QCD Evolution Workshop

Phenomenology of TMDs using SCET

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Outline

- Brief review of the Collins TMD evolution
- Our approach to TMD Evolution
- Fit of Drell-Yan data
- Conclusions

Brief review of the Collins TMD evolution

A brief review of the Collins tmd evolution

- The Collins tmd evolution equation can be written[*] as:

$$\tilde{F}(x, b_T; \zeta_F, \mu_f) = \tilde{R}^C(b_T; \zeta_i, \mu_i, \zeta_f, \mu_f)\, \tilde{F}(x, b_T; \zeta_i, \mu_i)$$

Output function at the scale ζ_f, μ_f in the impact parameter space

Input function at the scale ζ_i, μ_i in the impact parameter space

Evolutor between final and initial scales

- ζ is the scale introduced to regulate the rapidity divergences, usually:

$$\zeta \equiv \mu^2 \equiv Q^2$$

[*]*Aybat, Collins, Qiu ,Rogers, PRD85, 034043 (2012), Collins, Foundations of perturbative QCD(2011)*

A brief review of the Collins tmd evolution

- The Collins evolutor can be easily rewritten in this form:

$$\tilde{R}^C(b_T;\zeta_i,\mu_i,\zeta_f,\mu_f) = \exp\left\{\int_{\mu_i}^{\mu_f}\frac{d\bar{\mu}}{\bar{\mu}}\gamma_F\left(\alpha_s(\bar{\mu}),\ln\frac{\zeta_f}{\bar{\mu}^2}\right)\right\}\left(\frac{\zeta_f}{\zeta_i}\right)^{-D(b_T,\mu_i)}$$

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- Anomalous dimension of F $\quad \dfrac{d\ln\tilde{F}(x,b_T;\zeta,\mu)}{d\ln\mu} = \gamma_F\left(\alpha_s(\mu),\ln\dfrac{\zeta}{\mu^2}\right)$

For instance at first order in the coupling constants:

$$\gamma_F\left(\alpha_s(\mu),\ln\frac{Q^2}{\mu^2}\right) = \alpha_s(\mu)\frac{C_F}{\pi}\left(\frac{3}{2}-\ln\frac{Q^2}{\mu^2}\right)$$

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$$\frac{dD(b_T,\mu)}{d\ln\mu} = \Gamma_{\text{cusp}} = \frac{1}{2}\gamma_K \qquad D(b_T,\mu) = -\frac{1}{2}\tilde{K}(b_T,\mu)$$

$$D(b_T,\mu) = D(b_*,\mu_{b_*}) + \int_{\mu_{b_*}}^{\mu}\frac{d\bar{\mu}}{\bar{\mu}}\Gamma_{\text{cusp}} + g_K(b_T)$$

$$b_*(b_T) \equiv \frac{b_T}{\sqrt{1+b_T^2/b_{\max}^2}} \qquad \mu_{b_*} = \frac{C_1}{b_*} \qquad C_1 = 2e^{-\gamma_E} \qquad g_K(b_T) = \frac{1}{2}g_2\,b_T^2$$

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$$D(b_T,\mu) = \int_{\mu_{b_*}}^{\mu}\frac{d\bar{\mu}}{\bar{\mu}}\Gamma_{\text{cusp}} + g_K(b_T) \qquad \Gamma_{\text{cusp}}(\mu) \simeq C_F\alpha_s(\mu) + \ldots$$

- A scale to control the non-perturbative part: one parameter b_{max}
- A non-perturbative function: at least one parameter g_2

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TMD evolution

TMD evolution

- Our main tmd evolution equation is:

$$\tilde{F}(x, b_T; Q_F, \mu_f) = \tilde{R}(b_T; Q_i, \mu_i, Q_f, \mu_f)\, \tilde{F}(x, b_T; Q_i, \mu_i)$$

Output function at the scale $\mathbf{Q_f, \mu_f}$ in the impact parameter space

Input function at the scale $\mathbf{Q_i, \mu_i}$ in the impact parameter space

Evolutor between final and initial scales

- We always set:

$$\mu_i \equiv Q_i$$
$$\mu_f \equiv Q_f$$

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The evolutor

$$\tilde{R}(b_T; Q_i, \mu_i, Q_f, \mu_f) = \exp\left\{ \int_{\mu_i}^{\mu_f} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_F \left(\alpha_s(\bar{\mu}), \ln \frac{Q_f^2}{\bar{\mu}^2} \right) \right\} \left(\frac{Q_f^2}{Q_i^2} \right)^{-D(b_T, \mu_i)}$$

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$$\tilde{R}(b_T; Q_i, \mu_i, Q_f, \mu_f) = \exp\left\{ \int_{\mu_i}^{\mu_f} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_F \left(\alpha_s(\bar{\mu}), \ln \frac{Q_f^2}{\bar{\mu}^2} \right) \right\} \left(\frac{Q_f^2}{Q_i^2} \right)^{-D(b_T, \mu_i)}$$

- As in the Collins case

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The evolutor

$$\tilde{R}(b_T; Q_i, \mu_i, Q_f, \mu_f) = \exp\left\{ \int_{\mu_i}^{\mu_f} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_F \left(\alpha_s(\bar{\mu}), \ln \frac{Q_f^2}{\bar{\mu}^2} \right) \right\} \left(\frac{Q_f^2}{Q_i^2} \right)^{-D(b_T, \mu_i)}$$

- As in the Collins case $\frac{dD}{d \ln \mu} = \Gamma_{\mathrm{cusp}}$

- Notice that if $\zeta \equiv \mu^2 \equiv Q^2$ this approach is identical to the Collins' one

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The evolutor

$$\tilde{R}(b_T;Q_i,\mu_i,Q_f,\mu_f) = \exp\left\{\int_{\mu_i}^{\mu_f}\frac{d\bar{\mu}}{\bar{\mu}}\gamma_F\left(\alpha_s(\bar{\mu}),\ln\frac{Q_f^2}{\bar{\mu}^2}\right)\right\}\left(\frac{Q_f^2}{Q_i^2}\right)^{-D(b_T,\mu_i)}$$

- However the RG evolution is treated differently, obtaining:

$$\begin{aligned}D^R(b_T;\mu) = &-\frac{\Gamma_0}{2\beta_0}\ln(1-X) + \frac{1}{2}\left(\frac{a_s}{1-X}\right)\left[-\frac{\beta_1\Gamma_0}{\beta_0^2}(X+\ln(1-X)) + \frac{\Gamma_1}{\beta_0}X\right]\\ &+\frac{1}{2}\left(\frac{a_s}{1-X}\right)^2\left[2d_2(0) + \frac{\Gamma_2}{2\beta_0}(X(2-X)) + \frac{\beta_1\Gamma_1}{2\beta_0^2}(X(X-2)-2\ln(1-X)) + \frac{\beta_2\Gamma_0}{2\beta_0^2}X^2\right.\\ &\left.+\frac{\beta_1^2\Gamma_0}{2\beta_0^3}(\ln^2(1-X)-X^2)\right].\end{aligned}$$

$$a_s = \frac{\alpha_s(\mu)}{4\pi} \qquad \mu_b = \frac{C_1}{b_T} \qquad L_\perp = \ln\left(\frac{\mu^2}{\mu_b^2}\right) \qquad X = a_s\beta_0 L_\perp$$

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The evolutor

$$D(b;\mu) = \sum_{n=1}^{\infty} d_n(L_\perp) \left(\frac{\alpha_s}{4\pi}\right)^n \qquad \Gamma_{\text{cusp}} = \sum_{n=1}^{\infty} \Gamma_{n-1} \left(\frac{\alpha_s}{4\pi}\right)^n$$

$$\frac{dD}{d\ln\mu} = \Gamma_{\text{cusp}}$$

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$$\frac{dD}{d\ln\mu} = \Gamma_{\text{cusp}}$$

$$\beta = -2\alpha_s \sum_{n=1}^{\infty} \beta_{n-1} \left(\frac{\alpha_s}{4\pi}\right)^n$$

$$d'_n(L_\perp) = \frac{1}{2}\Gamma_{n-1} + \sum_{m=1}^{n-1} m\beta_{n-1-m} d_m(L_\perp)$$

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The evolutor

$$D(b;\mu) = \sum_{n=1}^{\infty} d_n(L_\perp)\left(\frac{\alpha_s}{4\pi}\right)^n \qquad \Gamma_{\rm cusp} = \sum_{n=1}^{\infty} \Gamma_{n-1}\left(\frac{\alpha_s}{4\pi}\right)^n$$

$$\frac{dD}{d\ln\mu} = \Gamma_{\rm cusp}$$

$$\beta = -2\alpha_s \sum_{n=1}^{\infty} \beta_{n-1}\left(\frac{\alpha_s}{4\pi}\right)^n$$

$$d_n'(L_\perp) = \frac{1}{2}\Gamma_{n-1} + \sum_{m=1}^{n-1} m\beta_{n-1-m} d_m(L_\perp)$$

$$2d_n(L_\perp) = (\beta_0 L_\perp)^n \left(\frac{\Gamma_0}{\beta_0}\frac{1}{n}\right) + (\beta_0 L_\perp)^{n-1}\left(\frac{\Gamma_0\beta_1}{\beta_0^2}\left(-1 + H_{n-1}^{(1)}\right)|_{n\geq 3} + \frac{\Gamma_1}{\beta_0}|_{n\geq 2}\right)$$
$$+ (\beta_0 L_\perp)^{n-2}\left((n-1)2d_2(0)|_{n\geq 2} + (n-1)\frac{\Gamma_2}{2\beta_0}|_{n\geq 3} + \frac{\beta_1\Gamma_1}{\beta_0^2}s_n|_{n\geq 4} + \frac{\beta_1^2\Gamma_0}{\beta_0^3}t_n|_{n\geq 5} + \frac{\beta_2\Gamma_0}{2\beta_0^2}(n-3)|_{n\geq 4}\right)$$

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The evolutor

$$D(b;\mu) = \sum_{n=1}^{\infty} d_n(L_\perp)\left(\frac{\alpha_s}{4\pi}\right)^n \qquad \Gamma_{\rm cusp} = \sum_{n=1}^{\infty} \Gamma_{n-1}\left(\frac{\alpha_s}{4\pi}\right)^n$$

$$\frac{dD}{d\ln\mu} = \Gamma_{\rm cusp}$$

$$\beta = -2\alpha_s \sum_{n=1}^{\infty} \beta_{n-1}\left(\frac{\alpha_s}{4\pi}\right)^n$$

$$d'_n(L_\perp) = \frac{1}{2}\Gamma_{n-1} + \sum_{m=1}^{n-1} m\beta_{n-1-m} d_m(L_\perp)$$

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$$+ (\beta_0 L_\perp)^{n-2}\left((n-1)2d_2(0)|_{n\geq 2} + (n-1)\frac{\Gamma_2}{2\beta_0}|_{n\geq 3} + \frac{\beta_1\Gamma_1}{\beta_0^2}s_n|_{n\geq 4} + \frac{\beta_1^2\Gamma_0}{\beta_0^3}t_n|_{n\geq 5} + \frac{\beta_2\Gamma_0}{2\beta_0^2}(n-3)|_{n\geq 4}\right)$$

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The evolutor

$$D(b;\mu) = \sum_{n=1}^{\infty} d_n(L_\perp) \left(\frac{\alpha_s}{4\pi}\right)^n \qquad \Gamma_{\text{cusp}} = \sum_{n=1}^{\infty} \Gamma_{n-1} \left(\frac{\alpha_s}{4\pi}\right)^n$$

$$\frac{dD}{d\ln\mu} = \Gamma_{\text{cusp}}$$

$$D^R(b;\mu_i) = \sum_{n=1}^{\infty} d_n(L_\perp) a^n =$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\left\{X^n\left(\frac{\Gamma_0}{\beta_0}\frac{1}{n}\right) + aX^{n-1}\left(\frac{\Gamma_0\beta_1}{\beta_0^2}\left(-1+H_{n-1}^{(1)}\right)|_{n\geq 3} + \frac{\Gamma_1}{\beta_0}|_{n\geq 2}\right)\right.$$

$$\left. + a^2X^{n-2}\left((n-1)2d_2(0)|_{n\geq 2} + (n-1)\frac{\Gamma_2}{2\beta_0}|_{n\geq 3} + \frac{\beta_1\Gamma_1}{\beta_0^2}s_n|_{n\geq 4} + \frac{\beta_1^2\Gamma_0}{\beta_0^3}t_n|_{n\geq 5} + \frac{\beta_2\Gamma_0}{2\beta_0^2}(n-3)|_{n\geq 4}\right) + \ldots\right\}$$

