma_p)_p$ be a related (O)-sequence. For each $t, l \in \mathbb{N}$, put $a_{t,l} = \sigma_l$. Note that $(a_{t,l})_{t,l}$ is a (D)-sequence, such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $p \in \mathbb{N}$, $p = \varphi(1)$, with $\sigma_p \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$, so that the (O)-limit in the condition (R1) of global regularity with respect to $(\sigma_p)_p$ is a (D)-limit with respect to the (D)-sequence $(a_{t,l})_{t,l}$. For every $k \in \mathbb{N}$ there are two sequences $(G_r^{(k)})_r$ and $(F_r^{(k)})_r$ in $\mathcal{G}^*$ and $\mathcal{H}^*$ respectively, with $F_r^{(k)} \subset F_{r+1}^{(k)} \subset E_k \subset G_{r+1}^{(k)} \subset G_r^{(k)}$ for all $r \in \mathbb{N}$, and such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $r_k \in \mathbb{N}$ with

$$\widetilde{m_n}(G_r^{(k)} \setminus F_r^{(k)}) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t+k)} \quad \text{whenever } r \geq r_k.$$

By Lemma 2.2 there is a (D)-sequence $(b_{t,l})_{t,l}$, with

$$\widetilde{m_n}(Q) \wedge \Big(\sum_{k=1}^{q}\Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+k)}\Big)\Big) \leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \quad \text{for all } q \in \mathbb{N} \text{ and } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

For each $k \in \mathbb{N}$, let $G_k := G_{r_k}^{(k)}$, $F_k := F_{r_k}^{(k)}$, and put $E := \bigcup_{k=1}^{\infty} E_k$, $F := \bigcup_{k=1}^{\infty} F_k$, $G' := \bigcup_{k=1}^{\infty} G_k$: observe that $G' \in \mathcal{G}^*$. By global σ-additivity of $\widetilde{m_n}$ there is a (D)-sequence $(c_{t,l})_{t,l}$, such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a natural number k_0 with

$$\widetilde{m_n}\Big(F \setminus \bigcup_{k=1}^{h} F_k\Big) \leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \quad \text{whenever } h \geq k_0. \tag{13}$$

Put $F' := \bigcup_{k=1}^{k_0} F_k$: note that $F' \in \mathcal{H}^*$. For every $t, l \in \mathbb{N}$, set $d_{t,l} = 2(b_{t,l} + c_{t,l})$. Observe that $(d_{t,l})_{t,l}$ is a (D)-sequence. We get:

$$\widetilde{m_n}(G' \setminus F') \leq \widetilde{m_n}(G' \setminus F) + \widetilde{m_n}(F \setminus F')$$

$$\leq \bigvee_{q\in\mathbb{N}} \Big(\widetilde{m_n}(Q) \wedge \Big(\sum_{k=1}^{q} \widetilde{m_n}(G_k \setminus F)\Big)\Big) + \widetilde{m_n}(F \setminus F') \leq$$

$$\leq \bigvee_{q\in\mathbb{N}} \Big(\widetilde{m_n}(Q) \wedge \Big(\sum_{k=1}^{q} \widetilde{m_n}(G_k \setminus F_k)\Big)\Big) + \widetilde{m_n}(F \setminus F') \leq$$

$$\leq \bigvee_{q\in\mathbb{N}} \Big(\widetilde{m_n}(Q) \wedge \Big(\sum_{k=1}^{q}\Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+k)}\Big)\Big)\Big) + \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \leq$$

