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An approach for rotational hardening in geotechnical modelling

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Abstract

The object of this paper is the definition of a purely kinematic rotational approach for the elastoplastic modelling of soils. Once the shape of the yield and plastic potential surfaces is assigned, their rotational kinematics in the stress space is ruled by a second order tensor directed along their surface axes. The proposed approach differs from others proposed in literature since it does not introduce any distortion of the surface and is suitable for the description of inherent or induced anisotropy as well as the behaviour of soils subjected to cyclic loading.

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1. Introduction

The description of soil behaviour requires the formulation of constitutive models based on both isotropic and kinematic hardening. In particular, kinematic hardening of yield and plastic potential surfaces is linked to inherent as well as induced anisotropy. The rotational kinematic nature of the behaviour of clays and sands is supported by many experimental investigations carried out in geotechnical literature [1-6].

The approaches proposed in literature to take into account anisotropic hardening consist in introducing an anisotropic second order tensor, which modifies the orientation of yield surfaces and plastic potentials by ruling a new

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direction of their axes. In particular, some models describe the rotational hardening by defining a unit tensor directed along the axis of the yield surface [7-11] while others are based on the definition of a purely deviatoric tensor [12-18]. Both the aforementioned approaches allow introducing anisotropic hardening of yield and plastic potential surfaces by redefining the invariants in such a way that the isotropic condition is a particular case of the general anisotropic case.

In this work, an alternative approach based on a rigid rotation is shown: it consists in introducing in the stress space a local reference system for yield and plastic potential surfaces. In this way, the gradients of the surfaces are computed in the local reference system and then transformed into the global representation.

2. The proposed approach

In this section, a new approach to kinematic rotational hardening in the stress space will be described. In order to define a rotated surface, two reference systems, one (rotated) local reference system (LRS) and one (fixed) global reference system (GRS), are used. The definition of the LRS must satisfy the following requirements: (i) the basis associated with the LRS is orthonormal and (ii) the local hydrostatic axis is parallel to tensor $\boldsymbol{\alpha}$. Once the bases associated with the two reference systems are determined, it is possible to establish the relationship between the tensor components associated with them through the computation of a suitable rotation matrix using the standard linear algebra methods. In the following, the procedure employed to evaluate the basis of the aforementioned LRS is shown. For the sake of clarity, two mathematical operators can be defined in order to simplify the mathematical formulation. For a second order tensor \mathbf{A} and a unit second order tensor \mathbf{B} , the following definition are useful:

$$\begin{aligned}\mathbf{N}(\mathbf{A}, \mathbf{B}) &= (\mathbf{A} \cdot \mathbf{B})\mathbf{B} \\ \hat{\mathbf{T}}(\mathbf{A}, \mathbf{B}) &= \frac{\mathbf{A} - \mathbf{N}(\mathbf{A}, \mathbf{B})}{\|\mathbf{A} - \mathbf{N}(\mathbf{A}, \mathbf{B})\|}\end{aligned}\quad (1)$$

Thus, \mathbf{N} is the projection of \mathbf{A} along the direction parallel to \mathbf{B} and $\hat{\mathbf{T}}$ is the normalised projection of \mathbf{A} along the direction orthogonal to \mathbf{B} . For the sake of simplicity, the mathematical formulation for computing the basis associated with the LRS is illustrated for the three-dimensional case in the space $\sigma_x, \sigma_y, \sigma_z$. The extension to the general six-dimensional case can be carried out using the same methodology reported in the following. This procedure is more easily handled if a conveniently defined auxiliary basis of tensors \mathbf{w}_i is introduced. Such a basis is obtained from the basis of the GRS by adopting the Gram-Schmidt procedure (e.g. [19]):

$$\begin{aligned}\mathbf{w}_1 &= \frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|} \\ \mathbf{w}_2 &= \frac{\mathbf{e}_2 - \mathbf{N}(\mathbf{e}_2, \mathbf{w}_1)}{\|\mathbf{e}_2 - \mathbf{N}(\mathbf{e}_2, \mathbf{w}_1)\|} \\ \mathbf{w}_3 &= \frac{\mathbf{e}_3 - \mathbf{N}(\mathbf{e}_3, \mathbf{w}_1) - \mathbf{N}(\mathbf{e}_3, \mathbf{w}_2)}{\|\mathbf{e}_3 - \mathbf{N}(\mathbf{e}_3, \mathbf{w}_1) - \mathbf{N}(\mathbf{e}_3, \mathbf{w}_2)\|}\end{aligned}\quad (2)$$

From the geometrical point of view, tensors \mathbf{w}_2 and \mathbf{w}_3 identify a plane orthogonal to tensor $\boldsymbol{\alpha}$, that is the local deviatoric plane.

