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Realized extreme quantile: A joint model for conditional quantiles and measures of volatility with EVT refinements

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Summary

We propose a new framework exploiting realized measures of volatility to estimate and forecast extreme quantiles. Our realized extreme quantile (REQ) combines quantile regression with extreme value theory and uses a measurement equation that relates the realized measure to the latent conditional quantile. Model estimation is performed by quasi maximum likelihood, and a simulation experiment validates this estimator in finite samples. An extensive empirical analysis shows that high-frequency measures are particularly informative of the dynamic quantiles. Finally, an out-of-sample forecast analysis of quantile-based risk measures confirms the merit of the REQ.

1 | INTRODUCTION

Quantitative financial risk management has become a fundamental tool for investment decisions, capital allocation, and regulation. The subprime mortgage crisis emphasized how the changing nature of financial risk requires accurate risk measures and models that respond quickly to the most recent events.

Value at risk (VaR) is considered the standard measure of *market risk*, and is used for both internal control of financial institutions and regulatory purposes. This measure owes its success to the fact that it quantifies risk in a single number, summarizing how much a portfolio of stocks can lose within a given time period, for a given confidence level. More formally, let $\{r_t\}_{t\in\mathbb{N}}$ be a time series of portfolio returns. The *conditional* VaR at level α is defined as the α-quantile of the conditional distribution of the portfolio returns:

$$\text{VaR}_t^{\alpha} = \inf\{x \in \mathbb{R} : \Pr(r_t \leq x|\mathcal{F}_{t-1}) \geq \alpha\},$$

where $\mathcal{F}_{t-1}$ denotes the information set available at $t-1$. Although VaR is a simple concept, its estimation poses very challenging problems. From a statistical point of view, it is sufficient to estimate the quantile of the conditional return distribution, but the fact that the latter changes over time makes the task difficult.

This paper proposes a novel dynamic approach to estimate VaR that combines elements from quantile regression, extreme value theory (EVT), and high-frequency (HF) financial econometrics. Engle and Manganelli (2004) propose a class of conditional autoregressive quantile models to estimate VaR called CAViaR. This approach allows one to model time-varying conditional quantiles without specifying the data-generating process; however, the lack of a distributional assumption makes inference unstable in the tails where data are sparse. A possible solution consists of integrating quantile regression with EVT (Chernozhukov & Fernández-Val, 2011; Wang, Li, & He, 2012). EVT provides limiting results for the largest and smallest values in an i.i.d. sample and allows for estimation of quantiles beyond the range of the observed data. To the extent that the quantile regression model is able to capture the time variation in the returns, a combined quantile–EVT model allows for more accurate estimates of extreme quantiles.

The contribution of this paper is twofold. First, our *realized extreme quantile* (REQ) approach boosts any quantile model with an EVT estimator for the extreme quantiles. Second, it exploits *realized measures* built from HF data to feed the dynamic system. Analogously to the realized generalized autoregressive conditional heteroskedasticity (GARCH) of Hansen, Huang, and Shek (2012), the realized measures are used as observable quantities to infer the behavior of the latent quantiles of the conditional return distribution. The intuition behind our approach is that, as long as the quantiles vary with the volatility level, the dynamic behavior of the volatility level should be informative of the dynamic behavior of the quantiles.

The remainder of the paper is organized as follows. Section 2 reviews several existing VaR methodologies; Section 3 presents the general REQ framework, along with estimation and inference therein; Section 4 presents a linear REQ model; Section 5 provides a simulation study; Section 6 contains the empirical analysis; Section 7 discusses future investigations. Proofs appear in the Supporting Information Appendix.

2 | A TAXONOMY OF VAR METHODOLOGIES

There exist several dynamic models for estimating VaR. We propose a simple taxonomy, summarized in Figure 1, where the models are classified with respect to the type of information they exploit (high frequency or low frequency) and the approach they take (volatility, EVT or quantile).

The *volatility approach* dates back to the ARCH class of models of Engle (1982), generalized to the GARCH class by Bollerslev (1986). These models completely define the conditional return distribution and express the conditional variance as the sum of autoregressive components plus past squared innovations. With the availability of HF data, new nonparametric estimators of the daily asset price variation, called realized measures, have been proposed in the financial econometrics literature. These measures are theoretically grounded in the powerful theory of quadratic variation (Barndorff-Nielsen & Shephard, 2002) and bear valuable information content to estimate the conditional variance. Engle (2002) includes the realized volatility in the standard GARCH equation and finds that it significantly contributes to explain the conditional volatility. Engle and Gallo (2006) and Shephard and Sheppard (2010), respectively, propose the multiplicative error model (MEM) and high-frequency-based volatility (HEAVY) frameworks that fully specify the conditional distributions of both the returns and the realized measures. In a different but related manner, Hansen et al. (2012) propose the realized GARCH class of models, where the realized measures are used within a measurement equation.

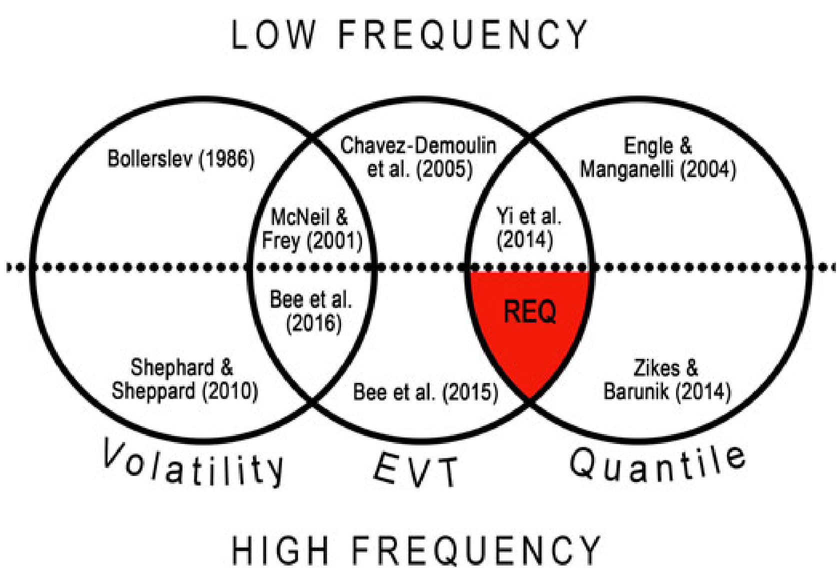

FIGURE 1 Taxonomy of VaR methodologies [Colour figure can be viewed at wileyonlinelibrary.com]

The *EVT approach* requires the specification of a dynamic model that can take into account the dependence and the time variation in the return distribution. A two-step procedure (McNeil & Frey, 2000) pre-whitens the returns with a GARCH model and then applies the peaks-over-threshold (POT) method of Davison and Smith (1990) to the tails of the estimated residuals. This mixed *volatility–EVT* approach outperforms several GARCH specifications in terms of VaR forecast. Bee, Dupuis, and Trapin (2016) extend this approach to several HF-based volatility models and find that using realized measures provides better VaR forecasts than McNeil and Frey (2000). A *pure EVT* approach appears in Chavez-Demoulin, Davison, and McNeil (2005) and Chavez-Demoulin, Embrechts, and Sardy (2014), where the POT model is extended to a dynamic framework. These dynamic POT models focus directly on the tails of the conditional return distribution and use past extreme returns as covariates in the extreme value models to account for the dependence in the returns. Building on this idea, Bee et al. (2015) propose the realized POT approach (RPOT), which introduces realized measures as covariates in the standard POT approach. They show that HF data add information beyond that conveyed by low-frequency (LF) data and that the RPOT provides better forecasts than dynamic POT models based on past exceedances.

The *quantile approach* focuses directly on the quantiles of the conditional return distribution. Engle and Manganelli (2004) propose the CAViaR model endowed with autoregressive components and covariates obtained as functions of past returns. Chen, Gerlach, Hwang, and McAleer (2012) and Jeon and Taylor (2013) further extend the CAViaR class of models and find that including other covariates, such as the daily range (Parkinson, 1980) and the implied volatility, provides additional information on the quantiles. Hua and Manzan (2013) and Žikeš and Baruník (2016) propose quantile models based on realized measures, and show that HF information allows for a much more reactive response to the impact of recent news.

Although quantile regression models perform well, inference in the tails is unstable because of their semi-parametric nature, particularly when the true process is heavy tailed. To overcome this issue, Manganelli and Engle (2004) extend the CAViaR suggesting a *quantile–EVT approach* where returns are divided by the estimated quantile at a specified level, and then an EVT estimator is applied to the tails of the estimated quantile residuals. Following these ideas, Yi, Feng, and Huang (2014) augment the conditional quantile model for GARCH processes of Xiao and Koenker (2009) with EVT. Our framework allows one to augment a general quantile model with EVT.

Figure 1 summarizes the current VaR modeling options. Although there exist quantile regression models based on HF data and quantile regression models with EVT based on LF data, a model that exploits HF data in the *quantile–EVT* framework is still missing and our REQ model fills this gap.

