Research Article

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p-fractional Hardy–Schrödinger–Kirchhoff systems with critical nonlinearities

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Abstract: This paper deals with the existence of nontrivial solutions for critical Hardy–Schrödinger–Kirchhoff systems driven by the fractional p-Laplacian operator. Existence is derived as an application of the mountain pass theorem and the Ekeland variational principle. The main features and novelty of the paper are the presence of the Hardy terms as well as critical nonlinearities.

Keywords: Integro-differential operators, Hardy–Schrödinger–Kirchhoff systems, Hardy terms, critical nonlinearities, variational methods

MSC 2010: 35D30, 35R11, 35A15, 47G20

1 Introduction

In this paper, we study the existence of solutions for an elliptic system of Hardy–Schrödinger–Kirchhoff type, involving the fractional p-Laplacian as well as critical nonlinearities. More precisely, we first consider the system in $\mathbb{R}^N$

$$\begin{cases} M(\|(u,v)\|^p)(\mathcal{L}_p^s u + V(x)|u|^{p-2}u) - \sigma \dfrac{|u|^{p-2}u}{|x|^{ps}} = \lambda H_u(x,u,v) + \dfrac{\alpha}{p_s^*}|v|^\beta |u|^{\alpha-2}u, \\ M(\|(u,v)\|^p)(\mathcal{L}_p^s v + V(x)|v|^{p-2}v) - \sigma \dfrac{|v|^{p-2}v}{|x|^{ps}} = \lambda H_v(x,u,v) + \dfrac{\beta}{p_s^*}|u|^\alpha |v|^{\beta-2}v, \end{cases} \tag{$\mathcal{S}$}$$

where $0 < s < 1 < p < \infty$, $sp < N$, $\alpha > 1$ and $\beta > 1$ with $\alpha + \beta = p_s^*$ and $p_s^* = \frac{Np}{N-sp}$. *The potential function* $V : \mathbb{R}^N \to \mathbb{R}^+$ *verifies*

$$V \in C(\mathbb{R}^N) \quad \textit{and} \quad \inf_{x\in\mathbb{R}^N} V(x) = V_0 > 0. \tag{$\mathcal{V}$}$$

The nonlocal fractional operator $\mathcal{L}_p^s$ is defined along any $\varphi \in C_0^\infty(\mathbb{R}^N)$ by

$$\mathcal{L}_p^s\varphi(x) = \lim_{\varepsilon\to0^+} 2 \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} |\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))K(x-y)\,dy$$

for $x \in \mathbb{R}^N$, where $B_\varepsilon(x)$ denotes the ball in $\mathbb{R}^N$ of radius $\varepsilon > 0$ at the center $x \in \mathbb{R}^N$ and, throughout the paper, $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ *is a measurable function such that*

(a) *there exists* $K_0 > 0$ *such that* $K(x) \geq K_0|x|^{-(N+ps)}$ *for any* $x \in \mathbb{R}^N \setminus \{0\}$,
(b) $mK \in L^1(\mathbb{R}^N)$, *where* $m(x) = \min\{|x|^p, 1\}$, $x \in \mathbb{R}^N$.

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A typical example of K is given by $K(x) = |x|^{-(N+sp)}$. In this case, the operator $\mathcal{L}_p^s$ simply reduces to the fractional p-Laplacian, denoted by $(-\Delta)_p^s$. In particular, $(-\Delta)_p^s$ is consistent with the fractional Laplacian $(-\Delta)^s$ as $p = 2$, and it is well known that $(-\Delta)_p^s$ reduces to the standard p-Laplacian as $s \uparrow 1$ in the limit sense of Bourgain–Brezis–Mironescu, as shown in [4].

For critical equations in $\mathbb{R}^N$ we refer the reader to [2, 7, 11, 14, 29] and references therein for the study of scalar problems with critical nonlinearities.

The main solution space of $(\mathcal{S})$ is $\mathbf{W} = W_{K,V}^{s,p}(\mathbb{R}^N) \times W_{K,V}^{s,p}(\mathbb{R}^N)$, with

$$W_{K,V}^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N, V) : \iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^p K(x-y)\,dx\,dy < \infty \right\}$$

and

$$L^p(\mathbb{R}^N, V) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} V(x)|u(x)|^p\,dx < \infty \right\},$$

with norm

$$|u|_{p,V} = \left(\int_{\mathbb{R}^N} V(x)|u(x)|^p\,dx \right)^{\frac{1}{p}}.$$

The space $\mathbf{W}$ is endowed with the norm

$$\|(u,v)\| = ([(u,v)]_{K,p}^p + |(u,v)|_{p,V}^p)^{\frac{1}{p}}$$

for all $(u,v) \in \mathbf{W}$, where

$$\begin{aligned} [(u,v)]_{K,p} &= ([u]_{K,p}^p + [v]_{K,p}^p)^{\frac{1}{p}} = \left(\iint_{\mathbb{R}^{2N}} \{|u(x)-u(y)|^p + |v(x)-v(y)|^p\} K(x-y)\,dx\,dy \right)^{\frac{1}{p}}, \\ |(u,v)|_{p,V} &= (|u|_{p,V}^p + |v|_{p,V}^p)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^N} V(x)\{|u(x)|^p + |v(x)|^p\}\,dx \right)^{\frac{1}{p}}. \end{aligned} \tag{1.1}$$

Then $\mathbf{W} = (\mathbf{W}, \|\cdot\|)$ is a separable reflexive real Banach space, see [17, 33] for more details.

Because of the presence of the Hardy terms in $(\mathcal{S})$, we assume that the system is *non-degenerate*. We recall that the degenerate case for $(\mathcal{S})$ corresponds to $M(0) = 0$. Hence, throughout the paper, we suppose that *the Kirchhoff function* $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is continuous and satisfies*

(M1) $\inf_{t \in \mathbb{R}_0^+} M(t) = a > 0$,

(M2) *there exists* $\theta \in [1, \frac{p_s^*}{p})$ *such that* $M(t)t \le \theta \mathscr{M}(t)$ *for all* $t \in \mathbb{R}_0^+$, *where* $\mathscr{M}(t) = \int_0^t M(\tau)\,d\tau$.

Usually, the existence of solutions of fractional Kirchhoff problems is derived, when M is also nondecreasing in $\mathbb{R}_0^+$. For more comments we refer, e.g., to [18, 31, 33]. However, (M1)–(M2) do not force M to be monotone as the example $M(t) = (1+t)^k + (1+t)^{-1}$ for $t \ge 0$, with $0 < k < 1$, shows. For details we refer to [1, 32].

The parameter σ is real and for the Hardy terms in $(\mathcal{S})$ it is important to recall the fractional Hardy–Sobolev inequality. By [25, Theorems 1 and 2], we know that

$$\begin{aligned} \|u\|_{p_s^*}^p &\le c_{N,p} \frac{s(1-s)}{(N-ps)^{p-1}} [u]_{s,p}^p, \qquad p_s^* = \frac{pN}{N-ps}, \quad N > ps, \\ \|u\|_{H_p}^p &\le c_{N,p} \frac{s(1-s)}{(N-ps)^p} [u]_{s,p}^p, \qquad \|u\|_{H_p} = \left(\int_{\mathbb{R}^N} |u(x)|^p \frac{dx}{|x|^{ps}} \right)^{\frac{1}{p}} \end{aligned} \tag{1.2}$$

for all $u \in D^{s,p}(\mathbb{R}^N)$, where the positive constant $c_{N,p}$ depends only on N and p and $D^{s,p}(\mathbb{R}^N)$ is the fractional Beppo–Levi space, that is, the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $[\,\cdot\,]_{s,p}$ defined as

$$[\varphi]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \right)^{\frac{1}{p}}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. Hence, denoting by $D_K^{s,p}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $[\,\cdot\,]_{K,p}$, then the best fractional Hardy–Sobolev constant, called $\mathcal{H}_p = \mathcal{H}(p, N, s, K)$, is given by

$$\mathcal{H}_p = \inf_{\substack{u\in D_K^{s,p}(\mathbb{R}^N)\\ u\neq 0}} \frac{[u]_{K,p}^p}{\|u\|_{H_p}^p}, \tag{1.3}$$

and $\mathcal{H}_p > 0$ thanks to (1.2) and (a).

Moreover, (1.2) and (a) imply the existence of a constant $C_{p_s^*} > 0$ such that

$$\|u\|_{p_s^*} \le C_{p_s^*}[u]_{K,p} \quad \text{for all } u \in D_K^{s,p}(\mathbb{R}^N). \tag{1.4}$$

Both (1.3) and (1.4) will be crucial for the statements of the main results.

The parameter λ in $(\mathcal{S})$ is strictly positive and the perturbed terms H_u, H_v are partial derivatives of *a Carathéodory function H satisfying the subcritical growth condition*:

Condition $(\mathcal{H})$. For a.e. $x \in \mathbb{R}^N$ it results $H(x,\cdot,\cdot) \in C^1(\mathbb{R}^2)$, $H(x,\cdot,\cdot) \ge 0$ in $\mathbb{R}^2$, $H_z(x,0,0) = (0,0)$ with $H_z = (H_u, H_v)$ and $z = (u, v)$. Furthermore, there exist μ and q such that $\theta p < \mu \le q < p_s^*$ and for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ for which the inequality

$$|H_z(x,z)| \le \mu\varepsilon|z|^{\mu-1} + qC_\varepsilon|z|^{q-1} \quad \text{for any } z \in \mathbb{R}^2,$$

where $|z| = \sqrt{u^2 + v^2}$, and also the inequalities

$$0 \le \mu H(x,z) \le H_z(x,z)\cdot z \quad \text{for all } z \in \mathbb{R}^2,$$

hold for a.e. $x \in \mathbb{R}^N$. Finally, for all measurable set E of $\mathbb{R}^N$, with positive Lebesgue measure, $H(x,u,v) > 0$ for a.e. $x \in E$ and $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Noncompact variational problems have attracted much attention since the late seventies. System $(\mathcal{S})$ is a reasonable useful generalization of popular elliptic problems, with singular potentials and critical nonlinearities, which naturally arise in quantum mechanics, astrophysics, as well as in Riemannian geometry in the so-called *scalar curvature problem* on the sphere $\mathbb{S}^N$. The loss of compactness is caused by the invariant action of the conformal group, or of one of its subgroups, leading to possible spikes formation. It is well known that the *Kazdan–Warner problem* of finding a conformal metric with prescribed scalar curvature $k(x)$ leads to finding positive solutions of

$$-\Delta u + V(x)u - \sigma\frac{u}{|x|^2} = k(x)|u|^{2^*-2}u.$$

This equation is a simplified version of the nonlinear *Wheeler–DeWitt equation*, which describes the quantum version of the Hamiltonian constraint using metric variables and combines mathematically the ideas of quantum mechanics and general relativity in quantum cosmology. The Wheeler–DeWitt equation is applied to model quantum states of the universe and is also used to investigate the qualitative behavior of the universe wave function. For a more detailed discussion and history we refer to the recent nice survey [19] and the references therein. In summary, equations with Hardy potentials arise from many physical contexts, such as molecular physics, quantum cosmology and linearization of combustion models. But, from the mathematical point of view, the main reason of interest in Hardy potentials relies in their criticality. Indeed, the non-compactness of the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N, |x|^{-2}\,dx)$, and in the context of $(\mathcal{S})$ of $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N, |x|^{-ps}\,dx)$, even locally in any neighborhood of zero, leads to other additional difficulties and, more importantly, to new phenomenon concerning the possibility of blow-up. Finally, the presence of the Hardy terms and of the fractional critical nonlinearities, as well as the fact that $(\mathcal{S})$ is studied in the entire space $\mathbb{R}^N$, cause, roughly speaking, a triple loss of compactness which produces new interesting difficulties.

Furthermore, concerning the Kirchhoff nature of $(\mathcal{S})$, we recall that, following [6], Fiscella and Valdinoci in [18] proposed a stationary Kirchhoff variational model, with critical nonlinear terms, in bounded regular domains of $\mathbb{R}^N$, which takes into account the *nonlocal* aspect of the tension arising from nonlocal measurements of the *fractional length of the string*. In [18], however, only the *non-degenerate* case was covered. Since then, several papers have been devoted to stationary fractional Kirchhoff problems involving critical nonlinearities in the *degenerate* case. For further comments we refer to [1, 7, 33] and the references therein. Let us recall that Kirchhoff problems, with Kirchhoff function M, are said to be *non-degenerate* if $M(0) > 0$, and *degenerate* if $M(0) = 0$. For example, existence of solutions for non-degenerate fractional Kirchhoff problems is treated in [18, 24] and for degenerate problems in [1, 7, 26, 33, 35].

For stationary Hardy–Kirchhoff fractional problems, with critical nonlinearities, even in the scalar case, very few contributions are known. We refer to [7, 15, 16] and the references therein. The main novelty of our paper is to treat $(\mathcal{S})$ in the setting of fractional p-Laplacian involving critical nonlinearities and Hardy terms. The results are new even in the case $M \equiv 1$.

Recall that throughout the paper $0 < s < 1 < p < \infty$, $sp < N$, $p_s^* = \frac{Np}{N-sp}$ and $\alpha > 1$, $\beta > 1$ with $\alpha + \beta = p_s^*$. In the superlinear case, that is, when $q \in (\theta p, p_s^*)$ as in $(\mathcal{H})$, we get the next existence result for $(\mathcal{S})$, which involves the main geometrical parameter $\kappa = \kappa(\mu, M, p)$ defined by

$$\kappa = \frac{a(\mu - \theta p)}{\theta(\mu - p)}, \tag{1.5}$$

similar to the one introduced in [7]. Clearly $\kappa \in (0, a]$, since $\theta \geq 1$ and $p \leq \theta p < \mu$ by $(\mathcal{H})$ and (M2). As shown in [7, Section 2] there are cases, besides the obvious one $M \equiv a$, in which $\kappa = a$, that is, $\theta = 1$ in (M2).

Theorem 1.1. *Suppose that* (a) *and* (b) *hold for K, that V satisfies* $(\mathcal{V})$*, that M verifies* (M1)–(M2) *and that H fulfils* $(\mathcal{H})$*. Then for any $\sigma \in (-\infty, \kappa \mathcal{H}_p)$ there exists $\lambda^* = \lambda^*(\sigma) > 0$ such that system* $(\mathcal{S})$ *admits at least one nontrivial solution $(u_{\sigma,\lambda}, v_{\sigma,\lambda})$ in $\mathbf{W}$ for all $\lambda \geq \lambda^*$. Moreover,*

$$\lim_{\lambda \to \infty} \|(u_{\sigma,\lambda}, v_{\sigma,\lambda})\| = 0 \tag{1.6}$$

holds.