The tensors of the new basis \mathbf{b}_i are determined by imposing that \mathbf{w}_1 is directed along the trisector of the LRS and that the basis $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ is orthonormal:

$$\mathbf{b}_i \cdot \mathbf{w}_1 = \frac{1}{\sqrt{3}} \quad i=1,2,3 \quad (3)$$

$$\mathbf{b}_i \cdot \mathbf{b}_i = 1 \quad (4)$$

$$\mathbf{b}_i \cdot \mathbf{b}_j = 0 \quad i \neq j \quad i, j = 1, 2, 3 \quad (5)$$

Tensors \mathbf{b}_i can be expressed as a linear combination of the auxiliary basis tensors:

$$\mathbf{b}_i = b_{1i}\mathbf{w}_1 + b_{2i}\mathbf{w}_2 + b_{3i}\mathbf{w}_3 \quad (6)$$

where b_{1i} , b_{2i} and b_{3i} are the components of tensors \mathbf{b}_i with respect to \mathbf{w}_i . From condition (3) it follows that:

$$b_{1i} = \frac{1}{\sqrt{3}} \quad (7)$$

Using equations (4), (5) and (7), the following equation is obtained

$$b_{2i}^2 + b_{3i}^2 = \frac{2}{3} \quad i = 1, 2, 3 \quad (8)$$

which represents a circumference of radius $\sqrt{2/3}$ in the plane b_{2i} , b_{3i} . In order to derive b_{2i} and b_{3i} , it is convenient to express (8) in a parametric form:

$$\begin{aligned} b_{i2} &= \sqrt{\frac{2}{3}} \sin \omega_i \\ b_{i3} &= \sqrt{\frac{2}{3}} \cos \omega_i \end{aligned} \quad i = 1, 2, 3 \quad (9)$$

Using the orthogonality condition (5), after some algebra, the following conditions are obtained:

$$\begin{aligned} \omega_3 &= \omega_2 + \frac{2}{3}\pi \\ \omega_1 &= \omega_2 + \frac{4}{3}\pi \end{aligned} \quad (10)$$

where ω_2 is the angle between \mathbf{w}_2 and the projection of \mathbf{b}_2 on the local deviatoric plane. From the constitutive point of view ω_2 is a hardening variable because it characterizes the rotation of the LRS about the local hydrostatic axis. In this work, a purely geometric constraint is imposed to define the value of ω_2 . For the sake of brevity, the expression of angle ω_2 is put in the form:

$$\omega_2 = Q(C_2, C_3) - Q(D_2, D_3) \quad (11)$$

where the function $Q(X, Y)$ assumes the values illustrated in Table 1 and

$$\begin{aligned}
C_2 &= \hat{\mathbf{T}}(-\boldsymbol{\delta}, \mathbf{w}_1) \cdot \mathbf{w}_2 \\
C_3 &= \hat{\mathbf{T}}(-\boldsymbol{\delta}, \mathbf{w}_1) \cdot \mathbf{w}_3 \\
D_2 &= \hat{\mathbf{T}}(-\boldsymbol{\alpha}, \mathbf{v}_1) \cdot \mathbf{v}_2 \\
D_3 &= \hat{\mathbf{T}}(-\boldsymbol{\alpha}, \mathbf{v}_1) \cdot \mathbf{v}_3
\end{aligned} \tag{12}$$

being $\boldsymbol{\delta}$ the tensor parallel to the global hydrostatic axis, while $\mathbf{v}_1, \mathbf{v}_2$ and \mathbf{v}_3 are the tensors of the auxiliary basis for $\boldsymbol{\alpha} = \boldsymbol{\delta} / \sqrt{3}$.

3. Application

In this section the proposed approach is employed to extend an existing multi-surface constitutive model by [11] to surfaces with non-circular deviatoric trace. The expressions of the yield and plastic potential surfaces are redefined as

$$f = q^2 - m^2 \rho^2 p^2 = 0 \tag{13}$$

$$g = q^2 - a^2 m^2 \rho^2 p^2 - k = 0 \tag{14}$$

where p and q are the mean effective pressure and the deviatoric invariant, in the local representation, respectively, m represents the size of the yield surface, a is a material parameter controlling the size of the plastic potential, k is a dummy variable evaluated by imposing that the stress point lies on the plastic potential surface, and, $\rho(\theta)$ is the Lode's dependence proposed by [21] being θ the Lode's angle in the local representation.

Table 1. Values of function $Q(X,Y)$.