3 | REALIZED EXTREME QUANTILE

3.1 | The general framework

Let r_t be the portfolio return at time t and x_t a realized measure observable at time t. We define realized measure as an estimator of the daily volatility, and not of the daily variance as it is common in the literature. Considering the square root of a standard estimator of the quadratic variation, the quantile and the realized measure are on the same scale. Let θ be the probability associated with the quantile regression model and let $(\beta(\theta), \gamma(\theta))$ be a vector of parameters associated respectively with past conditional quantiles and the realized measure. The general structure of the REQ model at the level θ, for $0 \leq \theta \leq 1$, is given by the following system of equations:

$$r_t = q_t^\theta + \epsilon_t^\theta, \tag{1}$$

$$q_t^\theta = f(q_{t-1}^\theta, \dots, q_{t-p}^\theta, x_{t-1}, \dots, x_{t-q}; \beta(\theta), \gamma(\theta)), \tag{2}$$

$$x_t = \omega(\theta) + \phi(\theta) q_t^\theta + \tau_1(\theta) z_t^\theta + \tau_2(\theta)[(z_t^\theta)^2 - 1] + u_t, \tag{3}$$

where ϵ_t^θ is such that $Q_\theta(\epsilon^\theta | \mathcal{F}_{t-1}) = 0$, with $Q_\theta(\cdot)$ the quantile function evaluated at the probability level θ, $\mathcal{F}_{t-1} = \sigma\{r_{t-1}, x_{t-1}, r_{t-2}, x_{t-2}, \dots\}$ is the information set available at time $t-1$, $z_t^\theta = r_t / q_t^\theta$, $u_t \sim N(0, \sigma_u^2)$, and $\omega, \phi, \tau_1, \tau_2$ are parameters possibly depending on θ. We refer to (Equations 1–3) respectively as the *return equation*, the *quantile equation*, and the *measurement equation*. No specific assumptions are made on the innovations of the return process, as it is standard in quantile regression. We add the return equation as it makes our quantile approach more intuitive.

The quantile equation (Equation 2) allows for a very general structure with the $\mathcal{F}_{t-1}$-measurable function, $f(\cdot)$, accommodating several possible specifications.

Example 1. Assuming that $f(\cdot)$ is a linear function and taking as realized measure the absolute returns $|r_t|$, one recovers the symmetric absolute value (SAV) model of Engle and Manganelli (2004), $q_t^\theta = \beta_0(\theta)+\beta_1(\theta)q_{t-1}^\theta+\gamma(\theta)|r_{t-1}|$.

Example 2. Neglecting the autoregressive component and considering as realized measures the realized volatility (RV) in a heterogeneous autoregressive (HAR) structure (Corsi, 2009), one obtains the heterogeneous autoregressive quantile (HARQ) model of Žikeš and Baruník (2016), $q_t^\theta = \beta_0(\theta)+\gamma_1(\theta)\mathrm{RV}_{t-1}+\gamma_2(\theta)\mathrm{RV}_{t-1,t-5}+\gamma_3(\theta)\mathrm{RV}_{t-1,t-22}$, where $\mathrm{RV}_{t-1,t-k} = k^{-1}\sum_{i=1}^{k}\mathrm{RV}_{t-i}$.

Note that the quantile equation (Equation 2) allows the parameters to depend on the probability θ, but in what follows we assume that the covariates have a constant effect on the quantiles q_t^θ for $\theta \in (0, \theta_c]$ with θ_c close to zero. This restricts the behavior in the lower tail, where we estimate the VaR, and leaves unspecified the behavior for $\theta \in (\theta_c, 1]$.

Assumption 1. Let $F_t^{z^\theta}$ be the conditional distribution of z_t^θ. We assume that $F_t^{z^\theta} = F^{z^\theta}$ for all $z_t^\theta > 1$.[1]

The main novelty of our quantile regression framework with respect to past quantile models used in a time series context is the inclusion of a measurement equation. Hansen et al. (2012) use a measurement equation to link an observed realized measure to the latent conditional volatility. We use the measurement equation (Equation 3) to link an observed realized measure to the latent conditional quantile. This is reasonable to the extent that the conditional quantile varies according to the degree of variation in the asset prices. Our use of a measurement equation differentiates our approach from that of Žikeš and Baruník (2016) where a noisy proxy of the volatility appears in the quantile equation. The following example further makes the point.

Example 3. Consider the close-to-close return process $r_t = \sigma_t\epsilon_t$, where σ_t is the latent volatility and ϵ_t an i.i.d. symmetric random variable. A quantile model for such a process can be defined as $r_t = q_t^\theta + \tilde{\epsilon}_t^\theta$, with $q_t^\theta = \sigma_t q^\theta$ and $\tilde{\epsilon}_t^\theta = \epsilon_t^\theta\sigma_t$, where $q^\theta = Q_\theta(\epsilon_t)$ and ϵ_t^θ is such that $Q_\theta(\epsilon_t^\theta) = 0$. In this case, the quantile varies according to σ_t and we can use the realized volatility as an observable for this latent quantity. However, the realized volatility x_t is a noisy estimator of σ_t, based on observations that cover only part of the day. The relationship between σ_t and x_t can thus be represented as measurement equation $x_t = f(\sigma_t) + \eta_t$ where $f(\cdot)$ is a functional and η_t a noise term, naturally linking the two quantities.

3.2 | Quasi-maximum likelihood quantile estimation

Consider a sample $y_1, \dots, y_n$ from the quantile regression model,

$$y_t = x_t'\beta + \epsilon_t^\theta, \quad Q_\theta(\epsilon_t^\theta) = 0,$$

where x_t is a vector of K regressors and $\beta \in \mathbb{R}^K$ is a vector of parameters. The regression quantile estimator $\hat{\beta}$ proposed by Koenker and Bassett (1978) and obtained as the solution to

$$\min_\beta \frac{1}{n}\sum_{t=1}^{n}\rho_t^\theta(y_t, f_t(\beta)),$$

where $\rho_t^\theta(y_t, f_t(\beta)) = [\theta - I\{y_t < f_t(\beta)\}][y_t - f_t(\beta)]$ is the *tick-loss* function, $f_t(\beta) = x_t'\beta$, and $I\{\cdot\}$ the indicator function, is consistent and asymptotically normal. Engle and Manganelli (2004) prove that these asymptotic results also hold in the CAViaR framework, under several specifications of the function $f(\cdot)$, provided that it satisfies some regularity conditions.

Adding realized measures built from HF data in the quantile regression, as in Hua and Manzan (2013) and Žikeš and Baruník (2016), does not affect the properties of the regression quantile estimator $\hat{\beta}$. In contrast, the REQ framework in Equations 1–3 requires performing a *joint estimation* of the quantile and measurement equations, and the regression quantile estimator $\hat{\beta}$ cannot be used.

For conditional quantiles, we consider the quasi-maximum likelihood (QML) estimator of Komunjer (2005). Based on the *tick-exponential* density, it provides a class of estimators consistent for the parameters of a correctly specified model of a given conditional quantile.

Definition 1. (Komunjer, 2005)
A probability measure on $\mathbb{R}$ with density ϕ_t^θ indexed by η, $\eta \in M_t \subset \mathbb{R}$ belongs to the tick-exponential family of order θ if for $y \in \mathbb{R}$:

[1] We are implicitly assuming that $q_t^\theta < 0$. Observations below q_t^θ therefore correspond to z_t^θ greater than one.

$$\phi_t^\theta(y,\eta) = \exp(-(1-\theta)[a_t(\eta) - b_t(y)]I\{y \le \eta\} + \theta[a_t(\eta) - c_t(y)]I\{y > \eta\}),$$

where $a_t : M_t \to \mathbb{R}$ is continuously differentiable and $b_t,\ c_t : \mathbb{R} \to \mathbb{R}$. The functions a_t, b_t, and c_t are such that for $\eta \in M_t$, ϕ_t^θ is a probability density and η is the θ-quantile of ϕ_t^θ.

Proposition 1. *Let* $a_t(\eta) = \frac{\eta}{\theta(1-\theta)}$ *and* $b_t(y) = c_t(y) = \frac{y}{\theta(1-\theta)}$, *then*

$$\log\phi_t^\theta(y,\eta) = -\frac{1}{\theta(1-\theta)}\rho_t^\theta(y,\eta).$$

Proof. See the Supporting Information Appendix. □

Proposition 1 states that the logarithm of the tick-exponential density is proportional to the function ρ_t^θ, yielding the regression quantile estimator $\hat{\beta}$ of Koenker and Bassett (1978). In particular, minimizing ρ_t^θ corresponds to maximizing $\log\phi_t^\theta$. We can thus write the quasi log-likelihood of the REQ model in Equations 1–3 as

$$\begin{aligned}\ell(r,x;\delta) = & \underbrace{-\sum_{t=1}^{n}[\theta(1-\theta)]^{-1}\left[\theta - I\left(r_t < q_t^\theta(\beta,\gamma)\right)\right]\left[r_t - q_t^\theta(\beta,\gamma)\right]}_{\ell(r;\delta)} \\ & \underbrace{-\frac{1}{2}\sum_{t=1}^{n}\left[\log(2\pi) + \log(\sigma_u^2) + u_t^2/\sigma_u^2\right]}_{\ell(x|r;\delta)},\end{aligned} \tag{4}$$

where $\delta = (\beta,\gamma,\omega,\phi,\tau,\sigma_u^2)$. The log-likelihood has two components: (i) $\ell(r;\delta)$, based on the tick-exponential density and corresponding to the quantile regression estimator of Koenker and Bassett (1978) plus a constant; (ii) the conditional Gaussian component $\ell(x|r;\delta)$, where the conditioning is with respect to the observed returns, as in the realized GARCH model of Hansen et al. (2012).

Consistency and asymptotic normality of the QML estimators based on the tick-exponential density are established in Komunjer (2005) under very general specifications of the quantile function f in Equation 2. The quasi log-likelihood function in Equation 4 depends on a tick-exponential component for the quantile equation and a Gaussian component for the measurement equation. This makes the asymptotic analysis of our model more complicated than that of the standard quantile regression. A full asymptotic analysis of this QML estimator is beyond the scope of this paper. However, we show below that the score is a martingale difference sequence. This result can be used to establish the asymptotic properties in future research.

Proposition 2. *Suppose that (i)* $E(u_t z_t|\mathcal{F}_{t-1}) = 0$, *(ii)* $E(z_t^2|\mathcal{F}_{t-1}) = 1$, *(iii)* $E(u_t^2|\mathcal{F}_{t-1}) = \sigma_u^2$, *then* $E\left(\frac{\partial\ell_t(r,x;\delta)}{\partial\delta}|\mathcal{F}_{t-1}\right) = 0$.

