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Research Article

Simplicial Approach to Fractal Structures

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A fractal lattice is defined by iterative maps on a simplex. In particular, Sierpinski gasket and von Koch flake are explicitly obtained by simplex transformations.

1. Introduction

Simplicial calculus [1–3] has been since the beginning a suitable tool for investigating discrete models in many physical problems such as discrete models in space-time [4–9] complex networks [10–13], molecular crystals, aggregates and diamond lattices [14–17], computer graphics [18, 19], and more recently signal processing and computer vision, such as stereo matching and image segmentation [20, 21].

In some recent papers [22–25], fractals [26–29] generated by simplexes, also called fractal lattices, were proposed for the analysis of nonconventional materials as some kind of polymers [24, 25] or nanocomposites [22, 23, 30, 31] having extreme physical and chemical properties. Moreover, the analysis of complex traffic on networks [32, 33] and image analysis [20, 21] based on fractal geometry and simplicial lattices has focussed on the importance of these methods in handling modern challenging problems.

However, only a few attempts were made in order to define the fractal lattice (structure) by an iterated system of functions on simplexes [34, 35]. The main scheme for affine contraction has been given in [35], whereas some generation of fractals by simplicial maps can be found in [34].

In this paper, we define a method based on simple algorithms for the generation of fractal-like structures by continuously deforming a simplex. This algorithm is based on a well-defined analytical map, which can be used to finitely describe fractals. Instead of recursive law, or nested maps (see, e.g., [1, 2, 15]), we propose a method which can be more easily implemented.

In the following, we will study an m-dimensional fractal structure defined by the transformation group of a simplicial complex. Starting from a simplex, it will define the group of transformation on it, so that the intrinsic (affine) metric remains scale invariant. The group of transformations (isometries and homotheties) will be characterized by matrices acting on the skeleton of the simplex. We will derive the basic properties of the fractal lattice and give a suitable definition of self-similarity on lattices. The concept of self-similarity is shown to be fulfilled by some classical transformation on simplices (homothety) and, simplicial based, fractals as the Sierpinski tessellations and the von Koch flake.

2. Euclidean Simplexes

In the ordinary Euclidean space $\mathbb{R}^n$, we assume that there exists a triangulation of $\mathbb{R}^n$, in the sense that there is at least a finite set of $n+1$ points geometrically independent (simplexes). A simplex will be considered both as a set of points and as the convex subspace of $\mathbb{R}^n$, defined by the geometrical support of the simplex. Union of n-adjacent simplexes is an n-polyhedron $\mathcal{P}$ [4, 18, 19].

The euclidean m-simplex σ^m, of independent vertices $V_0, V_1, \ldots, V_m$, is defined [1–3] as the subset of $\mathbb{R}^n$,

$$\sigma^m \overset{\text{def}}{=} \left\{ P \in \mathbb{R}^n \mid P \sum_{i=0}^{m} \lambda^i V_i \ \text{ with } \sum_{i=0}^{m} \lambda^i = 1,\ 0 \le \lambda^i \le 1 \right\}. \tag{2.1}$$

Let us denote with $[\sigma^m] = [V_0, V_1, \ldots, V_m]$ the set of points which form the *skeleton* of σ^m, and let $\#\sigma^m = m+1$ be the cardinality of the set of points. The p-face of σ^m, with $p \le m$, is any simplex σ^p such that $[\sigma^p] \cap [\sigma^m] \ne \emptyset$, and we write $\sigma^p \preceq \sigma^m$.

The number of p-faces of σ^m is $\binom{m+1}{p+1}$.

The m-dimensional *simplicial complex* Σ^m is defined as the finite set of p simplexes ($p \le m$) such that

(1) for all $\sigma^k \in \Sigma^m$ if $\sigma^h \preceq \sigma^k$, then $\sigma^h \in \Sigma^m$,

(2) for all $\sigma^k, \sigma^h \in \Sigma^m$, then either $[\sigma^h] \cap [\sigma^k] = \emptyset$ or $[\sigma^h] \cap [\sigma^k] = [\sigma^j]$ with $\sigma^j \in \Sigma^m$.

The set of points P such that $P \in \sigma^p$, $p \le m$, and $\sigma^p \in \Sigma^m$ is the geometric support of Σ^m also called m-polyhedron $\mathcal{M}^m$. The p-skeleton of Σ^m is $[\Sigma^m]^p \overset{\text{def}}{=} [\sigma^p]$ for all $\sigma^p \in \Sigma^m$. The *boundary* $\partial\Sigma^m$ of Σ^m is the complex Σ^{m-1} such that each $\sigma^{m-1} \in \Sigma^{m-1}$ is face of only one m-simplex of Σ^m. A finite set of simplexes is also called lattice (or tessellation).