X	Y	$Q(X,Y)$
$X \geq 0$	$Y \geq 0$	$\arccos(X)$
$X \geq 0$	$Y < 0$	$\pi - \arccos(X)$
$X < 0$	$Y \geq 0$	$\pi + \arccos(X)$
$X < 0$	$Y < 0$	$2\pi - \arccos(X)$

For the sake of simplicity, in this work the well-known deviatoric section of [20] is adopted for both yield surfaces and plastic potentials. When using cones with circular sections the response obtained by using the proposed model is the same as that obtained by using the original model by [11]. The influence of the yield surface deviatoric shape is shown in Figure 1, which is related to the results obtained by imposing a repeated hexagonal stress path (10 cycles). Figure 1a shows the initial configuration of the yield surfaces, while Figures 1b-e show their evolution during the first cycle of the hexagonal path. Finally, Figure 1f compares the plastic strain paths in the deviatoric representation obtained from the simulation related to circular and non-circular deviatoric shapes.

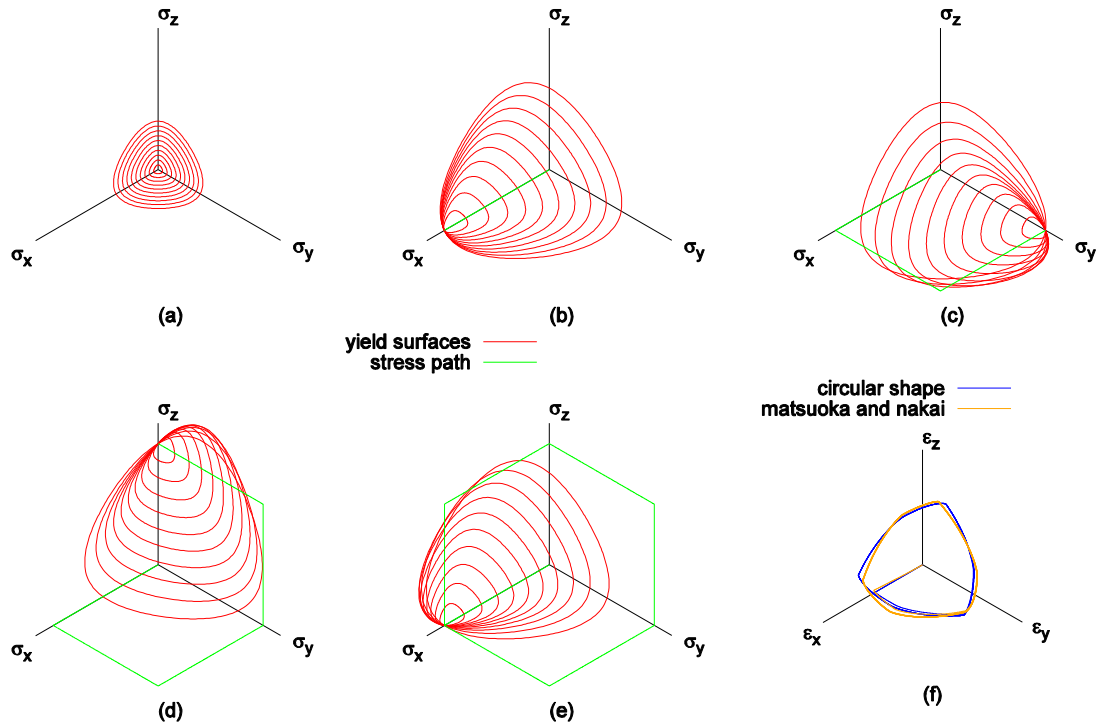


Fig. 1. (a) initial configuration of the yield surfaces; (b) configuration after the first stage; (c) configuration after the third stage; (d) configuration after the fifth stage (e) configuration at the end of hexagonal stress path; (f) comparison between the strain paths obtained using the circular and the non-circular deviatoric shapes.

4. Conclusions

In this paper, a purely kinematic approach for rotational hardening elastoplasticity has been proposed, which guarantees a rigid rotation of the yield and plastic potential surfaces in the stress space. This approach is based on the definition of a local basis, which links the components of the stress tensor in the global reference system to the corresponding components in a local reference system where yield surfaces and plastic potentials are formulated. The relationship between the two reference systems is determined by using the Gram-Schmidt procedure. The proposed approach was implemented in a numerical code and, for comparison purposes, the constitutive model by [11] was extended to non-circular deviatoric shapes. The results of the simulations show that notwithstanding the different predicted strain path using different deviatoric traces of the surfaces (circular and Matsuoka-Nakai like) the condition of plastic adaption is preserved.

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