- For |X|<1 the series can be summed…

$$a_s = \frac{\alpha_s(\mu)}{4\pi} \qquad \mu_b = \frac{C_1}{b_T} \qquad L_\perp = \ln\left(\frac{\mu^2}{\mu_b^2}\right) \qquad X = a_s\beta_0 L_\perp$$

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$$\frac{dD}{d\ln\mu} = \Gamma_{\text{cusp}}$$

$$D^R(b_T;\mu) = -\frac{\Gamma_0}{2\beta_0}\ln(1-X) + \frac{1}{2}\left(\frac{a_s}{1-X}\right)\left[-\frac{\beta_1\Gamma_0}{\beta_0^2}(X+\ln(1-X)) + \frac{\Gamma_1}{\beta_0}X\right]$$
$$+\frac{1}{2}\left(\frac{a_s}{1-X}\right)^2\left[2d_2(0) + \frac{\Gamma_2}{2\beta_0}(X(2-X)) + \frac{\beta_1\Gamma_1}{2\beta_0^2}(X(X-2)-2\ln(1-X)) + \frac{\beta_2\Gamma_0}{2\beta_0^2}X^2\right.$$
$$\left.+\frac{\beta_1^2\Gamma_0}{2\beta_0^3}(\ln^2(1-X)-X^2)\right].$$

- For $|X|<1$ the series can be summed and analytically continued for $X \rightarrow -\infty$

$$a_s = \frac{\alpha_s(\mu)}{4\pi} \qquad \mu_b = \frac{C_1}{b_T} \qquad L_\perp = \ln\left(\frac{\mu^2}{\mu_b^2}\right) \qquad X = a_s\beta_0 L_\perp$$

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$$D(b;\mu) = \sum_{n=1}^{\infty} d_n(L_\perp) \left(\frac{\alpha_s}{4\pi}\right)^n \qquad \Gamma_{\rm cusp} = \sum_{n=1}^{\infty} \Gamma_{n-1} \left(\frac{\alpha_s}{4\pi}\right)^n$$

$$\frac{dD}{d\ln\mu} = \Gamma_{\rm cusp}$$

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$$+\frac{1}{2}\left(\frac{a_s}{1-X}\right)^2\left[2d_2(0) + \frac{\Gamma_2}{2\beta_0}(X(2-X)) + \frac{\beta_1\Gamma_1}{2\beta_0^2}(X(X-2) - 2\ln(1-X)) + \frac{\beta_2\Gamma_0}{2\beta_0^2}X^2\right.$$
$$\left.+\frac{\beta_1^2\Gamma_0}{2\beta_0^3}(\ln^2(1-X) - X^2)\right].$$

- For $|X|<1$ the series can be summed and analytically continued for $X \to -\infty$

$$a_s = \frac{\alpha_s(\mu)}{4\pi} \qquad \mu_b = \frac{C_1}{b_T} \qquad L_\perp = \ln\left(\frac{\mu^2}{\mu_b^2}\right) \qquad X = a_s\beta_0 L_\perp$$

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The evolutor

- Alternative derivation at LO:

$$\frac{dD}{d\ln\mu} = \Gamma_{\text{cusp}} \qquad \Gamma_{\text{cusp}} = \sum_{n=1}^{\infty} \Gamma_{n-1} \left(\frac{\alpha_s}{4\pi}\right)^n$$

$$D\left(b;Q_i\right) = D\left(b;\mu_b\right) + \int_{\mu_b}^{Q_i} \frac{d\bar{\mu}}{\bar{\mu}} \Gamma_{\text{cusp}}$$

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$$\frac{dD}{d\ln\mu} = \Gamma_{\text{cusp}} \qquad\qquad \Gamma_{\text{cusp}} = \sum_{n=1}^{\infty} \Gamma_{n-1} \left(\frac{\alpha_s}{4\pi}\right)^n$$

$$D\left(b; Q_i\right) = D\left(b; \mu_b\right) + \int_{\mu_b}^{Q_i} \frac{d\bar{\mu}}{\bar{\mu}} \Gamma_{\text{cusp}}$$

$$D(b; Q_i) = -\frac{\Gamma_0}{2\beta_0} \ln \frac{\alpha_s(Q_i)}{\alpha_s(\mu_b)}$$

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$$\frac{dD}{d\ln\mu} = \Gamma_{\rm cusp} \qquad \Gamma_{\rm cusp} = \sum_{n=1}^{\infty} \Gamma_{n-1}\left(\frac{\alpha_s}{4\pi}\right)^n$$

$$D(b;Q_i) = D(b;\mu_b) + \int_{\mu_b}^{Q_i} \frac{d\bar{\mu}}{\bar{\mu}} \Gamma_{\rm cusp}$$

$$D(b;Q_i) = -\frac{\Gamma_0}{2\beta_0} \ln\frac{\alpha_s(Q_i)}{\alpha_s(\mu_b)}$$

$$\alpha_s(\mu_b) = \alpha_s(Q_i)/(1-X)$$

$$X = \frac{\alpha_s(Q_i)}{4\pi}\beta_0 \ln(Q_i^2/\mu_b^2)$$

$$a_s = \frac{\alpha_s(\mu)}{4\pi}$$

$$\mu_b = \frac{C_1}{b_T}$$

$$L_\perp = \ln\left(\frac{\mu^2}{\mu_b^2}\right)$$

$$X = a_s\beta_0 L_\perp$$

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$$D\left(b; Q_i\right) = D\left(b; \mu_b\right) + \int_{\mu_b}^{Q_i} \frac{d\bar{\mu}}{\bar{\mu}} \Gamma_{\text{cusp}}$$

$$D(b; Q_i) = -\frac{\Gamma_0}{2\beta_0} \ln \frac{\alpha_s(Q_i)}{\alpha_s(\mu_b)}$$

$$\alpha_s(\mu_b) = \alpha_s(Q_i)/(1 - X)$$

$$D(b; Q_i) = -\frac{\Gamma_0}{2\beta_0} \ln(1 - X)$$

$$a_s = \frac{\alpha_s(\mu)}{4\pi}$$

$$\mu_b = \frac{C_1}{b_T}$$

$$L_\perp = \ln\left(\frac{\mu^2}{\mu_b^2}\right)$$

$$X = a_s \beta_0 L_\perp$$

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The evolutor

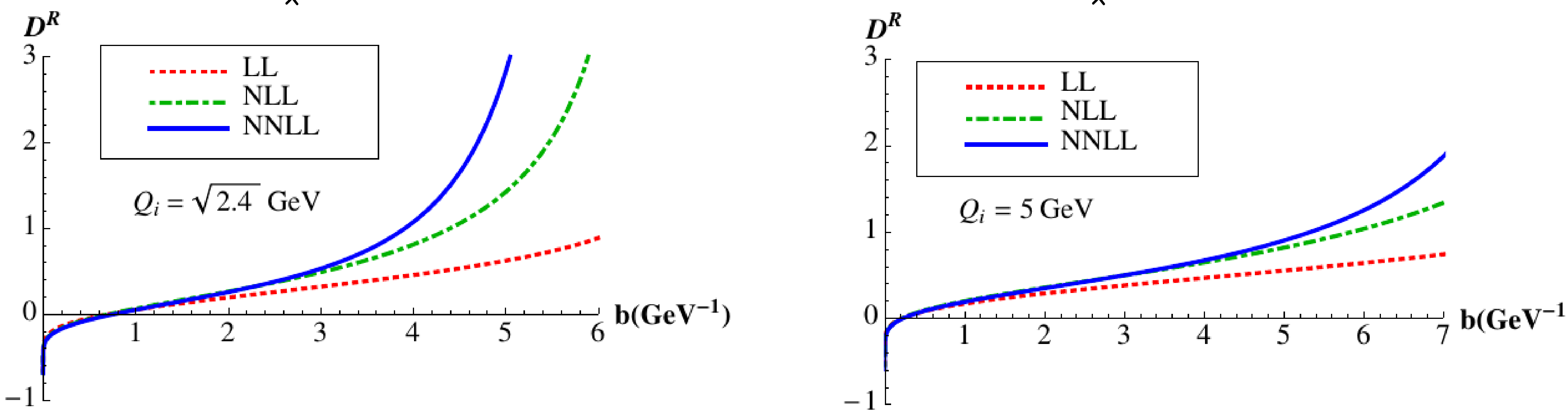
- The resummed series is valid up to X=1. At first order this correspond to

$$b_X = \frac{C_1}{\mu_i} \exp\left(\frac{2\pi}{\beta_0 \alpha_s(\mu_i)}\right)$$

- In practice the convergence of D deteriorate approaching X=1, however appearing with a minus sing in the exponent of the evolutor, R goes to zero enough fast provided the final scale is enough bigger then the initial scale.

The evolutor

b_x~6.8 GeV^{-1} b_x~11 GeV^{-1}



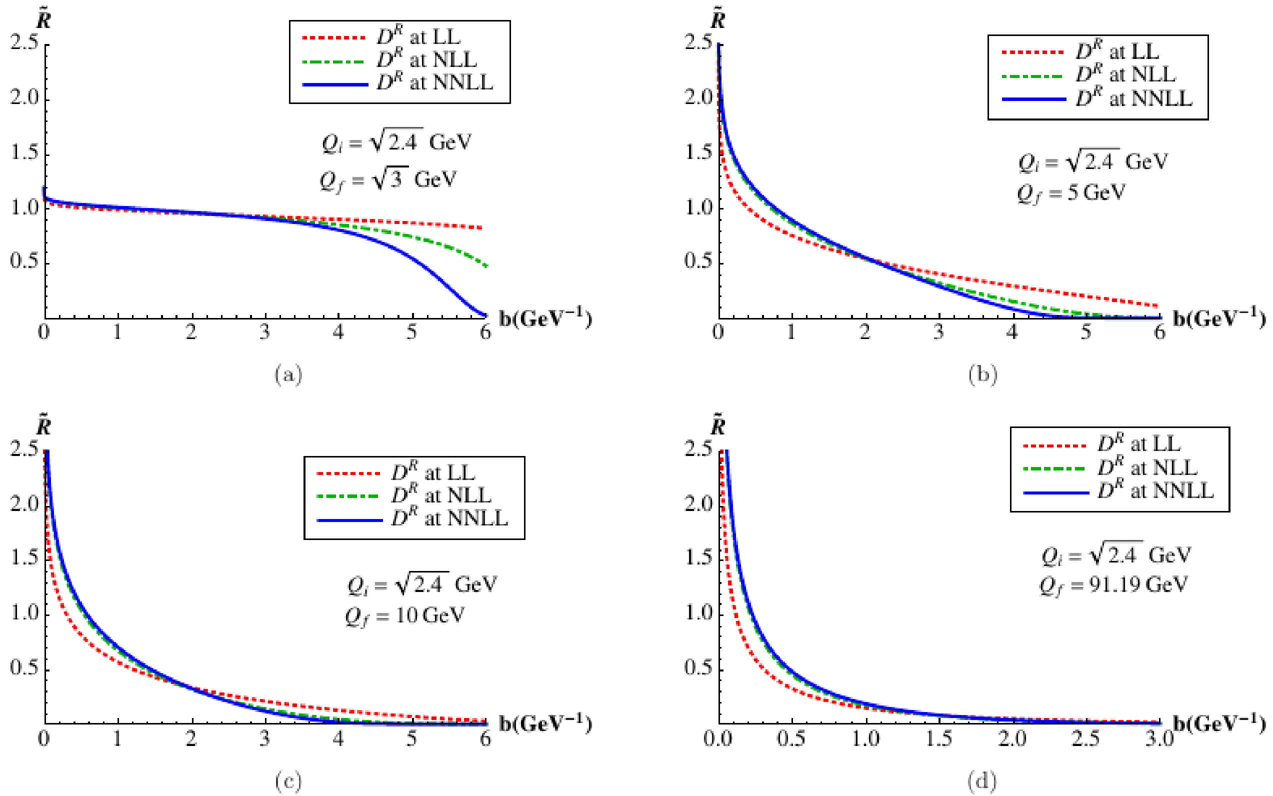
(a) (b)

Resummed D at $Q_i = \sqrt{2.4}$ GeV *with* $n_f = 4$ *(a) and* $Q_i = 5$ GeV *with* $n_f = 5$ *(b).*

- The convergence deteriorate approaching b_x
- Increasing Q_i, b_X increases and a good convergence is obtained at larger b

$$b_X = \frac{C_1}{\mu_i} \exp\left(\frac{2\pi}{\beta_0 \alpha_s(\mu_i)}\right)$$