2.1. Barycentric Coordinates and Barycentric Bases

In each simplex, it is possible to define the *barycentric basis* as follows: given the m-simplex σ^m with vertices $V_0, \ldots, V_m$, the barycentric basis is the set of $(m+1)$ vectors

$$\mathbf{e}_i \overset{\text{def}}{=} V_i - \mathcal{G}^m, \tag{2.2}$$

based on the *barycenter*

$$\mathcal{G}^m \overset{\text{def}}{=} \mathcal{G}(\sigma^m) = \sum_{i=0}^{m} \frac{1}{m+1} V_i. \tag{2.3}$$

These vectors $\mathbf{e}_i$ belong to the n-dimensional vector space E isomorphic to $\mathbb{R}^n$. Moreover, they are linearly dependent, since according to their definition, it is

$$\sum_{i=0}^{m} \mathbf{e}_i = \mathbf{0}. \tag{2.4}$$

Each point $P \in \sigma^m$ can be characterized by a set of *barycentric coordinates* $(\lambda^0, \ldots, \lambda^m)$ such that

$$0 \leq \lambda^i \leq 1, \quad \sum_{i=0}^{m} \lambda^i = 1, \quad i = 0, \ldots, m, \tag{2.5}$$

and $P - \mathcal{G}^m = \sum_{i=0}^{m} \lambda^i \mathbf{e}_i \overset{(2.2),(2.4)}{\Longrightarrow} P = \sum_{i=0}^{m} \lambda^i V_i$. Therefore, each point of σ^m can be formally expressed as a linear combination of the skeleton $[\sigma^m]$.

The dual space is defined as the linear map of the vector space E into $\mathbb{R}$ as

$$\left\langle \mathbf{e}^i, \mathbf{e}_k \right\rangle = \tilde{\delta}^i_k, \tag{2.6}$$

with [14]

$$\tilde{\delta}^i_k \overset{\text{def}}{=} \delta^i_k - \frac{1}{m+1} = \begin{cases} -\dfrac{1}{m+1}, & i \neq k, \\ +\dfrac{m}{m+1}, & i = k, \end{cases} \tag{2.7}$$

δ^i_k being the Kroneker symbol. According to the definition (2.7), it is

$$\sum_{i=0}^{m} \tilde{\delta}^i_k = \sum_{k=0}^{m} \tilde{\delta}^i_k = 0. \tag{2.8}$$

In addition, the metric tensor in σ^m is defined as [5]

$$\tilde{g}_{ij} \overset{\text{def}}{=} -\frac{1}{2} \tilde{\delta}^h_i \tilde{\delta}^k_j \ell^2_{hk}, \quad (i, j, h, k = 0, 1, \ldots, m) \tag{2.9}$$

being $\ell^2_{hk} \overset{\text{def}}{=} (V_k - V_h)^2 = (\mathbf{e}_k - \mathbf{e}_h)^2$.

2.2. Measures of the m-Simplex

Let

$$\mathbf{L}_{ij} \overset{\text{def}}{=} V_j - V_i \quad (= \mathbf{e}_j - \mathbf{e}_i), \qquad l_{ij} \overset{\text{def}}{=} \langle \mathbf{L}_{ij}, \mathbf{L}_{ij} \rangle, \tag{2.10}$$

by using the ordinary wedge product of the vectors $\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_p}$, we can define the p-form ω,

$$\omega = \frac{1}{p!} \sum_{j_1,\ldots,j_m} \omega^{j_1\cdots j_p} \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_p}, \tag{2.11}$$

whose affine components are $\omega^{j_1\cdots j_p} \overset{\text{def}}{=} \langle \omega, \mathbf{e}_{j_1} \wedge \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_p} \rangle$ [14].

The euclidean measure of the m-simplex σ (volume) is [14]

$$\varepsilon\Omega^2 \overset{\text{def}}{=} \frac{1}{m!} |\mathbf{L}_{01} \wedge \cdots \wedge \mathbf{L}_{0m}|, \tag{2.12}$$

from where, it follows that

$$\Omega^2 = \left(\frac{1}{m!}\right)^3 \sum_{\substack{j_1,\ldots,j_m \\ k_1,\ldots,k_m}} \varepsilon^{j_1\ldots j_m} \varepsilon^{k_1\ldots k_m} \prod_{a=1}^{m} l^2_{j_a k_a}, \tag{2.13}$$

being

$$\varepsilon^{j_1\cdots j_m} \overset{\text{def}}{=} \pm 1, \tag{2.14}$$

according to the even/odd permutation $j_0, j_1, \ldots, j_m$ of the indices $0, 1, \ldots, m$.

In particular, the volume of each p-face $\sigma_{i_1\ldots i_{m-p}}$ (see also [9]) is

$$\Omega^2_{i_1\ldots i_{m-p}} = \left(-\frac{1}{2}\right)^p \left(\frac{1}{p!}\right)^3 \sum_{\substack{j_1,\ldots,j_p \\ k_1,\ldots,k_p}} \varepsilon^{j_1\ldots j_p} \varepsilon^{k_1\ldots k_p} \prod_{a=1}^{p} l^2_{j_a k_a} \quad (0 < p \leq m), \tag{2.15}$$

where $j_1, \ldots, j_p, k_1, \ldots, k_p \neq i_1, \ldots, i_{m-p}$.