A very natural appealing open problem is to prove existence of nontrivial solutions for $(\mathcal{S})$, when $M(0) = 0$ and $M(t) > 0$ for all $t > 0$. However, Theorem 1.1 was recently established in [27, Theorem 1.1], without the Hardy terms, that is, in the case $\sigma = 0$, but in the degenerate case. Because of the lack of compactness, due to the presence of the Hardy terms, Theorem 1.1 is more delicate to prove than in [27] and a tricky step in the proof is necessary to overcome this new difficulty. Theorem 1.1 extends to entire solutions the existence results recently obtained for fractional systems, with critical nonlinear terms, but in bounded domains, in [9, 10, 12, 13, 20, 22, 28], and generalizes to the fractional Hardy–Schrödinger–Kirchhoff case the systems driven by the p-Laplacian operator studied in [23]. However, in the systems treated in [17] the fractional p-Laplacian operator is replaced by two possibly different fractional Laplacian operators and H is not required to satisfy the Ambrosetti–Rabinowitz growth condition as assumed in $(\mathcal{H})$. Finally, Theorem 1.1 extends in a broad sense [34, Theorem 1.1].

In what follows, we shall study system $(\mathcal{S})$ under the solely assumption (M1) on the Kirchhoff function M. We first prove the next addition to Theorem 1.1.

Theorem 1.2. *Suppose that* (a) *and* (b) *hold for K, that V satisfies* $(\mathcal{V})$*, that M verifies* (M1)*, that H fulfils* $(\mathcal{H})$ *with $p < \mu \leq q < p_s^*$ and that*

$$pM(0) < \mu a. \tag{1.7}$$

Then for any $\epsilon \in (M(0), \frac{a\mu}{p})$ and for any $\sigma \in (-\infty, \kappa_\epsilon \mathcal{H}_p)$, where

$$\kappa_\epsilon = \frac{a\mu - p\epsilon}{\mu - p} > 0, \tag{1.8}$$

there exists $\lambda^* = \lambda^*(\epsilon, \sigma) > 0$ such that system $(\mathcal{S})$ *admits at least one nontrivial solution $(u_{\sigma,\lambda}, v_{\sigma,\lambda})$ in $\mathbf{W}$ for all $\lambda \geq \lambda^*$. Furthermore,* (1.6) *continues to hold.*

Clearly the request (1.7) is automatic whenever $M(0) = a$, being $\mu > p$ by $(\mathcal{H})$. The assumption $M(0) = a$, together with monotonicity of M, was assumed in [18, 31] in the scalar case, as well as in numerous papers. A very interesting open problem is to construct a nontrivial solution $(u_{\sigma,\lambda}, v_{\sigma,\lambda})$ of $(\mathcal{S})$ when $\mu a \leq pM(0)$ and the growth condition on M stated in (M2) does not hold; in other words, when both Theorems 1.1 and (1.2) cannot be applied. Finally, the number κ_ϵ in (1.8) is in the open interval $(0, a)$, since $\epsilon > M(0) \geq a$ by (M1).

Theorem 1.2 extends in several directions [15, Theorem 1.3 (ii)] given in the scalar case. We refer to [15] for further comments.

In the sublinear case, that is, when $q \in (1, p)$, we continue to assume on M solely (M1) but, following [27], we take H of the special separated form $H(x, u, v) = h(x)f(u, v)$. Hence, we deal with the following new system in $\mathbb{R}^N$:

$$\begin{cases} M(\|(u,v)\|^p)(\mathcal{L}_p^s u + V(x)|u|^{p-2}u) - \sigma \dfrac{|u|^{p-2}u}{|x|^{ps}} = h(x)f_u(u,v) + \gamma \dfrac{\alpha}{p_s^*}|v|^\beta |u|^{\alpha-2}u, \\ M(\|(u,v)\|^p)(\mathcal{L}_p^s v + V(x)|v|^{p-2}v) - \sigma \dfrac{|v|^{p-2}v}{|x|^{ps}} = h(x)f_v(u,v) + \gamma \dfrac{\beta}{p_s^*}|u|^\alpha |v|^{\beta-2}v, \end{cases} \tag{$\mathcal{S}'$}$$

where V *satisfies* $(\mathcal{V})$, $\gamma > 0$ *and* f *verifies*

(f1) $f \in C^1(\mathbb{R}^2, \mathbb{R}^+)$ *and there exist* $C > 0$ *and* $q \in (1, p)$ *such that*

$$|f_z(z)| \leq C|z|^{q-1} \quad \text{for all } z = (u, v) \in \mathbb{R}^2,$$

where $f_z = (f_u, f_v)$ *and* f_u, f_v *denote the partial derivatives of* f *with respect to the first and second variable*,

(f2) *there exist* $a_0 > 0$, $\delta > 0$ *and* $q_1 \in (1, p)$ *such that*

$$f(z) \geq a_0|z|^{q_1} \quad \textit{for all } z \in \mathbb{R}^2 \textit{ with } |z| \leq \delta.$$

Concerning the function h in $(\mathcal{S}')$, we assume from now on that h *verifies*

(h) $0 \leq h \in L^{p_s^*/(p_s^*-q)}(\mathbb{R}^N)$ *and there exists a nonempty open subset* Ω *of* $\mathbb{R}^N$ *such that* $\inf_{x\in\Omega} h(x) > 0$.

In order to cover the more interesting case when $\gamma > 0$ in $(\mathcal{S}')$, we need a further assumption on h. Fix $\sigma < a\mathcal{H}_p$ and set

$$\eta(t) = \frac{1}{2p}\Big(a - \frac{\sigma^+}{\mathcal{H}_p}\Big)t^p - \frac{C_{p_s^*}^{p_s^*}}{p_s^*} t^{p_s^*}$$

for all $t \geq 0$. Since $p < p_s^*$, the positive number

$$\rho_0 = \Bigg(\frac{a\mathcal{H}_p - \sigma^+}{2\mathcal{H}_p C_{p_s^*}^{p_s^*}}\Bigg)^{\frac{1}{p_s^*-p}}$$

is such that

$$\eta(\rho_0) = \max_{t\geq 0} \eta(t) = \Big(\frac{1}{2p} - \frac{1}{2p_s^*}\Big)\Big(a - \frac{\sigma^+}{\mathcal{H}_p}\Big)^{\frac{p_s^*}{p_s^*-p}} (2C_{p_s^*}^{p_s^*})^{\frac{p}{p-p_s^*}} > 0.$$

We are now able to state the existence result for $(\mathcal{S}')$.

Theorem 1.3. *Assume that* (a) *and* (b) *hold for* K, *that* V *satisfies* $(\mathcal{V})$, *that* M *verifies* (M1), *that* f *fulfils* (f1)–(f2) *and that* h *satisfies* (h). *Let* σ *be in* $(-\infty, a\mathcal{H}_p)$. *Then* $(\mathcal{S}')$ *admits at least one nontrivial solution* $(u_{\sigma,\gamma}, v_{\sigma,\gamma})$ *in* $\mathbf{W}$ *for all* $\gamma \leq 0$. *If* $\gamma > 0$ *and* h, *depending on* σ^+, *satisfies*

$$\eta(\rho_0) > \Big[\frac{1}{2p}\Big(a - \frac{\sigma^+}{\mathcal{H}_p}\Big)\Big]^{\frac{q}{q-p}} (CC_{p_s^*}^q \|h\|_{\frac{p_s^*}{p_s^*-q}})^{\frac{p}{p-q}}, \tag{1.9}$$

where C *and* q *are introduced in* (f1) *and* $C_{p_s^*} > 0$ *in* (1.4), *then there exists* $\gamma^{**} = \gamma^{**}(\sigma, h) > 0$ *such that system* $(\mathcal{S}')$ *admits at least one nontrivial solution* $(u_{\sigma,\gamma}, v_{\sigma,\gamma})$ *in* $\mathbf{W}$ *for all* $\gamma \in (0, \gamma^{**})$.

Clearly, condition (h) simply requires that h is nontrivial and (1.9) that the norm of h in $L^{\frac{p_s^*}{p_s^*-q}}(\mathbb{R}^N)$ is sufficiently small.

Theorem 1.3 was recently established in a weaker form in [27, Theorem 1.2] when $\sigma = 0$, that is, without the Hardy terms. Again, the nontrivial presence of the Hardy terms makes Theorem 1.3 more difficult to handle than in [27]. Furthermore, Theorem 1.3 generalizes the existence results obtained in [9, 10, 12, 13, 17, 20, 22, 28] in several directions. Finally, Theorem 1.3 extends in a broad sense the recent [34, Theorem 1.2].

However, as far as we know, Theorems 1.1–1.3 are new even when $M \equiv 1$ and $p = 2$. The paper is structured in the following way. In Section 2, we present some preliminary results, which are useful for the next main sections. In Section 3, we establish the key compactness theorems, particularly helpful to apply the mountain pass lemma at a special mountain pass level and to prove Theorems 1.1 and 1.2, that is, the existence of a nontrivial solution for $(\mathcal{S})$. Finally, Section 4 is devoted to the proof of Theorem 1.3 via the Ekeland variational principle.

2 Variational framework

In this section we briefly recall the relevant definitions and notations related to real uniformly convex Banach space $\mathbf{W}$ and for a complete treatment, we refer to [2, 17, 32, 33].

Combining the results of [7, Lemmas 4.1 and 5.1] and [33, Theorem 2.1], we get [17, Lemma 2.2], that is, thanks to [17, Section 5] we have the next properties.

Lemma 2.1. *Under* (a)–(b) *on K and* $(\mathcal{V})$ *on V, the embeddings*

$$\mathbf{W} \hookrightarrow W_K^{s,p}(\mathbb{R}^N) \times W_K^{s,p}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N)$$

are continuous for all $\nu \in [p, p_s^*]$, and

$$\|(u,v)\|_\nu \le \|u\|_\nu + \|v\|_\nu \le C_\nu \|(u,v)\| \quad \text{for all } (u,v) \in \mathbf{W},$$

where C_ν depends on ν, N, s, K_0 and p.

The next result can be proved similarly to the arguments used for [8, Lemma 2.2].

Lemma 2.2. *Assume* (a)–(b) *on K and* $(\mathcal{V})$ *on V. Let $\{(u_n, v_n)\}_n$ and (u,v) be in $\mathbf{W}$ such that $(u_n, v_n) \rightharpoonup (u,v)$ weakly in $\mathbf{W}$ and $(u_n, v_n) \to (u,v)$ a.e. in $\mathbb{R}^N$. Then $(u_n, v_n) \to (u,v)$ strongly in $L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N)$ as $n \to \infty$ for any $\nu \in (p, p_s^*)$.*

Let us present a technical lemma, which will play a crucial role in the study of compactness property of functional I. This result was proved in the scalar case in [30, Lemma 3.2] when $K(x) = |x|^{-N-ps}$. For the sake of completeness, we report here the proof.

Lemma 2.3. *Assume* (a)–(b) *on K and* $(\mathcal{V})$ *on V. Let $\{(u_n, v_n)\}_n$ and (u,v) be in $\mathbf{W}$ such that $(u_n, v_n) \rightharpoonup (u,v)$ weakly in $\mathbf{W}$ and $(u_n, v_n) \to (u,v)$ a.e. in $\mathbb{R}^N$. Then*

$$\|(u_n - u, v_n - v)\|^p = \|(u_n, v_n)\|^p - \|(u,v)\|^p + o(1) \quad \text{as } n \to \infty.$$

Proof. Let us define $\omega_n : \mathbb{R}^{2N} \to \mathbb{R}^+$ by

$$\omega_n(x,y) = |\{|(u_n(x) - u(x)) - (u_n(y) - u(y))|^p - |u_n(x) - u_n(y)|^p + |u(x) - u(y)|^p\}K(x-y)|.$$

We want to prove that

$$\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \omega_n(x,y)\,dx\,dy = 0. \tag{2.1}$$

Given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\big||a+b|^p - |a|^p\big| \le \varepsilon |a|^p + C_\varepsilon |b|^p \quad \text{for any } a, b \in \mathbb{R}.$$

Thus, putting $a = (u_n(x) - u(x)) - (u_n(y) - u(y))$ and $b = u(x) - u(y)$, we get

$$\omega_n(x,y) \le \varepsilon |(u_n(x) - u(x)) - (u_n(y) - u(y))|^p K(x-y) + C_\varepsilon |u(x) - u(y)|^p K(x-y). \tag{2.2}$$

Define $\omega_n^\varepsilon : \mathbb{R}^{2N} \to \mathbb{R}^+$ by

$$\omega_n^\varepsilon(x,y) = \big(\omega_n(x,y) - \varepsilon |(u_n(x) - u(x)) - (u_n(y) - u(y))|^p K(x-y)\big)^+.$$

By (2.2) and (b) we have

$$\omega_n^\varepsilon(x,y) \le C_\varepsilon |u(x)-u(y)|^p K(x-y) \in L^1(\mathbb{R}^N).$$

Since in particular $u_n \to u$ a.e. in $\mathbb{R}^N$, clearly $\omega_n^\varepsilon \to 0$ a.e. in $\mathbb{R}^{2N}$, so that

$$\lim_{n\to\infty} \iint_{\mathbb{R}^N} \omega_n^\varepsilon(x,y)\,dx\,dy = 0$$

by the dominated convergence theorem. Then, by (2.2),

$$\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \omega_n(x,y)\,dx\,dy \le \varepsilon \limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} |(u_n(x)-u(x))-(u_n(y)-u(y))|^p K(x-y)\,dx\,dy = \varepsilon C,$$

since $([u_n]_{K,p})_n$ is bounded, thanks to the fact that $u_n \rightharpoonup u$ in $W^{s,p}_{K,V}(\mathbb{R}^N)$, so that in particular $u_n \rightharpoonup u$ in $D^{s,p}_K(\mathbb{R}^N)$. Consequently, since $\varepsilon > 0$ is arbitrary, the claim (2.1) holds true and it implies that

$$[u_n]^p_{K,p} = [u_n - u]^p_{K,p} + [u]^p_{K,p} + o(1) \quad \text{as } n \to \infty.$$

A similar argument shows that

$$[v_n]^p_{K,p} = [v_n - v]^p_{K,p} + [v]^p_{K,p} + o(1) \quad \text{as } n \to \infty$$

and

$$|u_n|^p_{V,p} = |u_n - u|^p_{V,p} + |u|^p_{V,p} + o(1), \quad |v_n|^p_{V,p} = |v_n - v|^p_{V,p} + |v|^p_{V,p} + o(1) \quad \text{as } n \to \infty.$$

This concludes the proof. □

Since α and $\beta > 1$ in problems $(\mathcal{S})$ and $(\mathcal{S}')$ are such that $\alpha + \beta = p_s^*$, the Hölder inequality and (1.4) yield

$$\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta\,dx \le \Big(\int_{\mathbb{R}^N} |u|^{p_s^*}\,dx\Big)^{\frac{\alpha}{p_s^*}} \Big(\int_{\mathbb{R}^N} |v|^{p_s^*}\,dx\Big)^{\frac{\beta}{p_s^*}} \le C_{p_s^*}^{p_s^*} [u]^\alpha_{K,p} [v]^\beta_{K,p} \le C_{p_s^*}^{p_s^*} [(u,v)]^{p_s^*}_{K,p} \tag{2.3}$$

for all $(u,v) \in \mathbf{W}$.