$$\leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)}.$$

By Theorem 2.3 we find an (O)-sequence $(\pi_p)_p$ in R with the property that for every $p \in \mathbb{N}$ there is $\varphi_p \in \mathbb{N}^{\mathbb{N}}$ with $\bigvee_{t=1}^{\infty} d_{t,\varphi_p(t)} \leq \pi_p$. Thus for each $p \in \mathbb{N}$ there are $G_p' \in \mathcal{G}^*$, $F_p' \in \mathcal{H}^*$, with $F_p' \subset E \subset G_p'$ and $\widetilde{m_n}(G_p' \setminus F_p') \leq \pi_p$. For each $p \in \mathbb{N}$, set $G_p := \bigcap_{i=1}^{p} G_i'$ and $F_p := \bigcup_{i=1}^{p} F_i'$. We get $F_p \subset F_{p+1} \subset E \subset G_{p+1} \subset G_p$, $G_p \in \mathcal{G}^*$, $F_p \in \mathcal{H}^*$, and $\widetilde{m_n}(G_p \setminus F_p) \leq \widetilde{m_n}(G_p' \setminus F_p') \leq \pi_p$ for all $p \in \mathbb{N}$. Thus the set E satisfies condition (R1). This proves the claim.

Thus the finitely additive measures $\widetilde{m_n}$, $n \in \mathbb{N}$, are globally (s)-bounded and globally regular on $\Sigma(Q)$. Arguing analogously as in the previous implication, by (D) II) used with $\Sigma(Q)$ and $\mathcal{I}_{\text{fin}}$, there is a finite set $M_{0,*} \subset M_0$, such

that the measures $\widetilde{m_n}$, $n \in M_* := M_0 \setminus M_{0,*}$, are globally uniformly (s)-bounded and globally uniformly regular on $\Sigma(Q)$. "Coming back" to Σ, we get global uniform (s)-boundedness of the measures m_n, $n \in M_*$, on Σ. Since $M_* \in \mathcal{F}(\mathcal{I})$, then it follows that (N) II) implies (BJ) II).

We now prove equivalence between (BJ) I), (VHS) I), (N) I) and (D) I).

We start with the implication (BJ) I) $\Longrightarrow$ (VHS) I). Let R be a Dedekind complete (ℓ)-group, Ω be as in Theorem 2.1 and $m_n : \Sigma \to R$, $n \geq 0$, be a sequence of τ-continuous finitely additive equibounded measures, satisfying together with λ property ($*$) with respect to R and $\mathcal{I}$. By Lemma 3.1, $\alpha\alpha\alpha$), there are a meager set $N_0 \subset \Omega$ and a set $M_0 \in \mathcal{F}(\mathcal{I})$ with the property that for every $\omega \in \Omega \setminus N_0$ the real-valued measures $m_n(\cdot)(\omega)$, $n \in M_0$, and $m_0(\cdot)(\omega)$ satisfy together with λ property ($*$) with respect to $\mathbb{R}$ and $\mathcal{I}_{\text{fin}}$. By (BJ) I) applied with $R = \mathbb{R}$ and $\mathcal{I} = \mathcal{I}_{\text{fin}}$, for each $\omega \in \Omega \setminus N_0$ there is a set $M_0^{(\omega)} \subset M_0$, such that $M_0 \setminus M_0^{(\omega)}$ is finite and the real-valued measures $m_n(\omega)$, $n \in M_0^{(\omega)}$, are uniformly (s)-bounded on Σ, that is

$$\lim_k (\sup_{n \in M_0^{(\omega)}} [\sup_{A \in \Sigma, A \subset C_k} |m_n(A)(\omega)|]) = 0 \tag{14}$$

for any disjoint sequence $(C_k)_k$ in Σ.

Fix arbitrarily $\varepsilon > 0$ and $\omega \in \Omega \setminus N_0$, and choose a disjoint sequence $(C_k)_k$ in Σ. Then, by (14), we find $k_0 \in \mathbb{N}$ with

$$\sup_{A \in \Sigma, A \subset C_k} |m_n(A)(\omega)| \leq \varepsilon \tag{15}$$

whenever $k \geq k_0$ and $n \in M_0^{(\omega)}$. Since the measures $m_n(\omega)$, $\omega \in \Omega \setminus N_0$, $n \in M_0 \setminus M_0^{(\omega)}$ are (s)-bounded and $M_0 \setminus M_0^{(\omega)}$ is finite, then for each $n \in M_0 \setminus M_0^{(\omega)}$ there is a natural number k_n with

$$\sup_{A \in \Sigma, A \subset C_k} |m_n(A)(\omega)| \leq \varepsilon \tag{16}$$

for all $k \geq k_n$. If $k^* := \max\{k_0, k_n : n \in M_0 \setminus M_0^{(\omega)}\}$, from (15) and (16) it follows that

$$\sup_{A \in \Sigma, A \subset C_k} |m_n(A)(\omega)| \leq \varepsilon \quad \text{for each } k \geq k^* \text{ and } n \in M_0.$$