3. m-Dimensional Homothety

Let $\mathcal{O}(\sigma_i)$ be the subspace of $\mathbb{R}^m$ to which σ_i belongs; it can be easily proved that [14]

$$\forall\ \mathbf{v} \in \mathcal{O}(\sigma_i) \qquad \left\langle \mathbf{n}^i, \mathbf{v} \right\rangle = 0 \quad (\ i \text{ fixed }), \tag{3.1}$$

where the normal vector $\mathbf{n}^i$ is defined as

$$\mathbf{n}^i \overset{\text{def}}{=} -\frac{m\Omega}{\Omega_i}\mathbf{e}^i, \qquad h^i \overset{\text{def}}{=} \frac{m\Omega}{\Omega_i}. \tag{3.2}$$

The above definition of vector orthogonal to a $(m-1)$-face allows us to characterize the *m-parallelism of simplexes* as follows. Let σ, $\widehat{\sigma}$ be two simplexes in $\mathbb{R}^m$; let $\sigma_i, \widehat{\sigma}_i$ be the ith $(m-1)$-faces of σ and $\widehat{\sigma}$, respectively, and let $\mathbf{n}^i$, $\widehat{\mathbf{n}}^i$ be their normal vectors, then we say that σ is m-parallel to $\widehat{\sigma}$ ($\sigma \|_m \widehat{\sigma}$) if and if only $\sigma_i \| \widehat{\sigma}_i$, that is, $\mathbf{n}^i = \widehat{\mathbf{n}}^i$ $(i = 0, \ldots, m)$.

Let φ be a map

$$\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^m, \quad \sigma \overset{\varphi}{\longmapsto} \widehat{\sigma} \tag{3.3}$$

such that

(1) φ is a bijective simplicial map on σ,

(2) the s-adjacent faces of σ correspond (under the map φ) to s-adjacent faces of $\widehat{\sigma}$,

(3) σ and $\widehat{\sigma}$ are m-parallel.

We also assume that this transformation depends on the edge vectors and in particular on the edge lengths, *so that any quantity, defined on the simplex, transforming under the action of* φ*, is a function of the edge lengths.* Furthermore, we assume the following conditions:

(4) there exists a fixed point under the action of φ:

$$\exists O \in \mathbb{R}^m \mid \varphi(O) \equiv O, \tag{3.4}$$

(5) each $(m-1)$-face σ_i translates of an amount $t \in [0, \infty)$.

Let us choose as a fixed point one of the vertices, for example, V_0. We define this bijective simplicial map applying any $P \in \sigma$ into $\widehat{P} \in \widehat{\sigma}$ $(t \in [0, \infty))$ as

$$\widehat{P} \overset{\text{def}}{=} P + t\frac{\Omega_0}{m\Omega}\sum_{i=0}^{m}\lambda^i \mathbf{L}_{0i}; \tag{3.5}$$

in particular, this function acts on any vertex V_i as

$$\widehat{V}_i = V_i + t\frac{\Omega_0}{m\Omega}\mathbf{L}_{0i}, \qquad \left(\widehat{V}_0 = V_0\right), \tag{3.6}$$

so that we can easily prove that all the previous conditions are easily satisfied [14]. According to the above equations, each edge transforms as

$$\widehat{\mathbf{L}}_{ij} = \left(1 + t\frac{\Omega_0}{m\Omega}\right)\mathbf{L}_{ij}, \tag{3.7}$$

where $\widehat{\mathbf{L}}_{ij} = \widehat{V}_j - \widehat{V}_i$.

3.1. Variation Law of the p-Faces of σ

The variation law of the edge lengths, resulting from (3.7), is given by the formula

$$\widehat{l}_{ij} = \left(1 + t\frac{\Omega_0}{m\Omega}\right) l_{ij}, \tag{3.8}$$

where l_{ij} is the length of the edge $\mathbf{L}_{ij}$, and $\widehat{l}_{ij}$ is the length of the edge $\widehat{\mathbf{L}}_{ij}$.

According to (2.13), the volume Ω is a homogeneous function of degree m of the $m(m+1)/2$ variable $\{l_{ij}^2\}_{i<j}$, so that its variation law is

$$\widehat{\Omega} = \left(1 + t\frac{\Omega_0}{m\Omega}\right)^m \Omega, \tag{3.9}$$

and for any p-face,

$$\widehat{\Omega}_{i_1 \ldots i_{m-p}} = \left(1 + t\frac{\Omega_0}{m\Omega}\right)^p \Omega_{i_1 \ldots i_{m-p}} \quad (0 < p < m); \tag{3.10}$$

analogously, taking into account the definition $(5.5)_2$, we have the transformation law of h_i:

$$\widehat{h}_i = m\left(1 + t\frac{\Omega_0}{m\Omega}\right) h_i. \tag{3.11}$$

There follows, for the fundamental vectors of $\widehat{\sigma}$, that

$$\begin{cases} \widehat{\mathbf{e}}_i \overset{\text{def}}{=} \widehat{V}_i - \widehat{\mathcal{G}} = \left(1 + t\dfrac{\Omega_0}{m\Omega}\right) \mathbf{e}_i, \\ \widehat{\mathbf{n}}^i = \mathbf{n}^i, \\ \widehat{\mathbf{e}}^i = \dfrac{\widehat{\Omega}_i / \Omega_i}{\widehat{\Omega} / \Omega} \mathbf{e}^i = \left(1 + t\dfrac{\Omega_0}{m\Omega}\right)^{-1} \mathbf{e}^i. \end{cases} \tag{3.12}$$

4. Self-Similar Structure

Let $(\mathbb{R}^n, d)$ be the complete metric space with the standard Euclidean metric d, and let $K(\mathbb{R}^n)$ be the set

$$K(\mathbb{R}^n) = \{K \subseteq \mathbb{R}^n : K \text{ is a nonempty compact set}\}. \tag{4.1}$$

The iterated function system (IFS)

$$\{w_i\} = (\mathbb{R}^n, d, w_1, w_2, \ldots, w_n) \tag{4.2}$$

is the finite set of contractions w_i on the complete metric space $(\mathbb{R}^n, d)$, being the contraction w defined as

$$d(w(x), w(y)) \leq cd(x, y), \quad \forall x, y \in \mathbb{R}^n, \tag{4.3}$$

with c contraction coefficient.