The evolutor



Evolution kernel from $Q_i = \sqrt{2.4}$ GeV *up to* $Q_f = \{\sqrt{3}, 5, 10, 91.19\}$ GeV

- The evolutor vanishes rapidly at large b if $Q_f \gg Q_i$

The input function

- The input function is the product of a perturbative function times a non-perturbative function :

$$\tilde{F}_{q/N}(x, b_T, Q_i, \mu_i) = \tilde{F}^{pert}_{q/N}(x, b_T, Q_i, \mu_i) \times \tilde{F}^{NP}_{q/N}(x, b_T, Q_i)$$

- The perturbative function can be written as usual as the convolution of Wilson coefficients times the collinear pdfs

$$\tilde{F}^{pert}_{q/N}(x, b_T, Q_i, \mu_i) = \left(\frac{Q_i^2}{\mu_b^2}\right)^{-D_R(b_T,\mu_i)} \sum_j \tilde{C}_{qj}(x, b_T, \mu_i) \otimes f_{j/N}(x; \mu_i)$$

- The Wilson coefficients contains logs: $L_\perp = \ln(\mu^2/\mu_b^2) = \ln(\mu^2 b_T^2/C_1^2)$

$$\tilde{C}_{q\leftarrow j} = \delta(1-x) + 2a_s C_F \left[+1 - x - \delta(1-x)\left(\frac{1}{2}L_\perp^2 - \frac{3}{2}L_\perp + \frac{\pi^2}{12}\right) - \mathcal{P}_{q\leftarrow j} L_\perp\right]$$

The input function

- We can resum these logs using the same trick used for the D

$$\frac{d\tilde{C}_{qj}(x,b_T,\mu)}{d\ln\mu} = (\Gamma_{\rm cusp}L_\perp - \gamma_V)\tilde{C}_{qj}(x,b_T,\mu) - \sum_i \tilde{C}_{qj}(x,b_T,\mu)\otimes\mathcal{P}_{ij}(x)$$

- Obtaining:

$$\tilde{C}_{qj}(x,b_T,\mu) = \exp(h_\Gamma - h_{\gamma_V})\hat{C}_{qj}(x,b_T,\mu)$$

$$h_\Gamma^R(b_T;\mu) = h_\Gamma(b_T;\mu_b) + \int_{\mu_b}^{\mu}\frac{d\bar{\mu}}{\bar{\mu}}\Gamma_{\rm cusp}L_\perp$$

$$h_\gamma^R(b_T;\mu) = h_\gamma(b_T;\mu_b) + \int_{\mu_b}^{\mu}\frac{d\bar{\mu}}{\bar{\mu}}\gamma^V$$

The input function

- We can resum these logs using the same trick used for the D

$$\frac{d\tilde{C}_{qj}(x,b_T,\mu)}{d\ln\mu} = (\Gamma_{\text{cusp}}L_\perp - \gamma_V)\tilde{C}_{qj}(x,b_T,\mu) - \sum_i \tilde{C}_{qj}(x,b_T,\mu)\otimes\mathcal{P}_{ij}(x)$$

- Obtaining:

$$\tilde{C}_{qj}(x,b_T,\mu) = \exp(h_\Gamma - h_{\gamma_V})\hat{C}_{qj}(x,b_T,\mu)$$

$$h^R_\Gamma(b_T;\mu) = \frac{\Gamma_0(X-(X-1)\ln(1-X))}{2a_s\beta_0^2} + \frac{\beta_1\Gamma_0\left(2X+\ln^2(1-X)+2\ln(1-X)\right) - 2\beta_0\Gamma_1(X+\ln(1-X))}{4\beta_0^3}$$
$$+\frac{a_s}{4\beta_0^4(1-X)}\left(\beta_0^2\Gamma_2X^2 - \beta_0(\beta_1\Gamma_1(X(X+2)+2\ln(1-X)) + \beta_2\Gamma_0((X-2)X+2(X-1)\ln(1-X)))\right.$$
$$\left.+\beta_1^2\Gamma_0(X+\ln(1-X))^2\right)\,. \qquad (2$$

The input function

- We can resume these logs using the same trick used for the D

$$\frac{d\tilde{C}_{qj}(x,b_T,\mu)}{d\ln\mu} = (\Gamma_{\rm cusp}L_\perp - \gamma_V)\tilde{C}_{qj}(x,b_T,\mu) - \sum_i \tilde{C}_{qj}(x,b_T,\mu)\otimes\mathcal{P}_{ij}(x)$$

- Obtaining:

$$\tilde{C}_{qj}(x,b_T,\mu) = \exp(h_\Gamma - h_{\gamma_V})\hat{C}_{qj}(x,b_T,\mu)$$

$$\begin{aligned} h_\gamma^R(b_T;\mu) = & -\frac{\gamma_0}{2\beta_0}\ln(1-X) + \frac{1}{2}\left(\frac{a_s}{1-X}\right)\left[-\frac{\beta_1\gamma_0}{\beta_0^2}(X+\ln(1-X)) + \frac{\gamma_1}{\beta_0}X\right] \\ & + \frac{1}{2}\left(\frac{a_s}{1-X}\right)^2\left[\frac{\gamma_2}{2\beta_0}(X(2-X)) + \frac{\beta_1\gamma_1}{2\beta_0^2}(X(X-2) - 2\ln(1-X)) + \frac{\beta_2\gamma_0}{2\beta_0^2}X^2\right. \\ & \left. + \frac{\beta_1^2\gamma_0}{2\beta_0^3}(\ln^2(1-X) - X^2)\right] \end{aligned}$$

The input function

- We can resum these logs using the same trick used for the D

$$\frac{d\tilde{C}_{qj}(x,b_T,\mu)}{d\ln\mu} = (\Gamma_{\rm cusp}L_\perp - \gamma_V)\tilde{C}_{qj}(x,b_T,\mu) - \sum_i \tilde{C}_{qj}(x,b_T,\mu)\otimes\mathcal{P}_{ij}(x)$$

- Obtaining:

$$\tilde{C}_{qj}(x,b_T,\mu) = \exp(h_\Gamma - h_{\gamma_V})\hat{C}_{qj}(x,b_T,\mu)$$

$$\hat{C}_{qj} = \delta(1-z)\delta_{qi} - a_s\left[\mathcal{P}^{(1)}_{q\leftarrow i}(z)\frac{L_\perp}{2} - \mathcal{R}^{(1)}_{q\leftarrow i}(z)\right]$$

$$\mathcal{P}^{(1)}_{q\leftarrow q}(z) = 4C_F\left(\frac{1+z^2}{1-z}\right)_+ , \qquad \mathcal{R}^{(1)}_{q\leftarrow q}(z) = 2C_F\left[1-z-\frac{\pi^2}{12}\delta(1-z)\right] ,$$

$$\mathcal{P}^{(1)}_{q\leftarrow g}(z) = 4T_F\left(z^2+(1-z)^2\right) , \qquad \mathcal{R}^{(1)}_{q\leftarrow g}(z) = 4T_F z(1-z) .$$

The input function

- The input function is the product of a perturbative function times a non-perturbative function :

$$\tilde{F}_{q/N}(x, b_T, Q_i, \mu_i) = \tilde{F}^{pert}_{q/N}(x, b_T, Q_i, \mu_i) \times \tilde{F}^{NP}_{q/N}(x, b_T, Q_i)$$

$$\tilde{F}^{pert}_{q/N}(x, b_T, Q_i, \mu_i) = \exp\left[-L_\perp D_R(b_T, \mu_i) + h_\Gamma - h_{\gamma_V}\right] \sum_j \hat{C}_{qj}(x, b_T, \mu_i) \otimes f_{j/N}(x; \mu_i)$$

Phenomenological analysis of Drell-Yan data

DY cross section

- We want to test our tmd evolution studying the Drell-Yan process at low and high energies. Our cross section reads:

$$\frac{d\sigma}{dQ^2 dy dq_T^2} = \sum_q \sigma_0^{\gamma/Z_0} |C_V(Q/\mu \equiv Q)|^2 \int \frac{d^2b}{4\pi} e^{-i\boldsymbol{q}_T \cdot \boldsymbol{b}_T}$$

$$\tilde{F}_{q/N_1}(x_1, b_T, Q, \mu = Q) \tilde{F}_{\bar{q}/N_2}(x_2, b_T, Q, \mu = Q)$$

DY cross section

- We want to test our tmd evolution studying the Drell-Yan process at low and high energies. Our cross section reads:

$$\frac{d\sigma}{dQ^2 dy dq_T^2} = \sum_q \sigma_0^{\gamma/Z_0} |C_V(Q/\mu \equiv Q)|^2 \int \frac{d^2b}{4\pi} e^{-i\boldsymbol{q}_T \cdot \boldsymbol{b}_T} \tilde{F}_{q/N_1}(x_1, b_T, Q, \mu = Q) \tilde{F}_{\bar{q}/N_2}(x_2, b_T, Q, \mu = Q)$$

Born cross section

DY cross section

- We want to test our tmd evolution studying the Drell-Yan process at low and high energies. Our cross section reads:

$$\frac{d\sigma}{dQ^2 dy dq_T^2} = \sum_q \sigma_0^{\gamma/Z_0} |C_V(Q/\mu \equiv Q)|^2 \int \frac{d^2b}{4\pi} e^{-i\boldsymbol{q}_T \cdot \boldsymbol{b}_T}$$

$$\tilde{F}_{q/N_1}(x_1, b_T, Q, \mu = Q) \tilde{F}_{\bar{q}/N_2}(x_2, b_T, Q, \mu = Q)$$

SCET hard matching coefficient

DY cross section

- We want to test our tmd evolution studying the Drell-Yan process at low and high energies. Our cross section reads:

$$\frac{d\sigma}{dQ^2 dy dq_T^2} = \sum_q \sigma_0^{\gamma/Z_0} |C_V(Q/\mu \equiv Q)|^2 \int \frac{d^2 b}{4\pi} e^{-i\boldsymbol{q}_T \cdot \boldsymbol{b}_T}$$

$$\tilde{F}_{q/N_1}(x_1, b_T, Q, \mu = Q) \tilde{F}_{\bar{q}/N_2}(x_2, b_T, Q, \mu = Q)$$

Fourier transform: our TMDs are defined up to b_X

$$\int \frac{d^2 b}{4\pi} e^{-i\boldsymbol{q}_T \cdot \boldsymbol{b}_T} \longrightarrow \frac{1}{2} \int_0^{b_X} db_T b_T J_0(b_T q_T)$$

$$b_X = \frac{C_1}{\mu_i} \exp\left(\frac{2\pi}{\beta_0 \alpha_s(\mu_i)}\right)$$

DY cross section

- We want to test our tmd evolution studying the Drell-Yan process at low and high energies. Our cross section reads:

$$\frac{d\sigma}{dQ^2 dy dq_T^2} = \sum_q \sigma_0^{\gamma/Z_0} |C_V(Q/\mu \equiv Q)|^2 \int \frac{d^2 b}{4\pi} e^{-i\boldsymbol{q}_T \cdot \boldsymbol{b}_T}$$
$$\tilde{F}_{q/N_1}(x_1, b_T, Q, \mu = Q) \tilde{F}_{\bar{q}/N_2}(x_2, b_T, Q, \mu = Q)$$

Evolved TMDs

$$\tilde{F}(x, b_T; Q_F, \mu_f) = \tilde{R}(b_T; Q_i, \mu_i, Q_f, \mu_f)\, \tilde{F}(x, b_T; Q_i, \mu_i)$$

$$\tilde{F}_{q/N}(x, b_T, Q_i, \mu_i) = \tilde{F}^{pert}_{q/N}(x, b_T, Q_i, \mu_i) \times \tilde{F}^{NP}_{q/N}(x, b_T, Q_i)$$

DY cross section

- We want to test our tmd evolution studying the Drell-Yan process at low and high energies. Our cross section reads:

$$\frac{d\sigma}{dQ^2 dy dq_T^2} = \sum_q \sigma_0^{\gamma/Z_0} |C_V(Q/\mu \equiv Q)|^2 \int \frac{d^2 b}{4\pi} e^{-i \boldsymbol{q}_T \cdot \boldsymbol{b}_T}$$
$$\tilde{F}_{q/N_1}(x_1, b_T, Q, \mu = Q) \tilde{F}_{\bar{q}/N_2}(x_2, b_T, Q, \mu = Q)$$

Evolved TMDs

$$\tilde{F}(x, b_T; Q_F, \mu_f) = \tilde{R}(b_T; Q_i, \mu_i, Q_f, \mu_f)\, \tilde{F}(x, b_T; Q_i, \mu_i)$$

$$\tilde{F}_{q/N}(x, b_T, Q_i, \mu_i) = \tilde{F}^{pert}_{q/N}(x, b_T, Q_i, \mu_i) \times \tilde{F}^{NP}_{q/N}(x, b_T, Q_i)$$