Lemma 2.4. *Assume* (a)–(b) *on K and* $(\mathcal{V})$ *on V. Let $\{(u_n, v_n)\}_n$ and (u,v) be in $\mathbf{W}$ such that $(u_n,v_n) \rightharpoonup (u,v)$ weakly in $\mathbf{W}$ and $(u_n,v_n) \to (u,v)$ a.e. in $\mathbb{R}^N$. Then, for any fixed $\alpha > 1$ and $\beta > 1$ with $\alpha + \beta = p_s^*$,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n - u|^\alpha |v_n - v|^\beta\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta\,dx - \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta\,dx$$

and

$$\begin{aligned} &|u_n|^{\alpha-2} u_n |v_n|^\beta \rightharpoonup |u|^{\alpha-2} u |v|^\beta \quad \text{in } L^{p_s^*/(p_s^*-1)}(\mathbb{R}^N),\\ &|u_n|^\alpha |v_n|^{\beta-2} v_n \rightharpoonup |u|^\alpha |v|^{\beta-2} v \quad \text{in } L^{p_s^*/(p_s^*-1)}(\mathbb{R}^N). \end{aligned} \tag{2.4}$$

Proof. The first part can be proved, with obvious changes, proceeding as in [21, proof of Lemma 2.1].

It remains to prove (2.4). The Hölder inequality and (1.4), since $\alpha > 1$, $\beta > 1$ and $\alpha + \beta = p_s^*$, yield

$$\int_{\mathbb{R}^N} \big||u_n|^{\alpha-1}|v_n|^\beta\big|^{\frac{p_s^*}{p_s^*-1}}\,dx \le \|u_n\|_{p_s^*}^{\frac{p_s^*(\alpha-1)}{p_s^*-1}} \|v_n\|_{p_s^*}^{\frac{p_s^*(p_s^*-\alpha)}{p_s^*-1}} \le C_{p_s^*}^{p_s^*} [u_n]_{K,p}^{\frac{p_s^*(\alpha-1)}{p_s^*-1}} [v_n]_{K,p}^{\frac{p_s^*(p_s^*-\alpha)}{p_s^*-1}} \le C_{p_s^*}^{p_s^*} [(u_n,v_n)]^{p_s^*}_{K,p} \le C$$

for a suitable constant $C > 0$. Similarly,

$$\int_{\mathbb{R}^N} \big||u_n|^\alpha |v_n|^{\beta-1}\big|^{\frac{p_s^*}{p_s^*-1}}\,dx \le C.$$

Thus, (2.4) holds by [3, Proposition A.8], since $(u_n,v_n) \to (u,v)$ a.e. in $\mathbb{R}^N$ and particular $(u_n,v_n) \rightharpoonup (u,v)$ weakly in $L^{p_s^*}(\mathbb{R}^N)$. □

3 Proof of Theorems 1.1 and 1.2

In this section, we first assume, without further mentioning, that the assumptions required in Theorem 1.1 are satisfied.

We say that the couple $(u,v)\in\mathbf{W}$ is a (weak) *solution* of problem $(\mathcal{S})$ if

$$
\begin{aligned}
&M(\|(u,v)\|^p)\langle (u,v),(\varphi,\psi)\rangle_{K,V,p}-\sigma(\langle u,\varphi\rangle_{H_p}+\langle v,\psi\rangle_{H_p})\\
&=\int_{\mathbb{R}^N}[H_u(x,u,v)\varphi+H_v(x,u,v)\psi]\,dx+\frac{\alpha}{p_s^*}\int_{\mathbb{R}^N}|u|^{\alpha-2}u|v|^{\beta}\varphi\,dx+\frac{\beta}{p_s^*}\int_{\mathbb{R}^N}|u|^{\alpha}|v|^{\beta-2}v\psi\,dx
\end{aligned}
$$

for any $(\varphi,\psi)\in\mathbf{W}$, where

$$
\begin{aligned}
\langle (u,v),(\varphi,\psi)\rangle_{K,V,p}&=\langle u,\varphi\rangle_{K,p}+\langle u,\varphi\rangle_{V,p}+\langle v,\psi\rangle_{K,p}+\langle v,\psi\rangle_{V,p},\\
\langle u,\varphi\rangle_{K,p}&=\iint_{\mathbb{R}^{2N}}|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy,\\
\langle u,\varphi\rangle_{V,p}&=\iint_{\mathbb{R}^{N}}V(x)|u(x)|^{p-2}u(x)\varphi(x)\,dx,\\
\langle u,\varphi\rangle_{H_p}&=\int_{\mathbb{R}^N}|u(x)|^{p-2}u(x)\varphi(x)\frac{dx}{|x|^{ps}}.
\end{aligned}
$$

Clearly, the entire (weak) solutions of $(\mathcal{S})$ are exactly the critical points of the Euler–Lagrange functional $I:\mathbf{W}\to\mathbb{R}$ associated with $(\mathcal{S})$, given for all $(u,v)\in\mathbf{W}$ by

$$
I(u,v)=\frac{1}{p}\mathscr{M}(\|(u,v)\|^p)-\frac{\sigma}{p}(\|u\|_{H_p}^p+\|v\|_{H_p}^p)-\lambda\int_{\mathbb{R}^N}H(x,u,v)\,dx-\frac{1}{p_s^*}\int_{\mathbb{R}^N}|u|^{\alpha}|v|^{\beta}\,dx,
$$

which is well defined and of class $C^1(\mathbf{W})$ by $(\mathcal{H})$ and the continuity of M.

We start by showing that the functional I has the geometric features required to apply the mountain pass theorem of Ambrosetti and Rabinowitz.

Lemma 3.1. *Fix $\sigma\in(-\infty,a\mathcal{H}_p)$ and any $\lambda>0$. Then there exists a couple $(e_1,e_2)\in C_0^\infty(\mathbb{R}^N)\times C_0^\infty(\mathbb{R}^N)$, independent of σ^+ and λ, such that $I(e_1,e_2)<0$, $\|(e_1,e_2)\|\geq 2$ and $\int_{\mathbb{R}^N}|e_1|^{\alpha}|e_2|^{\beta}\,dx>0$. Furthermore, there exist $\jmath=\jmath(\sigma,\lambda)>0$ and $\rho=\rho(\sigma,\lambda)\in(0,1]$ such that $I(u,v)\geq\jmath$ for any $(u,v)\in\mathbf{W}$ with $\|(u,v)\|=\rho$.*

Proof. Fix $\sigma\in(-\infty,a\mathcal{H}_p)$ and $\lambda>0$. Now $(u,v)\in C_0^\infty(\mathbb{R}^N)\times C_0^\infty(\mathbb{R}^N)$ such that $\|(u,v)\|=1$ and

$$
\int_{\mathbb{R}^N}|u|^{\alpha}|v|^{\beta}\,dx>0.
$$

Assumption (M2) implies that

$$
\mathscr{M}(t)\leq\mathscr{M}(1)t^{\theta}\quad\text{for all } t\geq 1. \tag{3.1}
$$

Thus, by $(\mathcal{H})$, (2.3) and (3.1), we have for $t\to\infty$,

$$
\begin{aligned}
I(tu,tv)&=\frac{1}{p}\mathscr{M}(\|t(u,v)\|^p)-\sigma\frac{t^p}{p}(\|u\|_{H_p}^p+\|v\|_{H_p}^p)-\lambda\int_{\mathbb{R}^N}H(x,tu,tv)\,dx-\frac{t^{p_s^*}}{p_s^*}\int_{\mathbb{R}^N}|u|^{\alpha}|v|^{\beta}\,dx\\
&\leq\mathscr{M}(1)\frac{t^{\theta p}}{p}+\sigma^-\frac{t^p}{p}(\|u\|_{H_p}^p+\|v\|_{H_p}^p)-\frac{t^{p_s^*}}{p_s^*}\int_{\mathbb{R}^N}|u|^{\alpha}|v|^{\beta}\,dx\to-\infty,
\end{aligned}\tag{3.2}
$$

since $p\leq\theta p<p_s^*$. Hence, taking $(e_1,e_2)=\tau_0(u,v)$ with $\tau_0>0$ sufficiently large, we obtain at once that $\|(e_1,e_2)\|\geq 2$, $\int_{\mathbb{R}^N}|e_1|^{\alpha}|e_2|^{\beta}\,dx>0$ and $I(e_1,e_2)<0$, as stated.

For the second part, we first note that $(\mathcal{H})$ gives for any $\varepsilon > 0$ the existence of $C_\varepsilon > 0$ such that

$$|H(x,z)| \le \varepsilon |z|^\mu + C_\varepsilon |z|^q \quad \text{for all } (x,z) \in \mathbb{R}^N \times \mathbb{R}^2$$

holds. Hence, (M1), (1.3), Lemma 2.1 and (2.3) imply that for all $(u,v) \in \mathbf{W}$, with $\|(u,v)\| \le 1$,

$$\begin{aligned} I(u,v) &\ge \frac{a}{p}\|(u,v)\|^p - \frac{\sigma^+}{p\mathcal{H}_p}([u]^p_{K,p} + [v]^p_{K,p}) - \lambda \int_{\mathbb{R}^N} \varepsilon(u^2+v^2)^{\frac{\mu}{2}}\,dx - \lambda \int_{\mathbb{R}^N} C_\varepsilon (u^2+v^2)^{\frac{q}{2}}\,dx - \frac{1}{p_s^*}\|u\|^\alpha_{p_s^*}\|v\|^\beta_{p_s^*} \\ &\ge \frac{1}{p}\Big(a - \frac{\sigma^+}{\mathcal{H}_p}\Big)\|(u,v)\|^p - \lambda\varepsilon C^\mu_\mu \|(u,v)\|^\mu - \lambda C_\varepsilon C^q_q \|(u,v)\|^q - \frac{C^{p_s^*}_{p_s^*}}{p_s^*}\|(u,v)\|^{p_s^*}. \end{aligned}$$

Clearly, there exists $\rho \in (0,1]$ such that

$$\max_{t\in[0,1]} y(t) = y(\rho) > 0,$$

where

$$y(t) = \frac{1}{p}\Big(a - \frac{\sigma^+}{\mathcal{H}_p}\Big)t^p - \lambda\varepsilon C^\mu_\mu t^\mu - \lambda C_\varepsilon C^q_q t^q - \frac{C^{p_s^*}_{p_s^*}}{p_s^*} t^{p_s^*},$$

since $p < \mu \le q < p_s^*$ and $\sigma < a\mathcal{H}_p$. Consequently, $I(u,v) \ge y(\rho) = \jmath$ for all $(u,v) \in \mathbf{W}$ with $\|(u,v)\| = \rho$, as desired. This concludes the proof. □

We recall in passing that, if X is a real Banach space, a $C^1(X)$ functional J satisfies the Palais–Smale condition at level $c \in \mathbb{R}$ if any Palais–Smale sequence $\{u_n\}_n$ at level c, that is, such that

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0 \quad \text{in } X' \text{ as } n \to \infty, \tag{3.3}$$

admits a convergent subsequence in X.

Now we discuss the compactness property for the functional I, given by the Palais–Smale condition at a suitable level. For this, we fix $\sigma \in (-\infty, a\mathcal{H}_p)$, $\lambda > 0$ and we set

$$c_{\sigma,\lambda} = \inf_{\xi\in\Gamma}\max_{t\in[0,1]} I(\xi(t)), \tag{3.4}$$

where

$$\Gamma = \{\xi \in C([0,1];\mathbf{W}) : \xi(0) = (0,0),\ I(\xi(1)) < 0\}.$$

Obviously, $c_{\sigma,\lambda} > 0$ thanks to Lemma 3.1, since in particular $\|(e_1,e_2)\| > \rho$. Before proving that I satisfies the Palais–Smale condition at level $c_{\sigma,\lambda}$, we introduce an asymptotic condition for the level $c_{\sigma,\lambda}$. This result was proved in [7, Lemma 2.3] in the scalar case and will be crucial to overcome the lack of compactness due to the presence of Hardy terms and critical nonlinearities.

Lemma 3.2. *For any $\sigma \in (-\infty, a\mathcal{H}_p)$ it results*

$$\lim_{\lambda\to\infty} c_{\sigma,\lambda} = 0.$$

Proof. Fix $\sigma \in (-\infty, a\mathcal{H}_p)$ and $\lambda > 0$. Let (e_1,e_2) be the couple determined in Lemma 3.1, which is independent of σ^+ and λ. Since I satisfies the mountain pass geometry at $(0,0)$ and (e_1,e_2), there exists $t_{\sigma,\lambda} > 0$ verifying $I(t_{\sigma,\lambda}e_1, t_{\sigma,\lambda}e_2) = \max_{t\ge0} I(te_1,te_2)$. Therefore, $\langle I'(t_{\sigma,\lambda}e_1, t_{\sigma,\lambda}e_2),(e_1,e_2)\rangle = 0$. Thus,

$$\begin{aligned} &t^{p-1}_{\sigma,\lambda}(M(\|t_{\sigma,\lambda}(e_1,e_2)\|^p)\|(e_1,e_2)\|^p - \sigma\|e_1\|^p_{H_p} - \sigma\|e_2\|^p_{H_p}) \\ &\quad = \lambda\int_{\mathbb{R}^N} H_u(x,t_{\sigma,\lambda}e_1,t_{\sigma,\lambda}e_2)e_1\,dx + \lambda\int_{\mathbb{R}^N} H_v(x,t_{\sigma,\lambda}e_1,t_{\sigma,\lambda}e_2)e_2\,dx + t^{p_s^*-1}_{\sigma,\lambda}\int_{\mathbb{R}^N}|e_1|^\alpha|e_2|^\beta\,dx \\ &\quad \ge t^{p_s^*-1}_{\sigma,\lambda}\int_{\mathbb{R}^N}|e_1|^\alpha|e_2|^\beta\,dx, \end{aligned} \tag{3.5}$$

by $(\mathcal{H})$, being $\lambda > 0$.