Thus the measures $m_n(\cdot)(\omega)$, $\omega \in \Omega \setminus N_0$, $n \in M_0$, are uniformly (s)-bounded.

Fix arbitrarily any decreasing sequence $(H_k)_k$ with τ-$\lim_k H_k = \emptyset$. By τ-continuity of m_n, $n \in \mathbb{N}$, we get $(O) \lim_k v(m_n)(H_k) = 0$ for all $n \in \mathbb{N}$. From this and Theorem 2.1 it follows that there is a meager set $N' \subset \Omega$, without loss of generality $N' \supset N_0$ (depending on $(H_k)_k$), such that

$$\lim_k [\sup_{A \in \Sigma, A \subset H_k} |m_n(A)(\omega)|] = \inf_k [\sup_{A \in \Sigma, A \subset H_k} |m_n(A)(\omega)|] = 0 \quad \text{for all } \omega \in \Omega \setminus N'. \tag{17}$$

By Lemma 2.7, we obtain

$$\lim_k (\sup_n [\sup_{A \in \Sigma, A \subset H_k} |m_n(A)(\omega)|]) = \inf_k (\sup_n [\sup_{A \in \Sigma, A \subset H_k} |m_n(A)(\omega)|]) = 0 \tag{18}$$

for every $\omega \in \Omega \setminus N'$.

From (18) and Theorem 2.1, proceeding with an analogous technique as in Boccuto and Candeloro (2010, Theorem 3.1), we find a meager set $N'' \subset \Omega$, without loss of generality $N'' \supset N'$, with

$$[\wedge_k (\vee_n [\vee_{A \in \Sigma, A \subset H_k} |m_n(A)|])](\omega) = 0$$

for each $\omega \in \Omega \setminus N''$. Since the complement of a meager subset of Ω is dense in Ω, we get

$$0 = \wedge_k (\vee_n [\vee_{A \in \Sigma, A \subset H_k} |m_n(A)|]) = \wedge_k (\vee_n v(m_n)(H_k)) = (O) \lim_k (\vee_n v(m_n)(H_k)),$$

and so we obtain uniform τ-continuity of the m_n's.

The proof of (BJ) I) $\Longrightarrow$ (D) I) is analogous to that of (BJ) I) $\Longrightarrow$ (VHS) I).

The proof of (VHS) I) $\Longrightarrow$ (N) I) is similar to that of (VHS) II) $\Longrightarrow$ (N) II).

We now prove (N) I) $\Longrightarrow$ (BJ) I). Assume that $m_n \colon \Sigma \to R$, $n \in \mathbb{N}$, is a sequence of equibounded finitely additive measures, satisfying property $(\ast)$ with respect to $\mathcal{I}$ and R. Let Ω be as in Theorem 2.1. By Lemma 3.1, $\alpha\alpha\alpha$), there exist a meager set $N_* \subset \Omega$ and $M_0 \in \mathcal{F}(\mathcal{I})$, such that for every $\omega \in \Omega \setminus N_*$ the real-valued measures $m_n(\cdot)(\omega)$, $n \in M_0$, are (s)-bounded and satisfy property $(\ast)$ with respect to $\mathbb{R}$ and $\mathcal{I}_{\text{fin}}$.

