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Multiple solutions for a class of perturbed second-order differential equations with impulses

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Abstract

The present paper is an attempt to investigate the existence of weak solutions for perturbed impulsive problems containing a Lipschitz nonlinear term. The study bases itself on the most recent variational approaches to the smooth functionals which are defined on reflexive Banach spaces. The findings of the study, finally, revealed that, under appropriate conditions, such problems possess at least three weak solutions. According to the results, these solutions are generated by impulses when the Lipschitz nonlinear term is zero.

Keywords: multiple solutions; perturbed impulsive differential equation; critical point theory; variational methods

1 Introduction

This paper attempts to study the existence of three weak solutions for the perturbed impulsive problem

$$\begin{cases} \ddot{u}(t) + V_u(t,u(t)) = h(u(t)), \quad t \in (s_{k-1}, s_k), \\ \Delta\dot{u}(s_k) = \lambda f_k(u(s_k)) + \mu g_k(u(s_k)), \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1}$$

where s_k, $k = 1, 2, \ldots, m$, are instants in which the impulses occur and $0 = s_0 < s_1 < s_2 \cdots < s_m < s_{m+1} = T$, $\Delta\dot{u}(s_k) = \dot{u}(s_k^+) - \dot{u}(s_k^-)$ with $\dot{u}(s_k^{\pm}) = \lim_{t\to s_k^{\pm}} \dot{u}(t)$, $f_k(\xi) = \operatorname{grad}_\xi F_k(\xi)$, $g_k(\xi) = \operatorname{grad}_\xi G_k(\xi)$, $h(\xi) = \operatorname{grad}_\xi H(\xi)$, $F_k, G_k, H \in C^1(\mathbb{R}^N, \mathbb{R})$, $V \in C^1([0,T] \times \mathbb{R}^N, \mathbb{R})$, $V_\xi(t,\xi) = \operatorname{grad}_\xi V(t,\xi)$, $h: \mathbb{R}^N \to \mathbb{R}^N$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, *i.e.*,

$$\left|h(\xi_1) - h(\xi_2)\right| \le L|\xi_1 - \xi_2|$$

for every $\xi_1, \xi_2 \in \mathbb{R}^N$ and $h(0) = 0$, and $\lambda > 0$ and $\mu \ge 0$ are two parameters.

Impulsive differential equations emerge from the real world problems and are acclimated to be employed as handy means for the description of the processes which are endowed with abrupt discontinuous jumps. As for this, these processes are used in such a vast array of fields as control theory, biology, impact mechanics, physics, chemistry, chemical engineering, population dynamics, biotechnology, economics, optimization theory, and the inspection process in operations research. That is why the theory of impulsive differential equations is now highly appreciated as a natural theoretical basis for the mathematical modeling of the natural phenomena of various kinds. For a comprehensive background in the theory and the applications of the impulsive differential equations, we hereby refer the interested reader to [1–11].

There is already a large body of research on the notion of impulsive differential equations in the literature. The findings of most of these studies are mainly achieved through some such theories as fixed point theory, topological degree theory (including continuation method and coincidence degree theory) and comparison method (including upper and lower solutions method and monotone iterative method) (see, for example, [12–15] and references therein). Recently, the existence and multiplicity of solutions for impulsive problems have been thoroughly investigated by [16–25] using variational methods and the critical point theory, the whole findings of which can be considered as nothing but generalizations of the corresponding ones for the second-order ordinary differential equations. Put differently, the aforementioned achievements can be applied to impulsive systems in the absence of the impulses and still give the existence of solutions in this situation. This is, somehow, to say that the nonlinear term V_u functions more significantly as compared to the role played by the impulsive terms f_k in guaranteeing the existence of solutions in these results. In [26], which is a probe into the existence of periodic and homoclinic solutions for a class of second-order differential equations of the form (1) in the case $\mu = 0$, via variational methods, the results signify that such a system enjoys at least one non-zero periodic solution as well as one non-zero homoclinic solution under appropriate conditions, and these solutions are generated by impulses when $f = 0$. Based on the variational methods and the critical point theory, [27] has examined problem (1) in the case $\mu = 0$, by means of which the authors have proved that such a problem admits at least one non-zero, two non-zeros, or an infinite number of periodic solutions as yielded by the impulses under different assumptions, respectively. Most particularly, using a smooth version of Theorem 2.1 in [28] which is a more precise version of Ricceri's variational principle ([29], Theorem 2.5) under some hypotheses on the behavior of the nonlinear terms at infinity, under conditions on the potentials of f_k and g_k, [30] has proved that the existence of definite intervals about λ and μ, in which problem (1) in the case $h \equiv 0$ admits an unbounded sequence of solutions generated by impulses. Moreover, it has been proved that replacing the conditions at infinity of the nonlinear terms with a similar one at zero admits the same results.

In the present paper, employing two sorts of three critical points theorems obtained in [31, 32], which we will recall in the next section (Theorems 2.1 and 2.2), we establish the existence of at least three weak solutions for problem (1). We also verify that these solutions are generated by impulses when $h \equiv 0$; see Theorems 3.1 and 3.2. We say that a solution of the problem (1) is called a solution generated by impulses if this solution is nontrivial when impulsive terms $f_k, g_k \not\equiv 0$ for some $1 \leq k \leq m$, but it is trivial when impulsive terms are zero. For example, if the problem (1) does not possess non-zero weak solution when $f_k = g_k \equiv 0$ for all $1 \leq k \leq m$, then a non-zero weak solution for problem (1) with $f_k, g_k \not\equiv 0$ for some $1 \leq k \leq m$ is called a weak solution generated by impulses. Along the same lines of reasoning, these theorems (Theorems 2.1 and 2.2) have been successfully employed by [33–35] to ensure the presence of at least three solutions for the perturbed boundary value problems.