For each $A \in K(\mathbb{R}^n)$, the (IFS) contracting mapping is

$$w : A \in K(\mathbb{R}^n) \longrightarrow w_1(A) \bigcup \cdots \bigcup w_n(A) \in K(\mathbb{R}^n), \tag{4.4}$$

with contraction coefficient $c = \max\{c_1, \ldots, c_n\}$. Each function w_i usually is linear, or more generally an affine transformation, but sometimes it can be nonlinear, including projective and Möbius transformations [27].

According to the Banach fixed-point theorem (see, e.g., [36]), every contraction mapping on a nonempty complete metric space has a unique fixed point, so that there exists a unique compact (i.e., closed and bounded) fixed set A such that $A = w(A)$. The set A is also known as the fixed set of the Hutchinson operator [28].

One way of constructing such fixed set is to start with an initial set A and by iterating the actions of w. Hence,

$$A = \bigcup_{i_1,\ldots,i_h=1,\ldots,n} w_{i_1} \circ \cdots \circ w_{i_h}(A), \tag{4.5}$$

so that A is a self-similar set, expressed as the finite union of its conformal copies, each one reduced by a factor c^h.

The attractor A of IFS is characterized by a similarity dimension as follows.

Definition 4.1. Given an IFS of n contraction mappings with the same contraction coefficient c, the similarity dimension is defined as

$$s = \frac{\log n}{\log 1/c} \left(= -\frac{\log n}{\log c} \right). \tag{4.6}$$

Sets having noninteger similarity dimensions are called fractal sets, or simply fractals. There follows that the iterated function systems are a method of constructing fractals; the resulting constructions are always self-similar such that $w(\mu x) = \mu^H w(x)$. Hence, each map w is also called a self-similar map [27].

5. Fractal Structures from Simplicial Maps

In this section, some examples of self-similar (scale invariant) structures obtained by IFS on simplexes are given in $\mathbb{R}^2$. In particular, the IFS will be defined by affine transformations, as conformal maps of the affine metrics.

In the following, we will introduce some self-similar maps defined both on 2-simplexes and 1-simplexes, so that, from (4.1),

$$K\left(\mathbb{R}^2\right) = \left\{\sigma^2;\ \sigma^1;\sigma^0\right\},$$
$$w : K(\mathbb{R}^2) \longmapsto K(\mathbb{R}^2). \tag{5.1}$$

In particular, let σ^2 be the simplex $[V_1, V_2, V_3]$, then it is

$$K\left(\mathbb{R}^2\right) = \{[V_1, V_2, V_3];\ [V_1, V_2],\ [V_1, V_3], [V_2, V_3]; [V_1], [V_2], [V_3]\}, \tag{5.2}$$

so that a map w on $K(\mathbb{R}^2)$ could be the more general function defined on any face of σ^2.

Examples. If the skeleton of σ^2 is the set of vertices $\{V_1, V_2, V_3\}$ with $V_1 = (x_1, y_1)$,$V_2 = (x_2, y_2)$, and $V_3 = (x_3, y_3)$, the affine map w is defined by the matrix

$$W = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{pmatrix} \tag{5.3}$$

and the constant vector

$$U = (u_1, u_2, u_3, u_4, u_5, u_6). \tag{5.4}$$

The function w maps a 2-simplex into a 2-simplex whereas, by a matrix product, the vector

$$X = (x_1, y_1, x_2, y_2, x_3, y_3) \tag{5.5}$$

is mapped into the vector

$$WX + U, \tag{5.6}$$

so that the skeleton of $w(\sigma^2)$ is given by the vector $WX + U$. For instance, a rotation with fixed point V_1 is given by the matrix

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{pmatrix}, \tag{5.7}$$

with

$$a_{33}a_{44} - a_{34}a_{43} = \pm 1, \qquad a_{55}a_{66} - a_{56}a_{65} = \pm 1, \tag{5.8}$$

and the vector $U = \{0,0,0,0,0,0\}$.