We claim that $\{t_{\sigma,\lambda}\}_\lambda$ is bounded in $\mathbb{R}$. Indeed, putting $\Lambda = \{\lambda > 0 : t_{\sigma,\lambda}\|(e_1, e_2)\| \ge 1\}$, from (M2), (1.3) and (3.1) we derive that

$$\begin{aligned} &t^p_{\sigma,\lambda}(M(\|t_{\sigma,\lambda}(e_1, e_2)\|^p)\|(e_1, e_2)\|^p - \sigma\|e_1\|^p_{H_p} - \sigma\|e_2\|^p_{H_p}) \\ &\le \theta\mathscr{M}(\|t_{\sigma,\lambda}(e_1, e_2)\|^p) + \frac{\sigma^-}{\mathcal{H}_p} t^p_{\sigma,\lambda}([e_1]^p_{K,p} + [e_2]^p_{K,p}) \\ &\le \Big(\theta\mathscr{M}(1) + \frac{\sigma^-}{\mathcal{H}_p}\Big) t^{\theta p}_{\sigma,\lambda}\|(e_1, e_2)\|^{\theta p} \end{aligned} \tag{3.6}$$

for any $\lambda \in \Lambda$, since $1 < p \le \theta p$. Therefore, (3.5) and (3.6) imply that

$$\Big(\theta\mathscr{M}(1) + \frac{\sigma^-}{\mathcal{H}_p}\Big)\|(e_1, e_2)\|^{\theta p} \ge t^{p^*_s - \theta p}_{\sigma,\lambda} \int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta \, dx \quad \text{for any } \lambda \in \Lambda,$$

which yields that $\{t_{\sigma,\lambda}\}_{\lambda\in\Lambda}$ is bounded since $\theta p < p^*_s$ and

$$\int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta \, dx > 0$$

by Lemma 3.1. It follows at once that $\{t_{\sigma,\lambda}\}_{\lambda>0}$ is bounded. This proves the claim.

Fix now a sequence $\{\lambda_n\}_n \subset \mathbb{R}^+$ such that $\lambda_n \to \infty$ as $n \to \infty$. Obviously $\{t_{\sigma,\lambda_n}\}_n$ is bounded. Thus, there exist a $t_0 \ge 0$ and subsequence of $\{\lambda_n\}_n$, still denoted by $\{\lambda_n\}_n$, such that $t_{\sigma,\lambda_n} \to t_0$. By the continuity of M, also $\{M(t^p_{\sigma,\lambda_n}\|(e_1, e_2)\|^p)\}_n$ is bounded, and so by (3.5) there exists $C_{\sigma^-} > 0$ such that, for any $n \in \mathbb{N}$,

$$\lambda_n\Big(\int_{\mathbb{R}^N} H_u(x, t_{\sigma,\lambda_n}e_1, t_{\sigma,\lambda_n}e_2)e_1 \, dx + \int_{\mathbb{R}^N} H_v(x, t_{\sigma,\lambda_n}e_1, t_{\sigma,\lambda_n}e_2)e_2 \, dx\Big) + t^{p^*_s-1}_{\sigma,\lambda_n} \int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta \, dx \le C_{\sigma^-}. \tag{3.7}$$

We assert that $t_0 = 0$. Otherwise, $(\mathcal{H})$ and the dominated convergence theorem yield

$$\begin{aligned} \int_{\mathbb{R}^N} H_u(x, t_{\sigma,\lambda_n}e_1, t_{\sigma,\lambda_n}e_2)e_1 \, dx &\to \int_{\mathbb{R}^N} H_u(x, t_0e_1, t_0e_2)e_1 \, dx, \\ \int_{\mathbb{R}^N} H_v(x, t_{\sigma,\lambda_n}e_1, t_{\sigma,\lambda_n}e_2)e_2 \, dx &\to \int_{\mathbb{R}^N} H_v(x, t_0e_1, t_0e_2)e_2 \, dx \end{aligned}$$

as $n \to \infty$. In particular, as $n \to \infty$

$$\begin{aligned} &\int_{\mathbb{R}^N} (H_u(x, t_{\sigma,\lambda_n}e_1, t_{\sigma,\lambda_n}e_2)e_1 + H_v(x, t_{\sigma,\lambda_n}e_1, t_{\sigma,\lambda_n}e_2)e_2) \, dx \\ &\qquad \to \int_{\mathbb{R}^N} (H_u(x, t_0e_1, t_0e_2)e_1 + H_v(x, t_0e_1, t_0e_2)e_2) \, dx > 0 \end{aligned}$$

by $(\mathcal{H})$ and the fact that $\int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta dx > 0$ as constructed in Lemma 3.1. Recalling that $\lambda_n \to \infty$, we get

$$\lim_{n\to\infty} \Big[\lambda_n\Big(\int_{\mathbb{R}^N} H_u(x, t_{\sigma,\lambda_n}e_1, t_{\sigma,\lambda_n}e_2)e_1 \, dx + \int_{\mathbb{R}^N} H_v(x, t_{\sigma,\lambda_n}e_1, t_{\sigma,\lambda_n}e_2)e_2 \, dx\Big) + t^{p^*_s-1}_{\sigma,\lambda_n} \int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta \, dx\Big] = \infty,$$

which contradicts (3.7). Thus $t_0 = 0$ and $t_{\sigma,\lambda} \to 0$ as $\lambda \to \infty$, since the sequence $\{\lambda_n\}_n$ is arbitrary.

Now the path $\xi(t) = t(e_1, e_2)$, $t \in [0, 1]$, belongs to Γ, so that Lemma 3.1 gives

$$\begin{aligned} 0 < c_{\sigma,\lambda} &\le \max_{t\ge0} I(\xi(t)) \\ &\le I(t_{\sigma,\lambda}e_1, t_{\sigma,\lambda}e_2) \\ &\le \frac{1}{p}\mathscr{M}(\|t_{\sigma,\lambda}(e_1, e_2)\|^p) + \frac{\sigma^-}{p} t^p_{\sigma,\lambda}(\|e_1\|^p_{H_p} + \|e_2\|^p_{H_p}). \end{aligned}$$

Moreover, $\mathscr{M}(\|t_{\sigma,\lambda}(e_1, e_2)\|^p) \to 0$ as $\lambda \to \infty$, by the continuity of $\mathscr{M}$ and the fact that (e_1, e_2) does not depend on λ. This completes the proof of the lemma. □

Now we are ready to prove the compactness property of I at the special level (3.4) and recall that the number $\kappa \in (0, a]$ was defined in (1.5).

Lemma 3.3. *For any $\sigma \in (-\infty, \kappa\mathcal{H}_p)$ there exists $\lambda^* = \lambda^*(\sigma) > 0$ such that for any $\lambda \geq \lambda^*$ the functional I satisfies the Palais–Smale condition at level $c_{\sigma,\lambda}$.*

Proof. Fix $\sigma \in (-\infty, \kappa\mathcal{H}_p)$ and let $\{(u_n, v_n)\}_n \subset \mathbf{W}$ be a Palais–Smale sequence of I at level $c_{\sigma,\lambda}$ for any $\lambda > 0$. By (M2) and ($\mathcal{H}$), we get

$$\begin{aligned}
I(u_n, v_n) &- \frac{1}{\mu}\langle I'(u_n, v_n), (u_n, v_n)\rangle \\
&= \frac{1}{p}\mathscr{M}(\|(u_n, v_n)\|^p) - \frac{1}{\mu}M(\|(u_n, v_n)\|^p)\|(u_n, v_n)\|^p - \sigma\Big(\frac{1}{p} - \frac{1}{\mu}\Big)(\|u_n\|^p_{H_p} + \|v_n\|^p_{H_p}) \\
&\qquad - \lambda \int_{\mathbb{R}^N} \Big(H(x, u_n, v_n) - \frac{1}{\mu}H_u(x, u_n, v_n)u_n - \frac{1}{\mu}H_v(x, u_n, v_n)v_n\Big)\,dx \\
&\qquad + \Big(\frac{1}{\mu} - \frac{1}{p_s^*}\Big)\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta\,dx \\
&\geq \Big(\frac{1}{\theta p} - \frac{1}{\mu}\Big)M(\|(u_n, v_n)\|^p)\|(u_n, v_n)\|^p - \frac{\sigma^+}{\mathcal{H}_p}\Big(\frac{1}{p} - \frac{1}{\mu}\Big)\|(u_n, v_n)\|^p,
\end{aligned} \tag{3.8}$$

since $p \leq \theta p < \mu < p_s^*$. Then, thanks to (1.3), (3.3), (3.8) and (M1), there exists $\gamma_{\sigma,\lambda} > 0$ such that as $n \to \infty$

$$c_{\sigma,\lambda} + \gamma_{\sigma,\lambda}\|(u_n, v_n)\| + o(1) \geq \nu_\sigma\|(u_n, v_n)\|^p, \quad \nu_\sigma = a\Big(\frac{1}{\theta p} - \frac{1}{\mu}\Big) - \frac{\sigma^+}{\mathcal{H}_p}\Big(\frac{1}{p} - \frac{1}{\mu}\Big) > 0, \tag{3.9}$$

since $\sigma < \kappa\mathcal{H}_p$. Therefore $\{(u_n, v_n)\}_n$ is bounded in the reflexive Banach space $\mathbf{W}$.

Thus, there exist $(u_{\sigma,\lambda}, v_{\sigma,\lambda}) \in \mathbf{W}$, nonnegative numbers $\kappa_{\sigma,\lambda}$, $\imath_{\sigma,\lambda}$, $\ell_{\sigma,\lambda}$ and $\delta_{\sigma,\lambda}$, and two functions $g_\mu \in L^\mu(\mathbb{R}^N)$ and $g_q \in L^q(\mathbb{R}^N)$ such that, up to a subsequence, still denoted by $\{(u_n, v_n)\}_n$, we have

$$\begin{gathered}
(u_n, v_n) \rightharpoonup (u_{\sigma,\lambda}, v_{\sigma,\lambda}) \quad \text{in } \mathbf{W}, \qquad \|(u_n, v_n)\| \to \kappa_{\sigma,\lambda}, \\
u_n \rightharpoonup u_{\sigma,\lambda} \quad \text{in } L^p(\mathbb{R}^N, |x|^{-ps}), \qquad \|u_n - u_{\sigma,\lambda}\|_{H_p} \to \imath_{\sigma,\lambda}, \\
v_n \rightharpoonup v_{\sigma,\lambda} \quad \text{in } L^p(\mathbb{R}^N, |x|^{-ps}), \qquad \|v_n - v_{\sigma,\lambda}\|_{H_p} \to \ell_{\sigma,\lambda}, \\
(u_n, v_n) \to (u_{\sigma,\lambda}, v_{\sigma,\lambda}) \quad \text{in } L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N), \qquad (u_n, v_n) \to (u_{\sigma,\lambda}, v_{\sigma,\lambda}) \quad \text{a.e. in } \mathbb{R}^N, \\
|(u_n, v_n)| \leq g_\mu \quad \text{a.e. in } \mathbb{R}^N, \qquad |(u_n, v_n)| \leq g_q \quad \text{a.e. in } \mathbb{R}^N \text{ and all } n \in \mathbb{N}, \\
\int_{\mathbb{R}^N} |u_n - u_{\sigma,\lambda}|^\alpha |v_n - v_{\sigma,\lambda}|^\beta\,dx \to \delta_{\sigma,\lambda}, \\
|u_n|^{\alpha-2}u_n|v_n|^\beta \rightharpoonup |u_{\sigma,\lambda}|^{\alpha-2}u_{\sigma,\lambda}|v_{\sigma,\lambda}|^\beta \quad \text{in } L^{p_s^*/(p_s^*-1)}(\mathbb{R}^N), \\
|u_n|^\alpha|v_n|^{\beta-2}v_n \rightharpoonup |u_{\sigma,\lambda}|^\alpha|v_{\sigma,\lambda}|^{\beta-2}v_{\sigma,\lambda} \quad \text{in } L^{p_s^*/(p_s^*-1)}(\mathbb{R}^N),
\end{gathered} \tag{3.10}$$

with $\nu \in (p, p_s^*)$, by (1.3), (2.3) and Lemmas 2.2 and 2.4.

Turning to (3.3), we have shown that

$$c_{\sigma,\lambda} + o(1) \geq \nu_\sigma\|(u_n, v_n)\|^p + \Big(\frac{1}{\mu} - \frac{1}{p_s^*}\Big)\int_{\mathbb{R}^N} |u_n|^\alpha|v_n|^\beta\,dx, \tag{3.11}$$

where ν_σ is given in (3.9).