The curious reader is also referred to [36–41], which have verified the existence of multiple solutions for boundary value problems. For a thorough study of the subject, we also refer the reader to [42–48].

The organization of the present paper is as follows. In Section 2 we recall some basic definitions and preliminary results, while Section 3 is devoted to the existence of multiple solutions for the impulsive differential problem (1).

2 Preliminaries

Our fundamental tool consists of three critical point theorems. In the first one, the coercivity of the functional $\Phi - \lambda\Psi$ is essential. In the second one, a proper sign hypothesis has been assumed.

Theorem 2.1 ([32], Theorem 2.6) *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$.*

Assume that there exist $r > 0$ and $\bar{v} \in X$, with $r < \Phi(\bar{v})$ such that

(a$_1$) $\frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} < \frac{\Psi(\bar{v})}{\Phi(\bar{v})}$;
(a$_2$) *for each $\lambda \in \Lambda_r :=]\frac{\Phi(\bar{v})}{\Psi(\bar{v})}, \frac{r}{\sup_{\Phi(u)\leq r} \Psi(u)}[$ the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2.2 ([31], Theorem 3.3) *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

1. $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
2. *for every $\lambda > 0$ and for every $u_1, u_2 \in X$ which are local minima for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has*

$$\inf_{s\in[0,1]} \Psi\big(su_1 + (1 - s)u_2\big) \geq 0.$$

Assume that there are two positive constants r_1, r_2 and $\bar{v} \in X$, with $2r_1 < \Phi(\bar{v}) < \frac{r_2}{2}$, such that

(b$_1$) $\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(\bar{v})}{\Phi(\bar{v})}$;
(b$_2$) $\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\bar{v})}{\Phi(\bar{v})}$.

Then, for each $\lambda \in]\frac{3}{2}\frac{\Phi(\bar{v})}{\Psi(\bar{v})}, \min\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}, \frac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}\}[$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(] - \infty, r_2[)$.

In this paper we consider the Hilbert space

$$X = \big\{u : [0, T] \to \mathbb{R}^N \mid u \text{ is absolutely continuous}, u(0) = u(T), \dot{u} \in \mathrm{L}^2\big([0, T], \mathbb{R}^N\big)\big\}$$

with the inner product

$$\langle u, v\rangle = \int_0^T \bigl[\bigl(u(t), v(t)\bigr) + \bigl(\dot{u}(t), \dot{v}(t)\bigr)\bigr]\,\mathrm{d}t \quad \text{for all } u, v \in X,$$

where $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^N$. Obviously, the corresponding norm into the above inner product is as follows:

$$\|u\| = \left(\int_0^T \bigl(\bigl|\dot{u}(t)\bigr|^2 + \bigl|u(t)\bigr|^2\bigr)\,\mathrm{d}t\right)^{\frac{1}{2}} \quad \text{for all } u \in X,$$

and X with this norm is a separable and uniformly convex Banach space.

Since the embedding $X \hookrightarrow \mathrm{C}([0,T], \mathbb{R}^N)$ is compact (see [49]), one has

$$C := \sup_{u \in X\setminus\{0\}} \frac{\max_{t\in[0,T]} |u(t)|}{\|u\|} < \infty. \tag{2}$$

We say that $u \in X$ is a weak solution of the problem (1) if

$$\int_0^T \bigl[\bigl(\dot{u}(t), \dot{v}(t)\bigr) - \bigl(V_u\bigl(t, u(t)\bigr), v(t)\bigr) + \bigl(h\bigl(u(t)\bigr), v(t)\bigr)\bigr]\,\mathrm{d}t + \lambda \sum_{k=1}^m \bigl(f_k\bigl(u(s_k)\bigr), v(s_k)\bigr)$$
$$+ \mu \sum_{k=1}^m \bigl(g_k\bigl(u(s_k)\bigr), v(s_k)\bigr) = 0$$

for every $v \in X$.

Moreover, set

$$G^\theta := \max_{|t|\le\theta}\left[-\sum_{k=1}^m G_k(t)\right]$$

for every $\theta > 0$ and

$$G_\eta := \inf_{[0,\eta]}\left[-\sum_{k=1}^m G_k(t)\right]$$

for every $\eta > 0$. It is obvious that $G^\theta \ge 0$ and $G_\eta \le 0$.

We consider the following assumptions on V:

(A1) V is continuously differentiable and there exist two positive constants $a_1, a_2 > 0$ so that $a_1|\xi|^2 \le -V(t,\xi) \le a_2|\xi|^2$ for all $(t,\xi) \in [0,T] \times \mathbb{R}^N$;

(A2) $-V(t,\xi) \le -(V_\xi(t,\xi),\xi) \le -2V(t,\xi)$ for all $(t,\xi) \in [0.T] \times \mathbb{R}^N$;

(A3) $V_{\xi_1-\xi_2}(t,\xi_1-\xi_2) = V_{\xi_1}(t,\xi_1) - V_{\xi_2}(t,\xi_2)$ for all $t \in [0,T]$ and $\xi_1, \xi_2 \in \mathbb{R}^N$.

We assume throughout and without further mention that the Lipschitz constant $L > 0$ of the function h meets the condition

$$\min\left\{\frac{1}{2}, a_1\right\} > TLC^2.$$

We require the proposition below in proving Theorem 3.1.

Proposition 2.3 *Let the assumptions* (A1), (A2), *and* (A3) *be satisfied and* $K : X \to X^*$ *be the operator defined by*

$$K(u)v = \int_0^T \bigl[\bigl(\dot{u}(t), \dot{v}(t)\bigr) - \bigl(V_u\bigl(t, u(t)\bigr), v(t)\bigr) + \bigl(h\bigl(u(t)\bigr), v(t)\bigr)\bigr]\,\mathrm{d}t.$$

Then K *admits a continuous inverse on* X^*.