Some more special maps will be given in the following where, in particular, we consider, without restriction, some special maps on the 1-faces of σ^2 such that

$$w\left(\sigma^2\right) = w_1\left(\sigma_1^2\right) \cup w_2\left(\sigma_2^2\right) \cup w_3\left(\sigma_3^2\right), \quad \#w\left(\sigma^2\right) = 3. \tag{5.9}$$

In this case, the matrix W, acting on σ^2, follows from the direct sum of lower-order matrices acting on σ^1 simplexes, as follows:

(a) the first vertex V_1 remains fixed, and the map w on σ^2 is a consequence of the transformation of the simplex $\sigma_1 = [V_2, V_3]$, that is, by defining

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad W_1 = \begin{pmatrix} a_{33} & a_{34} & a_{35} & a_{36} \\ a_{43} & a_{44} & a_{45} & a_{46} \\ a_{53} & a_{54} & a_{55} & a_{56} \\ a_{63} & a_{64} & a_{65} & a_{66} \end{pmatrix}, \tag{5.10}$$

it is

$$W = I \oplus W_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & a_{35} & a_{36} \\ 0 & 0 & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & 0 & a_{53} & a_{54} & a_{55} & a_{56} \\ 0 & 0 & a_{63} & a_{64} & a_{65} & a_{66} \end{pmatrix}, \tag{5.11}$$

(b) the second vertex V_2 remains fixed, and the map w on σ^2 is a consequence of the transformation of the simplex $\sigma_1 = [V_1, V_3]$, so that

$$W_2 = \begin{pmatrix} a_{11} & a_{12} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{25} & a_{26} \\ a_{51} & a_{52} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{65} & a_{66} \end{pmatrix},$$
$$W = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & a_{15} & a_{16} \\ a_{21} & a_{22} & 0 & 0 & a_{25} & a_{26} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & a_{55} & a_{56} \\ a_{61} & a_{62} & 0 & 0 & a_{65} & a_{66} \end{pmatrix}, \tag{5.12}$$

(c) the third vertex V_3 remains fixed, and the map w on σ^2 is a consequence of the transformation of the simplex $\sigma_1 = [V_1, V_2]$, that is,

$$W = W_3 \oplus I = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.13}$$

being

$$W_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}. \tag{5.14}$$

In the following, we will characterize the transformation on a 2-simplex as a result of iterative maps on its boundary 1-simplexes. These maps on 1-simplexes are defined by the matrices $W_1, W_2,$ and W_3, applied to the vectors of coordinates of $[V_2, V_3]$, $[V_1, V_3]$, and $[V_1, V_2]$, respectively.

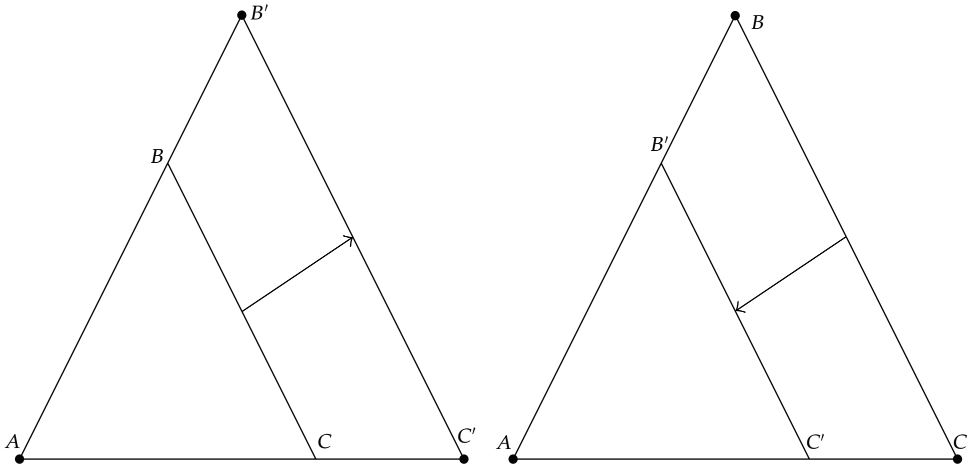

Figure 1: Homothety map.

5.1. Homothety

Let us consider the 2-simplex $\sigma^2 = \{A, B, C\}$ and the map (Figure 1)

$$\sigma^2 = \{A, B, C\} \Longrightarrow w\left(\sigma^2\right) = \{A, B', C'\}, \tag{5.15}$$

such that $\mathbf{n}_C = \pm\mathbf{n}_{C'}$. This map, according to (5.9), is obtained as a combination of 3 maps acting on the faces of σ^2, since

$$w\left(\sigma^2\right) = w_1([A, B]) \cup w_2([B, C]) \cup w_3([A, C]). \tag{5.16}$$

This map is a scale invariant, since there results

$$\ell_{AB}^2 = \lambda \ell_{A'B'}^2, \quad (0 \le \lambda), \tag{5.17}$$

$\ell_{BC}^2 = \lambda \ell_{B'C'}^2$, and $\ell_{AC}^2 = \lambda \ell_{A'C'}^2$, as well.

So that when $\lambda < 1$, we have a contraction and a dilation when $\lambda > 1$.

Moreover, according to (2.9), the metric $\widetilde{g}'_{ij}$ of the transformed simplex is given by a conformal transformation $\widetilde{g}'_{ij} = \lambda \widetilde{g}_{ij}$.

5.2. Sierpinski Gasket

As a first example of fractal defined by IFS on simplexes, we will consider the Sierpinski gasket. To this end, let us introduce an orthogonal coordinate system $0xy$ in $\mathbb{R}^2$ and three homothety maps w_1, w_2, and w_3. Each w_i is uniquely and completely determined once we know as it acts on the paired points $A = (x_A, y_A)$, $B = (x_B, y_B)$, and $C = (x_C, y_C)$, vertices of the 2-simplex $[\sigma^2] = [A, B, C]$.