Proof. See the Supporting Information Appendix. □

Proposition 2 can be used to adapt the necessary conditions for consistency in Theorem 2 of Komunjer (2005). If the conditions in Theorems 2.1, 7.2, and 7.3 of Newey and McFadden (1994) are satisfied, we have consistency and asymptotic normality of the QML estimator and consistency of the plug-in estimator of the asymptotic covariance, so that

$$\sqrt{n}\mathcal{I}_\delta^{-1/2}\mathcal{J}_\delta\left(\hat{\delta}-\delta\right) \to N(0,1), \quad \hat{\mathcal{J}}(\hat{\delta}) \xrightarrow{p} \mathcal{J}_\delta, \quad \hat{\mathcal{I}}(\hat{\delta}) \xrightarrow{p} \mathcal{I}_\delta,$$

where $\mathcal{J} = E\left(\frac{\partial^2\ell_t(r,x;\delta)}{\partial\delta\partial\delta}\right)$ and $\mathcal{I} = E\left(\frac{\partial\ell_t(r,x;\delta)}{\partial\delta}\frac{\partial\ell_t(r,x;\delta)'}{\partial\delta}\right)$.

In Section 4, we provide the analytical expressions of both the score and the Hessian matrix for a linear REQ model. We validate the QML estimation procedure via simulations in Section 5.

3.3 | The REQ estimator

The system of equations (Equations 1–3) provides an HF extension of the CAViaR framework of Engle and Manganelli (2004), alternative to the one proposed in Hua and Manzan (2013) and Žikeš and Baruník (2016), as it also takes into account the measurement error intrinsic in the realized measures. Following the taxonomy of Figure 1, we are still in the pure quantile approach. We make our methodology *extreme* by modeling z_t^θ with an EVT-based model.

Assumption 2. F^{z^θ} is regularly varying with extreme value index $\xi > 0$; that is, it satisfies

$$\bar{F}^{z^\theta}(z) = \mathcal{L}(z) z^{-\frac{1}{\xi}},$$

where $\mathcal{L}(z)$ is such that $\lim_{z\to\infty} \mathcal{L}(tz)/\mathcal{L}(z) = 1 + k(t)\phi(z) + o(\phi(z))$ for each $t > 0$, with $\phi(z) > 0$ and $\phi(z) \to 0$ as $z \to \infty$, and $\lim_{z\to\infty} \phi(tz)/\phi(z) = z^{\varrho}$ with $\varrho \leq 0$. as in Smith (1987).

Assumptions 1 and 2 define the asymptotic behaviour of the upper tail of F^{z^θ}. In particular, Assumption 2 implies that F^{z^θ} is a heavy-tailed distribution (Embrechts, Klüppelberg, & Mikosch, 1997). This is a quite plausible assumption given the better fit of heavy-tailed conditional distributions in the volatility literature (Bollerslev, 1987; Nelson, 1991). Furthermore, the condition on $\mathcal{L}(z)$ allows for a very general functional form of the tail and is standard in the extreme value literature (De Haan & Ferreira, 2006).

Proposition 3. *Let $\hat{z}_1^\theta, \ldots, \hat{z}_n^\theta$ be the quantile residuals and $\hat{z}_{(1)}^\theta > \cdots > \hat{z}_{(n)}^\theta$ the corresponding descending-order statistics. Under Assumptions 1 and 2, the REQ estimator for VaR_t^α, with $\alpha < \theta$, is defined as*

$$\widehat{\mathrm{VaR}}_t^\alpha = \hat{q}_t^\theta \hat{z}_{(k)}^\theta \left(\frac{k}{n\alpha}\right)^{\hat{\xi}}, \tag{5}$$

where $\hat{q}_t^\theta$ is the QML θ-quantile estimator,

$$\hat{\xi} = \frac{1}{k}\sum_{j=1}^{k} \log\left(\frac{\hat{z}_{(j)}^\theta}{\hat{z}_{(k)}^\theta}\right), \tag{6}$$

and k is an integer such that $k = k_n \to \infty$ and $k/n \to 0$ as $n \to \infty$.

Proof. See the Supporting Information Appendix. □

Asymptotic theory for models combining quantile regression and EVT has been developed in Chernozhukov (2005), Chernozhukov and Fernández-Val (2011), and Wang et al. (2012). Asymptotic properties of the REQ estimator can be obtained from Theorems 1 and 2 of Wang et al. under an explicit functional form for $k(t)$ and $\phi(z)$. Intuitively, $z_t^\theta = r_t/q_t^\theta$ can be approximated by $r_t/\hat{q}_t^\theta$, with $q_t^\theta = f(\cdot, \delta)$ and $\hat{q}_t^\theta = f(\cdot, \hat{\delta})$. To the extent that $\sup_{1\leq t\leq n} |\frac{r_t}{f(\cdot,\hat{\delta}) - f(\cdot;\delta)}|$ is small enough to satisfy a condition equivalent to (S.1) in the supplementary material of Wang et al., then under Assumptions 1 and 2 and other regularity conditions one has that

$$\sqrt{k}(\hat{\xi} - \xi) \xrightarrow{d} N\left(\frac{\varpi}{1-\varrho}, \xi^2\right),$$

where ϱ controls the second-order behavior of the tail of F^{z^θ} and ϖ is a bias term. Furthermore, under the conditions of Theorem 2 of Wang et al. (2012), one has that $\left(\frac{k}{n\alpha}\right)^{\hat{\xi}} \frac{\widehat{z_{(k)}}}{Q_\alpha(z^\theta)} = 1 + \frac{\sqrt{k}}{\log\{k/(n\alpha)\}}(W_n + o_p(1))$ with $W_n \sim N\left(\frac{\varpi}{1-\varrho}, \xi^2\right)$ and, assuming that $\sup_{1\leq t\leq n} |\frac{\hat{q}_t^\theta}{q_t^\theta} - 1| = \sup_{1\leq t\leq n} |\frac{f(\cdot;\hat{\delta})}{f(\cdot;\delta)} - 1|$ is small, from Proposition 1 of Wang et al. we have that, for $\frac{\sqrt{k}}{\log\{k/(n\alpha)\}} \to 0$,

$$\frac{\sqrt{k}}{\log\{k/(n\alpha)\}}\left\{\frac{\widehat{\mathrm{VaR}}_t^\alpha}{\mathrm{VaR}_t^\alpha} - 1\right\} \xrightarrow{p} N\left(\frac{\varpi}{1-\varrho}, \xi^2\right).$$

The asymptotic results for the REQ estimator thus strongly depend on the function f in Equation 2 and the properties of the estimator $\hat{\delta}$. For example, for the case of the GARCH quantile model, Yi et al. (2014) were able to establish the asymptotic properties of $\hat{\xi}$ and $\widehat{VaR}_t^\alpha$. Note that the asymptotic bias $\frac{\varpi}{1-\varrho}$ arises as a consequence of Assumption 2, which specifies a very general tail decay of the distribution of z_t^θ. We can make this bias disappear if we are willing to use the stronger assumption that the tail has a remainder term without second-order behavior, assuming, for instance, that asymptotically the tail follows a Pareto distribution. See Wang et al. (2012) for a similar discussion.

Assumption 2 requires that the conditional return distribution be heavy tailed; however, this may not always be true. This condition can be replaced by the following more general assumption.

Assumption 3. F^{z^θ} is in the maximum domain of attraction of an extreme value distribution G_ξ, written $F \in D(G_\xi)$, where ξ is the extreme value index.

Note that this assumption coincides with Assumption 2 if $\xi > 0$. When ξ is either equal to or less than zero, one obtains a distribution with exponential decay and a thin-tailed distribution respectively. Under Assumption 3, the REQ estimator in Equation 5 is still valid but we cannot use Equation 6 to estimate ξ. Another estimator could be used, such as the moment estimator of De Haan and Ferreira (2006), though this is less efficient (Wang & Li, 2013).

4 | A LINEAR REQ MODEL

A REQ model with a linear specification can be defined by the following quantile and measurement equations:

$$\begin{aligned} q_t^\theta &= \beta_0 + \beta_1 q_{t-1}^\theta + \gamma' x_{t-1}, \\ x_t &= \omega + \phi q_t^\theta + \tau_1 z_t^\theta + \tau_2((z_t^\theta)^2 - 1) + u_t, \end{aligned} \tag{7}$$

where x_t is a realized measure, $(\beta_0, \beta_1, \gamma, \omega, \phi, \tau_1, \tau_2)$ are parameters with $\beta_0 < 0, \beta_1 > 0, \gamma < 0$, to guarantee that $q_t^\theta < 0$, and $u_t \sim N(0, \sigma_u^2)$. The structure of this model is very appealing in finance, as it allows the current quantile to depend smoothly on the previous quantile, reflecting the common dynamics found in the volatility literature. When $\tau_1 = \tau_2 = 0$, the quantile equation can be rewritten in a compact CAViaR form as

$$q_t^\theta = \tilde{\beta}_0 + \tilde{\beta}_1 q_{t-1}^\theta + \gamma' u_t,$$

with $\tilde{\beta}_0 = (\beta_0 + \gamma\omega)$ and $\tilde{\beta}_1 = (\beta_1 + \gamma\phi)$.