Proof Since $|h(\xi_1) - h(\xi_2)| \leq L|\xi_1 - \xi_2|$ for every $\xi_1, \xi_2 \in \mathbb{R}^N$, using the Cauchy-Schwarz inequality one has $-L|\xi_1 - \xi_2|^2 \leq (h(\xi_1) - h(\xi_2), \xi_1 - \xi_2) \leq L|\xi_1 - \xi_2|^2$ for every $\xi_1, \xi_2 \in \mathbb{R}^N$. So, taking (2) into account, bearing in mind that $h(0) = 0$, we have

$$\begin{aligned}
\bigl\langle K(u), u\bigr\rangle &= \int_0^T \bigl(\bigl|\dot{u}(t)\bigr|^2 - \bigl(V_u\bigl(t, u(t)\bigr), u(t)\bigr) + \bigl(h\bigl(u(t)\bigr), u(t)\bigr)\bigr)\,\mathrm{d}t \\
&\geq \int_0^T \bigl(\bigl|\dot{u}(t)\bigr|^2 + a_1\bigl|u(t)\bigr|^2\bigr)\,\mathrm{d}t - L\int_0^T \bigl|u(t)\bigr|^2\,\mathrm{d}t \\
&\geq \bigl(\min\{1, a_1\} - TLC^2\bigr)\|u\|^2,
\end{aligned}$$

and because $\min\{1, a_1\} \geq \min\{\frac{1}{2}, a_1\} > TLC^2$, we have $\lim_{u\to\infty} \frac{\langle K(u),u\rangle}{\|u\|} = +\infty$, that is, K is coercive. For any $u, v \in X$ one has

$$\begin{aligned}
\bigl\langle K(u) - K(v), u - v\bigr\rangle &= \int_0^T \bigl(\dot{u}(t) - \dot{v}(t), \dot{u}(t) - \dot{v}(t)\bigr)\,\mathrm{d}t \\
&\quad - \int_0^T \bigl(V_u\bigl(t, u(t)\bigr) - V_v\bigl(t, v(t)\bigr), u(t) - v(t)\bigr)\,\mathrm{d}t \\
&\quad + \int_0^T \bigl(h\bigl(u(t)\bigr) - h\bigl(v(t)\bigr), u(t) - v(t)\bigr)\,\mathrm{d}t \\
&\geq \int_0^T \bigl|\dot{u}(t) - \dot{v}(t)\bigr|^2\,\mathrm{d}t + \int_0^T a_1\bigl|u(t) - v(t)\bigr|^2\,\mathrm{d}t \\
&\quad - L\int_0^T \bigl|u(t) - v(t)\bigr|^2\,\mathrm{d}t \\
&\geq \bigl(\min\{1, a_1\} - TLC^2\bigr)\|u - v\|^2,
\end{aligned}$$

so K is uniformly monotone. By Theorem 26.A(d) in [50], K^{-1} exists and is continuous on X^*. □

3 Main results

In this section, we show our main results of the existence of at least three weak solutions for the problem (1).

To obtain our first result, we take the two positive constants θ and η in such a way that

$$\frac{(a_2 + LTC^2)T\eta^2}{-\sum_{k=1}^m F_k(\eta)} < \frac{(a_3 - LTC^2)\theta^2}{C^2 \max_{|t|\leq\theta}[-\sum_{k=1}^m F_k(t)]}$$

and taking

$$\lambda \in \Lambda_1 := \biggl]\frac{(a_2 + LTC^2)T\eta^2}{-\sum_{k=1}^m F_k(\eta)}, \frac{(a_3 - LTC^2)\theta^2}{C^2 \max_{|t|\leq\theta}[-\sum_{k=1}^m F_k(t)]}\biggl[,$$

set

$$\delta_\lambda = \min\left\{\frac{\theta^2 - \frac{C^2\lambda}{(a_3-LTC^2)}\max_{|t|\leq\theta}[-\sum_{k=1}^m F_k(t)]}{\frac{C^2}{(a_3-LTC^2)}G^\theta}, \frac{\eta^2 - \frac{\lambda}{(a_2+LTC^2)T}[-\sum_{k=1}^m F_k(\eta)]}{\frac{1}{(a_2+LTC^2)T}G_\eta}\right\} \tag{3}$$

and

$$\bar{\delta}_\lambda := \min\left\{\delta_\lambda, \frac{1}{\max\{0, \frac{C^2}{(a_3-LTC^2)}\limsup_{|t|\to\infty}\frac{\sup\sum_{k=1}^m[-G_k(t)]}{|t|^2}\}}\right\}, \tag{4}$$

where we say $\rho/0 = +\infty$, so that, for example, $\bar{\delta}_\lambda = +\infty$ when

$$\limsup_{|t|\to\infty}\frac{\sup\sum_{k=1}^m[-G_k(t)]}{|t|^2} \leq 0$$

and $G_\eta = G^\theta = 0$.

Theorem 3.1 *Suppose that V satisfies the assumptions* (A1), (A2), *and* (A3). *Assume that there exist two positive constants θ and η such that $\theta < \sqrt{T}C\eta$ and*

(A4) $\frac{\max_{|t|\leq\theta}[-\sum_{k=1}^m F_k(t)]}{\theta^2} < \frac{a_3-LTC^2}{C^2(a_2+LTC^2)T}\frac{-\sum_{k=1}^m F_k(\eta)}{\eta^2}$, *where* $a_3 = \min\{\frac{1}{2}, a_1\}$;

(A5) $\limsup_{|t|\to+\infty}\frac{\sum_{k=1}^m[-F_k(t)]}{|t|^2} \leq 0$.