First we assert that

$$\lim_{\lambda\to\infty} \kappa_{\sigma,\lambda} = 0. \tag{3.12}$$

Otherwise, $\limsup_{\lambda\to\infty} \kappa_{\sigma,\lambda} = \kappa_\sigma > 0$. Hence there is a sequence $j \mapsto \lambda_j \uparrow \infty$ such that $\kappa_{\sigma,\lambda_j} \to \kappa_\sigma$ as $j \to \infty$. Then, letting $j \to \infty$, we get from (3.11) and Lemma 3.2 that

$$0 \geq \nu_\sigma\kappa_\sigma^p > 0.$$

This contradiction proves the assertion (3.12). Moreover,

$$\|(u_{\sigma,\lambda}, v_{\sigma,\lambda})\| \le \kappa_{\sigma,\lambda},$$

since $(u_n, v_n) \rightharpoonup (u_{\sigma,\lambda}, v_{\sigma,\lambda})$ in $\mathbf{W}$, so that $(\mathcal{V})$, (1.3), (2.3) and (3.12) imply that

$$\lim_{\lambda\to\infty} \|u_{\sigma,\lambda}\|_{H_p} = \lim_{\lambda\to\infty} \|v_{\sigma,\lambda}\|_{H_p} = \lim_{\lambda\to\infty} \int_{\mathbb{R}^N} |u_{\sigma,\lambda}|^\alpha |v_{\sigma,\lambda}|^\beta \, dx = \lim_{\lambda\to\infty} \|(u_{\sigma,\lambda}, v_{\sigma,\lambda})\| = 0. \tag{3.13}$$

Let us prove that $\{(u_n, v_n)\}_n$, up to a possibly further beyond subsequence, converges strongly to $(u_{\sigma,\lambda}, v_{\sigma,\lambda})$ in $\mathbf{W}$. To this end, let $\{\mathcal{U}_n\}_n$ be the sequence defined in $\mathbb{R}^{2N} \setminus \mathrm{Diag}(\mathbb{R}^{2N})$ by

$$(x, y) \mapsto \mathcal{U}_n(x, y) = |u_n(x) - u_n(y)|^{p-2}[u_n(x) - u_n(y)] \cdot K(x-y)^{\frac{1}{p'}}.$$

Then $\{\mathcal{U}_n\}_n$ is bounded in $L^{p'}(\mathbb{R}^{2N})$ by (b), as well as $\mathcal{U}_n \to \mathcal{U}_{\sigma,\lambda}$ a.e. in $\mathbb{R}^{2N}$, where

$$\mathcal{U}_{\sigma,\lambda}(x, y) = |u_{\sigma,\lambda}(x) - u_{\sigma,\lambda}(y)|^{p-2}[u_{\sigma,\lambda}(x) - u_{\sigma,\lambda}(y)] \cdot K(x-y)^{\frac{1}{p'}}.$$

Thus, going if necessary to a further subsequence, we get that $\mathcal{U}_n \rightharpoonup \mathcal{U}_{\sigma,\lambda}$ in $L^{p'}(\mathbb{R}^{2N})$ as $n \to \infty$. Hence,

$$\langle u_n, \varphi\rangle_{K,p} \to \langle u_{\sigma,\lambda}, \varphi\rangle_{K,p} \tag{3.14}$$

for any $\varphi \in W^{s,p}_{K,V}(\mathbb{R}^N)$, since $(x, y) \mapsto |\varphi(x) - \varphi(y)| \cdot K(x-y)^{\frac{1}{p}} \in L^p(\mathbb{R}^{2N})$ by (b). Furthermore, we have that $|u_n|^{p-2}u_n \rightharpoonup |u_{\sigma,\lambda}|^{p-2}u_{\sigma,\lambda}$ in $L^{p'}(\mathbb{R}^N, V)$ by [3, Proposition A.8]. From this,

$$\langle u_n, \varphi\rangle_{V,p} \to \langle u_{\sigma,\lambda}, \varphi\rangle_{V,p} \tag{3.15}$$

for any $\varphi \in W^{s,p}_{K,V}(\mathbb{R}^N)$, since $\varphi \in L^p(\mathbb{R}^N, V)$. In the same way, (3.10) and [3, Proposition A.8] imply that $|u_n|^{p-2}u_n \rightharpoonup |u_{\sigma,\lambda}|^{p-2}u_{\sigma,\lambda}$ in $L^{p'}(\mathbb{R}^N, |x|^{-ps})$ as $n \to \infty$. Consequently,

$$\langle u_n, \varphi\rangle_{H_p} \to \langle u_{\sigma,\lambda}, \varphi\rangle_{H_p} \tag{3.16}$$

for any $\varphi \in W^{s,p}_{K,V}(\mathbb{R}^N)$. A similar argument shows that the sequence $\{\mathcal{V}_n\}_n$, defined in $\mathbb{R}^{2N} \setminus \mathrm{Diag}(\mathbb{R}^{2N})$ by

$$(x, y) \mapsto \mathcal{V}_n(x, y) = |v_n(x) - v_n(y)|^{p-2}[v_n(x) - v_n(y)] \cdot K(x-y)^{\frac{1}{p'}},$$

is bounded in $L^{p'}(\mathbb{R}^{2N})$ as well as $\mathcal{V}_n \to \mathcal{V}_{\sigma,\lambda}$ a.e. in $\mathbb{R}^{2N}$, where

$$\mathcal{V}_{\sigma,\lambda}(x, y) = |v_{\sigma,\lambda}(x) - v_{\sigma,\lambda}(y)|^{p-2}[v_{\sigma,\lambda}(x) - v_{\sigma,\lambda}(y)] \cdot K(x-y)^{\frac{1}{p'}}.$$

Hence, going if necessary to a further subsequence, we have

$$\langle v_n, \psi\rangle_{K,p} \to \langle v_{\sigma,\lambda}, \psi\rangle_{K,p}, \quad \langle v_n, \psi\rangle_{V,p} \to \langle v_{\sigma,\lambda}, \psi\rangle_{V,p}, \quad \langle v_n, \psi\rangle_{H_p} \to \langle v_{\sigma,\lambda}, \psi\rangle_{H_p} \tag{3.17}$$

for all $\psi \in W^{s,p}_{K,V}(\mathbb{R}^N)$.

By $(\mathcal{H})$, with $\varepsilon = 1$, and (3.10), the Hölder inequality gives

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} [H_u(x, u_n, v_n)(u_n - u_{\sigma,\lambda}) + H_v(x, u_n, v_n)(v_n - u_{\sigma,\lambda})] \, dx \right| \\
&\quad \le \int_{\mathbb{R}^N} [\mu |(u_n, v_n)|^{\mu-1} |(u_n, v_n) - (u_{\sigma,\lambda}, v_{\sigma,\lambda})| + qC_1 |(u_n, v_n)|^{q-1} |(u_n, v_n) - (u_{\sigma,\lambda}, v_{\sigma,\lambda})|] \, dx \\
&\quad \le C(\|(u_n, v_n) - (u_{\sigma,\lambda}, v_{\sigma,\lambda})\|_\mu + \|(u_n, v_n) - (u_{\sigma,\lambda}, v_{\sigma,\lambda})\|_q) \to 0
\end{aligned} \tag{3.18}$$

as $n \to \infty$, by Lemma 2.2 since $p \le \theta p < \mu \le q < p_s^*$, for a suitable constant $C > 0$. While $(\mathcal{H})$ and the use of the dominated convergence theorem yield for any $(\varphi, \psi) \in \mathbf{W}$,

$$\begin{aligned}
\int_{\mathbb{R}^N} H_u(x, u_n, v_n)\varphi \, dx &\to \int_{\mathbb{R}^N} H_u(x, u_{\sigma,\lambda}, v_{\sigma,\lambda})\varphi \, dx, \\
\int_{\mathbb{R}^N} H_v(x, u_n, v_n)\psi \, dx &\to \int_{\mathbb{R}^N} H_v(x, u_{\sigma,\lambda}, v_{\sigma,\lambda})\psi \, dx
\end{aligned} \tag{3.19}$$

as $n \to \infty$. Consequently, (3.3), (3.10), (3.14)–(3.17) and (3.19) give at once that $(u_{\sigma,\lambda}, v_{\sigma,\lambda})$ satisfies the identity

$$\begin{aligned} M(\kappa_{\sigma,\lambda})\langle(u_{\sigma,\lambda}, v_{\sigma,\lambda}), (\varphi, \psi)\rangle_{K,V,p} - \sigma(\langle u_{\sigma,\lambda}, \varphi\rangle_{H_p} + \langle v_{\sigma,\lambda}, \psi\rangle_{H_p}) \\ = \int_{\mathbb{R}^N} [H_u(x, u_{\sigma,\lambda}, v_{\sigma,\lambda})\varphi + H_v(x, u_{\sigma,\lambda}, v_{\sigma,\lambda})\psi]\, dx + \frac{\alpha}{p_s^*}\int_{\mathbb{R}^N} |u_{\sigma,\lambda}|^{\alpha-2}u_{\sigma,\lambda}|v_{\sigma,\lambda}|^{\beta}\varphi\, dx \\ + \frac{\beta}{p_s^*}\int_{\mathbb{R}^N} |u_{\sigma,\lambda}|^{\alpha}|v_{\sigma,\lambda}|^{\beta-2}v_{\sigma,\lambda}\psi\, dx \end{aligned}$$

for any $(\varphi, \psi) \in \mathbf{W}$. In other words, $(u_{\sigma,\lambda}, v_{\sigma,\lambda})$ is a critical point of the $C^1(\mathbf{W})$ functional

$$I_{\kappa_{\sigma,\lambda}}(u, v) = \frac{1}{p}M(\kappa_{\sigma,\lambda}^p)\|(u, v)\|^p - \frac{\sigma}{p}(\|u\|_{H_p}^p + \|v\|_{H_p}^p) - \lambda\int_{\mathbb{R}^N} H(x, u, v)\, dx - \frac{1}{p_s^*}\int_{\mathbb{R}^N} |u|^{\alpha}|v|^{\beta}\, dx. \tag{3.20}$$

From (3.3), (3.10), (3.14)–(3.18) and (3.20) we deduce that

$$\begin{aligned} o(1) &= \langle I'(u_n, v_n) - I'_{\kappa_{\sigma,\lambda}}(u_{\sigma,\lambda}, v_{\sigma,\lambda}), (u_n, v_n) - (u_{\sigma,\lambda}, v_{\sigma,\lambda})\rangle \\ &= M(\|(u_n, v_n)\|^p)\|(u_n, v_n)\|^p - M(\|(u_n, v_n)\|^p)\langle(u_n, v_n), (u_{\sigma,\lambda}, v_{\sigma,\lambda})\rangle_{K,V,p} \\ &\quad - M(\kappa_{\sigma,\lambda}^p)\langle(u_{\sigma,\lambda}, v_{\sigma,\lambda}), (u_n, v_n)\rangle_{K,V,p} + M(\kappa_{\sigma,\lambda}^p)\|(u_{\sigma,\lambda}, v_{\sigma,\lambda})\|^p \\ &\quad - \sigma\int_{\mathbb{R}^N} \frac{(|u_n|^{p-2}u_n - |u_{\sigma,\lambda}|^{p-2}u_{\sigma,\lambda})(u_n - u_{\sigma,\lambda})}{|x|^{ps}}\, dx \\ &\quad - \sigma\int_{\mathbb{R}^N} \frac{(|v_n|^{p-2}v_n - |v_{\sigma,\lambda}|^{p-2}v_{\sigma,\lambda})(v_n - v_{\sigma,\lambda})}{|x|^{ps}}\, dx \\ &\quad - \lambda\int_{\mathbb{R}^N} [H_u(x, u_n, v_n) - H_u(x, u_{\sigma,\lambda}, v_{\sigma,\lambda})](u_n - u_{\sigma,\lambda})\, dx \\ &\quad - \lambda\int_{\mathbb{R}^N} [H_v(x, u_n, v_n) - H_v(x, u_{\sigma,\lambda}, v_{\sigma,\lambda})](v_n - v_{\sigma,\lambda})\, dx \\ &\quad - \frac{\alpha}{p_s^*}\int_{\mathbb{R}^N} (|v_n|^{\beta}|u_n|^{\alpha-2}u_n - |v_{\sigma,\lambda}|^{\beta}|u_{\sigma,\lambda}|^{\alpha-2}u_{\sigma,\lambda})(u_n - u_{\sigma,\lambda})\, dx \\ &\quad - \frac{\beta}{p_s^*}\int_{\mathbb{R}^N} (|u_n|^{\alpha}|v_n|^{\beta-2}v_n - |u_{\sigma,\lambda}|^{\alpha}|v_{\sigma,\lambda}|^{\beta-2}v_{\sigma,\lambda})(v_n - v_{\sigma,\lambda})\, dx \\ &= M(\kappa_{\sigma,\lambda}^p)[\kappa_{\sigma,\lambda}^p - \|(u_{\sigma,\lambda}, v_{\sigma,\lambda})\|^p] - \sigma[\|u_n\|_{H_p}^p + \|v_n\|_{H_p}^p - \|u_{\sigma,\lambda}\|_{H_p}^p - \|u_{\sigma,\lambda}\|_{H_p}^p] \\ &\quad - \int_{\mathbb{R}^N} |u_n|^{\alpha}|v_n|^{\beta}\, dx + \int_{\mathbb{R}^N} |u_{\sigma,\lambda}|^{\alpha}|v_{\sigma,\lambda}|^{\beta}\, dx + o(1), \end{aligned}$$

by continuity of M, since $\|(u_n, v_n)\| \to \kappa_{\sigma,\lambda}$ as $n \to \infty$, and $\alpha + \beta = p_s^*$. Therefore, as $n \to \infty$,

$$\begin{aligned} o(1) = M(\kappa_{\sigma,\lambda}^p)[\kappa_{\sigma,\lambda}^p - \|(u_{\sigma,\lambda}, v_{\sigma,\lambda})\|^p] - \sigma[\|u_n\|_{H_p}^p + \|v_n\|_{H_p}^p - \|u_{\sigma,\lambda}\|_{H_p}^p - \|u_{\sigma,\lambda}\|_{H_p}^p] \\ - \int_{\mathbb{R}^N} |u_n|^{\alpha}|v_n|^{\beta}\, dx + \int_{\mathbb{R}^N} |u_{\sigma,\lambda}|^{\alpha}|v_{\sigma,\lambda}|^{\beta}\, dx + o(1). \end{aligned} \tag{3.21}$$