To estimate this model we rely on the QML quantile estimator described in Section 3.2. As discussed in Komunjer (2005), estimation of quantile regression models is complicated by the fact that the likelihood function includes a nonsmooth component, that is, the indicator function. We use an optimization routine similar to that of Engle and Manganelli (2004) to estimate the parameters of the model. We start by fitting the following linear realized GARCH model:

$$\begin{aligned} r_t &= \sigma_t \tilde{\epsilon}_t, \\ \sigma_t &= \tilde{\beta}_0 + \tilde{\beta}_1 \sigma_{t-1} + \tilde{\gamma} x_{t-1}, \\ x_t &= \tilde{\omega} + \tilde{\phi}\sigma_t + \tilde{\tau}_1 \tilde{\epsilon}_t + \tilde{\tau}_2(\tilde{\epsilon}_t^2 - 1) + u_t, \end{aligned}$$

where $\tilde{\epsilon}_t \sim N(0,1)$ and $u_t \sim N(0, \tilde{\sigma}_u^2)$, and obtain estimates $\hat{\tilde{\beta}}_0$, $\hat{\tilde{\beta}}_1$, $\hat{\tilde{\gamma}}$, $\hat{\tilde{\omega}}$, $\hat{\tilde{\phi}}$, $\hat{\tilde{\tau}}_1$, $\hat{\tilde{\tau}}_2$, $\tilde{\sigma}_u^2$. Under this model we have $q_t^\theta = \sigma_t \mathcal{Q}_\theta(\epsilon)$ and we can recover model in Equation 7 if we multiply the linear realized GARCH model by $\mathcal{Q}_\theta(\epsilon)$ and adjust the parameters. Starting values (sv) for the QML quantile estimator are thus obtained as

$$\begin{aligned} &\beta_0^{\text{sv}} = \hat{\tilde{\beta}}_0 N^{-1}(\theta), \quad \beta_1^{\text{sv}} = \hat{\tilde{\beta}}_1, \quad \gamma^{\text{sv}} = \hat{\tilde{\gamma}} N^{-1}(\theta), \quad \omega^{\text{sv}} = \hat{\tilde{\omega}}, \\ &\phi^{\text{sv}} = \hat{\tilde{\phi}}/N^{-1}(\theta), \quad \tau_1^{\text{sv}} = -\hat{\tilde{\tau}}_1, \quad \tau_2^{\text{sv}} = \hat{\tilde{\tau}}_2, \quad \sigma_u^{2,\text{sv}} = \tilde{\sigma}_u^2, \end{aligned}$$

where $N^{-1}(\theta)$ is the Gaussian quantile function evaluated at the probability level θ. We use these starting values to initialize the Nelder–Mead simplex algorithm. We then use the estimated parameters to feed a quasi-Newton algorithm and take the new optimal parameters as input values to run the simplex algorithm a second time. We repeat this procedure until some convergence criterion is satisfied. Tolerance level for the function and the parameter values is set to 10^{-8}. All the computations are done with the *optim* function of R, with the core loops coded in Rcpp.

Standard error estimates for the model parameters are obtained by plugging the parameter estimates into the asymptotic covariance matrix. Analytical expressions for the score and the Hessian matrix are given in the following proposition.

Proposition 4. *Let* $\delta = (\lambda', \psi')$, $\lambda = (\beta_0, \beta_1, \gamma)$, $\psi = (\omega, \phi, \tau_1, \tau_2)$, $g_t = (1, q_t^\theta, u_t)$, *and* $v_t = (1, q_t^\theta, z_t^\theta, (z_t^\theta)^2 - 1)$. *The score of the model in Equation 7 is* $\frac{\partial \ell}{\partial \delta} = \sum_{t=1}^n \frac{\partial \ell_t}{\partial \delta}$ *with*

$$\frac{\partial \ell_t}{\partial \delta} = \left\{ \left([\theta(1-\theta)]^{-1}[\theta - I\{r_t \le q_t^\theta\}] - \frac{u_t}{\sigma_u^2}\dot{u}_t \right) \dot{q}_t, \frac{u_t}{\sigma_u^2} v_t, -\frac{1}{2}\left(\frac{\sigma_u^2 - u_t^2}{\sigma_t^4} \right) \right\},$$

where $\dot{u}_t = \frac{\partial u_t}{\partial q_t^\theta} = -\phi - (\tau_1 + 2\tau_2)\dot{z}_t$, $\dot{z}_t = \frac{\partial z_t}{\partial q_t^\theta} = -\frac{r_t}{(q_t^\theta)^2}$, $\dot{q}_t = \frac{\partial q_t^\theta}{\partial \lambda} = \left(\beta_1 - \gamma' \dot{u}_{t-1}\right)\dot{q}_{t-1} + g_t$ *and* $\dot{q}_1 = 0$. *The Hessian matrix is given by* $\frac{\partial^2 \ell}{\partial \delta \partial \delta'} = \sum_{t=1}^n \frac{\partial^2 \ell_t}{\partial \delta \partial \delta'}$ *with*

$$\frac{\partial^2 \ell_t}{\partial \delta \partial \delta'} = \begin{pmatrix} -\left\{ \frac{f(q_t^\theta)}{\theta(1-\theta)} + \frac{\dot{u}_t^2 + u_t \ddot{u}_t}{\sigma_u^2} \right\} \dot{q}_t \dot{q}_t' + \left\{ \frac{[\theta - I\{r_t \le q_t^\theta\}]}{\theta(1-\theta)} - \frac{u_t}{\sigma_u^2}\dot{u}_t \right\} \ddot{q}_t & \bullet & \bullet \\ \frac{\dot{u}_t}{\sigma_u^2} v_t \dot{q}_t' + \frac{u_t}{\sigma_u^2} b_t \dot{q}_t' & -\frac{1}{\sigma_u^2} v_t v_t' & \bullet \\ \frac{u_t}{\sigma_u^4} \dot{u}_t \dot{q}_t' & \frac{u_t}{\sigma_u^2} v_t' & \frac{1}{2}\left(\frac{\sigma_u^2 - 2u_t^2}{\sigma_u^6} \right) \end{pmatrix},$$

where $\ddot{u}_t = \frac{\partial^2 u}{\partial q_t^\theta \partial q_t^\theta} = -\tau_1 \dot{z}_t - 2\tau_2(\dot{z}_t^2 + z_t \ddot{z}_t)$, $\ddot{z}_t = \frac{\partial^2 z}{\partial q_t^\theta \partial q_t^\theta} = 2\frac{r_t}{(q_t^\theta)^3}$ *and* $b_t = (0, 1, \dot{z}_t, 2z_t\dot{z}_t)$.

Proof. See the Supporting Information Appendix. □

In the next section, we perform a simulation experiment for this simple linear model to assess the small-sample properties of our QML estimator and the REQ estimator for the VaR.

5 | SIMULATION

We consider the linear REQ model in Equation 7 and assume that the conditional random process is t distributed with 6 degrees of freedom, $t(6)$, implying that $\xi = 0.17$. More specifically, define the return process as

$$\begin{aligned} r_t &= \sigma_t \epsilon_t, \\ \sigma_t &= 0.02 + 0.6\sigma_{t-1} + 0.15x_{t-1}, \\ x_t &= 0.1 + 0.9\sigma_t - 0.02\epsilon_t + 0.02(\epsilon_t^2 - 1) + u_t, \end{aligned} \tag{8}$$

where $u_t \sim N(0, 0.0009)$. This process is similar to the linear realized GARCH of Hansen et al. (2012), but models σ_t instead of σ_t^2. Multiplying the process in Equation 8 by the θ quantile of the $t(6)$ and adjusting the parameters, we obtain the corresponding quantile regression model:

$$\begin{aligned} q_t^\theta &= -0.023 + 0.6q_{t-1}^\theta - 0.17x_{t-1}, \\ x_t &= 0.1 - 0.76q_t^\theta + 0.02z_t^\theta + 0.02((z_t^\theta)^2 - 1) + u_t, \end{aligned} \tag{9}$$

where ϵ_t^θ is such that $Q_\theta(\epsilon_t^\theta | \mathcal{F}_{t-1}) = 0$ and $z_t^\theta = r_t / q_t^\theta$. We consider $\theta = 0.1$ and generate $B = 500$ replications of $N = 4,000$ observations drawn from the process in Equation 9. We then use the QML quantile estimator described in Section 3.2 to estimate the parameters of the model and apply the REQ estimator to obtain the VaR estimates.

Table 1 reports the results for the QML estimator. The averages of the estimated parameters are in line with the true values, confirming that our estimation procedure performs well. The left-hand panel of Figure 2 displays the estimates of the parameter ξ obtained with the estimator in Equation 6, when setting $k = N \cdot p$ with $p = 0.01$. Consistently with the theoretical results discussed in Section 3.3, the histogram resembles a normal distribution centered on a value greater than the true value of ξ. Indeed, the average of the estimates $\hat{\xi}$ is equal to 0.22 instead of 0.17. This is a byproduct of the fact that the quantile residuals follow a Student's t-distribution. Although this is a heavy-tailed distribution, the tails do not follow an exact Pareto distribution asymptotically. Referring to Assumption 2, the Student's t-distribution with ν degrees of freedom has a tail with second-order behavior controlled by $\varrho = -2/\nu$ (see Wang et al., 2012). The right-hand panel of Figure 2 compares the true VaR at level $\alpha = 0.01$ and the VaR obtained with the REQ estimator for one simulated sample. If the REQ estimator perfectly recovers the true VaR, then the dots in the figure should lie on the 45° line. The REQ performs well and this is confirmed by the box plot of the ratio of estimated and true VaR for all samples at the bottom of Figure 2.

TABLE 1 QML quantile estimator

Parameter	β_0	β_1	γ_1	ω	ϕ	$\tau_1 \times 10^2$	$\tau_2 \times 10^2$	$\sigma_u^2 \times 10^2$
True	−0.023	0.600	−0.170	0.100	−0.760	2.000	2.000	0.090
Mean	−0.023	0.590	−0.170	0.100	−0.790	2.000	1.900	0.090
SD	0.006	0.033	0.022	0.012	0.092	0.120	0.120	0.002
Mean ASE	0.004	0.036	0.019	0.011	0.076	0.070	0.090	0.002

Note. QML results for 500 replications of the process in Equation 9. Parameter values used in the simulation (True), mean of the estimated parameters (Mean), standard deviation of the estimated parameters (SD), mean of the plug-in estimates of the asymptotic standard errors (Mean ASE).