Then, for each $\lambda \in \Lambda_1$ *and for each arbitrary function* $G_k \in C^1(\mathbb{R}^N, \mathbb{R})$ *denoting* $g_k(\xi) = \operatorname{grad}_\xi G_k(\xi)$ *for each* $\xi \in \mathbb{R}^N$ *for* $k = 1, 2, \ldots, m$, *fulfilling the condition*

$$\limsup_{|t|\to\infty}\frac{\sum_{k=1}^m[-G_k(t)]}{|t|^2} < +\infty,$$

there exists $\bar{\delta}_\lambda > 0$ *given by* (4) *such that, for each* $\mu \in [0, \bar{\delta}_\lambda[$, *the problem* (1) *admits at least three distinct weak solutions in* X.

Proof Fix λ, G_k for $k = 1, 2, \ldots, m$ and μ as in the conclusion. Our aim is applying Theorem 2.1 for the functionals $\Phi, \Psi : X \to \mathbb{R}$, defined by

$$\Phi(u) = \int_0^T\left[\frac{1}{2}\left|\dot{u}(t)\right|^2 - V\big(t, u(t)\big)\right]\mathrm{d}t + \int_0^T H\big(u(t)\big)\,\mathrm{d}t$$

and

$$\Psi(u) = -\left(\sum_{k=1}^m F_k\big(u(s_k)\big) + \frac{\mu}{\lambda}\sum_{k=1}^m G_k\big(u(s_k)\big)\right).$$

It is easily observable that Ψ is a Gâteaux differentiable functional and sequentially weakly upper semicontinuous whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)v = -\left(\sum_{k=1}^m\big(f_k\big(u(s_k)\big), v(s_k)\big) + \frac{\mu}{\lambda}\sum_{k=1}^m\big(g_k\big(u(s_k)\big), v(s_k)\big)\right),$$

and $\Psi' : X \to X^*$ is a compact operator. Moreover, Φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)v = \int_0^T \bigl[\bigl(\dot{u}(t), \dot{v}(t)\bigr) - \bigl(V_u\bigl(t, u(t)\bigr), v(t)\bigr)\bigr]\,\mathrm{d}t + \int_0^T \bigl(h\bigl(u(t)\bigr), v(t)\bigr)\,\mathrm{d}t$$

for every $v \in X$, while Proposition 2.3 shows that Φ' admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous. Indeed, let $u_n \in X$ with $u_n \to u$ weakly in X, taking weakly lower semicontinuity of the norm, we have $\liminf_{n\to+\infty} \|u_n\| \geq \|u\|$ and $u_n \to u$ uniformly on $[0, T]$. Hence, since V and H are continuous, we have

$$\lim_{n\to+\infty} \frac{1}{2} \int_0^T \bigl[\bigl|\dot{u_n}(t)\bigr|^2 - V\bigl(t, u_n(t)\bigr)\bigr]\,\mathrm{d}t + \int_0^T H\bigl(u_n(t)\bigr)\,\mathrm{d}t \geq \frac{1}{2} \int_0^T \bigl[\bigl|\dot{u}(t)\bigr|^2 - V\bigl(t, u(t)\bigr)\bigr]\,\mathrm{d}t + \int_0^T H\bigl(u(t)\bigr)\,\mathrm{d}t.$$

Thus $\liminf_{n\to+\infty} \Phi(u_n) \geq \Phi(u)$, that is, Φ is sequentially weakly lower semicontinuous. Like the proof of Lemma 1 of [26], we observe that the weak solutions of the problem (1) are concisely the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$. Since $-L|\xi| \leq |h(\xi)| \leq L|\xi|$ for every $\xi \in \mathbb{R}^N$, we have $|H(\xi)| \leq L|\xi|^2$ for all $\xi \in \mathbb{R}^N$. In parallel lines with the assumption (A1),

$$\bigl(a_3 - LTC^2\bigr)\|u\|^2 \leq \Phi(u) \leq \bigl(a_4 + LTC^2\bigr)\|u\|^2, \tag{5}$$

where $a_4 = \min\{\frac{1}{2}, a_2\}$. Put $r := \frac{\theta^2(a_3-LTC^2)}{C^2}$ and $w(t) := \eta$ for every $t \in [0, T]$. Because $\min\{\frac{1}{2}, a_1\} > TLC^2$, we have $\min\{1, a_1\} > TLC^2$, which means $a_3 - LTC^2 > 0$, and so $r > 0$. It is clear that $w \in X$ and

$$\|w\|^2 = T\eta^2.$$

Since $\theta < \sqrt{T}C\eta$, using (5), we have $0 < r < \Phi(w)$. Taking (2) into account, from (5) we observe that

$$\begin{aligned}\Phi^{-1}\bigl(]-\infty, r[\bigr) &= \bigl\{u \in X; \Phi(u) \leq r\bigr\} \\ &\subseteq \bigl\{u \in X; \bigl(a_3 - LTC^2\bigr)\|u\|^2 \leq r\bigr\} \\ &\subseteq \bigl\{u \in X; \bigl|u(t)\bigr| \leq \theta \text{ for each } t \in [0, T]\bigr\},\end{aligned}$$

and it follows that

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r])} \left[-\sum_{k=1}^m F_k\bigl(u(s_k)\bigr) - \frac{\mu}{\lambda}\sum_{k=1}^m G_k\bigl(u(s_k)\bigr)\right] \\ &\leq \max_{|\xi|\leq\theta}\left[-\sum_{k=1}^m F_k(\xi)\right] + \frac{\mu}{\lambda}G^\theta.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\Psi(w) &= -\sum_{k=1}^{m} F_k\bigl(w(t)\bigr) - \frac{\mu}{\lambda}\sum_{k=1}^{m} G_k\bigl(w(t)\bigr)\\ &\geq -\sum_{k=1}^{m} F_k(\eta) + \frac{\mu}{\lambda} G_\eta.\end{aligned}$$