Let Q be the Stone space associated with Σ, $\mathcal{Q}$ be the algebra of all clopen subsets of Q and $\Sigma(\mathcal{Q})$ be the σ-algebra generated by $\mathcal{Q}$. For each $n \geq 0$ and $\omega \in \Omega \setminus N_*$, let $\widetilde{m_{n,\omega}} \colon \Sigma(\mathcal{Q}) \to \mathbb{R}$ be the Stone extensions of $m_n(\cdot)(\omega)$, and let $\overline{\widetilde{\lambda}} \colon \Sigma(\mathcal{Q}) \to \mathbb{R}$ be the Stone extension of λ.

Fix $\omega \in \Omega \setminus N_*$. By (N) used with the ideal $\mathcal{I}_{\text{fin}}$, $\mathbb{R}$ and $\Sigma(\mathcal{Q})$, we find a set $M_0^{(\omega)} \subset M_0$ such that $M_0 \setminus M_0^{(\omega)}$ is finite and the measures $\widetilde{m_{n,\omega}}$, $n \in M_0^{(\omega)}$, are uniformly σ-additive. Proceeding analogously as in the proof of the implication (N) II) $\Longrightarrow$ (BJ) II) and "coming back" to Σ, we obtain that the measures $m_n(\cdot)(\omega)$, $n \in M_0^{(\omega)}$, are uniformly (s)-bounded on Σ. Thus for every disjoint sequence $(C_k)_k$ in Σ we get

$$\lim_k \left(\sup_{n \in M_0^{(\omega)}} \left[\sup_{A \in \Sigma, A \subset C_k} |m_n(A)(\omega)| \right] \right) = 0. \tag{19}$$

From (19), arguing analogously as in (15-16), we obtain

$$\lim_k \left(\sup_{n \in M_0} \left[\sup_{A \in \Sigma, A \subset C_k} |m_n(A)(\omega)| \right] \right) = 0. \tag{20}$$

Since $\omega \in \Omega \setminus N_*$ was chosen arbitrarily, by Theorem 2.1 and by a density argument, arguing analogously as in the proof of (BJ) I) $\Longrightarrow$ (VHS) I) and in Boccuto and Candeloro (2010, Theorem 3.1), from (20) we get $(O) \lim_k \left[\bigvee_{n \in M_0} v(m_n)(C_k) \right] = 0$. By arbitrariness of $(C_k)_k$, the measures m_n, $n \in M_0$, are uniformly (s)-bounded on Σ. This proves that (N) I) implies (BJ) I).

The proof of (D) I) $\Longrightarrow$ (BJ) I) is analogous to that of (N) I) $\Longrightarrow$ (BJ) I). Indeed, using the same notations and proceeding analogously as in the previous implication, we get that the measures $\widetilde{m_{n,\omega}}$, $n \in M_0^{(\omega)}$, are σ-additive, and hence also (s)-bounded and regular: indeed, it is enough to argue analogously as in the proof of the implication (D) II) $\Longrightarrow$ (BJ) II) with $R = \mathbb{R}$ (see also Billingsley, 1968, Theorem 1.1). By (D) I), these measures are uniformly (s)-bounded and uniformly regular. "Coming back" to Σ, we obtain that the measures $m_n(\cdot)(\omega)$, $n \in M_0^{(\omega)}$, are uniformly (s)-bounded on Σ. From this, arguing similarly as in (15-16), we get that the measures $m_n(\cdot)(\omega)$, $n \in M_0$, are uniformly (s)-bounded on Σ. Proceeding analogously as in the proof (N) I) $\Longrightarrow$ (BJ) I), we obtain that the measures m_n, $n \in M_0$, are uniformly (s)-bounded on Σ. This ends the proof. $\square$

Remarks 3.8 (a) By means of techniques similar to those used in the implications (BJ) I) $\Longrightarrow$ (VHS) I) and (BJ) II) $\Longrightarrow$ (VHS) II), it is possible also to prove directly the implications (BJ) I) $\Longrightarrow$ (N) I) and (BJ) II) $\Longrightarrow$ (N) II).