In order to define the Sierpinski gasket by IFS of maps, we consider a sequence of maps that, at each step, shrink the area of σ^2 by a factor 0.25 and move the edges by a suitable homothety (Figure 2). In particular, the 3 maps are explicitly defined as follows:

$$w_1: \begin{pmatrix} x_A \\ y_A \\ x_B \\ y_B \\ x_C \\ y_C \end{pmatrix} \Longrightarrow M \cdot \begin{pmatrix} x_A \\ y_A \\ x_B \\ y_B \\ x_C \\ y_C \end{pmatrix},$$

$$w_2: \begin{pmatrix} x_A \\ y_A \\ x_B \\ y_B \\ x_C \\ y_C \end{pmatrix} \Longrightarrow M \cdot \begin{pmatrix} x_A \\ y_A \\ x_B \\ y_B \\ x_C \\ y_C \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 1/2 \\ 0 \\ 1/2 \end{pmatrix}, \tag{5.18}$$

$$w_3: \begin{pmatrix} x_A \\ y_A \\ x_B \\ y_B \\ x_C \\ y_C \end{pmatrix} \Longrightarrow M \cdot \begin{pmatrix} x_A \\ y_A \\ x_B \\ y_B \\ x_C \\ y_C \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix},$$

where M is the matrix

$$M = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix}. \tag{5.19}$$

Once we get the vertices of $w(\sigma^2)$, we can easily define the map for each point P of the σ^2 convex domain.

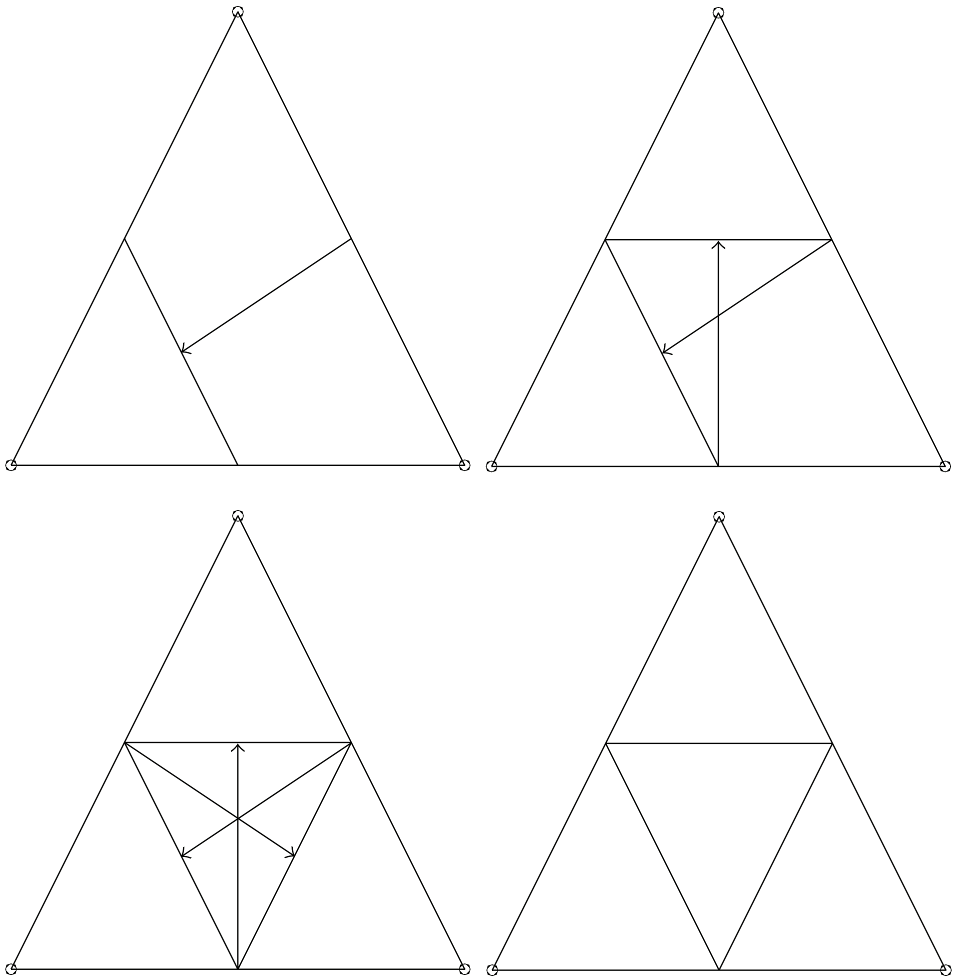

Figure 2: Fundamental maps.