DY cross section

- Two free parameters, no x or Q^2 dependence, exp form

$$\tilde{F}^{NP}_{q/N}(x, b_T, Q_i) \equiv \tilde{F}^{NP}(b_T) = \exp(-h_1 b_T)(1 + h_2 b_T^2)$$

- Another important choice is the choice of the initial scale Q_i:

$$Q_i = Q_0 + q_T \quad \text{with} \quad Q_0 = 2 \text{ GeV}$$

Drell-Yan data selection

- Z_0 production at Tevatron (98 points)

	CDF Run I	D0 Run I	CDF Run II	D0 Run II
points	32	16	41	9
$\sqrt{s}$	1.8 TeV	1.8 TeV	1.96 TeV	1.96 TeV
σ	248 ± 11 pb	221 ± 11.2 pb	256 ± 15.2 pb	255.8 ± 16.7 pb

- Low energy Drell-Yan experiments (125 points)

	E288 200	E288 300	E288 400	R209
points	35	35	49	6
$\sqrt{s}$	19.4 GeV	23.8 GeV	27.4 GeV	62 GeV
E_{beam}	200 GeV	300 GeV	400 GeV	-
Beam/Target	p Cu	p Cu	p Cu	p p
M range used	4-9 GeV	4-9 GeV	5-9 and 10.5-14 GeV	5-8 and 11-25 GeV
Other kin. var	y=0.4	y=0.21	y=0.03	
Observable	$Ed^3\sigma/d^3\mathbf{p}$	$Ed^3\sigma/d^3\mathbf{p}$	$Ed^3\sigma/d^3\mathbf{p}$	$d\sigma/dq_T^2$

Drell-Yan data FIT

- Z_0 production at Tevatron + low energy DY (223 points)
- MSTW08 PDFs (but we also tried CTEQ10 with similar results)
- NNLL and NLL fits
- 2 free parameters + 2 normalization parameters and Q_i=2 GeV+ q_T
  - @tevatron to reduce errors (important only for the run I)

$$\frac{1}{\sigma_{exp}}\left(\frac{d\sigma}{dq_T}\right)_{exp} \qquad \frac{1}{\sigma_{teo}}\left(\frac{d\sigma}{dq_T}\right)_{teo}$$

  - For E288 and R209 two normalization parameters

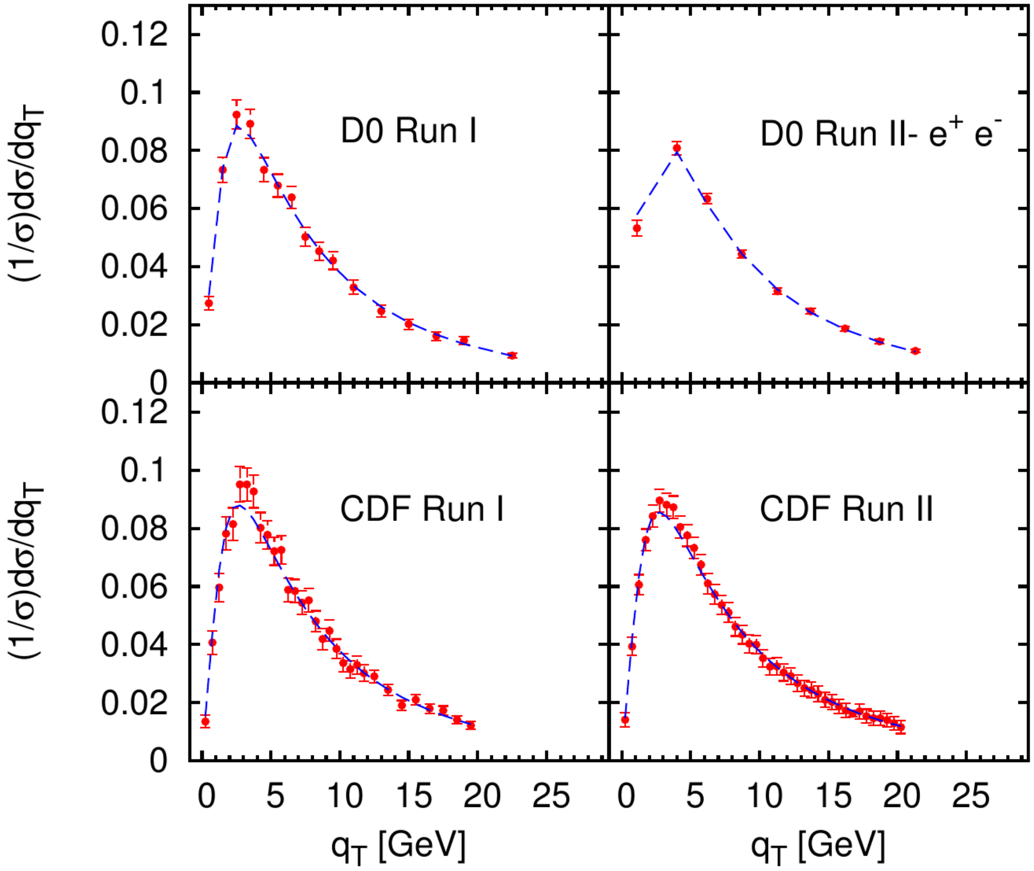
$$N_{E288} \qquad N_{R209}$$

Results

NNLL	223 points	$\chi^2/d.o.f = 1.12$
	$h_1 = 0.33 \pm 0.05$	$h_2 = 0.13 \pm 0.03$
	$N_{E288} = 0.85 \pm 0.04$	$N_{R209} = 1.5 \pm 0.2$

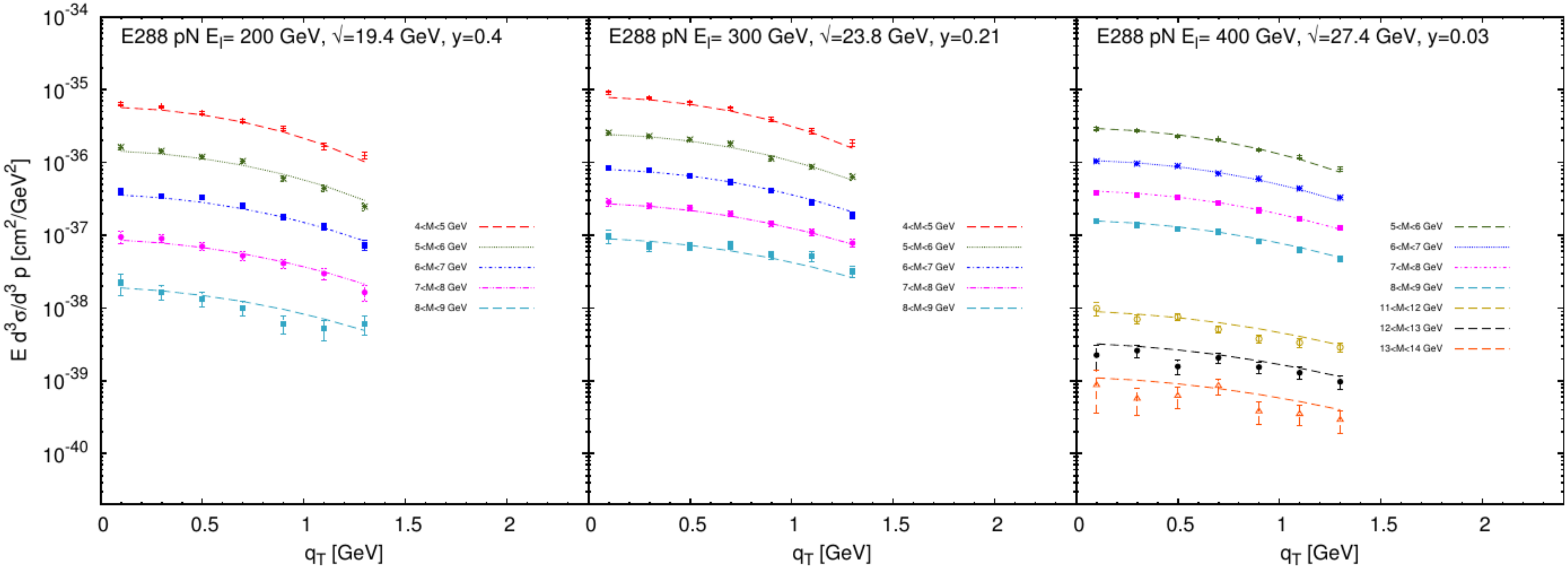
Results

NNLL	223 points	$\chi^2/d.o.f = 1.12$
	$h_1 = 0.33 \pm 0.05$	$h_2 = 0.13 \pm 0.03$
	$N_{E288} = 0.85 \pm 0.04$	$N_{R209} = 1.5 \pm 0.2$



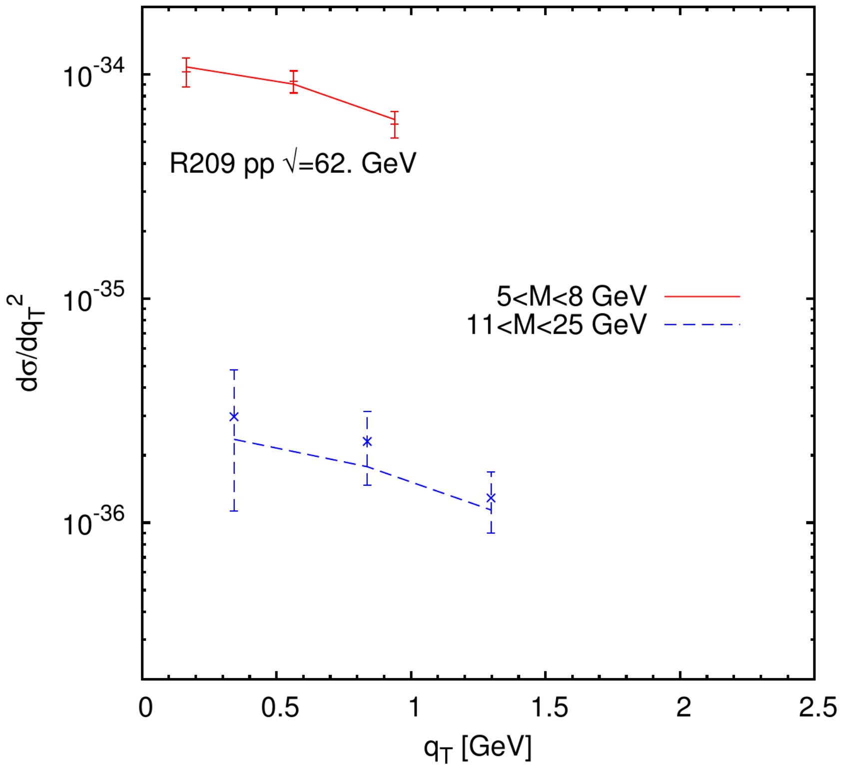
Results

NNLL	223 points	$\chi^2/d.o.f = 1.12$
	$h_1 = 0.33 \pm 0.05$	$h_2 = 0.13 \pm 0.03$
	$N_{E288} = 0.85 \pm 0.04$	$N_{R209} = 1.5 \pm 0.2$



Results

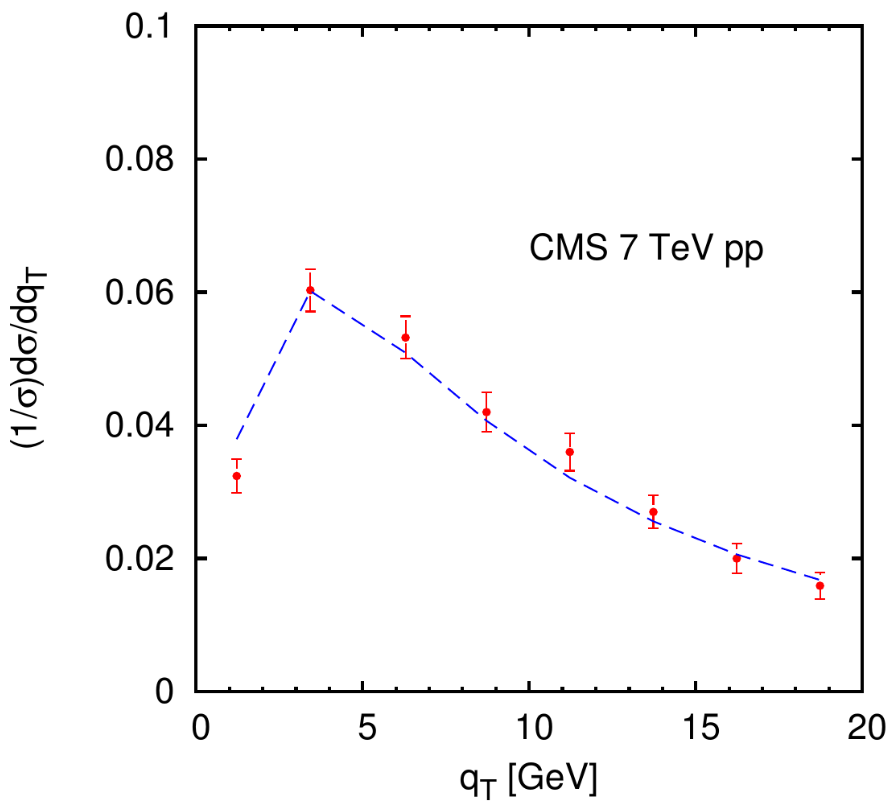
NNLL	223 points	$\chi^2/d.o.f = 1.12$
	$h_1 = 0.33 \pm 0.05$	$h_2 = 0.13 \pm 0.03$
	$N_{E288} = 0.85 \pm 0.04$	$N_{R209} = 1.5 \pm 0.2$