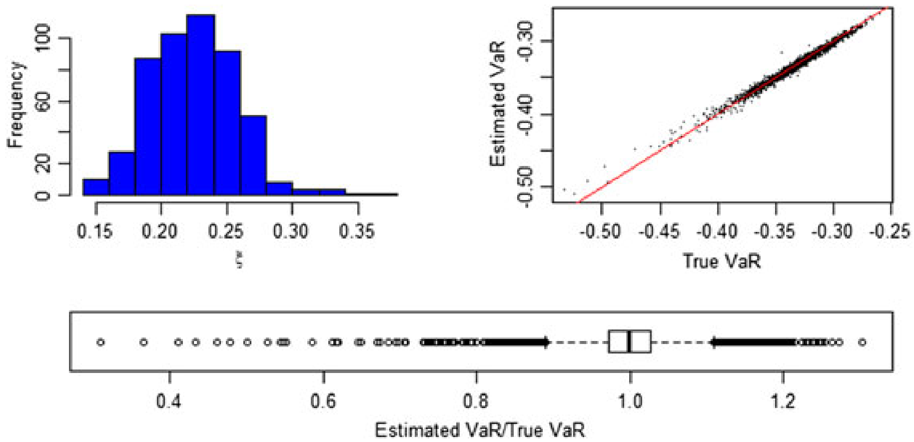

FIGURE 2 REQ estimator performance. The left-hand panel reports the estimates of ξ for 500 replications of the process in Equation 9. The right-hand panel reports the 4,000 estimated and true VaR for one sample with the 45° line. The bottom panel reports the ratio of the estimated and true VaR for all 500 replications [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 2 Data description

Asset	Abbr.	*T*	Asset	Abbr.	*T*
Amsterdam Exchange Index	AEX	3,816	IBEX 35	IBX	3,782
All Ordinaries Index	AOI	3,743	IPC Mexico	IPC	3,748
Bovespa Index	BVP	3,664	Korea Composite Index	KCI	3,690
CAC 40	CAC	3,817	Nasdaq 100	NSQ	3,747
DAX 30	DAX	3,795	Nikkei 225	NK	3,630
Dow Jones Industrial	DJ	3,746	Russel 2000 Index	RUS	3,745
Euro Stoxx 50	ESX	3,794	SP 500	SPX	3,744
FTSE MIB	MIB	3,778	Swiss Market Index	SMI	3,749
FTSE 100	FT	3,764			

Note. List of time series considered (Asset), the abbreviation (Abbr.) used throughout the paper, and their length (T). The starting date and the ending date of the samples are, respectively, January 2, 2000 and December 31, 2014. Stock exchanges respect different holidays and the number of observations T subsequently differs.

6 | EMPIRICAL ANALYSIS

The empirical analysis is based on the Oxford-Man Institute "Realized Library" version 0.2 (Heber, Lunde, Shephard, & Sheppard, 2009) recording the daily observations of several realized measures for 17 different stock indices from the beginning of 2000 to the end of 2014 (see Table 2). We consider the following five realized measures: the absolute returns (AR), the daily range (DR) of Parkinson (1980), the realized volatility (RV) of Andersen, Bollerslev, Diebold, & Labys (2001), the bipower variation (BV) of Barndorff-Nielsen and Shephard (2004), and the realized kernel (RK) of Barndorff-Nielsen, Hansen, Lunde, & Shephard (2008). Table 3 reports details regarding the measures used in the analysis.

6.1 | In-sample fit

In this section we assess the in-sample fit of the linear REQ model at the quantile level $\theta = 0.1$. The choice of quantile level is justified by the following argument: We need to select a quantile level that allows for a reliable estimation of the quantile model, but that at the same time guarantees the validity of Assumption 1; that is, that observations below the specified quantile level belong to the same distribution. A well-known result in EVT states that the distribution of the exceedances over a high threshold of a sample drawn from an i.i.d. distribution goes to a generalized Pareto (GP) as the threshold goes to the upper bound of the underlying distribution (Pickands, 1975). It is also standard to model the size of exceedances over some sufficiently high threshold using the GP. Under Assumption 1, the latter practice can be restated in our dynamic context as the distribution of $-(r_t - q_t)$ given that r_t falls below q_t should be GP. The validity of our choice

TABLE 3 Realized measures

Name	Type	Frequency	Formula	Details
AR	LF	Daily	$\text{AR}_t = \lvert r_t \rvert$	r_t is the difference of log-prices between consecutive days
DR	LF	Daily	$\text{DR}_t = \sqrt{\frac{(h_t - l_t)}{4\log 2}}$	h_t and l_t are the max and min log-prices recorded on day t.
RV	HF	5 min	$\text{RV}_t = \sqrt{\sum_{i=1}^{1/\Delta} r^2_{t-1+i\cdot\Delta}}$	r_Δ is the Δ-period intra-day return. We use $\Delta = 5$ min
BV	HF	5 min	$\text{BV}_t = \frac{\pi}{2}\sqrt{\sum_{i=2}^{1/\Delta} \lvert r_{t-1+(i-1)\cdot\Delta}\rvert \lvert r_{t-1+i\cdot\Delta}\rvert}$	
RK	HF	All	$\text{RK}_t = \sqrt{\sum_{h=-H}^{H} k\left(\frac{h}{H+1}\right)\gamma_h}$	$\gamma_h = \sum_{i=\lvert h\rvert+1}^{1/\Delta} r_{t-1+i\cdot\Delta} r_{t-1+(i-\lvert h\rvert)\cdot\Delta}$, $k(x)$ is a weight function

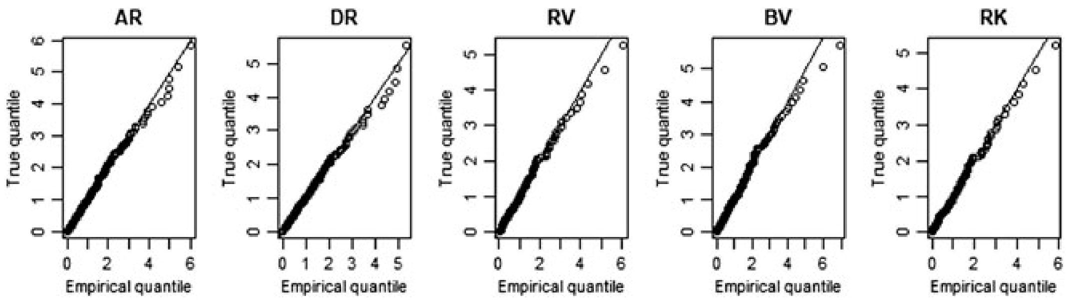

FIGURE 3 In-sample fit of linear REQ model for S&P 500. Q–Q plot for the GP distribution fitted to the residuals $-(r_t - q_t^\theta)$ obtained at level $\theta = 0.1$ using the realized measures defined in Table 3

of θ can therefore be checked by assessing the goodness of fit of the GP distribution to the latter $-(r_t - q_t)$ using a Q–Q plot. Figure 3 reports the Q–Q plots for the S&P 500, and the plots show that choosing $\theta = 0.1$ yields a good fit. Unreported plots for the other series show similar patterns.

Table 4 reports the QML estimates of both the quantile and measurement error equations, the value of the partial log-likelihood for the quantile model $\ell(r)$, and the tail index estimates obtained for the 17 equity indices listed in Table 2. For the sake of conciseness, we report the detailed results for the S&P 500 and the average results obtained across all the indices. We now highlight some key results from Table 4.

To assist with the interpretation, consider the case where the realized volatility captures all of the conditional volatility and the error term is normally distributed. Following the notation in Section 4, this means that $\tilde{\phi} = 1$. We would then obtain $\phi = \tilde{\phi}/N^{-1}(\theta) = -0.78$. Estimates in Table 4 are close to this value but show some departure in the case of DR.

The parameter γ is significant regardless of the realized measure used. The parameters γ associated with the HF-based realized measures have higher magnitude, and the value of β_1 is reduced when the HF-based measures are included in the model. The first two rows of plots in Figure 4 show this very well. This behavior is also observed in Žikeš and Baruník (2016) and signals the additional information content conveyed by the HF-based realized measures.

Another interesting aspect is that the leverage parameter τ_1 is always positive and significant. The z_t^θ, which we use to account for the leverage effect, are positive when a negative shock occurs, so one should observe a positive τ_1 if negative shocks have a higher impact on the volatility than positive shocks. The results in Table 4 and the last row of Figure 4 confirm that this is the case, consistently with what is typically found in the volatility literature.

The importance of adding a leverage component is shown in Figure 5, where the scatter plots of the residuals $\left(\hat{z}_t^\theta, \hat{u}_t\right)$, obtained from the linear REQ specification with and without leverage component, appear. The asymmetric effect left in the residuals of the model without the leverage component is not negligible, and regressing $\hat{z}_t^\theta$ on $\hat{u}_t$ returns a strongly significant positive slope coefficient equal to 0.0007. This finding is coherent with the observations of Hansen et al. (2012) and it is relevant for the validity of condition (i) in Proposition 2.