(b) Observe that in the classical case, namely when the involved ideal is $\mathcal{I}_{\text{fin}}$, our results about the Brooks-Jewett, Vitali-Hahn-Saks, Nikodým convergence and Dieudonné theorems hold, even if we drop the condition of λ-uniform $\mathcal{I}_{\text{fin}}$-exhaustiveness, which however in general is essential when $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$. In this framework, when it is dealt with respect to a common (O)-sequence, the Vitali-Hahn-Saks theorem, when the involved Fréchet-Nikodým topology is generated by a finitely additive extended real-valued positive measure, is proved in Boccuto and Candeloro (2002, Corollary 5.7); the Nikodým convergence theorem (N) is demonstrated in Boccuto and Candeloro (2002, Corollary 5.5), and some versions of the Dieudonné theorem are presented in Boccuto and Candeloro (2001/2002, Theorem 3.3). When the concept of (s)-boundedness and those related with it are intended not necessarily with respect to a common order sequence, the Brooks-Jewett, Nikodým and Vitali-Hahn-Saks theorem are proved in Boccuto and Candeloro (2010, Theorem 3.1), Boccuto, Dimitriou, and Papanastassiou (2010a, Theorem 3.6) and Boccuto and Dimitriou (2011b, Theorem 3.6) respectively, while some versions of the Dieudonné theorem are given in Boccuto and Candeloro (2010, Theorem 5.1).

(c) Observe that, when $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$, in general the condition of λ-uniform $\mathcal{I}$-exhaustiveness cannot be dropped. Indeed, let $\Sigma = \mathcal{P}(\mathbb{N})$, $R = \mathbb{R}$, $\mathcal{I}$ be any fixed admissible and not maximal ideal of $\mathbb{N}$, and $\mathcal{F}$ be its dual filter. As seen in §2.2, the ideal $\mathcal{I}_\delta$ of the subsets of $\mathbb{N}$ having asymptotical density 0 is not maximal.

Define $\lambda \colon \Sigma \to \mathbb{R}$ by $\lambda(A) = \sum_{n \in A} \dfrac{1}{2^n}$, $A \in \Sigma$. It is easy to see that λ is a σ-additive measure, and Σ is λ-separable (indeed, the set $\mathcal{I}_{\text{fin}}$ of all finite subsets of $\mathbb{N}$ is countable and dense in Σ with respect to the Fréchet-Nikodým topology generated by λ).

Let us define the Dirac measures δ_n, $n \in \mathbb{N}$, as follows. For every $A \in \Sigma$ and $n \in \mathbb{N}$, set $\delta_n(A) = 1$ if $n \in A$, and $\delta_n(A) = 0$ if $n \in \mathbb{N} \setminus A$. It is not difficult to see that δ_n is a σ-additive measure on Σ for all $n \in \mathbb{N}$, and that $\lim_n \delta_n(W) = 0$, and hence $(\mathcal{I}) \lim_n \delta_n(W) = 0$, for each $W \in \mathcal{I}_{\text{fin}}$.

However, observe that for every $\vartheta > 0$ there is a cofinite set $Z_\vartheta \subset \mathbb{N}$, with $\lambda(Z_\vartheta) < \vartheta$, and hence $|\lambda(E) - \lambda(F)| = \lambda(E \Delta F) \leq \lambda(Z_\vartheta) < \vartheta$ whenever $E \cup F \subset Z_\vartheta$. For each $M \in \mathcal{F}$ it is possible to find an integer $\overline{n} \in M$ large enough and two sets $E, F \in \Sigma$, with $E \cup F \subset Z_\vartheta$ and $\overline{n} \in E \setminus F$, so that $|\delta_{\overline{n}}(E) - \delta_{\overline{n}}(F)| = \delta_{\overline{n}}(E) = 1$. Thus the measures δ_n, $n \in \mathbb{N}$, are not λ-uniformly $\mathcal{I}$-exhaustive on Σ.