Results

NNLL	223 points	$\chi^2/d.o.f = 1.12$
	$h_1 = 0.33 \pm 0.05$	$h_2 = 0.13 \pm 0.03$
	$N_{E288} = 0.85 \pm 0.04$	$N_{R209} = 1.5 \pm 0.2$

- Prediction CMS



Results

NNLL	223 points	$\chi^2/d.o.f = 1.12$
	$h_1 = 0.33 \pm 0.05$	$h_2 = 0.13 \pm 0.03$
	$N_{E288} = 0.85 \pm 0.04$	$N_{R209} = 1.5 \pm 0.2$

NLL	223 points	$\chi^2/d.o.f = 1.51$
	$h_1 = 0.26 \pm 0.05$	$h_2 = 0.13 \pm 0.03$
	$N_{E288} = 0.89 \pm 0.04$	$N_{R209} = 1.3 \pm 0.2$

Conclusions

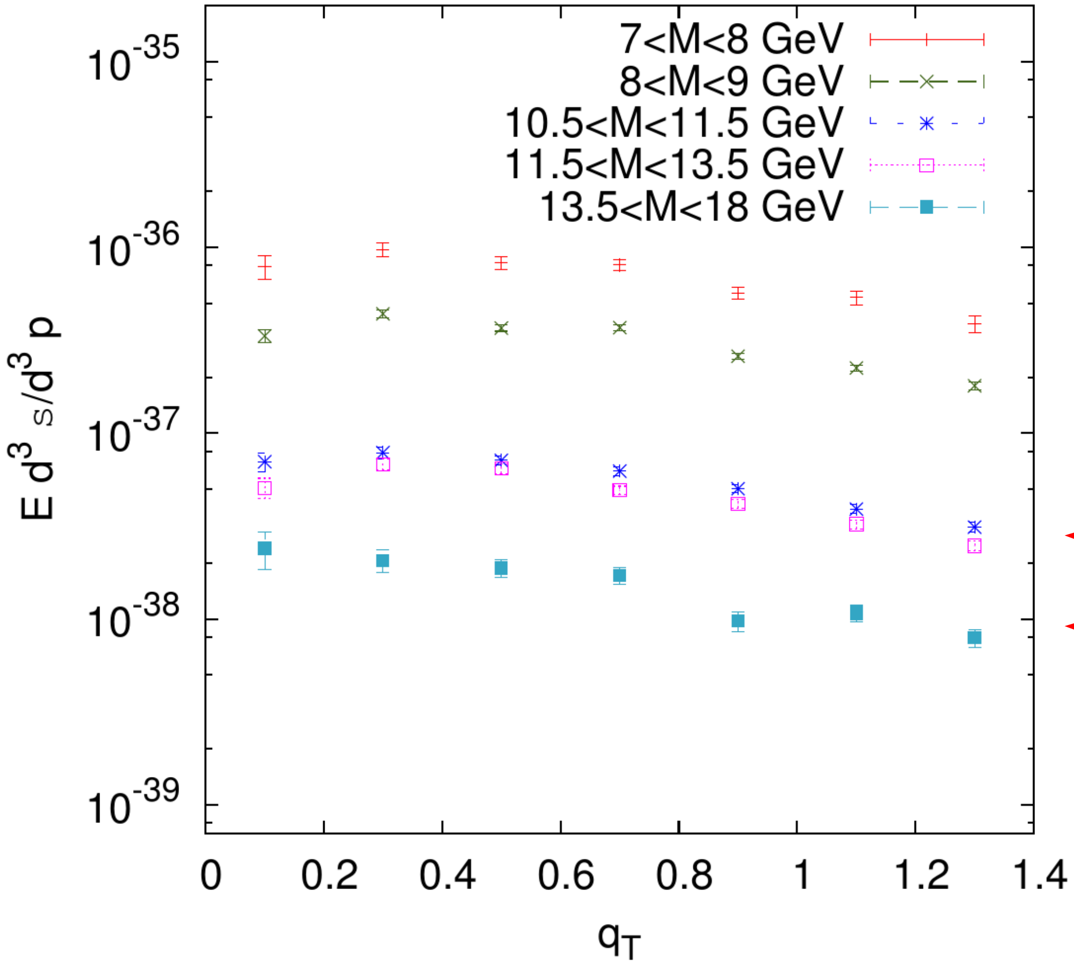
- The approach illustrated here tries to maximize the perturbative content of the TMDs
- We are able to fit successfully the low and high energy DY data with few parameters
- Low energy sector suffers many uncertainties (experimental and theoretical)
- High energy sector more under control (see pred. CMS)

appunti

	points	$\chi^2/points$	N_{exp}	h_1, h_2
NNLL	223	1.10		0.33 ± 0.05 , 0.13 ± 0.03
E288 200	35	1.53		
E288 300	35	1.50	$N_{E288} = 0.85 \pm 0.04$	
E288 400	49	2.07		
R209	6	0.16	$N_{R209} = 1.5 \pm 0.2$	
CDF Run I	32	0.74	-	
D0 Run I	16	0.43	-	
CDF Run II	41	0.30	-	
D0 Run II	9	0.61	-	

	points	$\chi^2/points$	N_{exp}	h_1, h_2
NLL	223	1.48		0.26 ± 0.05 , 0.13 ± 0.03
E288 200	35	2.60		
E288 300	35	1.12	$N_{E288} = 0.89 \pm 0.04$	
E288 400	49	1.79		
R209	6	0.25	$N_{R209} = 1.2 \pm 0.2$	
CDF Run I	32	1.31	-	
D0 Run I	16	1.44	-	
CDF Run II	41	0.62	-	
D0 Run II	9	2.40	-	

E605



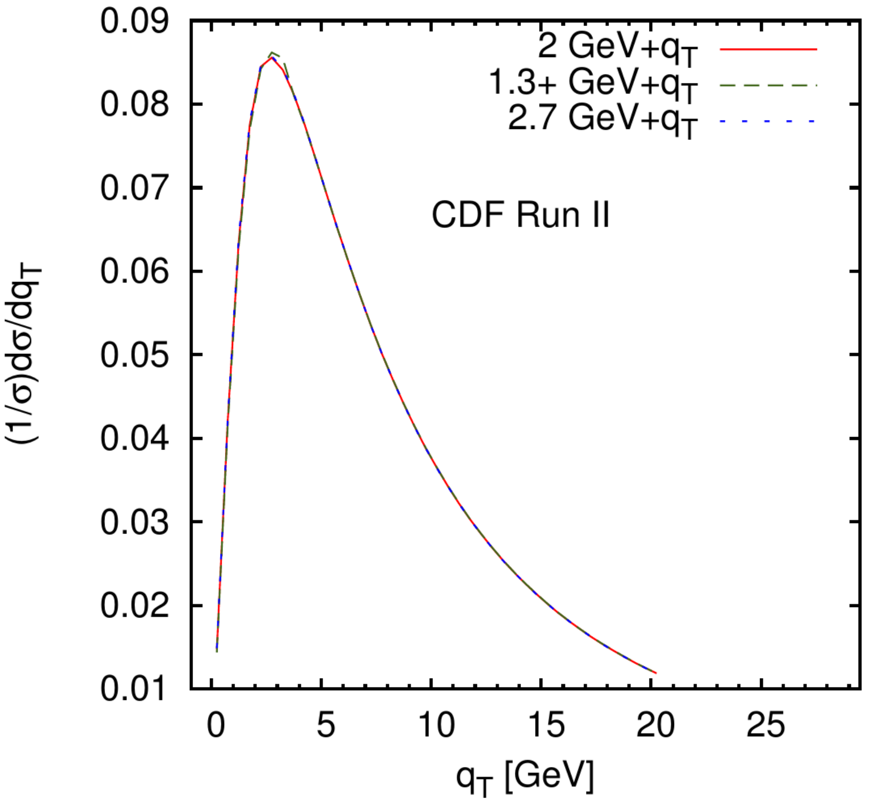
Can be fitted with a common normalization

These two bins largely overlap

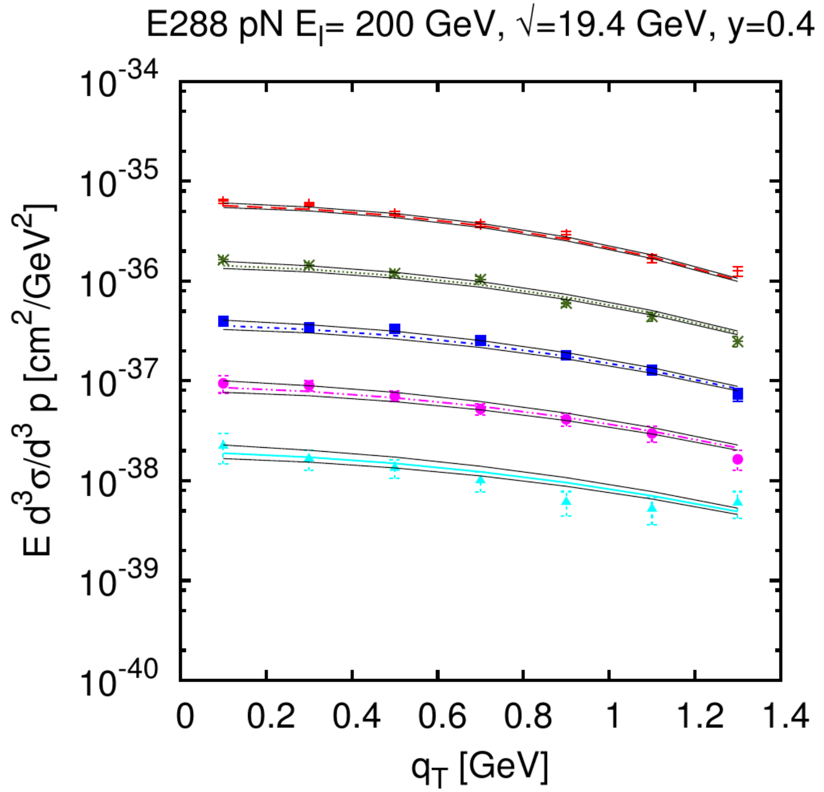
In general last 3 bins have a different normalization compared to first two

Data seem not to scale as $1/M^2$

Scale Error

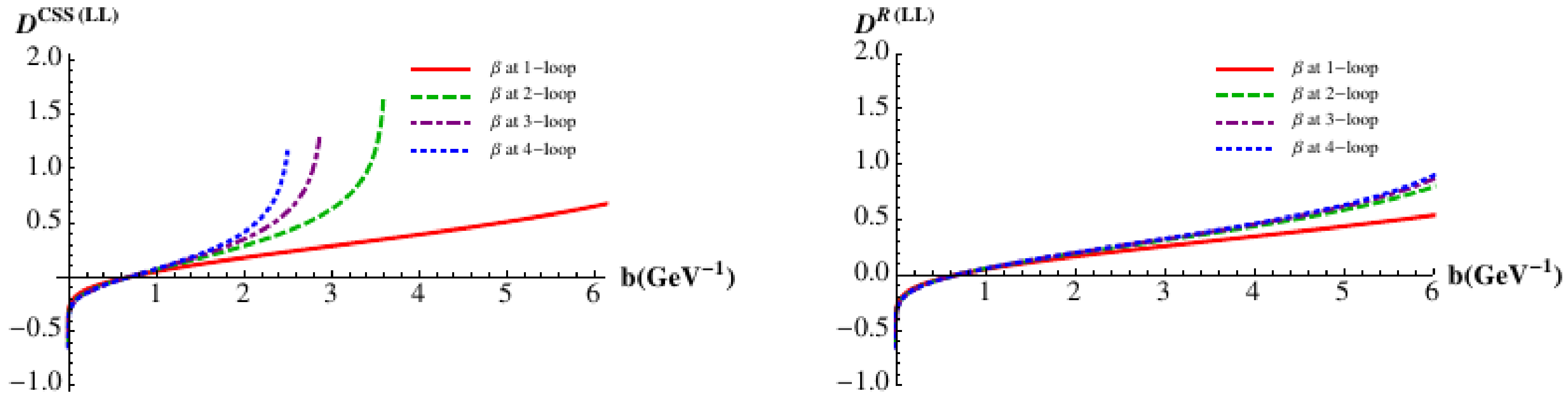


2 GeV+q_T
1.3+ GeV+q_T
2.7 GeV+q_T
CDF Run II
$(1/\sigma)d\sigma/dq_T$
q_T [GeV]



E288 pN E_l= 200 GeV, √=19.4 GeV, y=0.4
$E\ d^3\sigma/d^3p$ [cm^2/GeV^2]
q_T [GeV]