To highlight the importance of adding HF-based realized measures, we perform an out-of-sample analysis of the partial log-likelihood, $\ell(r)$. We use the partial log-likelihood to compare the REQ model across the different realized measures, as the full log-likelihood is misleading given that the time series of the realized measures are different. We split the available

TABLE 4 In-sample fit

RM	$\beta_0 \times 10^3$	$\beta_1 \times 10^3$	$\gamma \times 10^3$	$\omega \times 10^3$	$\phi \times 10^3$	$\tau_1 \times 10^3$	$\tau_2 \times 10^3$	σ_u	$\ell(\mathbf{r})$	ξ
Details for S&P 500										
AR	−1.000	850	−70	−4	−920	−0.300	3.600	0.010	−8.095	0.230
	0.100	0.010	0.100	0.900	0.010	0.200	0.190	0.002		0.050
DR	−3.000	730	−200	−5	−980	0.200	4.100	0.008	−8.003	0.200
	0.080	0.010	0.100	0.370	0.010	0.100	0.380	0.001		0.040
RV	−4.000	460	−680	−3	−700	0.900	1.500	0.007	−7.857	0.190
	0.070	2.200	1.800	0.320	0.390	0.110	0.250	0.001		0.040
BV	−2.00	470	−710	−1	−670	0.900	0.700	0.007	−7.883	0.170
	0.050	1.600	1.200	0.220	0.210	0.090	0.150	0.001		0.030
RK	−3.000	500	−610	−3	−730	1.000	1.800	0.009	−7.897	0.190
	0.050	1.700	1.500	0.250	0.25	0.120	0.210	0.001		0.040
Average across all assets										
AR	−3.400	700	−100	−0.600	−7.80	−0.190	5.850	0.013	−8.430	0.200
	0.210	1.400	3.200	0.290	0.120	0.170	0.640	0.001		0.040
DR	−2.900	710	−200	−5.400	−1040	0.320	4.090	0.009	−8.307	0.180
	0.090	0.900	1.200	0.480	0.140	0.100	0.420	0.001		0.030
RV	−3.300	530	−530	−3.400	−800	0.770	1.530	0.008	−8.238	0.170
	0.060	1.020	0.920	0.350	0.140	0.100	0.210	0.001		0.030
BV	−3.700	500	−590	−3.800	−770	0.800	1.350	0.007	−8.310	0.170
	0.060	1.100	0.001	0.390	0.150	0.100	0.210	0.001		0.030
RK	−3.200	540	−510	−3.600	−820	0.730	1.610	0.009	−8.206	0.180
	0.060	1.010	0.900	0.370	0.130	0.110	0.220	0.001		0.030

Note. QML estimates and standard errors for the quantile and measurement equation parameters, partial log-likelihood value $\ell(r)$ and estimates of the tail index ξ for the linear REQ model.

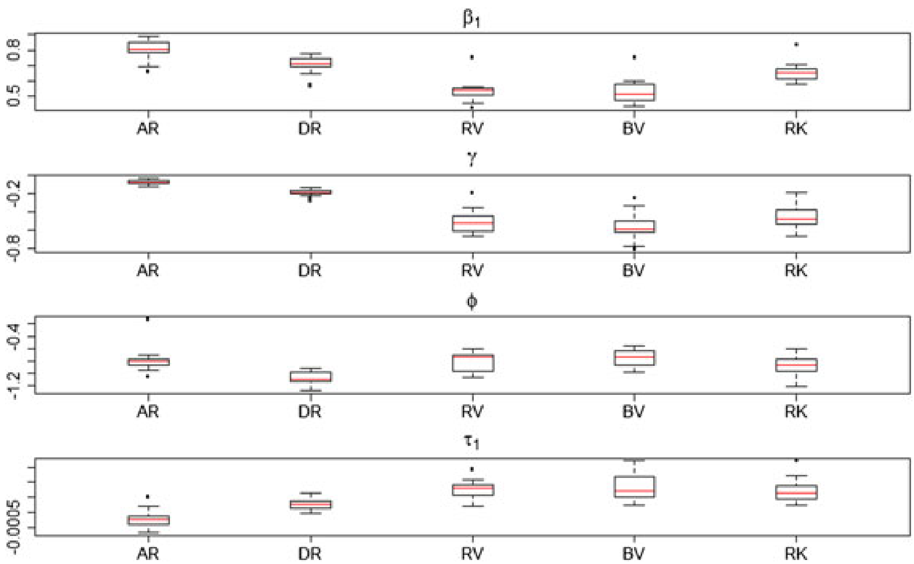

FIGURE 4 QML estimates of quantile equation parameters β_1 and γ, and measurement equation parameters ϕ and τ_1, for the 17 indices when using realized measures defined in Table 3 [Colour figure can be viewed at wileyonlinelibrary.com]

time series into three subsamples of 5 years: 2000–2004, 2005–2009, and 2010–2014. We fit the REQ model on the first subsample and then use the estimated parameters to produce the values of the partial log-likelihood on the other two samples. Figure 6 reports the gain in terms of log-likelihood obtained across the 17 time series when a high-frequency realized measure is used instead of the daily range (DR). Like Brownlees and Gallo (2010) and Gerlach and Chen (2016), we use DR as the benchmark since it is well known that it provides highly accurate estimates of the volatility compared

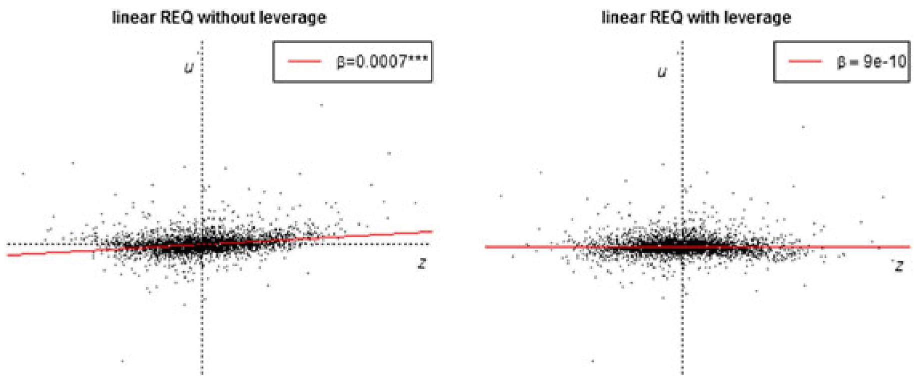

FIGURE 5 Leverage effect. Residuals $\left(\hat{z}_t^{\theta}, \hat{u}_t\right)$ from the linear REQ model with and without leverage component. Results are for the returns and RV of the S&P 500 [Colour figure can be viewed at wileyonlinelibrary.com]

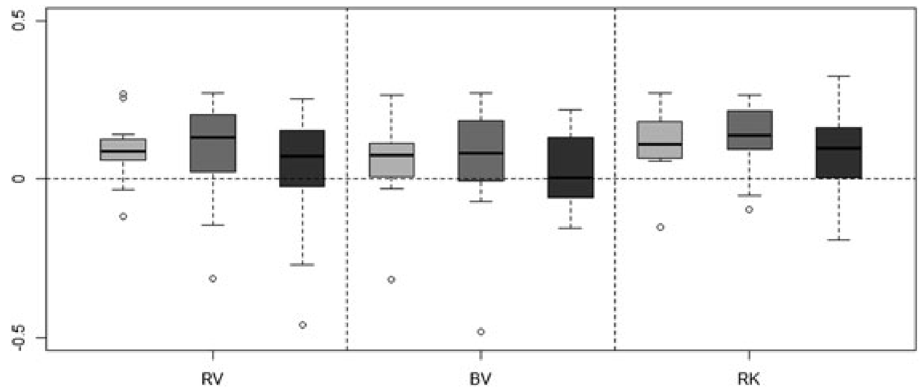

FIGURE 6 Partial log-likelihood. Difference between the partial return log-likelihood obtained when using the high-frequency realized measures RV, BV, and RK, instead of the daily range for the 17 series listed in Table 2. The in-sample fit (2000–2004) is in light gray. The out-of-sample fits (2005–2009 and 2010–2014) are in medium and dark gray, respectively

to other daily estimators (Alizadeh, Brandt, & Diebold, 2002; Parkinson, 1980). The predominantly positive average gains in Figure 6 confirm that HF measures contain extra information about the dynamics of the daily quantile.

Finally, we also use $\ell(r)$ to compare our quantile model with measurement equation to the corresponding quantile model without measurement equation, trying to assess the relevance of accounting for the measurement error. A comparison is difficult, however, as $\ell(r)$ is only the partial log-likelihood in the first case, and in the second case $\ell(r)$, which we note $\ell(q)$ in the empirical results, is the optimized object. Moreover, the contribution of the measurement component to the quantile likelihood dwarfs that of the quantile component (see Table 5). Given this, while the standard quantile model presents slightly larger log-likelilood values (see Table 5), we think that the inclusion of a measurement equation is worthwhile when adding a valid proxy for the latent volatility. We can see that the use of a high-frequency realized measure makes the difference between $\ell(r)$ and $\ell(q)$ close to zero.

6.2 | Out-of-sample forecasting

In this section, we compare the out-of-sample forecasting performance of several specifications of our quantile model, each one encompassing a different class according to our taxonomy in Figure 1. We consider linear quantile models as in Equation 7 and denote them as realized quantile (RQ) models. Next, we consider linear REQ models, using the estimator in Equation 5 to produce the EVT-based VaR estimate. In both cases, we employ either LF or HF measures. Our goal is to assess the value added by EVT and HF information to a quantile model against pure quantile models and mixed quantile–EVT models with LF measures.