Furthermore, (3.10) and the celebrated Brézis and Lieb lemma of [5] give

$$\|u_n\|_{H_p}^p = \|u_n - u_{\sigma,\lambda}\|_{H_p}^p + \|u_{\sigma,\lambda}\|_{H_p}^p + o(1), \quad \|v_n\|_{H_p}^p = \|v_n - v_{\sigma,\lambda}\|_{H_p}^p + \|v_{\sigma,\lambda}\|_{H_p}^p + o(1) \tag{3.22}$$

as $n \to \infty$, while again (3.10) and Lemma 2.3 yield

$$\|(u_n, v_n)\|^p = \|(u_n, v_n) - (u_{\sigma,\lambda}, v_{\sigma,\lambda})\|^p + \|(u_{\sigma,\lambda}, v_{\sigma,\lambda})\|^p + o(1) \tag{3.23}$$

as $n \to \infty$. Hence, from (3.10), (3.21), (3.22), (3.23) and Lemma 2.4 we obtain

$$\begin{aligned} M(\kappa^p_{\sigma,\lambda}) \lim_{n\to\infty} \|(u_n, v_n) - (u_{\sigma,\lambda}, v_{\sigma,\lambda})\|^p &= \sigma \lim_{n\to\infty} (\|u_n - u_{\sigma,\lambda}\|^p_{H_p} + \|v_n - v_{\sigma,\lambda}\|^p_{H_p}) \\ &\quad + \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n - u_{\sigma,\lambda}|^\alpha |v_n - v_{\sigma,\lambda}|^\beta \, dx \\ &= \sigma(\imath^p_{\sigma,\lambda} + \ell^p_{\sigma,\lambda}) + \delta_{\sigma,\lambda} \end{aligned} \tag{3.24}$$

as $n \to \infty$. By (3.11) and Lemma 2.4 again, we get as $n \to \infty$,

$$c_{\sigma,\lambda} + o(1) \geq \left(\frac{1}{\mu} - \frac{1}{p^*_s}\right)\left[\delta_{\sigma,\lambda} + \int_{\mathbb{R}^N} |u_{\sigma,\lambda}|^\alpha |v_{\sigma,\lambda}|^\beta \, dx\right] + o(1).$$

Then Lemma 3.2 and (3.13) imply that

$$\lim_{\lambda\to\infty} \delta_{\sigma,\lambda} = 0.$$

Since $\sigma < \kappa\mathcal{H}_p \leq a\mathcal{H}_p$ there exists $c \in [0, 1)$ such that $\sigma^+ = ca\mathcal{H}_p$. Of course, (3.24) can be rewritten as

$$(1-c)M(\kappa^p_{\sigma,\lambda}) \lim_{n\to\infty} \|(u_n - u_{\sigma,\lambda}, v_n - v_{\sigma,\lambda})\|^p + cM(\kappa^p_{\sigma,\lambda}) \lim_{n\to\infty} \|(u_n - u_{\sigma,\lambda}, v_n - v_{\sigma,\lambda})\|^p = \sigma(\imath^p_{\sigma,\lambda} + \ell^p_{\sigma,\lambda}) + \delta_{\sigma,\lambda}.$$

Thus, by (1.3) combined with (1.1) and $(\mathcal{V})$, using (2.3) with $(u, v) = (u_n - u_{\sigma,\lambda}, v_n - v_{\sigma,\lambda})$ and (M1), we get

$$\delta_{\sigma,\lambda} + \sigma^+(\imath^p_{\sigma,\lambda} + \ell^p_{\sigma,\lambda}) \geq (1-c)aC^{-p^*_s}_{p^*_s} \delta^{\frac{p}{p^*_s}}_{\sigma,\lambda} + ca\mathcal{H}_p(\imath^p_{\sigma,\lambda} + \ell^p_{\sigma,\lambda})$$

for all $\lambda > 0$, being $c \in [0, 1)$. Therefore, since $\sigma^+ = ca\mathcal{H}_p$,

$$\delta_{\sigma,\lambda} \geq (1-c)aC^{-p^*_s}_{p^*_s} \delta^{\frac{p}{p^*_s}}_{\sigma,\lambda}. \tag{3.25}$$

Consequently, (3.13) and (3.25) imply at once that there exists $\lambda^* = \lambda^*(\sigma) > 0$ such that $\delta_{\sigma,\lambda} = 0$ for all $\lambda \geq \lambda^*$. In other words,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n - u_{\sigma,\lambda}|^\alpha |v_n - v_{\sigma,\lambda}|^\beta \, dx = 0$$

for all $\lambda \geq \lambda^*$.

Now, assume by contradiction that there exists $\lambda \geq \lambda^*$ such that $\imath_{\sigma,\lambda} + \ell_{\sigma,\lambda} > 0$. Then, by (M1) and (1.3), since $\sigma < a\mathcal{H}_p$, we have

$$\begin{aligned} M(\kappa^p_{\sigma,\lambda}) \lim_{n\to\infty} \|(u_n - u_{\sigma,\lambda}, v_n - v_{\sigma,\lambda})\|^p &\leq \sigma \lim_{n\to\infty} (\|u_n - u_{\sigma,\lambda}\|^p_{H_p} + \|v_n - v_{\sigma,\lambda}\|^p_{H_p}) \\ &< a\mathcal{H}_p \lim_{n\to\infty} (\|u_n - u_{\sigma,\lambda}\|^p_{H_p} + \|v_n - v_{\sigma,\lambda}\|^p_{H_p}) \\ &\leq M(\kappa^p_{\sigma,\lambda}) \lim_{n\to\infty} \|(u_n - u_{\sigma,\lambda}, v_n - v_{\sigma,\lambda})\|^p, \end{aligned}$$

which gives a contradiction.

Thus, $\imath_{\sigma,\lambda} + \ell_{\sigma,\lambda} = 0$ for all $\lambda \geq \lambda^*$. By (3.24) this yields

$$\lim_{n\to\infty} \|(u_n, v_n) - (u_{\sigma,\lambda}, v_{\sigma,\lambda})\| = 0$$

thanks to (M1). In conclusion, $(u_n, v_n) \to (u_{\sigma,\lambda}, v_{\sigma,\lambda})$ as $n \to \infty$ in $\mathbf{W}$, as required. □

Proof of Theorem 1.1. Lemmas 3.1 and 3.3 guarantee that for any $\sigma \in (-\infty, \kappa\mathcal{H}_p)$ there exists $\lambda^* = \lambda^*(\sigma)$ such that for any $\lambda \geq \lambda^*$, the functional I satisfies all assumptions of the mountain pass theorem at level $c_{\sigma,\lambda}$. Hence, there exists a critical point $(u_{\sigma,\lambda}, v_{\sigma,\lambda}) \in \mathbf{W}$ of I at level $c_{\sigma,\lambda}$. Clearly, we have $(u_{\sigma,\lambda}, v_{\sigma,\lambda}) \neq (0, 0)$, since $I(u_{\sigma,\lambda}, v_{\sigma,\lambda}) = c_{\sigma,\lambda} > 0 = I(0, 0)$. Moreover, the asymptotic behavior (1.6) is a direct consequence of (3.13). □

We conclude this section proving the non-degenerate result stated in Theorem 1.2. For this, we need a truncation argument, as in [1, 18], in order to control the growth of the elliptic part of $(\mathcal{S})$. From here until the end of the section we assume that the hypotheses of Theorem 1.2 are satisfied.

Proof of Theorem 1.2. Take $\epsilon \in \mathbb{R}$ with $0 < a \le M(0) < \epsilon < \frac{a\mu}{p}$, which is possible by (1.7). Put for all $t \in \mathbb{R}_0^+$,

$$M_\epsilon(t) = \begin{cases} M(t), & \text{if } M(t) \le \epsilon, \\ \epsilon, & \text{if } M(t) > \epsilon, \end{cases} \qquad \text{so that} \qquad M_\epsilon(0) = M(0), \quad \min_{t\in\mathbb{R}_0^+} M_\epsilon(t) = a. \tag{3.26}$$

Let us consider the following auxiliary system in $\mathbb{R}^N$:

$$\begin{cases} M_\epsilon(\|(u,v)\|^p)(\mathcal{L}_p^s u + V(x)|u|^{p-2}u) - \sigma \dfrac{|u|^{p-2}u}{|x|^{ps}} = \lambda H_u(x,u,v) + \dfrac{\alpha}{p_s^*}|v|^\beta |u|^{\alpha-2}u, \\ M_\epsilon(\|(u,v)\|^p)(\mathcal{L}_p^s v + V(x)|v|^{p-2}v) - \sigma \dfrac{|v|^{p-2}v}{|x|^{ps}} = \lambda H_v(x,u,v) + \dfrac{\beta}{p_s^*}|u|^\alpha |v|^{\beta-2}v. \end{cases} \tag{3.27}$$

We are going to solve (3.27), using a mountain pass argument as done in the proof of Theorem 1.1, but replacing the Kirchhoff function M with M_ϵ.

To this end, fix $\lambda > 0$ and $\sigma \in (-\infty, \kappa_\epsilon \mathcal{H}_p)$, where κ_ϵ is given in (1.8). Clearly, (3.27) can be thought as the Euler–Lagrange system of the $C^1(\mathbf{W})$ functional

$$I_\epsilon(u,v) = \frac{1}{p}\mathscr{M}_\epsilon(\|(u,v)\|^p) - \frac{\sigma}{p}(\|u\|_{H_p}^p + \|v\|_{H_p}^p) - \lambda \int_{\mathbb{R}^N} H(x,u,v)\,dx - \frac{1}{p_s^*}\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta\,dx$$

for all $(u,v) \in \mathbf{W}$, where $\mathscr{M}_\epsilon$ denotes the primitive of M_ϵ. First let us note that for the functional I_ϵ Lemmas 3.1 and 3.2 continue to hold. Indeed, for Lemma 3.1 it is enough to observe that (3.2) is now replaced by

$$\begin{aligned} I_\epsilon(tu,tv) &= \frac{1}{p}\mathscr{M}_\epsilon(\|t(u,v)\|^p) - \sigma\frac{t^p}{p}(\|u\|_{H_p}^p + \|v\|_{H_p}^p) - \lambda\int_{\mathbb{R}^N} H(x,tu,tv)\,dx - \frac{t^{p_s^*}}{p_s^*}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,dx \\ &\le \epsilon\frac{t^p}{p} + \sigma^-\frac{t^p}{p}(\|u\|_{H_p}^p + \|v\|_{H_p}^p) - \frac{t^{p_s^*}}{p_s^*}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,dx \to -\infty \end{aligned}$$

as $t \to \infty$, thanks to the definition (3.26) and the fact that $p < p_s^*$. Similarly, also Lemma 3.2 can be proved in a simpler way, by observing that by (3.26), now (3.6) becomes

$$t_{\sigma,\lambda}^p(M_\epsilon(\|t_{\sigma,\lambda}(e_1,e_2)\|^p)\|(e_1,e_2)\|^p - \sigma\|e_1\|_{H_p}^p - \sigma\|e_2\|_{H_p}^p) \le \Big(\epsilon + \frac{\sigma^-}{\mathcal{H}_p}\Big)t_{\sigma,\lambda}^p\|(e_1,e_2)\|^p$$

for any $\lambda \in \Lambda$. Therefore, by using also (3.5), we get

$$\Big(\epsilon + \frac{\sigma^-}{\mathcal{H}_p}\Big)\|(e_1,e_2)\|^p \ge t_{\sigma,\lambda}^{p_s^*-p}\int_{\mathbb{R}^N}|e_1|^\alpha|e_2|^\beta\,dx \quad \text{for any } \lambda \in \Lambda,$$

which yields that $\{t_{\sigma,\lambda}\}_{\lambda\in\Lambda}$ is bounded since $p < p_s^*$ and $\int_{\mathbb{R}^N}|e_1|^\alpha|e_2|^\beta\,dx > 0$ by Lemma 3.1. It follows at once that $\{t_{\sigma,\lambda}\}_{\lambda>0}$ is bounded. The rest of the proof is unchanged. Hence Lemmas 3.1 and 3.2 are valid for I_ϵ and it remains to prove for I_ϵ the main Lemma 3.3.

Fix a Palais–Smale sequence $\{(u_n,v_n)\}_n \in \mathbf{W}$ of I_ϵ at level $c_{\sigma,\lambda}$. Proceeding as in the proof of Lemma 3.3, by (M1) and (3.26) now (3.9) becomes

$$c_{\sigma,\lambda} + \gamma_{\sigma,\lambda}\|(u_n,v_n)\| + o(1) \ge \nu_{\epsilon,\sigma}\|(u_n,v_n)\|^p \quad \text{with} \quad \nu_{\epsilon,\sigma} = \frac{a}{p} - \frac{\epsilon}{\mu} - \frac{\sigma^+}{\mathcal{H}_p}\Big(\frac{1}{p} - \frac{1}{\mu}\Big) > 0,$$

by (1.8), since $\epsilon < \frac{a\mu}{p}$. Consequently, we get that $\{(u_n,v_n)\}_n$ is bounded in $\mathbf{W}$ and so we derive again (3.10). Therefore,

$$c_{\sigma,\lambda} + o(1) \ge \nu_{\epsilon,\sigma}\|(u_n,v_n)\|^p + \Big(\frac{1}{\mu} - \frac{1}{p_s^*}\Big)\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta\,dx. \tag{3.28}$$

While the other key formulas hold true with no relevant modifications, just considering that $p < \mu \le q < p_s^*$ and $\sigma < \kappa_\epsilon\mathcal{H}_p < a\mathcal{H}_p$. Thus, arguing as before, we find that the sequence $\{(u_n,v_n)\}_n$, up to a subsequence, still denoted by $\{(u_n,v_n)\}_n$, strongly converges in $\mathbf{W}$ to some $(u_{\sigma,\lambda},v_{\sigma,\lambda}) \in \mathbf{W}$ for all λ sufficiently large.

In conclusion, we have shown that for any $\epsilon \in (M(0), \frac{a\mu}{p})$ and any $\sigma \in (-\infty, \kappa_\epsilon \mathcal{H}_p)$ there exists a suitable $\lambda_0 = \lambda_0(\epsilon, \sigma) > 0$ such that system (3.27) admits a nontrivial solution $(u_{\sigma,\lambda}, v_{\sigma,\lambda}) \in \mathbf{W}$ with $I_\epsilon(u_{\sigma,\lambda}, v_{\sigma,\lambda}) = c_{\sigma,\lambda}$. Hence, (3.28) implies that for all $\lambda \geq \lambda_0$,

$$c_{\sigma,\lambda} \geq \nu_{\epsilon,\sigma} \|(u_{\sigma,\lambda}, v_{\sigma,\lambda})\|^p \quad \text{with} \quad \nu_{\epsilon,\sigma} > 0,$$

so that (1.6) follows at once by Lemma 3.2.