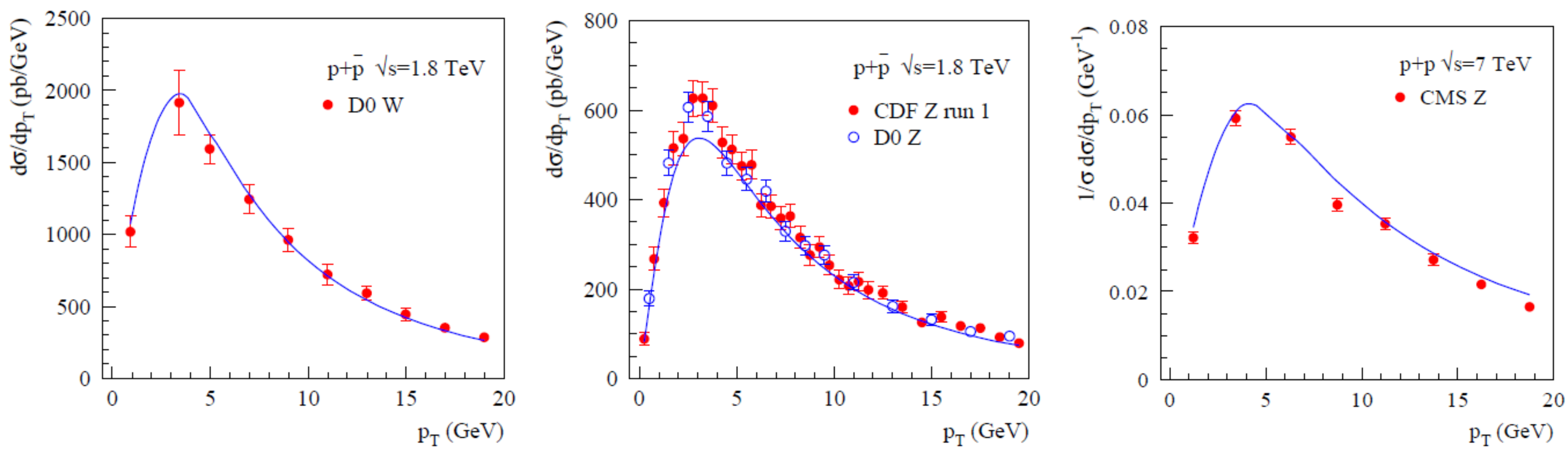
D^{CSS} vs D^R



Resummed $D(b; Q_i = \sqrt{2.4})$ *at LL* .