We perform an out-of-sample analysis across the 17 indices, using a rolling window scheme with a window of size $S = 2,000$ observations and daily updates. We estimate the VaR at level α as follows: QML estimation of RQ models is

TABLE 5 Quantile log-likelihood decomposition

	AR			DR			RV			BV			RK		
	$\ell(\boldsymbol{q})$	$\ell(\boldsymbol{r})$	$\ell(\boldsymbol{x}\|\boldsymbol{r})$	$\ell(\boldsymbol{q})$	$\ell(\boldsymbol{r})$	$\ell(\boldsymbol{x}\|\boldsymbol{r})$	$\ell(\boldsymbol{q})$	$\ell(\boldsymbol{r})$	$\ell(\boldsymbol{x}\|\boldsymbol{r})$	$\ell(\boldsymbol{q})$	$\ell(\boldsymbol{r})$	$\ell(\boldsymbol{x}\|\boldsymbol{r})$	$\ell(\boldsymbol{q})$	$\ell(\boldsymbol{r})$	$\ell(\boldsymbol{x}\|\boldsymbol{r})$
AEX	−7.470	−7.990	19,889.960	−7.380	−7.720	20,449.890	−7.300	−7.490	20,800.790	−7.300	−7.480	20,857.360	−7.300	−7.540	20,571.250
AOI	−5.150	−5.400	20,876.380	−5.110	−5.380	22,188.980	−5.080	−5.180	22,002.490	−5.080	−5.170	22,529.860	−5.080	−5.190	21,637.110
BVP	−11.600	−12.080	17,942.590	−11.470	−11.730	18,663.300	−11.420	−11.530	18,668.180	−11.410	−11.480	19,095.200	−11.420	−11.530	18,329.190
CAC	−8.070	−8.500	19,944.080	−7.970	−8.280	20,321.900	−7.900	−8.740	20,396.050	−7.900	−8.260	20,525.260	−7.900	−8.280	20,261.850
DAX	−8.620	−9.050	19,499.540	−8.500	−8.810	19,834.840	−8.410	−8.500	19,954.960	−8.410	−9.290	20,171.680	−8.400	−8.620	19,737.490
DJ	−7.270	−8.210	18,383.940	−7.160	−7.860	20,259.310	−7.060	−7.550	20,227.100	−7.040	−7.060	20,318.120	−7.070	−7.200	20,078.160
ESX	−8.790	−9.270	19,529.460	−8.640	−9.020	19,941.340	−8.600	−9.010	20,005.010	−8.600	−9.440	20,268.060	−8.590	−8.850	19,684.180
MIB	−8.400	−8.700	19,450.750	−8.310	−8.680	20,190.950	−8.240	−8.760	20,191.230	−8.230	−8.670	20,402.720	−8.220	−8.750	20,209.320
FT	−6.180	−6.460	18,549.390	−6.100	−6.350	20,672.720	−6.070	−6.140	20,796.080	−6.080	−6.090	20,937.190	−6.060	−6.230	20,710.020
IBX	−8.400	−8.690	18,831.770	−8.330	−8.710	20,044.810	−8.270	−9.050	20,228.850	−8.280	−9.330	20,356.170	−8.270	−8.910	20,156.830
IPC	−8.710	−9.150	19,007.050	−8.600	−9.250	20,093.900	−8.610	−9.400	21,027.180	−8.610	−9.850	21,256.670	−8.670	−9.260	20,858.340
KCI	−7.980	−8.350	19,367.510	−7.800	−8.150	19,513.560	−7.760	−7.980	19,889.810	−7.760	−8.520	19,942.320	−7.750	−7.780	19,696.130
NSQ	−8.710	−9.240	19,307.140	−8.530	−8.840	19,446.860	−8.460	−9.500	19,844.790	−8.450	−9.530	19,746.420	−8.430	−8.910	19,696.410
NK	−7.340	−7.860	18,483.190	−7.260	−7.990	19,626.310	−7.220	−7.500	20,094.390	−7.200	−7.400	20,201.330	−7.210	−8.600	19,802.110
RUS	−9.360	−9.700	19,219.340	−9.210	−9.570	19,742.590	−9.270	−9.320	20,141.680	−9.280	−9.320	20,167.940	−9.220	−9.790	19,940.750
SPY	−7.640	−8.100	17,674.570	−7.490	−8.000	20,213.450	−7.380	−7.860	20,286.090	−7.370	−7.480	20,346.230	−7.400	−7.900	19,933.490
SMI	−6.280	−6.570	20,309.620	−6.140	−6.790	20,787.650	−6.090	−6.570	21,272.130	−6.100	−6.890	21,308.230	−6.130	−6.160	21,380.880

Note. In-sample values of the quantile log-likelihood, $\ell(r)$, and the log-likelihood of the measurement equation, $\ell(x|r)$, as per (4) evaluated at the optimum over the period 2000–2004. In-sample values of the maximum log-likelihood, noted $\ell(q)$, of corresponding model without measurement equation is also shown for each realized measure.

TABLE 6 *Note.* Evaluation of VaR forecasts at 1% level

	UC-RQ					UC-REQ				
	AR	**DR**	**RV**	**BV**	**RK**	**AR**	**DR**	**RV**	**BV**	**RK**
AEX	**0.000**	**0.031**	0.669	0.442	0.513	0.782	0.669	0.844	0.513	0.669
AOI	**0.000**	**0.000**	0.201	0.727	0.405	**0.019**	0.087	0.201	0.134	0.087
BVP	**0.000**	0.422	**0.018**	**0.039**	**0.018**	0.569	0.681	0.139	0.229	0.504
CAC	**0.000**	**0.000**	0.382	0.601	0.190	**0.0189**	0.052	**0.011**	**0.006**	**0.019**
DAX	**0.000**	0.353	**0.008**	**0.019**	0.133	0.805	0.990	0.353	0.481	0.481
DJ	**0.000**	**0.034**	**0.024**	**0.000**	0.164	0.136	**0.055**	0.715	0.294	0.897
ESX	**0.000**	0.8220	0.218	**0.008**	**0.039**	0.352	0.479	0.352	0.249	0.479
MIB	**0.000**	**0.037**	**0.004**	**0.004**	**0.009**	0.244	0.690	0.516	0.244	0.877
FT	**0.000**	**0.002**	0.349	0.958	0.496	**0.001**	**0.004**	**0.001**	**0.001**	**0.000**
IBX	**0.000**	**0.001**	0.781	0.461	0.844	**0.000**	**0.008**	**0.008**	**0.025**	**0.025**
IPC	**0.000**	0.206	0.095	0.095	**0.000**	0.296	0.296	0.163	0.094	0.095
KCI	**0.000**	0.615	**0.015**	**0.006**	0.068	0.123	**0.015**	0.123	0.123	0.320
NSQ	**0.000**	**0.011**	0.899	0.387	0.909	0.909	0.387	0.720	0.543	0.717
NK	**0.000**	**0.008**	0.161	0.394	0.512	0.863	0.863	0.677	0.557	0.940
RUS	**0.000**	**0.000**	**0.011**	0.051	0.051	**0.003**	**0.033**	0.913	0.713	0.293
SPX	**0.000**	**0.001**	**0.001**	**0.000**	0.725	0.0877	0.202	0.292	0.406	0.088
SMI	0.090	0.716	0.094	**0.004**	0.094	**0.034**	0.716	0.413	0.413	0.720
	IND-RQ					IND-REQ				
	AR	DR	RV	BV	RK	AR	DR	RV	BV	RK
AEX	0.078	0.349	0.504	0.112	0.483	0.149	0.504	0.526	0.483	0.504
AOI	0.684	0.722	0.309	0.586	0.253	0.081	0.371	0.433	0.413	0.393
BVP	**0.010**	0.485	0.781	0.754	0.781	0.507	0.601	0.702	0.676	0.626
CAC	0.154	0.088	0.462	0.594	0.423	0.332	0.366	0.315	0.299	0.332
DAX	0.097	0.459	0.789	0.763	0.687	0.524	0.546	0.459	0.480	0.480
DJ	0.798	0.435	0.760	0.865	0.683	0.338	0.059	0.518	0.453	0.540
ESX	0.638	0.568	0.663	0.789	0.737	0.459	0.480	0.459	0.439	0.480
MIB	0.195	0.359	0.813	0.813	0.787	0.660	0.588	0.612	0.660	0.565
FT	**0.018**	0.279	0.097	0.544	0.613	0.249	0.294	0.249	0.264	0.236
IBX	**0.038**	0.642	0.197	0.479	**0.009**	0.133	0.310	0.531	0.344	0.460
IPC	0.074	**0.035**	0.709	0.709	0.865	**0.002**	0.279	0.683	0.709	0.709
KCI	0.562	0.208	0.782	0.809	0.730	0.704	0.782	0.704	0.704	0.653
NSQ	0.490	0.322	0.540	0.634	0.574	0.563	0.634	0.586	0.610	0.517
NK	0.323	0.504	0.061	0.090	0.217	0.168	0.168	0.192	0.107	0.147
RUS	0.143	0.772	0.786	0.734	0.734	0.579	0.436	0.563	0.517	0.453
SPX	0.645	0.155	0.839	0.865	0.586	0.370	**0.036**	0.453	0.474	0.394
SMI	0.394	0.587	0.709	0.812	0.709	0.357	0.587	0.475	0.475	0.518
	CC-RQ					CC-REQ				
	AR	DR	RV	BV	RK	AR	DR	RV	BV	RK
AEX	**0.000**	0.063	0.722	0.208	0.624	0.337	0.722	0.794	0.624	0.722
AOI	**0.000**	**0.000**	0.259	0.804	0.363	**0.014**	0.153	0.320	0.230	0.158
BVP	**0.000**	0.561	0.058	0.113	0.058	0.675	0.794	0.308	0.441	0.704
CAC	**0.000**	**0.000**	0.515	0.750	0.303	**0.039**	0.099	**0.023**	**0.013**	**0.039**
DAX	**0.000**	0.488	**0.028**	0.060	0.296	0.783	0.824	0.488	0.601	0.601
DJ	**0.000**	0.076	0.077	**0.002**	0.347	0.206	**0.027**	0.751	0.429	0.813
ESX	**0.000**	0.821	0.423	**0.028**	0.113	0.487	0.600	0.487	0.377	0.600
MIB	**0.000**	0.074	**0.014**	**0.014**	**0.034**	0.457	0.790	0.706	0.457	0.829
FT	**0.000**	**0.005**	0.162	0.822	0.692	**0.001**	**0.010**	**0.001**	**0.002**	**0.001**
IBX	**0.000**	**0.005**	0.414	0.586	**0.033**	**0.000**	**0.018**	**0.025**	0.051	0.061
IPC	**0.000**	**0.049**	0.229	0.229	**0.002**	**0.005**	0.318	0.345	0.229	0.229
KCI	**0.000**	0.395	0.051	**0.022**	0.177	0.282	0.051	0.282	0.282	0.547
NSQ	**0.000**	**0.024**	0.814	0.609	0.840	0.832	0.609	0.801	0.723	0.751
NK	**0.000**	**0.024**	0.064	0.164	0.372	0.377	0.377	0.387	0.228	0.344
RUS	**0.000**	**0.001**	**0.037**	0.140	0.140	**0.011**	0.075	0.832	0.749	0.428
SPX	**0.000**	**0.001**	**0.006**	**0.002**	0.803	0.153	**0.048**	0.427	0.542	0.159
SMI	0.163	0.800	0.228	**0.016**	0.228	0.068	0.800	0.548	0.548	0.753

Note. p-values for the unconditional coverage (UC), independence (IND) and conditional coverage (CC) tests of Christoffersen (1998) obtained with the realized quantile (RQ) and REQ models. Rejections at the 5% level are in bold.