Comment. In fact, let λ_1, λ_2, and λ_3 be the barycentric coordinates of a given point P inside σ^2, as given by (2.1), then we can write the barycentric expansion of $P \equiv (x, y)$ in terms of the coordinates of vertices $A, B,$ and C as

$$x = \lambda_1 x_A + \lambda_2 x_B + \lambda_3 x_C,$$
$$y = \lambda_1 y_A + \lambda_2 y_B + \lambda_3 y_C. \tag{5.20}$$

Substituting $\lambda_3 = 1 - \lambda_1 - \lambda_2$ into the above and rearranging, this linear transformation can be written as

$$H \cdot \Lambda = P - C, \tag{5.21}$$

where Λ is the vector of barycentric coordinates, and H is the matrix

$$H = \begin{pmatrix} x_A - x_C & x_B - x_C \\ y_A - y_C & y_B - y_C \end{pmatrix}. \tag{5.22}$$

Since H is invertible, we can easily obtain the barycentric coordinates of $P = (x, y)$:

$$\begin{gathered} \lambda_1 = \frac{(y_B - y_C)(x - x_C) + (x_C - x_B)(y - y_C)}{(y_B - y_C)(x_A - x_C) + (x_C - x_B)(y_A - y_C)}, \\ \lambda_2 = \frac{(y_C - y_A)(x - x_C) + (x_A - x_C)(y - y_C)}{(y_C - y_A)(x_B - x_C) + (x_A - x_C)(y_B - y_C)}, \\ \lambda_3 = 1 - \lambda_1 - \lambda_2. \end{gathered} \tag{5.23}$$

According to (5.9) each map $w_i,\ i = \{1, 2, 3\}$ is a contraction (dilation) of the σ^2 faces, such that the union gives rise to a 2-simplex (Figure 2). Any $P \in [\sigma^2]$ is mapped into $\widehat{P} \in \widehat{\sigma} = w_i([\sigma^2])$ as

$$\widehat{P} = P - \frac{1}{2}\sum_{i=0}^{m} \lambda^i \mathbf{L}_{0i}. \tag{5.24}$$

Moreover, each vertex in the $w_i([\sigma^2]),\ i = 1, 2, 3$ can be expressed as in (3.6)

$$\widehat{V}_i = V_i - \frac{1}{2}\mathbf{L}_{0i}, \tag{5.25}$$

so that

$$\begin{gathered} \widehat{\mathbf{L}}_{ij} = \frac{1}{2}\mathbf{L}_{ij}, \qquad \widehat{l}_{ij} = \frac{1}{2} l_{ij}, \\ \widehat{\Omega} = \frac{1}{4}\Omega. \end{gathered} \tag{5.26}$$

Reiterating this process for each remaining triangle, at the step k, we will obtain the compact set T_k given by 3^k triangles whose edges are contracted by $(1/2^k)$. In other words,

$$\begin{gathered} \widehat{\mathbf{L}}_{ij} = \frac{1}{2^k}\mathbf{L}_{ij}; \qquad \widehat{l}_{ij} = \frac{1}{2^k} l_{ij}, \\ \widehat{\Omega} = \frac{1}{2^{2k}}\Omega. \end{gathered} \tag{5.27}$$

Finally, we note that through the three simplicial maps in $\mathbb{R}^2$, provided with the natural metric d, we are able to construct the IFS $(\mathbb{R}^2, d, w_1, w_2 w_3)$ that has the well-known Sierpinski

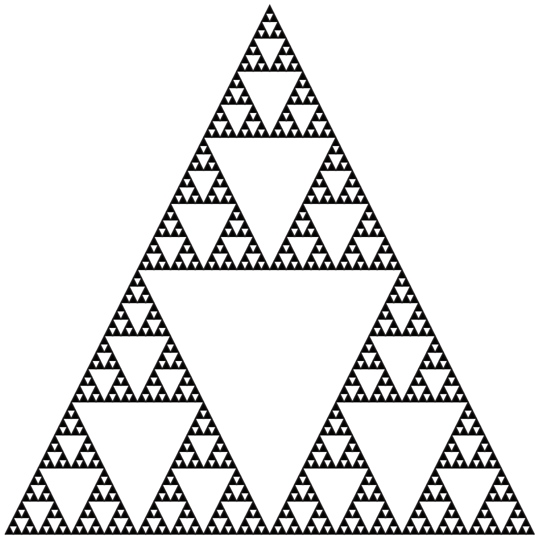

Figure 3: Sierpinski gasket.

gasket $T = \bigcap_k T_k$ as fractal attractor. So, we have obtained the Sierpinski gasket, as the combination of homothety maps (Figure 2). The iterating function will generate the known fractal-shaped curve (Figure 3).

The Sierpinski gasket supplies one of the most simple cases of construction of fractals through simplicial maps. In fact, the fractal structure is obtained acting on the 2-simplex only with homothetic transformations. Sometimes a fractal object can be constructed not only acting on simplexes with one map, but considering the compositions of different suitable transformations. Hereafter, in order to obtain another fractal object, we will consider, in details, some more elementary maps: the translation and the rotation (which are special cases of the matrix W).

6. Von Koch Curve

The von Koch curve [27, 28] can be obtained as a combination of homothety, translation, and rotation maps, so that the von Koch snowflake is obtained by their iteration.

6.1. Translation

Let the translation operator be defined as the operator

$$T : \mathbb{R}^m \longrightarrow \mathbb{R}^m, \quad \sigma \stackrel{T}{\longmapsto} \widehat{\sigma} \tag{6.1}$$

such that

$$T(P) = P + \mathbf{v}, \tag{6.2}$$

where $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ is a given vector of $\mathbb{R}^m$, then the image of a simplex σ under the function T is the translation of σ by T so that any vertex V_i is transformed into

$$\widehat{V}_i = V_i + \mathbf{v}\cdot \tag{6.3}$$

Since in a Euclidean space, any translation is an isometry, we have no variation of the edge lengths of σ.