Fix $\epsilon \in (M(0), \frac{a\mu}{p})$ and $\sigma \in (-\infty, \kappa_\epsilon \mathcal{H}_p)$. By (1.6) and the continuity of M,

$$a \leq M(0) = M_\epsilon(0) = \lim_{\substack{\lambda\to\infty \\ \lambda \geq \lambda_0}} M_\epsilon(\|(u_{\sigma,\lambda}, v_{\sigma,\lambda})\|^p).$$

Therefore, there exists $\lambda^* = \lambda^*(\epsilon, \sigma) \geq \lambda_0$ such that

$$a \leq M_\epsilon(\|(u_{\sigma,\lambda}, v_{\sigma,\lambda})\|^p) < \epsilon \quad \text{for all } \lambda \geq \lambda^*.$$

In closing, for any $\epsilon \in (M(0), \frac{a\mu}{p})$ and for any $\sigma \in (-\infty, \kappa_\epsilon \mathcal{H}_p)$ there exists a threshold $\lambda^* = \lambda^*(\epsilon, \sigma) > 0$ such that for any $\lambda \geq \lambda^*$ the mountain pass solution $(u_{\sigma,\lambda}, v_{\sigma,\lambda})$ of (3.27) is also a solution of system $(\mathcal{S})$. □

4 Proof of Theorem 1.3

In this section we assume that the hypotheses of Theorem 1.3 are fulfilled. System $(\mathcal{S}')$ has a variational structure and the underlying functional is $\mathcal{J} : \mathbf{W} \to \mathbb{R}$, given by

$$\mathcal{J}(u, v) = \frac{1}{p}\mathscr{M}(\|(u, v)\|^p) - \frac{\sigma}{p}(\|u\|^p_{H_p} + \|v\|^p_{H_p}) - \int_{\mathbb{R}^N} h(x) f(u, v)\, dx - \frac{\gamma}{p_s^*}\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta\, dx.$$

Clearly, $(\mathcal{V})$, (f1), (h) and the continuity of M imply that $\mathcal{J}$ is of class $C^1(\mathbf{W})$. We first show that $\mathcal{J}$ has a useful geometrical profile and recall that $\sigma \in (-\infty, a\mathcal{H}_p)$ and that, when $\gamma > 0$, we require also (1.9) on h, that is, h may depend on σ^+.

Lemma 4.1. *Let $\sigma \in (-\infty, a\mathcal{H}_p)$ and $\gamma \leq 1$. Then there exist positive numbers ρ_0 and $\jmath$ such that $\mathcal{J}(u, v) \geq \jmath$ for any $(u, v) \in \mathbf{W}$ with $\|(u, v)\| = \rho_0$. Moreover,*

$$m_{\sigma,\gamma} = \inf_{(u,v)\in\overline{B}_{\rho_0}} \mathcal{J}(u, v) < 0,$$

where $B_{\rho_0} = \{(u, v) \in \mathbf{W} : \|(u, v)\| < \rho_0\}$.

Proof. Fix $\sigma \in (-\infty, a\mathcal{H}_p)$ and $\gamma \leq 1$. By (M1), (f1), (h), (1.3), Lemma 2.1 and (2.3) we obtain for all $(u, v) \in \mathbf{W}$,

$$\begin{aligned} \mathcal{J}(u, v) &\geq \frac{a}{p}\|(u, v)\|^p - \frac{\sigma^+}{p\mathcal{H}_p}([u]^p_{K,p} + [v]^p_{K,p}) - C\int_{\mathbb{R}^N} h(x)|(u, v)|^q\, dx - \frac{\gamma^+}{p_s^*}\|u\|^\alpha_{p_s^*}\|v\|^\beta_{p_s^*} \\ &\geq \frac{1}{p}\Big(a - \frac{\sigma^+}{\mathcal{H}_p}\Big)\|(u, v)\|^p - CC^q_{p_s^*}\|h\|_{\frac{p_s^*}{p_s^*-q}}\|(u, v)\|^q - \frac{\gamma^+}{p_s^*}C^{p_s^*}_{p_s^*}\|(u, v)\|^{p_s^*}. \end{aligned} \tag{4.1}$$

Therefore, if $\gamma \leq 0$, for $\rho_0 > 0$ sufficiently large we have

$$\mathcal{J}(u, v) \geq \rho_0^q\Big[\frac{1}{p}\Big(a - \frac{\sigma^+}{\mathcal{H}_p}\Big)\rho_0^{p-q} - CC^q_{p_s^*}\|h\|_{\frac{p_s^*}{p_s^*-q}}\Big] = \jmath > 0$$

for all $(u, v) \in \mathbf{W}$ with $\|(u, v)\| = \rho_0$, since $1 < q < p$.

If $\gamma \in (0, 1]$, then the Young inequality yields for any $\varepsilon > 0$,

$$CC^q_{p_s^*}\|h\|_{\frac{p_s^*}{p_s^*-q}}\|(u, v)\|^q \leq \varepsilon\|(u, v)\|^p + \varepsilon^{-\frac{q}{p-q}}\Big(CC^q_{p_s^*}\|h\|_{\frac{p_s^*}{p_s^*-q}}\Big)^{\frac{p}{p-q}},$$

since $1 < q < p$. Thus, for $\varepsilon = \frac{a\mathcal{H}_p - \sigma^+}{2p\mathcal{H}_p} > 0$ inequality (4.1) implies that

$$\mathcal{J}(u, v) \geq \varepsilon\|(u, v)\|^p - \varepsilon^{\frac{q}{q-p}}\Big(CC^q_{p_s^*}\|h\|_{\frac{p_s^*}{p_s^*-q}}\Big)^{\frac{p}{p-q}} - \frac{C^{p_s^*}_{p_s^*}}{p_s^*}\|(u, v)\|^{p_s^*},$$

since $0 < \gamma \le 1$. Let us consider the function

$$\eta(t) = \varepsilon t^p - \frac{C_{p_s^*}^{p_s^*}}{p_s^*} t^{p_s^*}, \quad t \ge 0.$$

Since $1 < p < p_s^*$ the number

$$\rho_0 = \left[\frac{a\mathcal{H}_p - \sigma^+}{2\mathcal{H}_p C_{p_s^*}^{p_s^*}}\right]^{\frac{1}{p_s^*-p}} > 0$$

is such that

$$\eta(\rho_0) = \max_{t\ge0} \eta(t) = \left(\frac{1}{2p} - \frac{1}{2p_s^*}\right)\left(a - \frac{\sigma^+}{\mathcal{H}_p}\right)^{\frac{p_s^*}{p_s^*-p}} (2C_{p_s^*}^{p_s^*})^{\frac{p}{p-p_s^*}} > 0.$$

Therefore, since h satisfies (1.9), we obtain for all $(u, v) \in \mathbf{W}$ with $\|(u, v)\| = \rho_0$,

$$\mathcal{I}(u, v) \ge \eta(\rho_0) - \left[\frac{1}{2p}\left(a - \frac{\sigma^+}{\mathcal{H}_p}\right)\right]^{\frac{q}{q-p}} \left(CC_{p_s^*}^q \|h\|_{\frac{p_s^*}{p_s^*-q}}\right)^{\frac{p}{p-q}} > 0,$$

which concludes the proof of the first part of the lemma.

Let $x_0 \in \Omega$ and let $r \in (0, 1)$ be sufficiently small so that $B(x_0, 2r) \subset \Omega$, where Ω is given in (h). Choose φ and ψ in $C_0^\infty(B(x_0, 2r))$ such that $0 \le \varphi \le \frac{1}{2}$ and $0 \le \psi \le \frac{1}{2}$ with $\|(\varphi, \psi)\| \le \rho_0$ and $\int_{B(x_0,2r)} |(\varphi, \psi)|^{q_1}\, dx > 0$. Let $\delta > 0$ be the number given in (f2). For all $t \in (0, \delta)$ then (f2), (h) and the continuity of M yield

$$\begin{aligned}\mathcal{I}(t\varphi, t\psi) &\le \frac{1}{p}\mathscr{M}(\|t(\varphi, \psi)\|^p) + \sigma^- \frac{t^p}{p}(\|\varphi\|_{H_p}^p + \|\psi\|_{H_p}^p) - \int_\Omega h(x)f(t\varphi, t\psi)\, dx - \gamma \frac{t^{p_s^*}}{p_s^*}\int_\Omega |\varphi|^\alpha|\psi|^\beta\, dx \\ &\le \frac{t^p}{p}\Big(\max_{\xi\in[0,(\delta\rho_0)^p]} M(\xi) + \frac{\sigma^-}{\mathcal{H}_p}\Big)\rho_0^p - t^{q_1} a_0 \inf_{x\in\Omega} h(x) \int_{B(x_0,2r)} |(\varphi, \psi)|^{q_1}\, dx + \gamma^- \frac{t^{p_s^*}}{p_s^*}\int_{B(x_0,2r)} |\varphi|^\alpha|\psi|^\beta\, dx.\end{aligned}$$

Hence, $\mathcal{I}(t\varphi, t\psi) < 0$ for $t \in (0, \delta)$ sufficiently small, since $1 < q_1 < p < p_s^*$ by (f2). This shows that $m_{\sigma,\gamma} < 0$ and completes the proof. □

By Lemma 4.1 and the Ekeland variational principle, there exists a sequence $(u_n, v_n) \subset B_{\rho_0}$ such that

$$m_{\sigma,\gamma} \le \mathcal{I}(u_n, v_n) \le m_{\sigma,\gamma} + \frac{1}{n} \quad \text{and} \quad \mathcal{I}(u, v) \ge \mathcal{I}(u_n, v_n) - \frac{1}{n}\|(u, v) - (u_n, v_n)\| \tag{4.2}$$

for all $(u, v) \in \overline{B}_{\rho_0}$. Then a standard procedure gives that $\{(u_n, v_n)\}_n$ is a Palais–Smale sequence of $\mathcal{I}$ at level $m_{\sigma,\gamma}$.

Lemma 4.2. *Let $\sigma \in (-\infty, a\mathcal{H}_p)$. Then there exists $\gamma^{**} = \gamma^{**}(\sigma) \in (0, 1]$ such that, up to a subsequence, $\{(u_n, v_n)\}_n$ in (4.2) strongly converges to some $(u_{\sigma,\gamma}, v_{\sigma,\gamma})$ in $\mathbf{W}$ for all $\gamma < \gamma^{**}$.*

Proof. Fix $\sigma \in (-\infty, a\mathcal{H}_p)$ and $\gamma \le 1$. Since $\{(u_n, v_n)\}_n$ constructed in (4.2) is in B_{ρ_0}, it follows that, by reasoning as in Lemma 3.3, there exist a subsequence of $\{(u_n, v_n)\}_n$, still denoted by $\{(u_n, v_n)\}_n$, and $(u_{\sigma,\gamma}, v_{\sigma,\gamma}) \in \overline{B}_{\rho_0}$ such that (3.10) holds true. Now we want to show that as $n \to \infty$,

$$\int_{\mathbb{R}^N} h(x)|(u_n, v_n) - (u_{\sigma,\gamma}, v_{\sigma,\gamma})|^q\, dx \to 0. \tag{4.3}$$

Fix $\varepsilon > 0$. Since $h \in L^{p_s^*/(p_s^*-q)}(\mathbb{R}^N)$ and $\{(u_n, v_n)\}_n$ is bounded in $\mathbf{W}$, there exists $R > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_R} h(x)|(u_n, v_n) - (u_{\sigma,\gamma}, v_{\sigma,\gamma})|^q\, dx \le \left(\int_{\mathbb{R}^N\setminus B_R} |h(x)|^{\frac{p_s^*}{p_s^*-q}}\, dx\right)^{\frac{p_s^*-q}{p_s^*}} \|(u_n, v_n) - (u_{\sigma,\gamma}, v_{\sigma,\gamma})\|_{p_s^*}^q \le \frac{\varepsilon}{2},$$

where B_R is the ball in $\mathbb{R}^N$ with radius $R > 0$ centered at point 0. Furthermore, for any measurable subset $E \subset B_R$, by the Hölder inequality

$$\int_E h(x)|(u_n, v_n) - (u_{\sigma,\gamma}, v_{\sigma,\gamma})|^q\, dx \le C\left(\int_E |h(x)|^{\frac{p_s^*}{p_s^*-q}}\, dx\right)^{\frac{p_s^*-q}{p_s^*}}.$$

Hence, $\{h(x)(u_n - u_{\sigma,\gamma}, v_n - v_{\sigma,\gamma})\}_n$ is equi-integrable and uniformly bounded in $L^1(B_R)$, thanks to (h). Thus, by (3.10) and the Vitali convergence theorem, there exists $n_0 > 0$ such that

$$\int_{B_R} h(x)|(u_n, v_n) - (u_{\sigma,\gamma}, v_{\sigma,\gamma})|^q \, dx \le \frac{\varepsilon}{2}$$

as $n \ge n_0$. Therefore, we arrive at

$$\begin{aligned}\int_{\mathbb{R}^N} h(x)|(u_n, v_n) - (u_{\sigma,\gamma}, v_{\sigma,\gamma})|^q \, dx &\le \int_{\mathbb{R}^N\setminus B_R} h(x)|(u_n, v_n) - (u_{\sigma,\gamma}, v_{\sigma,\gamma})|^q \, dx \\ &\quad + \int_{B_R} h(x)|(u_n, v_n) - (u_{\sigma,\gamma}, v_{\sigma,\gamma})|^q \, dx \\ &\le \varepsilon\end{aligned}$$

for all $n \ge n_0$. This proves (4.3). By (f1) and the Hölder inequality, we have

$$\begin{aligned}\bigg|\int_{\mathbb{R}^N} h(x)[f_u(u_n, v_n)(u_n - u_{\sigma,\gamma}) + f_v(u_n, v_n)(v_n - v_{\sigma,\gamma})] \, dx\bigg| &\le C \int_{\mathbb{R}^N} h(x)|(u_n, v_n)|^{q-1}|(u_n, v_n) - (u_{\sigma,\gamma}, v_{\sigma,\gamma})| \, dx \\ &\le C\bigg(\int_{\mathbb{R}^N} h(x)|(u_n, v_n) - (u_{\sigma,\gamma}, v_{\sigma,\gamma})|^q \, dx\bigg)^{\frac{1}{q}}\end{aligned}$$

for a suitable constant $C > 0$. Thus, by (4.3) it follows that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h(x)[f_u(u_n, v_n)(u_n - u_{\sigma,\gamma}) + f_v(u_n, v_n)(v_n - v_{\sigma,\gamma})] \, dx = 0. \tag{4.4}$$

Similarly, by using again (h) and (f1), we have as $n \to \infty$,

$$\begin{aligned}\int_{\mathbb{R}^N} h(x)f_u(u_n, v_n)\varphi \, dx &\to \int_{\mathbb{R}^N} h(x)f_u(u_{\sigma,\gamma}, v_{\sigma,\gamma})\varphi \, dx, \\ \int_{\mathbb{R}^N} h(x)f_v(u_n, v_n)\psi \, dx &\to \int_{\mathbb{R}^N} h(x)f_v(u_{\sigma,\gamma}, v_{\sigma,\gamma})\psi \, dx\end{aligned} \tag{4.5}$$

for any $(\varphi, \psi) \in \mathbf{W}$.