EIKV phenomenology

Z and W-Boson Production

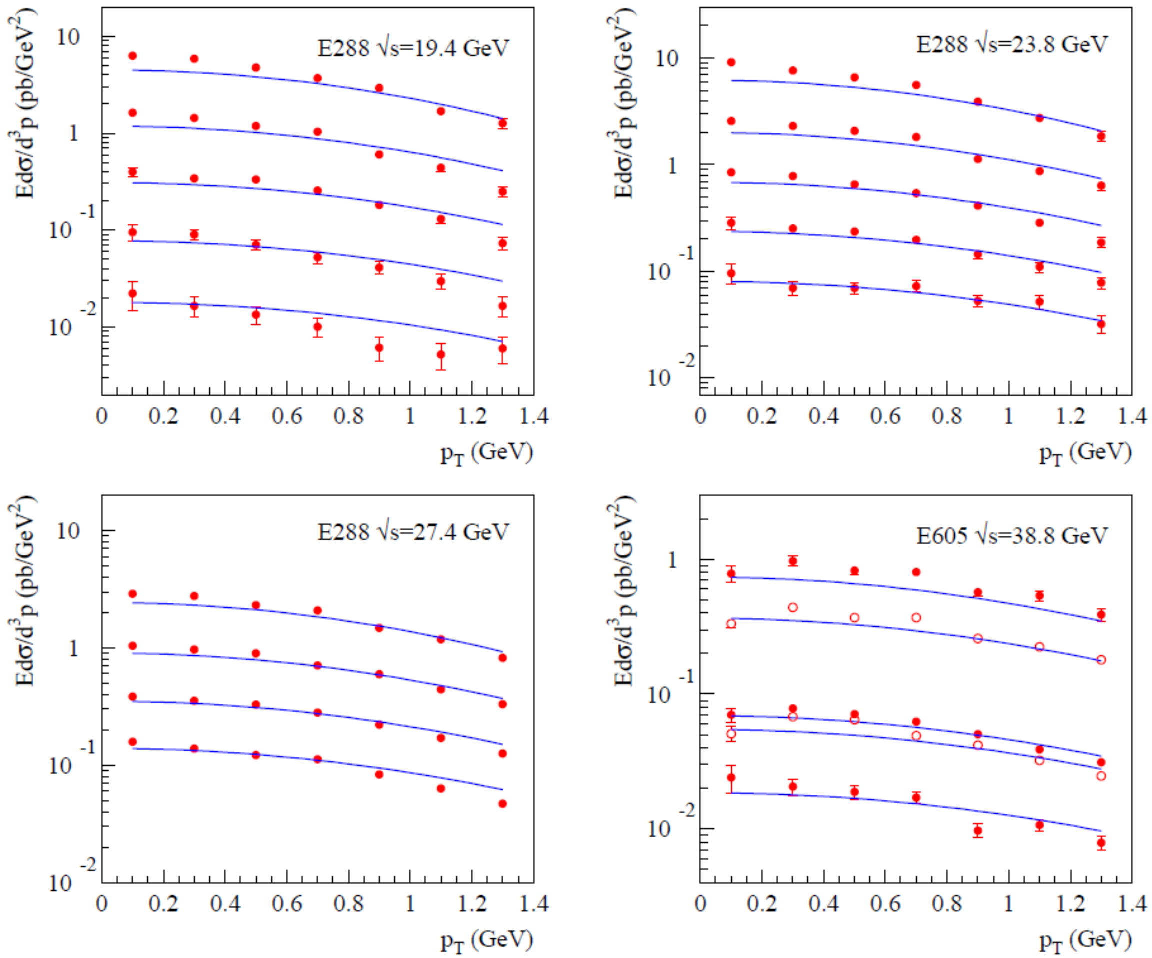


MSTW2008 PDF

Echevarria, Idilbi, Kang, Vitev Arxiv: 1401.5078

EIKV phenomenology

Low energy Drell-Yan

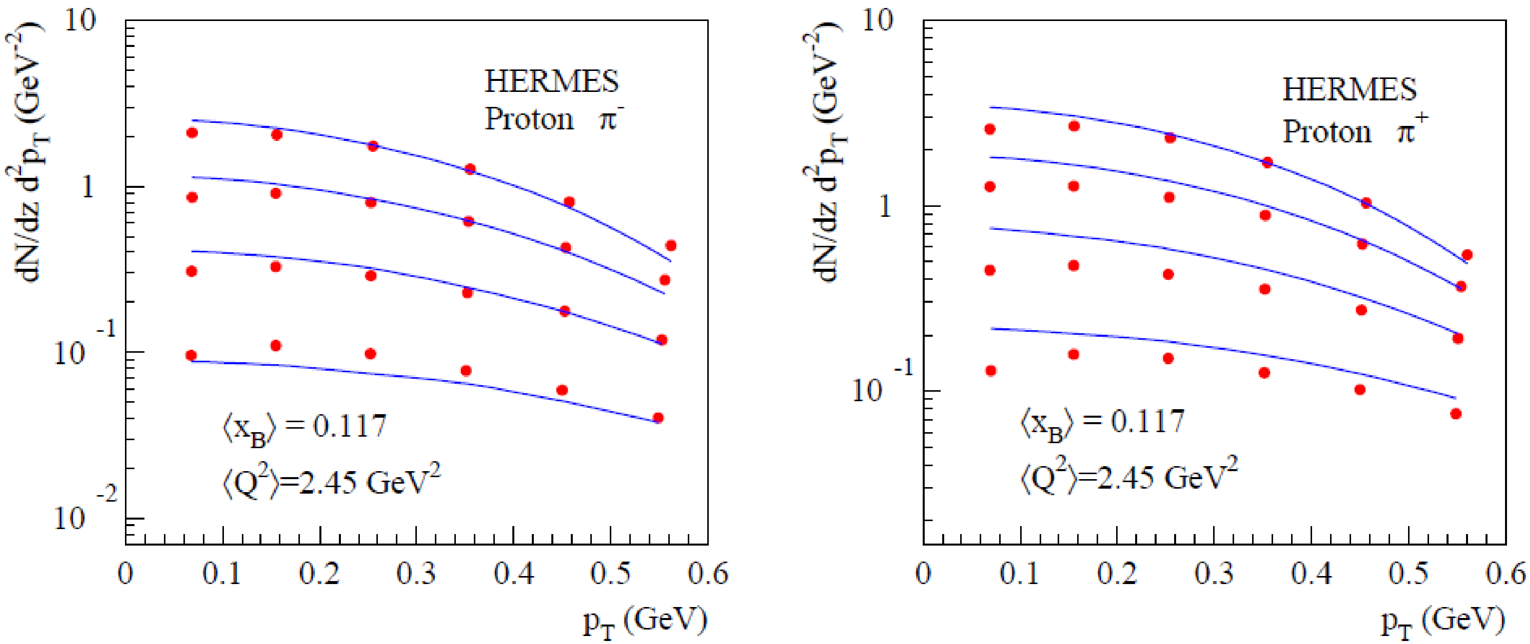


EKS98 Cu PDF

Echevarria, Idilbi, Kang, Vitev Arxiv: 1401.5078

EIKV phenomenology

HERMES SIDIS data

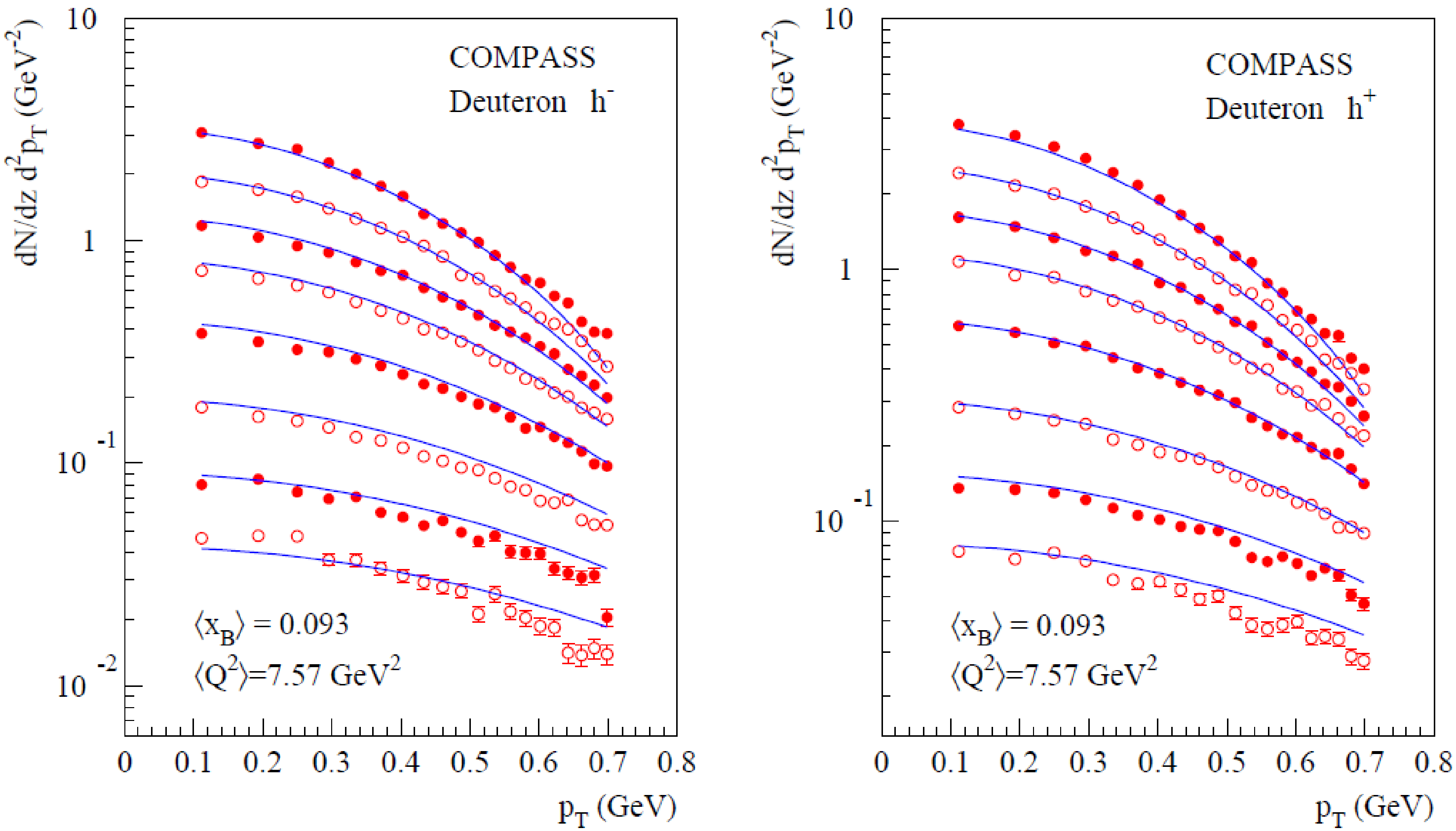


MSTW2008 PDF and DSS

Echevarria, Idilbi, Kang, Vitev Arxiv: 1401.5078

EIKV phenomenology

(some...) COMPASS SIDIS data



MSTW2008 PDF and DSS

Echevarria, Idilbi, Kang, Vitev Arxiv: 1401.5078

CSS Phenomenology

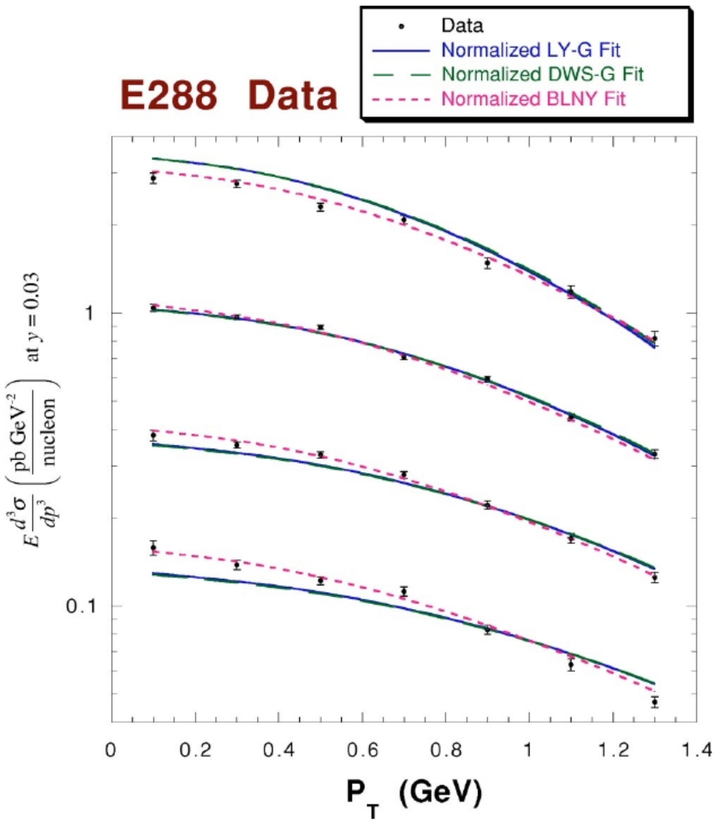
Nadolsky et al. Analyzed low energy DY data and Z boson production data Using different parametrizations

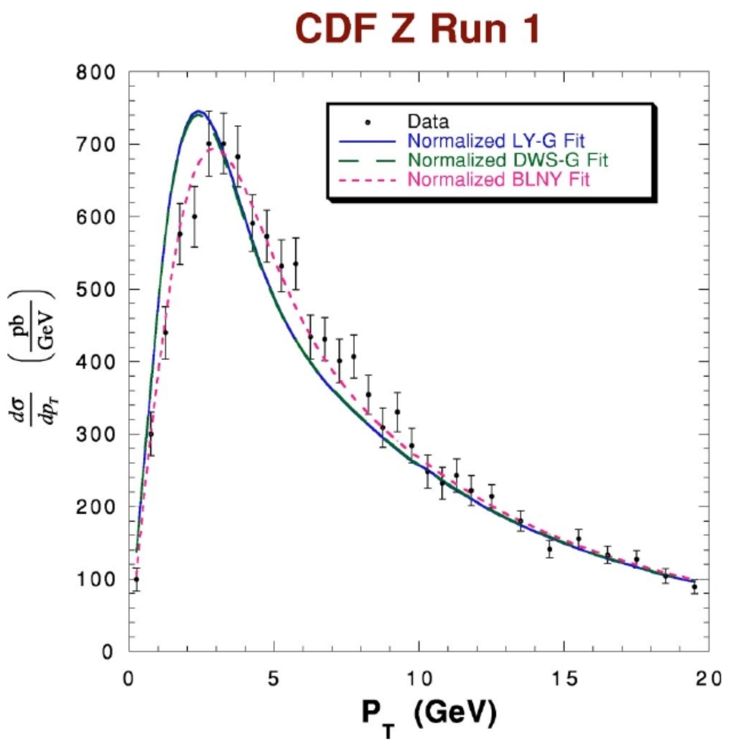
Parameter	DWS-G fit	LY-G fit	BLNY fit
g_1	0.016	0.02	0.21
g_2	0.54	0.55	0.68
g_3	0.00	-1.50	-0.60
CDF Z Run-0 N_{fit}	1.00 (fixed)	1.00 (fixed)	1.00 (fixed)
R209 N_{fit}	1.02	1.01	0.86
E605 N_{fit}	1.15	1.07	1.00
E288 N_{fit}	1.23	1.28	1.19
DØ Z Run-1 N_{fit}	1.01	1.01	1.00
CDF Z Run-1 N_{fit}	0.89	0.90	0.89
χ^2	416	407	176
χ^2/DOF	3.47	3.42	1.48

$$b_{max} = 0.5\ \text{GeV}^{-1}$$

See,Nadolsky et al., Phys.Rev. D67,073016 (2003)

CSS Phenomenology



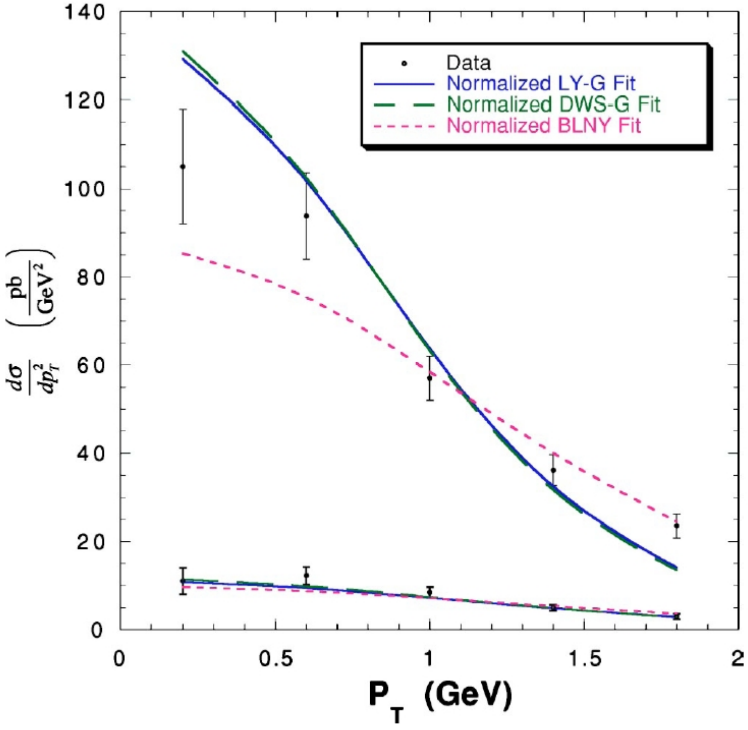


$b_{max} = 0.5\ \text{GeV}^{-1}$

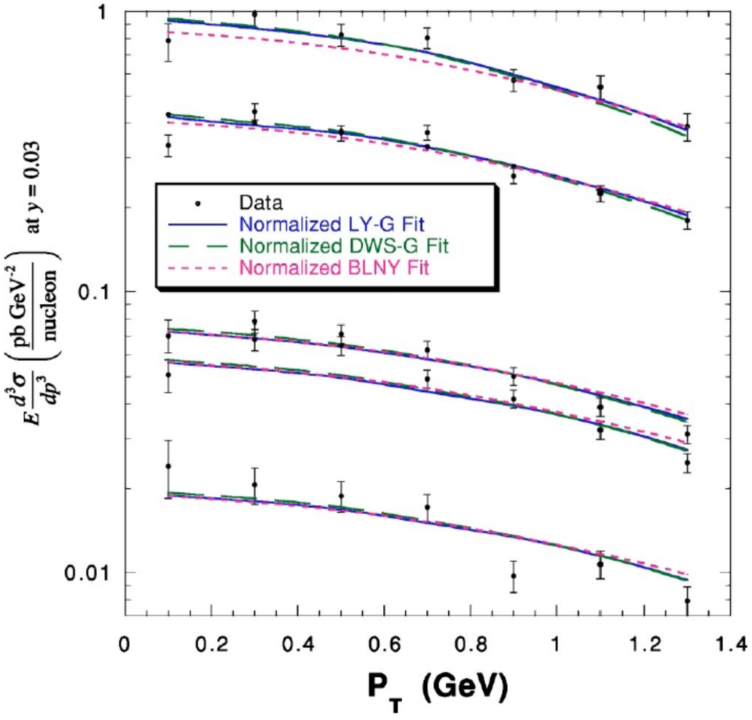
See,Nadolsky et al., Phys.Rev. D67,073016 (2003)

CSS Phenomenology

R209 Data



E605 Data

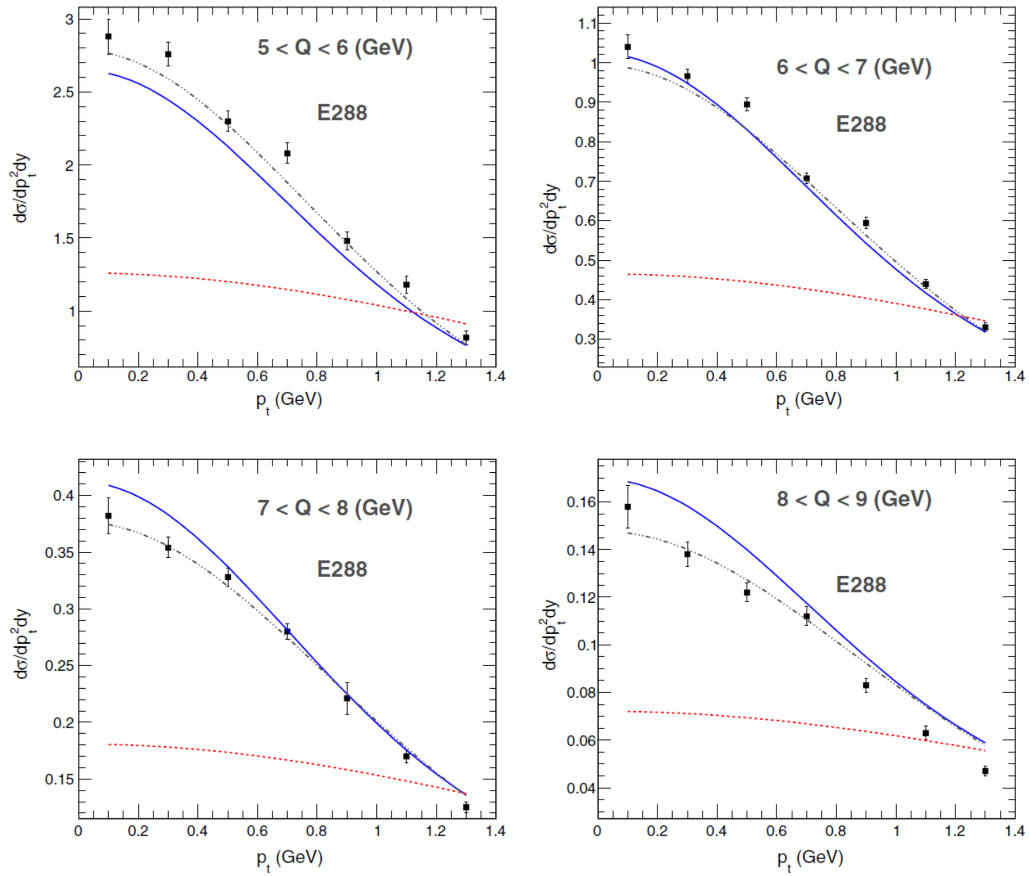


$$b_{max} = 0.5 \text{ GeV}^{-1}$$

See,Nadolsky et al., Phys.Rev. D67,073016 (2003)