So, we obtain

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}[-\sum_{k=1}^{m}[F_k(u(s_k)) + \frac{\mu}{\lambda}G_k(u(s_k))]]}{r}\\ &\leq \frac{\max_{|\xi|\leq\theta}[-\sum_{k=1}^{m}F_k(\xi)] + \frac{\mu}{\lambda}G^\theta}{\frac{\theta^2(a_3-LTC^2)}{C^2}}\end{aligned} \tag{6}$$

and

$$\begin{aligned}\frac{\Psi(w)}{\Phi(w)} &\geq \frac{-\sum_{k=1}^{m}F_k(w(t)) - \frac{\mu}{\lambda}\sum_{k=1}^{m}G_k(w(t))}{\frac{(a_2+LTC^2)\eta^2}{C^2}}\\ &\geq \frac{-\sum_{k=1}^{m}F_k(\eta) + \frac{\mu}{\lambda}G_\eta}{\frac{(a_2+LTC^2)\eta^2}{C^2}}.\end{aligned} \tag{7}$$

Since $\mu < \delta_\lambda$, one has

$$\mu < \frac{\theta^2 - \frac{C^2}{a_3-LTC^2}\lambda\max_{|\xi|\leq\theta}[-\sum_{k=1}^{m}F_k(\xi)]}{\frac{C^2}{(a_3-LTC^2)}G^\theta},$$

this means

$$\frac{\max_{|\xi|\leq\theta}[-\sum_{k=1}^{m}F_k(\xi)] + \frac{\mu}{\lambda}G^\theta}{\frac{\theta^2(a_3-LTC^2)}{C^2}} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{\eta^2 - \frac{C^2}{a_3-LTC^2}\lambda[-\sum_{k=1}^{m}F_k(\eta)]}{\frac{C^2}{a_3-LTC^2}G_\eta},$$

this means

$$\frac{-\sum_{k=1}^{m}F_k(\eta) + \frac{\mu}{\lambda}G_\eta}{\frac{\eta^2(a_3-LTC^2)}{C^2}} > \frac{1}{\lambda}.$$

Then

$$\frac{\max_{|\xi|\leq\theta}[-\sum_{k=1}^{m}F_k(\xi)] + \frac{\mu}{\lambda}G^\theta}{\frac{\theta^2(a_3-LTC^2)}{C^2}} < \frac{1}{\lambda} < \frac{-\sum_{k=1}^{m}F_k(\eta) + \frac{\mu}{\lambda}G_\eta}{\frac{\eta^2(a_3-LTC^2)}{C^2}}. \tag{8}$$

Hereupon, from (6)-(8) we infer that the condition (a_1) of Theorem 2.1 is achieved. Eventually, since $\mu < \bar{\delta}_\lambda$, we can fix $l > 0$ in such a manner that

$$\limsup_{|\xi|\to\infty} \frac{\sum_{k=1}^m [-G_k(\xi)]}{|\xi|^2} < l$$

and $\mu l < \frac{a_3 - LTC^2}{C^2}$. Therefore, there exists a constant q such that

$$\sum_{k=1}^m \bigl[-G_k(u)\bigr] \le l|u|^2 + q \quad \text{for all } u \in \mathbb{R}^N \tag{9}$$

for $k = 1, 2, \ldots, m$. Now, fix $0 < \varepsilon < \frac{a_3 - LTC^2}{C^2\lambda} - \frac{\mu l}{\lambda}$. Owing to the assumption (A4) there is a constant q_ε such that

$$\sum_{k=1}^m \bigl[-F_k(u)\bigr] \le \varepsilon|u|^2 + q_\varepsilon \quad \text{for all } u \in \mathbb{R}^N \tag{10}$$

for $k = 1, 2, \ldots, m$. Due to (5), (9), and (10) we have

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \int_0^T \left[\frac{1}{2}\bigl|\dot{u}(t)\bigr|^2 - V\bigl(t, u(t)\bigr) + H\bigl(u(t)\bigr)\right] \mathrm{d}t \\
&\quad - \lambda\left[-\sum_{k=1}^m \left[F_k\bigl(u(s_k)\bigr) + \frac{\mu}{\lambda} G\bigl(u(s_k)\bigr)\right]\right] \\
&\ge \bigl(a_3 - LTC^2\bigr)\|u\|^2 - \lambda\varepsilon|u|^2 - \lambda q_\varepsilon - \mu l|u|^2 - \mu q \\
&\ge \bigl(a_3 - LTC^2 - \lambda C^2\varepsilon - \mu C^2 l\bigr)\|u\|^2 - \lambda q_\varepsilon - \mu q.
\end{aligned}$$

This means that the functional $\Phi - \lambda\Psi$ is coercive, and the assumption (a_2) of Theorem 2.1 is verified. From (6) and (8),

$$\lambda \in \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\le r} \Psi(u)}\right[$$

and Theorem 2.1 (with $\bar{v} = w$) ensures that the problem (1) possesses at least three weak solutions in X. □

We now offer another version of Theorem 3.1 within which no asymptotic condition on the nonlinear term is necessary; contrarily, each constituent of f_k and g_k for $k = 1, 2, \ldots, m$ is considered to be negative.

Fix positive constants θ_1, θ_2, and η in such a way that

$$\begin{aligned}
&\frac{3}{2}\frac{(a_2 + LTC^2)T\eta^2}{[-\sum_{k=1}^m F(\eta)]} \\
&\quad < \frac{a_3 - LTC^2}{C^2} \min\left\{\frac{\theta_1^2}{\max_{|\xi|\le\theta_1}[-\sum_{k=1}^m F_k(\xi)]}, \frac{\theta_2^2}{2\max_{|\xi|\le\theta_2}[-\sum_{k=1}^m F_k(\xi)]}\right\}
\end{aligned}$$

and put

$$\begin{aligned}\Lambda_2 := \Bigg]&\frac{3}{2}\frac{(a_2+LTC^2)T\eta^2}{[-\sum_{k=1}^m F(\eta)]},\\ &\frac{a_3-LTC^2}{C^2}\min\left\{\frac{\theta_1^2}{\max_{|\xi|\le\theta_1}[-\sum_{k=1}^m F_k(\xi)]},\frac{\theta_2^2}{2\max_{|\xi|\le\theta_2}[-\sum_{k=1}^m F_k(\xi)]}\right\}\Bigg[.\end{aligned}$$

By the above symbolization, we obtain the following multiplicity result.