Moreover, it is not true that the limit $(\mathcal{I}) \lim_n \delta_n(A)$ exists in $\mathbb{R}$ for all $A \subset \mathbb{N}$. Indeed, since $\mathcal{I}$ is not maximal, there is a set $C \subset \mathbb{N}$ with $C \notin \mathcal{I}$ and $\mathbb{N} \setminus C \notin \mathcal{I}$, and so we get $\delta_n(C) = 1$ if and only if $n \in C$ and $\delta_n(C) = 0$ if and only if $n \notin C$. Let now $l \neq 0$ and $\varepsilon_0 := |l|/2 > 0$. Then for each $n \in \mathbb{N} \setminus C$ we have $|\delta_n(C) - l| = |l| > \varepsilon_0$, and so $\{n \in \mathbb{N} : |\delta_n(C) - l| > \varepsilon_0\} \notin \mathcal{I}$, because it contains $\mathbb{N} \setminus C$ and $\mathbb{N} \setminus C \notin \mathcal{I}$. In the case $l = 0$, take $\varepsilon_0 = 1/2$. For every $n \in C$, $|\delta_n(C)| = 1 > \varepsilon_0$. Thus, $\{n \in \mathbb{N} : |\delta_n(C)| > \varepsilon_0\} \notin \mathcal{I}$, since it contains C and $C \notin \mathcal{I}$. Hence, the limit $(\mathcal{I}) \lim_n \delta_n(C)$ does not exist in $\mathbb{R}$. Furthermore, given any infinite subset $M \subset \mathbb{N}$ and $k \in M$, we get $\sup_{n \in M} \delta_n(\{k\}) = 1$, and so the measures δ_n, $n \in M$, are not uniformly (s)-bounded on Σ (see also Boccuto & Dimitriou, 2011b, Example 3.11).

(d) In general, it is not possible to formulate versions of limit theorems, for instance Brooks-Jewett or Nikodým-type theorems, similar to the classical ones, when the classical pointwise convergence of the measures involved is replaced with the weaker pointwise ideal convergence, even when $R = \mathbb{R}$. Indeed, as soon as $\mathcal{I}$ is any admissible ideal of $\mathbb{N}$, different from $\mathcal{I}_{\text{fin}}$, we have the following example.

Let $H := \{h_1 < \ldots < h_s < h_{s+1} < \ldots\}$ be an infinite set belonging to $\mathcal{I}$ and such that $\mathbb{N} \setminus H$ is infinite. Since $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$, the set H does exist. For every $n \notin H$ and $E \subset \mathbb{N}$, set $m_n(E) = 0$. For any $s \in \mathbb{N}$ and $E \subset \mathbb{N}$, put $m_{h_s}(E) = 1$ if $s \in E$ and 0 otherwise. Note that $m_0(E) := (\mathcal{I}) \lim_n m_n(E) = 0$ for each $E \subset \mathbb{N}$. Moreover, it is easy to check that the m_n's are σ-additive positive equibounded measures. Indeed, given $n \in \mathbb{N}$ and any disjoint sequence $(C_j)_j$ of subsets of $\mathbb{N}$, the quantity $m_n(C_j)$ can be different from zero (and in this case is equal to 1) at most for one index j, since, for every $s \in \mathbb{N}$, $m_n(\{s\}) \neq 0$ if and only if $n = h_s$.

For every $j \in \mathbb{N}$, set $C_j := \{j\}$. We get: $1 \geq \sup_{n \in \mathbb{N}} m_n(C_j) \geq m_{h_j}(C_j) = 1$, and so it is not true that $(\mathcal{I}) \lim_j [\sup_{n \in \mathbb{N}} m_n(C_j)] = 0$. $\hfill\square$

Open Problems

(a) Find similar results about limit theorems and their equivalence considering (D)-convergence and/or some other classes of ideals/filters of $\mathbb{N}$.

(b) Prove some other similar theorems involving σ-additivity and (s)-boundedness not necessarily with respect to a common order sequence.

(c) Ask whether equivalence-type results remain valid when absolute continuity is not formulated in topological terms.

(d) Find similar results in case of other notions of measure regularity.

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