Yuan-Sun phenomenolgy

- Gaussian parametrization for the PDF and the fragmentation function at the scale of HERMES.
- Parameters g_0 and g_h as in Schweitzer et al, Phys. ReV. D81,094019 (2010)



Roger-Aybat

Yuan-Sun

CSS

Sun and Yuan, Phys. Rev. D88, 034016 (2013), Phys. Rev. D88, 114012 (2013)

TMD Collins

TMD evolution formalism

- The simplest version of the Collins TMD evolution equation can be summarized by the following expression:

$$\widetilde{F}(x, \boldsymbol{b}_T; Q) = \widetilde{F}(x, \boldsymbol{b}_T; Q_0)\, \widetilde{R}(Q, Q_0, b_T)\ \exp\left\{ -g_K(b_T) \ln \frac{Q}{Q_0} \right\}$$

Corresponding to Eq. 44 of Ref [*] with $\tilde{K}$=0 and : $\mu^2 = \zeta_F = \zeta_D = Q^2$

- [*]S. M. Aybat, J. C. Collins, J.-W. Qiu and T.C. Rogers, arXiv:1110.6428 [hep-ph]

TMD evolution formalism

- At LO the evolution equation can be summarized by the following expression:

$$\widetilde{F}(x, \boldsymbol{b}_T; Q) = \widetilde{F}(x, \boldsymbol{b}_T; Q_0)\, \widetilde{R}(Q, Q_0, b_T)\; \exp\left\{-g_K(b_T)\ln\frac{Q}{Q_0}\right\}$$

Output function at the scale **Q** in the impact parameter space

Input function at the scale $\mathbf{Q_0}$ in the impact parameter space

Evolution kernel

TMD evolution formalism

- At LO the evolution equation can be summarized by the following expression:

$$\widetilde{F}(x, \boldsymbol{b}_T; Q) = \widetilde{F}(x, \boldsymbol{b}_T; Q_0)\, \widetilde{R}(Q, Q_0, b_T)\, \exp\left\{-g_K(b_T) \ln \frac{Q}{Q_0}\right\}$$

- **Perturbative** part of the evolution kernel

TMD evolution formalism

- At LO the evolution equation can be summarized by the following expression:

$$\widetilde{F}(x, \boldsymbol{b}_T; Q) = \widetilde{F}(x, \boldsymbol{b}_T; Q_0)\, \widetilde{R}(Q, Q_0, b_T)\, \exp\left\{-g_K(b_T) \ln \frac{Q}{Q_0}\right\}$$

- **Perturbative** part of the evolution kernel

$$\widetilde{R}(Q, Q_0, b_T) \equiv \exp\left\{\ln \frac{Q}{Q_0} \int_{Q_0}^{\mu_b} \frac{\mathrm{d}\mu'}{\mu'} \gamma_K(\mu') + \int_{Q_0}^{Q} \frac{\mathrm{d}\mu}{\mu} \gamma_F\left(\mu, \frac{Q^2}{\mu^2}\right)\right\}$$

TMD evolution formalism

➢ At LO the evolution equation can be summarized by the following expression:

$$\widetilde{F}(x, \boldsymbol{b}_T; Q) = \widetilde{F}(x, \boldsymbol{b}_T; Q_0)\, \widetilde{R}(Q, Q_0, b_T)\, \exp\left\{-g_K(b_T)\ln\frac{Q}{Q_0}\right\}$$

➢ **Perturbative** part of the evolution kernel

$$\widetilde{R}(Q, Q_0, b_T) \equiv \exp\left\{\ln\frac{Q}{Q_0}\int_{Q_0}^{\mu_b}\frac{d\mu'}{\mu'}\gamma_K(\mu') + \int_{Q_0}^{Q}\frac{d\mu}{\mu}\gamma_F\left(\mu, \frac{Q^2}{\mu^2}\right)\right\}$$

$$\gamma_K(\mu) = \alpha_s(\mu)\,\frac{2\,C_F}{\pi}$$

$$\gamma_F(\mu; \frac{Q^2}{\mu^2}) = \alpha_s(\mu)\,\frac{C_F}{\pi}\left(\frac{3}{2} - \ln\frac{Q^2}{\mu^2}\right)$$

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Scale that separates the perturbative region
from the non perturbative one

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$$\mu_b = \frac{C_1}{b_*(b_T)} \qquad b_*(b_T) \equiv \frac{b_T}{\sqrt{1 + b_T^2/b_{\max}^2}} \qquad C_1 = 2e^{-\gamma_E}$$

One of the possible prescription to separate the perturbative region from the non perturbative one

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- **Non Perturbative** (scale independent) part of the evolution kernel that needs to be empirically modeled

$$g_K(b_T) = \frac{1}{2}\, g_2\, b_T^2$$

Common choice used in the unpolarized DY data analyses in the CSS formalism

Parametrization of the input functions

$$\widetilde{F}(x, \boldsymbol{b}_T; Q) = \widetilde{F}(x, \boldsymbol{b}_T; Q_0) \widetilde{R}(Q, Q_0, b_T) \, \exp\left\{ -g_K(b_T) \ln \frac{Q}{Q_0} \right\}$$

- Model/parametrization: Different parametrizations here can give very different answers!
- Our approach: Let us apply our standard parametrizations i.e. gaussians factorized among collinear and transverse degree of freedom. It is not a unique choice or the best one!

Parametrization of the input functions

- TMD evolution equations using a gaussian model::

$$\widetilde{f}_{q/p}(x,b_T;Q) = f_{q/p}(x,Q_0)\,\widetilde{R}(Q,Q_0,b_T)\;\exp\left\{-b_T^2\left(\alpha^2 + \frac{g_2}{2}\ln\frac{Q}{Q_0}\right)\right\}$$

$$\widetilde{D}_{h/q}(z,b_T;Q) = \frac{1}{z^2}D_{h/q}(z,Q_0)\,\widetilde{R}(Q,Q_0,b_T)\;\exp\left\{-b_T^2\left(\beta^2 + \frac{g_2}{2}\ln\frac{Q}{Q_0}\right)\right\}$$

$$\widetilde{f}'^{\perp}_{1T}(x,b_T;Q) = -2\,\gamma^2\,f^{\perp}_{1T}(x;Q_0)\,\widetilde{R}(Q,Q_0,b_T)\,b_T\;\exp\left\{-b_T^2\left(\gamma^2 + \frac{g_2}{2}\ln\frac{Q}{Q_0}\right)\right\}$$

Collins TMD evolution of the Sivers function (PRD85,2012)

$$\tilde{F}'^{\perp f}_{1T}(x, b_T; \mu, \zeta_F) = \tilde{F}'^{\perp f}_{1T}(x, b_T; \mu_0, Q_0^2) \exp\left\{ \ln \frac{\sqrt{\zeta_F}}{Q_0} \tilde{K}(b_*; \mu_b) + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(g(\mu'); 1) - \ln \frac{\sqrt{\zeta_F}}{\mu'} \gamma_K(g(\mu')) \right] + \int_{\mu_0}^{\mu_b} \frac{d\mu'}{\mu'} \ln \frac{\sqrt{\zeta_F}}{Q_0} \gamma_K(g(\mu')) - g_K(b_T) \ln \frac{\sqrt{\zeta_F}}{Q_0} \right\}. \quad (44)$$

$$\tilde{F}'^{\perp f}_{1T}(x, b_T; \mu, \zeta_F) = \sum_j \frac{M_p b_T}{2} \int_x^1 \frac{d\hat{x}_1\, d\hat{x}_2}{\hat{x}_1\, \hat{x}_2} \tilde{C}^{\text{Sivers}}_{f/j}(\hat{x}_1, \hat{x}_2, b_*; \mu_b^2, \mu_b, g(\mu_b))\, T_{F\, j/P}(\hat{x}_1, \hat{x}_2, \mu_b)$$
$$\times \exp\left\{ \ln \frac{\sqrt{\zeta_F}}{\mu_b} \tilde{K}(b_*; \mu_b) + \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(g(\mu'); 1) - \ln \frac{\sqrt{\zeta_F}}{\mu'} \gamma_K(g(\mu')) \right] \right\} \times \exp\left\{ -g^{\text{Sivers}}_{f/P}(x, b_T) - g_K(b_T) \ln \frac{\sqrt{\zeta_F}}{Q_0} \right\}. \quad (47)$$

Collins TMD evolution of the unpolarized PDF (PRD83,114042,2011)

$$
\begin{aligned}
\tilde{F}_{f/P}(x, \mathbf{b}_T; \mu, \zeta_F) = {} & \overbrace{\sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{f/j}(x/\hat{x}, b_*; \mu_b^2, \mu_b, g(\mu_b)) f_{j/P}(\hat{x}, \mu_b)}^{A} \\
& \times \overbrace{\exp\left\{ \ln\frac{\sqrt{\zeta_F}}{\mu_b} \tilde{K}(b_*; \mu_b) + \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(g(\mu'); 1) - \ln\frac{\sqrt{\zeta_F}}{\mu'} \gamma_K(g(\mu')) \right] \right\}}^{B} \\
& \times \overbrace{\exp\left\{ g_{j/P}(x, b_T) + g_K(b_T) \ln\frac{\sqrt{\zeta_F}}{\sqrt{\zeta_{F,0}}} \right\}}^{C},
\end{aligned}
\tag{26}
$$

TMD evolution of the Sivers function

$$
\begin{aligned}
\frac{\tilde{f}'^{\perp}_{1T}(x,b_T,Q,Q)}{\tilde{f}'^{\perp}_{1T}(x,b_T,Q_0,Q_0)} &= \exp\left\{\int_Q^{Q_0}\frac{d\kappa}{\kappa}\left[\gamma_F(\kappa;1)-\gamma_K(\kappa)\ln\left(Q/\kappa\right)\right]\right\} \\
&\quad \exp\left[-\int_{\mu_b}^{Q_0}\frac{d\kappa}{\kappa}\gamma_K(\kappa)\ln\left(Q/Q_0\right)\right]\exp\left[-g_K(b_T)\ln\left(Q/Q_0\right)\right] \\
&= \tilde{R}(Q,Q_0,b_T)\exp\left[-g_K(b_T)\ln\left(Q/Q_0\right)\right]
\end{aligned}
$$

Notice that:

$$
\frac{\tilde{f}'^{\perp}_{1T}(x,b_T,Q,\zeta_F)}{\tilde{f}'^{\perp}_{1T}(x,b_T,Q_0,\zeta_{F0})} = \frac{\tilde{f}_1(x,b_T,Q,\zeta_F)}{\tilde{f}_1(x,b_T,Q_0,\zeta_{F0})} \equiv \frac{\tilde{F}(x,b_T,Q,\zeta_F)}{\tilde{F}(x,b_T,Q_0,\zeta_{F0})}
$$

Aybat, Collins, Qiu, Rogers, Phys. Rev. D85, 034043 (2012)

TABLE I. Resummation scheme

order	H	$\hat{C}_{q\leftarrow j}$	$\Gamma_{\rm cusp}$	γ^V	D^R	h_Γ^R	h_γ^R
LL	tree	tree	α_s^1	α_s^0	D^{R0}	h_Γ^{R0}	h_γ^{R0}
NLL	tree	tree	α_s^2	α_s^1	D^{R1}	h_Γ^{R1}	h_γ^{R1}
NNLL	NLO	NLO	α_s^3	α_s^2	D^{R2}	h_Γ^{R2}	h_γ^{R2}