Theorem 3.2 *Order the Banach space X by the positive cone X^+ (see Section 5.4 of [51]), and suppose that V satisfies in the assumptions (A1), (A2), and (A3), $F_k \in C^1(\mathbb{R}^N,\mathbb{R})$, each component of $f_k(\xi) = \operatorname{grad}_\xi F_k(\xi)$ for $k = 1,2,\dots,m$ is negative and there exist three positive constants θ_1, θ_2, and η such that $\theta_1 < C\sqrt{\frac{T}{2}}\eta < \frac{\theta_2}{2}\sqrt{\frac{a_3-LTC^2}{a_2+LTC^2}}$ where $a_3 = \min\{\frac{1}{2},a_1\}$ and*

$$\begin{aligned}\text{(B1)}\quad &\max\left\{\frac{\max_{|\xi|\le\theta_1}[-\sum_{k=1}^m F_k(\xi)]}{\theta_1^2},\frac{2\max_{|\xi|\le\theta_2}[-\sum_{k=1}^m F_k(\xi)]}{\theta_2^2}\right\}\\ &\quad<\frac{2}{3}\frac{a_3-LTC^2}{C^2(a_2+LTC^2)T}\frac{[-\sum_{k=1}^m F(\eta)]}{\eta^2}.\end{aligned}$$

Then, for each $\lambda \in \Lambda_2$ and for every arbitrary function $G_k \in C^1(\mathbb{R}^N,\mathbb{R})$ such that each component of $g_k(\xi) = \operatorname{grad}_\xi G_k(\xi)$ for every $\xi \in \mathbb{R}^N$ is negative for $k = 1,2,\dots,m$, there exists $\delta_\lambda^* > 0$ defined by

$$\begin{aligned}\min\Bigg\{&\frac{(a_3-LTC^2)\theta_1^2 - C^2\lambda\max_{|\xi|\le\theta_1}[-\sum_{k=1}^m F(\xi)]}{C^2G^{\theta_1}},\\ &\frac{(a_3-LTC^2)\theta_2^2 - 2C^2\lambda\max_{|\xi|\le\theta_2}[-\sum_{k=1}^m F_k(\xi)]}{2C^2G^{\theta_2}}\Bigg\}\end{aligned}$$

such that, for each $\mu \in [0,\delta_\lambda^*[$, the problem (1) possesses at least three weak solutions u_1, u_2, and u_3 such that $u_i(t) \in X^+$ (or $u_i(t) \ge 0$) for all $t \in [0,T]$ and $i = 1,2,3$.

Proof Fix λ, G_k for $k = 1,2,\dots,m$ and μ as in the conclusion and take X, Φ, and Ψ as in the proof of Theorem 3.1. Obviously, the regularity assumptions of Theorem 2.2 on Φ and Ψ are satisfied. Our goal is to check (b_1) and (b_2). For this purpose, put $w(t) = \eta$ for every $t \in [0,T]$,

$$r_1 := \frac{(a_3-LTC^2)\theta_1^2}{C^2}$$

and

$$r_2 := \frac{(a_3-LTC^2)\theta_2^2}{C^2}.$$

According to condition $\theta_1 < C\sqrt{\frac{T}{2}}\eta < \frac{\theta_2}{2}\sqrt{\frac{a_3-LTC^2}{a_2+LTC^2}}$, and from (5), we get

$$2r_1 < \Phi(w) < \frac{r_2}{2}.$$

Since $\mu < \delta_\lambda^*$ and $G_\eta = 0$, one has

$$\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1])}\Psi(u)}{r_1} &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1])}[-\sum_{k=1}^m[F_k(u(s_k))+\frac{\mu}{\lambda}G_k(u(s_k))]]}{r_1}\\
&\le \frac{\max_{|\xi|\le\theta_1}[-\sum_{k=1}^m F_k(\xi)]+\frac{\mu}{\lambda}G^{\theta_1}}{\frac{(a_3-LTC^2)\theta_1^2}{C^2}}\\
&< \frac{1}{\lambda} < \frac{2}{3}\frac{[-\sum_{k=1}^m F_k(\eta)]+\frac{\mu}{\lambda}G\eta}{\frac{(a_2+LTC^2)T\eta^2}{C^2}}\\
&\le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{2\sup_{u\in\Phi^{-1}(]-\infty,r_2])}\Psi(u)}{r_2} &= \frac{2\sup_{u\in\Phi^{-1}(]-\infty,r_2])}[-\sum_{k=1}^m[F_k(u(s_k))+\frac{\mu}{\lambda}G_k(u(s_k))]]}{r_2}\\
&\le \frac{2\sup_{|\xi|\le\theta_2}[-\sum_{k=1}^m F_k(\xi)]+2\frac{\mu}{\lambda}G^{\theta_2}}{\frac{(a_3-LTC^2)\theta_2^2}{C^2}}\\
&< \frac{1}{\lambda} < \frac{2}{3}\frac{[-\sum_{k=1}^m F_k(\eta)]+\frac{\mu}{\lambda}G_\eta}{\frac{(a_2+LTC^2)T\eta^2}{C^2}}\\
&\le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.
\end{aligned}$$