As in Lemma 3.3, we easily get (3.14)–(3.17). Hence, by also (3.10), (4.2) and (4.5) we can prove that $(u_{\sigma,\gamma}, v_{\sigma,\gamma})$ verifies the identity

$$\begin{aligned}&M(\kappa_{\sigma,\lambda})\langle(u_{\sigma,\lambda}, v_{\sigma,\lambda}), (\varphi, \psi)\rangle_{K,V,p} - \sigma(\langle u_{\sigma,\lambda}, \varphi\rangle_{H_p} + \langle v_{\sigma,\lambda}, \psi\rangle_{H_p}) \\ &\quad = \int_{\mathbb{R}^N} h(x)[f_u(u_{\sigma,\gamma}, v_{\sigma,\gamma})\varphi + f_v(u_{\sigma,\gamma}, v_{\sigma,\gamma})\psi] \, dx + \frac{\alpha}{p_s^*}\int_{\mathbb{R}^N} |u_{\sigma,\lambda}|^{\alpha-2}u_{\sigma,\lambda}|v_{\sigma,\lambda}|^\beta \varphi \, dx \\ &\qquad + \frac{\beta}{p_s^*}\int_{\mathbb{R}^N} |u_{\sigma,\lambda}|^\alpha |v_{\sigma,\lambda}|^{\beta-2}v_{\sigma,\lambda}\psi \, dx\end{aligned} \tag{4.6}$$

for any $(\varphi, \psi) \in \mathbf{W}$, namely $(u_{\sigma,\gamma}, v_{\sigma,\gamma})$ is a critical point of the $C^1(\mathbf{W})$ functional

$$\mathcal{J}_{\kappa_{\sigma,\gamma}}(u, v) = \frac{1}{p}M(\kappa_{\sigma,\gamma})\|(u, v)\|^p - \frac{\sigma}{p}(\|u\|_{H_p}^p + \|v\|_{H_p}^p) - \int_{\mathbb{R}^N} h(x)f(u, v) \, dx - \frac{\gamma}{p_s^*}\int_{\mathbb{R}^N} |u|^\alpha|v|^\beta \, dx.$$

Thus, by sending $n \to \infty$ in (4.2) and by (3.10), (4.4) and (4.6), we get (3.21). Moreover, by the Brézis and Lieb lemma we have (3.22) and by Lemma 2.3 we obtain (3.23). Finally, combining (3.21)–(3.23) together with Lemma 2.4, we derive the main formula

$$\begin{aligned}M(\kappa_{\sigma,\gamma}^p)\lim_{n\to\infty}\|(u_n, v_n) - (u_{\sigma,\gamma}, v_{\sigma,\gamma})\|^p &= \sigma \lim_{n\to\infty}(\|u_n - u_{\sigma,\gamma}\|_{H_p}^p + \|v_n - v_{\sigma,\gamma}\|_{H_p}^p) \\ &\quad + \gamma\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n - u_{\sigma,\gamma}|^\alpha|v_n - v_{\sigma,\gamma}|^\beta \, dx \\ &= \sigma(\imath_{\sigma,\gamma}^p + \ell_{\sigma,\gamma}^p) + \gamma\delta_{\sigma,\gamma}.\end{aligned} \tag{4.7}$$

Let us first consider the case $\gamma \le 0$. Assume by contradiction that $\iota^p_{\sigma,\gamma} + \ell^p_{\sigma,\gamma} > 0$. Then, from (4.7) we get

$$\begin{aligned} M(\kappa^p_{\sigma,\gamma}) \lim_{n\to\infty} \|(u_n - u_{\sigma,\gamma}, v_n - v_{\sigma,\gamma})\|^p &\le \sigma \lim_{n\to\infty} (\|u_n - u_{\sigma,\gamma}\|^p_{H_p} + \|v_n - v_{\sigma,\gamma}\|^p_{H_p}) \\ &< a\mathcal{H}_p \lim_{n\to\infty} (\|u_n - u_{\sigma,\gamma}\|^p_{H_p} + \|v_n - v_{\sigma,\gamma}\|^p_{H_p}) \\ &\le M(\kappa^p_{\sigma,\gamma}) \lim_{n\to\infty} \|(u_n - u_{\sigma,\gamma}, v_n - v_{\sigma,\gamma})\|^p, \end{aligned}$$

since $\sigma < a\mathcal{H}_p$. This is impossible. Therefore, $\iota^p_{\sigma,\gamma} + \ell^p_{\sigma,\gamma} = 0$ for all $\gamma \le 0$ and so (4.7) and (M1) imply that

$$\lim_{n\to\infty} \|(u_n, v_n) - (u_{\sigma,\gamma}, v_{\sigma,\gamma})\| = 0, \tag{4.8}$$

as required.

Let us now consider the case $\gamma \in (0, 1]$. Since $\sigma < a\mathcal{H}_p$, there exists $c \in [0, 1)$ such that $\sigma^+ = ca\mathcal{H}_p$. Of course, (4.7) can be rewritten as

$$(1 - c)M(\kappa^p_{\sigma,\gamma}) \lim_{n\to\infty} \|(u_n - u_{\sigma,\gamma}, v_n - v_{\sigma,\gamma})\|^p + cM(\kappa^p_{\sigma,\gamma}) \lim_{n\to\infty} \|(u_n - u_{\sigma,\gamma}, v_n - v_{\sigma,\gamma})\|^p = \sigma(\iota^p_{\sigma,\gamma} + \ell^p_{\sigma,\gamma}) + \gamma\delta_{\sigma,\gamma}.$$

Thus, combining (1.3) with (1.1), $(\mathcal{V})$ and (M1), and using (2.3) with $(u, v) = (u_n - u_{\sigma,\gamma}, v_n - v_{\sigma,\gamma})$, we get

$$\gamma\delta_{\sigma,\gamma} + \sigma^+(\iota^p_{\sigma,\gamma} + \ell^p_{\sigma,\gamma}) \ge (1 - c)C^{-p^*_s}_{p^*_s} a\delta^{\frac{p}{p^*_s}}_{\sigma,\gamma} + ca\mathcal{H}_p(\iota^p_{\sigma,\gamma} + \ell^p_{\sigma,\gamma})$$

for all $\gamma \in (0, 1]$, being $c \in [0, 1)$. Therefore,

$$\gamma\delta_{\sigma,\gamma} \ge (1 - c)C^{-p^*_s}_{p^*_s} a\delta^{\frac{p}{p^*_s}}_{\sigma,\gamma}, \tag{4.9}$$

since $\sigma^+ = ca\mathcal{H}_p$.

Let us define

$$\gamma^{**} = \begin{cases} \inf\{\gamma \in (0, 1] : \delta_{\sigma,\gamma} > 0\}, & \text{if there exists } \gamma \in (0, 1] \text{ such that } \delta_{\sigma,\gamma} > 0, \\ 1, & \text{if } \delta_{\sigma,\gamma} = 0 \text{ for all } \gamma \in (0, 1]. \end{cases}$$

We claim that $\gamma^{**} > 0$ if there exists $\gamma \in (0, 1]$ such that $\delta_{\sigma,\gamma} > 0$. Otherwise, there exists a sequence $\{\gamma_k\}_k$ with $\delta_{\sigma,\gamma_k} > 0$ such that $\gamma_k \to 0$ as $k \to \infty$. Thus, (4.9) implies that

$$\gamma_k \delta^{1-\frac{p}{p^*_s}}_{\sigma,\gamma_k} \ge (1 - c)C^{-p^*_s}_{p^*_s} a > 0.$$

This is an obvious contradiction, since $\{\delta_{\sigma,\gamma}\}_{\gamma\in(0,1]}$ is uniformly bounded above by (2.3). Indeed, we have $\{(u_n, v_n)\}_n \subset B_{\rho_0}$, $(u_{\sigma,\gamma}, v_{\sigma,\gamma}) \in \overline{B}_{\rho_0}$ and ρ_0, given in Lemma 4.1, is independent of γ.

Hence, $\delta_{\sigma,\gamma} = 0$ for any $\gamma \in (0, \gamma^{**})$. Therefore, for all $\gamma \in (0, \gamma^{**})$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n - u_{\sigma,\gamma}|^\alpha |v_n - v_{\sigma,\gamma}|^\beta \, dx = 0.$$

Now, assume by contradiction that there exists $\gamma \in (0, \gamma^{**})$ such that $\iota_{\sigma,\gamma} + \ell_{\sigma,\gamma} > 0$. Since $\sigma < a\mathcal{H}_p$, arguing as in the previous case, we obtain from (M1) and (1.3) that

$$\begin{aligned} M(\kappa^p_{\sigma,\gamma}) \lim_{n\to\infty} \|(u_n - u_{\sigma,\gamma}, v_n - v_{\sigma,\gamma})\|^p &= \sigma \lim_{n\to\infty} (\|u_n - u_{\sigma,\gamma}\|^p_{H_p} + \|v_n - v_{\sigma,\gamma}\|^p_{H_p}) \\ &< a\mathcal{H}_p \lim_{n\to\infty} (\|u_n - u_{\sigma,\gamma}\|^p_{H_p} + \|v_n - v_{\sigma,\gamma}\|^p_{H_p}) \\ &\le M(\kappa^p_{\sigma,\gamma}) \lim_{n\to\infty} \|(u_n - u_{\sigma,\gamma}, v_n - v_{\sigma,\gamma})\|^p, \end{aligned}$$

which gives a contradiction. Thus, $\iota_{\sigma,\gamma} + \ell_{\sigma,\gamma} = 0$ for any $\gamma \in (0, \gamma^{**})$. Now (3.24) and (M1) imply again the validity of (4.8).

In conclusion, $(u_n, v_n) \to (u_{\sigma,\gamma}, v_{\sigma,\gamma})$ as $n \to \infty$ in $\mathbf{W}$ for all $\gamma < \gamma^{**}$, as required. □

Proof of Theorem 1.3. Fix $\sigma \in (-\infty, a\mathcal{H}_p)$. For any $\gamma \le 1$, Lemma 4.1 and the Ekeland variational principle give the existence of a Palais–Smale sequence $\{(u_n, v_n)\}_n$ in $\mathbf{W}$ at level $m_{\sigma,\gamma}$. Moreover, by Lemma 4.2 there exists $\gamma^{**} = \gamma^{**}(\sigma) > 0$ such that, up to a subsequence, $\{(u_n, v_n)\}_n$ strongly converges to $(u_{\sigma,\gamma}, v_{\sigma,\gamma})$ in $\mathbf{W}$ with $m_{\sigma,\gamma} = \mathcal{I}(u_{\sigma,\gamma}, v_{\sigma,\gamma}) < 0$ and $\mathcal{I}'(u_{\sigma,\gamma}, v_{\sigma,\gamma}) = 0$ for any $\gamma < \gamma^{**}$. Consequently, $(u_{\sigma,\gamma}, v_{\sigma,\gamma})$ is a nontrivial solution of system $(\mathcal{S}')$. □

As in [35], we can conclude by giving an example, which illustrates a very simple application of Theorems 1.1–1.3. To this end, consider the following prototype system in $\mathbb{R}^N$:

$$\begin{cases} (a + \theta b\|(u,v)\|^{(\theta-1)p})[(-\Delta)_p^s u + V(x)|u|^{p-2}u] - \sigma \dfrac{|u|^{p-2}u}{|x|^{ps}} = \lambda h(x)|(u,v)|^{q-2}u + \gamma \dfrac{\alpha}{p_s^*}|u|^{\alpha-2}u|v|^{\beta}, \\ (a + \theta b\|(u,v)\|^{(\theta-1)p})[(-\Delta)_p^s v + V(x)|v|^{p-2}v] - \sigma \dfrac{|v|^{p-2}v}{|x|^{ps}} = \lambda h(x)|(u,v)|^{q-2}v + \gamma \dfrac{\beta}{p_s^*}|v|^{\beta-2}v|u|^{\alpha}, \end{cases} \tag{4.10}$$

where $a > 0$, $b \geq 0$, $1 < q < p_s^*$, $\alpha > 1$, $\beta > 1$ with $\alpha + \beta = p_s^*$, $0 \leq h \in L^{p_s^*/(p_s^*-q)}(\mathbb{R}^N)$ with $\inf_{x\in\Omega} h(x) > 0$, where Ω is a nonempty open subset of $\mathbb{R}^N$, and finally λ is a positive number and γ is a real parameter. Here, $M(t) = a + \theta b t^{\theta-1}$, $H(x,u,v) = \frac{h(x)|(u,v)|^q}{q}$, $\mu = q$ and

$$I(u,v) = \frac{a}{p}\|(u,v)\|^p + \frac{b}{p}\|(u,v)\|^{\theta p} - \frac{\sigma}{p}(\|u\|_{H_p}^p + \|v\|_{H_p}^p) - \frac{\lambda}{q}\int_{\mathbb{R}^N} h(x)|(u,v)|^q\,dx - \frac{\gamma}{p_s^*}\int_{\mathbb{R}^N} |u|^{\alpha}|v|^{\beta}\,dx.$$

If $\theta \in [1, \frac{p_s^*}{p})$, then M satisfies conditions (M1)–(M2), so that for all $q \in (\theta p, p_s^*)$, Theorem 1.1 can be applied to system (4.10) for all $\gamma > 0$. While M satisfies (M1) for all $\theta \geq 1$, so that for all $q \in (p, p_s^*)$, Theorem 1.2 can be applied to system (4.10) for all $\gamma > 0$, since clearly $M(0) = a$.

Finally, for all $\theta \geq 1$ and all $q \in (1, p)$, Theorem 1.3 can be applied for all $\lambda > 0$ and $\gamma \leq 1$.

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