TABLE 7 DM tests on VaR at 1% level

	RQ vs. REQ					LF vs. HF		
	AR	DR	RV	BV	RK	RV	BV	RK
AEX	**0.007**	0.195	**0.024**	0.210	0.312	0.178	0.191	0.149
AOI	0.311	**0.023**	**0.032**	**0.009**	0.082	0.226	0.091	0.207
BVP	**0.012**	0.377	0.306	0.104	**0.013**	0.077	0.056	0.060
CAC	**0.047**	**0.043**	**0.014**	0.151	0.134	0.201	0.231	0.191
DAX	0.136	0.209	0.276	0.061	0.211	0.270	0.161	0.268
DJ	0.124	0.298	0.075	**0.009**	**0.043**	**0.007**	**0.005**	**0.012**
ESX	**0.028**	0.438	**0.001**	0.064	**0.000**	0.291	0.126	0.497
MIB	0.079	**0.087**	**0.005**	**0.000**	**0.000**	0.158	0.146	0.098
FT	**0.007**	0.245	0.131	0.441	0.250	0.190	0.181	0.144
IBX	0.172	0.242	**0.045**	0.167	0.133	0.067	0.091	0.054
IPC	0.558	**0.042**	**0.005**	**0.040**	**0.002**	**0.007**	**0.004**	**0.048**
KCI	**0.025**	0.087	**0.000**	**0.000**	**0.002**	0.052	0.171	0.133
NSQ	**0.020**	**0.080**	0.161	**0.042**	**0.034**	0.293	0.192	0.192
NK	0.086	0.101	0.250	0.084	**0.024**	0.084	0.109	**0.046**
RUS	**0.002**	0.057	**0.008**	0.065	**0.018**	**0.032**	**0.037**	**0.028**
SPX	0.170	0.556	0.088	0.154	0.142	**0.037**	**0.041**	0.058
SMI	0.807	0.420	**0.022**	0.058	**0.028**	0.144	0.161	0.188

Note. p-values of DM tests: (RQ vs. REQ) realized quantile (RQ) models against REQ models using the same realized measure. Rejection of the null hypothesis supports superior performances of the REQ; (LF vs. HF) REQ model with DR against REQ models with HF measures. Rejection of the null supports the use of a high-frequency measure. Rejections at the 0.05 significance level appear in bold.

performed at the α-quantile; QML estimation of REQ models is performed at the 0.1-quantile and then we apply the REQ estimator in Equation 5 at level α, setting the threshold z_k at the probability level $\alpha_k = 0.025$.

Table 6 reports the p-values from the unconditional coverage (UC), independence (IND), and conditional coverage (CC) tests of Christoffersen (1998) computed on the VaR forecasts at level $\alpha = 1\%$. The quantile models with LF measures directly fitted at the 1% quantile perform quite poorly as they often reject the null of correct unconditional coverage. RQ models with HF measures perform better, recording seven rejections with RV, eight with BV, and just four with RK. REQ models perform definitely better than RQ models in terms of unconditional coverage. These results are closely mimicked by the conditional coverage test, while all the models perform well with respect to the independence of the violations.

To emphasize the added value of EVT and the contribution of the HF measures, we perform a series of Diebold–Mariano (DM) tests using the tick-loss function $\rho_t^\theta(r_t, q_t^\theta)$ as loss criterion (Brownlees & Gallo, 2010). We first test the null of equal predictive accuracy between RQ and REQ models under the same realized measure against the alternative that REQ outperforms RQ. Results reported in Table 7 for the VaR forecasts at the 1% level show that the null is often rejected at the 5% level, thus favoring the REQ models. We then use a DM test to assess whether REQ models based on an HF measure outperform the REQ model based on daily range, our LF benchmark. The results in Table 7 show that the null of equal predictive accuracy of the forecasts at the level $\alpha = 1\%$ is rejected at least four times out of 17 on each HF measure, and in general the p-values are quite low. In conclusion, HF measures apport meaningful information also from the forecasting perspective. The conclusions drawn from the VaR at the 1% level carry over at level $\alpha = 0.5\%$ (results available upon request).

Given the similarity between our model and the realized GARCH framework of Hansen et al. (2012), we perform an out-of-sample forecast analysis using a realized GARCH(1, 1) model with the realized kernel (RK) and applying extreme value theory to the tails of the estimated residuals. We use again a rolling window scheme with a window size of $S = 2,000$ observations to produce VaR forecasts at level $\alpha = 1\%$. Table 8 reports the p-values of the performance statistics for VaR forecasts at level $\alpha = 1\%$, and they are similar to those reported for the REQ model with RK in Table 6. We also perform a Diebold and Mariano (1995) test of equal predictive accuracy against the alternative that the realized GARCH outperforms the REQ model. The p-values in the rightmost column of Table 8 show that the null is never rejected. Unreported forecasts at level $\alpha = 0.5\%$ confirm the results. The realized GARCH with EVT refinements and the REQ model thus have similar a performance on the time series considered.

TABLE 8 Realized GARCH VaR forecasts at level $\alpha = 1\%$

	UC	IND	CC	DM
AEX	0.844	0.526	0.794	0.334
AOI	0.549	0.609	0.727	0.301
BVP	0.504	0.626	0.704	0.130
CAC	**0.042**	0.367	0.099	0.429
DAX	0.820	0.568	0.820	0.410
DJ	0.911	0.563	0.833	0.145
ESX	0.803	0.523	0.783	0.159
MIB	0.517	0.612	0.706	0.691
FT	**0.025**	0.344	**0.050**	0.237
IBX	0.461	0.247	0.385	0.213
IPC	**0.004**	0.812	**0.016**	0.237
KCI	0.461	0.488	0.593	0.223
NSQ	0.089	0.404	0.164	0.793
NK	0.557	0.107	0.229	0.141
RUS	0.055	0.375	0.106	0.362
SPX	0.088	0.403	0.162	0.134
SMI	0.539	0.610	0.721	0.651

Note. p-values for the unconditional coverage (UC), independence (IND) and conditional coverage (CC) tests of Christoffersen (1998) and the Diebold and Mariano (1995) test (DM) on the null of equal predictive accuracy of the REQ model versus realized GARCH. Rejection of the null in DM favors the realized GARCH. Realized measure is RK for both REQ and realized GARCH. Rejections at the 0.05 significance level appear in bold.

The fact that our approach can compete with a very good model such as the realized GARCH with EVT tails suggests that it is a valid alternative. While both models perform similarly when the true quantile is a scalar times the volatility scaling, the REQ should perform better when the latter relationship does not hold as it models the quantile directly.

7 | CONCLUSIONS

The financial econometrics literature provides a long list of models delivering accurate estimates of quantile-based risk measures. In this paper, we fill a gap in the literature by proposing a novel quantile regression approach integrated with EVT and measures built from HF data.

Model estimation is performed via QML and it is validated through a simulation study. A large empirical analysis confirms the good in-sample fit and out-of-sample forecasts of this model, which are in line with the models used in the literature.

There are several open questions that need to be carefully addressed in future research. First, an advantage of our framework compared to the CAViaR class of models is the possibility of producing multi-day-ahead predictions, as we model the dynamics of both the quantile and the realized measure. It would therefore be interesting to assess the forecast performance of our model on longer horizons. Second, Hansen and Huang (2016) find that augmenting the realized GARCH using multiple realized measures provides better estimates of the volatility. An analogous extension could also be useful in the REQ framework. In a preliminary simulation study we find that the QML estimation of a linear REQ model with multiple realized measures provides consistent estimates. Similarly, the REQ estimator for the VaR keeps performing well. Third, our REQ estimator assumes the tail index ξ to be constant; however, nothing prevents it from being covariate dependent, that is, $\xi = \xi(x)$, adapting the framework of Wang and Li (2013) to our context. Recent findings by Bollerslev and Todorov (2014) and Kelly and Jiang (2014) provide evidence of time variation in the tail index. This feature may have important implications in the estimation of quantile-based risk measures, and the merit of including a covariate-dependent tail index should be investigated.

Other important issues relate to the multivariate dimension of this model. White, Kim, and Manganelli (2015) propose a multivariate extension of the CAViaR framework that provides an estimate of dynamic tail dependence. This approach gives important insights into the spillovers between quantiles and can be used to measure systemic risk. It could be interesting to extend the REQ framework in this direction.

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SUPPORTING INFORMATION

Additional Supporting Information may be found online in the supporting information tab for this article.

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