Therefore, (b_1) and (b_2) of Theorem 2.2 are fulfilled. In the following, we show that $\Phi - \lambda\Psi$ satisfies the assumption 2 of Theorem 2.2. Let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and, thus, they are weak solutions for the problem (1). We want to show that they are nonnegative. Let u_0 be a nontrivial weak solution of problem (1). Arguing by a contradiction, assume that the set $\mathcal{A} = \{t \in [0,T] : u_0(t) < 0\} = \{t \in [0,T] : 0 - u_0(t) \in X^+, u_0(t) \neq 0\}$ is non-empty and its measure is positive. Put

$$\bar{v}(t) = \begin{cases} 0, & 0 \le u_0(t),\\ u_0(t), & u_0(t) < 0\end{cases}$$

for all $t \in [0,T]$. Clearly, $\bar{v} \in X$. Since u_0 is a weak solution of (1) we have

$$\begin{aligned}
&\int_0^T \bigl[\bigl(\dot{u}_0(t),\dot{\bar{v}}(t)\bigr) - \bigl(V_{u_0}\bigl(t,u_0(t)\bigr),\bar{v}(t)\bigr) + \bigl(h\bigl(u_0(t),\bar{v}(t)\bigr)\bigr)\bigr]\,\mathrm{d}t\\
&\quad= -\lambda\sum_{k=1}^m\bigl(f_k\bigl(u_0(s_k)\bigr),\bar{v}(s_k)\bigr) - \mu\sum_{k=1}^m\bigl(g_k\bigl(u_0(s_k)\bigr),\bar{v}(s_k)\bigr).
\end{aligned}$$

Thus, from our sign assumptions on the data, since $-L|\xi|^2 \le (h(\xi),\xi) \le L|\xi|^2$ for every $\xi \in \mathbb{R}^N$, we have

$$\begin{aligned}
0 &\le \bigl(\min\{1,a_1\} - TLC^2\bigr)\|u_0\|_{X(\mathcal{A})}^2 \le \int_{\mathcal{A}}\bigl(\bigl|\dot{u}_0(t)\bigr|^2 + a_1\bigl|u_0(t)\bigr|^2 - L\bigl|u_0(t)\bigr|^2\bigr)\,\mathrm{d}t\\
&\le \int_0^T\bigl[\bigl(\dot{u}_0(t),\dot{u}_0(t)\bigr) - \bigl(V_{u_0}\bigl(t,u_0(t)\bigr),u_0(t)\bigr) + \bigl(h\bigl(u_0(t)\bigr),u_0(t)\bigr)\bigr]\,\mathrm{d}t \le 0.
\end{aligned}$$

Hence, $u_0 = 0$ in $\mathcal{A}$ and this is absurd. Then we conclude $u_1(t) \geq 0$ and $u_2(t) \geq 0$ for every $t \in [0, T]$. Thus, it follows that $su_1 + (1-s)u_2 \geq 0$ for all $s \in [0,1]$, and that

$$-(\lambda f_k + \mu g_k)\big(su_1 + (1-s)u_2\big) \geq 0 \quad \text{for } k = 1,2,\dots,m,$$

and consequently, $\Psi(su_1 + (1-s)u_2) \geq 0$, for every $s \in [0,1]$. Hence, since all the hypotheses of Theorem 2.2 are satisfied, it follows that, for every

$$\lambda \in \left]\frac{3}{2}\frac{\Phi(w)}{\Psi(w)}, \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}, \frac{r_2/2}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points u_i for $i = 1,2,3$, such that $0 \leq u_i(t) < \theta_2$ for all $t \in [0,T]$ and $i = 1,2,3$, which are the weak solutions of the problem (1), and the favorable result is achieved. □

In the following, we present a special case of Theorem 3.1.

Corollary 3.3 *Suppose that V satisfies the assumptions* (A1), (A2), *and* (A3), *and*

$$\liminf_{\xi\to 0}\frac{\max_{|t|\leq\xi}[-\sum_{k=1}^m F_k(t)]}{\xi^2} = \limsup_{\xi\to+\infty}\frac{\sum_{k=1}^m[-F_k(\xi)]}{\xi^2} = 0.$$

Then there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ and every arbitrary function $G_k \in C^1(\mathbb{R}^N, \mathbb{R})$, denoting $g_k(\xi) = \operatorname{grad}_\xi G_k(\xi)$ for every $\xi \in \mathbb{R}^N$ for $k = 1,2,\dots,m$, satisfying the asymptotical condition

$$\limsup_{|t|\to\infty}\frac{\sum_{k=1}^m[-G_k(t)]}{|t|^2} < +\infty,$$

there exists $\delta_\lambda^* > 0$ such that, for each $\mu \in [0, \delta_\lambda^*[$, the problem (1) *admits at least three distinct weak solutions in X.*

Proof Fix $\lambda > \lambda^* := \frac{(a_2+LTC^2)T\eta^2}{-\sum_{k=1}^m F_k(\eta)}$ for some $\eta > 0$. Recalling

$$\liminf_{\xi\to 0}\frac{\max_{|t|\leq\xi}[-\sum_{k=1}^m F_k(t)]}{\xi^2} = 0,$$

there exists a sequence $\{\theta_n\} \subset]0,+\infty[$ with this feature that $\lim_{n\to\infty}\theta_n = 0$ and

$$\lim_{n\to\infty}\frac{\max_{|t|\leq\theta_n}[-\sum_{k=1}^m F_k(t)]}{\theta_n^2} = 0.$$

Hence, there exists $\bar{\theta} > 0$ such that

$$\frac{\max_{|t|\leq\bar{\theta}}[-\sum_{k=1}^m F_k(t)]}{\bar{\theta}^2} < \min\left\{\frac{a_3 - LTC^2}{C^2(a_2+LTC^2)T}\frac{-\sum_{k=1}^m F_k(\eta)}{\eta^2}; \frac{a_3 - LTC^2}{\lambda C^2}\right\}$$

and $\bar{\theta} < \sqrt{T}C\eta$. The conclusion follows from Theorem 3.1. □

Now, as an example, we present the following consequence of Theorem 3.2 with $m = T = N = 1$.

Corollary 3.4 *Suppose that V satisfies the assumptions* (A1), (A2), *and* (A3), $f_1 : \mathbb{R} \to \mathbb{R}$ *is a negative continuous function and* $h : \mathbb{R} \to \mathbb{R}$ *is a Lipschitz continuous function with the Lipschitz constant* $L > 0$ *such that* $h(0) = 0$, $\min\{1, a_1\} > 2L$, *and* $a_2 + 2L < 16(a_3 - 2L)$ *where* $a_3 = \min\{\frac{1}{2}, a_1\}$. *Furthermore, assume that*

$$\lim_{\xi \to 0^+} \frac{f_1(\xi)}{\xi} = 0$$

and

$$\int_0^{1/2} f_1(x)\,\mathrm{d}x < \frac{3}{32} \frac{a_2 + 2L}{a_3 - 2L} \int_0^4 f_1(x)\,\mathrm{d}x.$$

Then, for every $\lambda \in]\frac{3}{8} \frac{a_2+2L}{-\int_0^{1/2} f_1(x)\,\mathrm{d}x}, \frac{4(a_3-2L)}{-\int_0^4 f_1(x)\,\mathrm{d}x}[$, *and for every arbitrary negative continuous function* $g_1 : \mathbb{R} \to \mathbb{R}$, *there exists* $\delta_\lambda^* > 0$ *such that, for each* $\mu \in [0, \delta_\lambda^*[$, *the problem*

$$\begin{cases} u''(t) + V_u(t, u(t)) = h(u(t)), & t \neq s_1, \\ \Delta u'(s_1) = \lambda f_1(u(s_1)) + \mu g_1(u(s_1)), \\ u(0) - u(1) = u'(0) - u'(1) = 0, \end{cases} \tag{11}$$

possesses at least three weak solutions u_1, u_2, *and* u_3 *such that* $0 \le u_i(t) < 4$ *for all* $t \in [0, T]$ *and* $i = 1, 2, 3$.

Proof Our goal is to use Theorem 3.2 by choosing $m = T = N = 1$, $\theta_2 = 4$ and $\eta = \frac{1}{2}$. Since $c = \sqrt{2}$, we observe that

$$\frac{3}{2} \frac{(a_2 + LTC^2)T\eta^2}{[-\sum_{k=1}^m F_k(\eta)]} = \frac{3}{8} \frac{a_2 + 2L}{-\int_0^{1/2} f_1(x)\,\mathrm{d}x}$$

and

$$\frac{a_3 - LTC^2}{C^2} \frac{\theta_2^2}{2\max_{|\xi| \le \theta_2}[-\sum_{k=1}^m F_k(\xi)]} = \frac{4(a_3 - 2L)}{-\int_0^4 f_1(x)\,\mathrm{d}x}.$$

Moreover, since $\lim_{\xi \to 0^+} \frac{f_1(\xi)}{\xi} = 0$, one has

$$\lim_{\xi \to 0^+} \frac{\int_0^\xi f_1(x)\,\mathrm{d}x}{\xi^2} = 0.$$

Then there exists a positive constant $\theta_1 < \frac{1}{2}$ such that

$$\frac{\int_0^{\theta_1} f_1(x)\,\mathrm{d}x}{\theta_1^2} > \frac{4}{3} \frac{a_3 - 2L}{a_2 + 2L} \int_0^{\frac{1}{2}} f_1(x)\,\mathrm{d}x$$

and

$$\frac{\theta_1^2}{\int_0^{\theta_1} f_1(x)\,\mathrm{d}x} < \frac{8}{\int_0^4 f_1(x)\,\mathrm{d}x}.$$

Finally, an easy calculation shows that all hypotheses of Theorem 3.2 are fulfilled, and the conclusion follows. □

Remark 3.1 From Assumptions (A1), (A2), and (A3), we can show, by the same reasoning as given in Theorem 4 of [26], that the problem (1) when $h \equiv 0$ does not possess any non-zero weak solution in the cases where impulsive terms are zero. Consequently, the ensured weak solutions for the problem (1) when $h \equiv 0$ in Theorems 3.1 and 3.2 and in Corollary 3.3 are generated by impulses when impulsive terms $f_k, g_k \not\equiv 0$ for some $1 \leq k \leq m$, as well as for the problem (11) when $h \equiv 0$ in Corollary 3.4 are generated by impulses when impulsive terms $f_1, g_1 \not\equiv 0$.

Remark 3.2 The methods used here can be applied studying discrete boundary value problems as in [52].

4 Concluding remarks

The theory of impulsive dynamic equations is generally thought to provide a natural framework for mathematical modeling of many real world phenomena such as chemotherapy, population dynamics, optimal control, ecology, industrial robotics, physics phenomena, *etc.* The impulsive effects can be broadly found in numerous evolution processes where their states may undergo abrupt changes at specific moments of time. As far as the second-order dynamic equations are concerned, we often take into account the impulses in terms of position and velocity. In the motion of spacecraft, on the contrary, we are supposed to consider instantaneous impulses depending on the position leading to jump discontinuities in velocity, but with no changes in terms of position. Impulsive problems such as problem (1) are considered as highly important for the description of quite a large number of real world phenomena including biology (biological phenomena involving thresholds), medicine (bursting rhythm models), pharmacokinetics, mechanics, and engineering. To this end, we have established, in this paper, the existence criteria of at least three solutions for the perturbed impulsive problem (1) based on variational methods and the critical point theory, under suitable hypotheses. The results of the study, finally, illustrated that these solutions are generated by impulses while $h \equiv 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this study equally and read and approved the final version of the manuscript.

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