According to the definitions (5.3), (5.4) in $\mathbb{R}^2$, it is

$$U = (v_1,\ v_2, v_1,\ v_2, v_1,\ v_2), \quad v_1 = Cnst.,\ v_2 = Cnst., \tag{6.4}$$

being W the zero matrix.

6.2. Rotation

Rotation is characterized by having a fixed point, however, like the translation which is an isometry. This is like the previous maps on simplexes that can be defined by a suitable matrix (5.3). $\mathbb{R}^2$ rotation is defined by (5.7), which however can be expressed by a single parameter (rotation angle). Hence, in two dimensions, a rotation with fixed point V_0 is the operator

$$R : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \sigma \overset{R}{\longmapsto} \widehat{\sigma} \tag{6.5}$$

such that

$$R(P) = V_0 + R_\theta(P - V_0), \tag{6.6}$$

where R_θ is the matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{6.7}$$

so that (5.7), when applied to the simplex $\sigma^2 = [V_0, V_1, V_2]$ with one fixed vertex, becomes

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}. \tag{6.8}$$

With respect to an orthogonal coordinate system with origin O, for any $P \in \sigma$, we define the rotation as the bijective simplicial map which applies $P \in \sigma$ into $\widehat{P} \in \widehat{\sigma}$,

$$\widehat{P} \overset{\text{def}}{=} [R_\theta(P - \mathbf{v})] + \mathbf{v}, \tag{6.9}$$

where $\mathbf{v} = V_0 - O$; in particular, the vertices $V_0,\ V_1,$ and V_2 are transformed into

$$\widehat{V}_0 = V_0, \qquad \widehat{V}_i = [R_\theta(\mathbf{L}_{0i} + O)] + (V_0 - O), \quad (i = 1,\ 2). \tag{6.10}$$

6.3. Von Koch Snowflake

Let $0xy$ be an orthogonal coordinate system for $\mathbb{R}^2$, and let $\sigma^2 = [A, B, C]$ be a two simplex under the homothety map. According to (5.16), this map can be realized by a composition of maps on the 1-simplexes $\sigma_1 = [A, B]$, $\sigma_1 = [B, C]$, and $\sigma_1 = [A, C]$. The coordinates of vertices are $A = (x_A, y_A)$, $B = (x_B, y_B)$, and $C = (x_C, y_C)$, respectively. In the following we will give both the construction of the Koch curve as IFS on σ_1 and the construction of the Koch snowflake as IFS on σ_2.

6.3.1. Von Koch Curve

Koch curve can be classically constructed by starting with a line segment, then recursively altering the shape as follows: divide the line segment into three segments of equal length; draw an equilateral triangle that has the middle segment from step 1 as its base and points outward; remove the line segment that is the base of the triangle from step 2 (see Figure 5).

Following the classical construction, we consider the following maps on 1-simplexes:

$$\begin{gathered} w_1 : \begin{pmatrix} x_A \\ y_A \\ x_B \\ y_B \end{pmatrix} \Longrightarrow M \cdot \begin{pmatrix} x_A \\ y_A \\ x_B \\ y_B \end{pmatrix}, \\ w_2 : \begin{pmatrix} x_A \\ y_A \\ x_B \\ y_B \end{pmatrix} \Longrightarrow M \cdot \begin{pmatrix} x_A \\ y_A \\ x_B \\ y_B \end{pmatrix} + \begin{pmatrix} 2/3 \\ 0 \\ 2/3 \\ 0 \end{pmatrix}, \end{gathered} \tag{6.11}$$

where M is the matrix

$$M = \begin{pmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix}. \tag{6.12}$$

Hence, w_1 is a factor of the homothety w having A as a fixed vertex, while w_2 leaves B unchanged:

$$\begin{aligned} w_1(P) &= P - \frac{2}{3}\lambda^i \mathbf{L}_{0i}, \quad \mathbf{L}_{0i} = V_i - A, \\ w_2(P) &= P - \frac{2}{3}\lambda^i \mathbf{L}_{0i}, \quad \mathbf{L}_{0i} = V_i - B. \end{aligned} \tag{6.13}$$

Moreover, each vertex in the $w_i([\sigma^1]),\ i = 1, 2$, can be expressed as in(3.6)

$$\widehat{V}_i = V_i - \frac{2}{3}\mathbf{L}_{0i}. \tag{6.14}$$

Let us now consider the transformation, on two steps, which first rotates $w_1([A, B])$ of an angle $\theta = 60°$ around the fixed point A, and then it translates the rotated simplex by a vector $\mathbf{v} = (1/3, 0)$.

So that we obtain

$$w_3 : \begin{pmatrix} x_A \\ y_A \\ x_B \\ y_B \end{pmatrix} \Longrightarrow M' \cdot \begin{pmatrix} x_A \\ y_A \\ x_B \\ y_B \end{pmatrix} + \begin{pmatrix} 1/3 \\ 0 \\ 1/3 \\ 0 \end{pmatrix}, \tag{6.15}$$

where M' is the matrix

$$M' = \begin{pmatrix} 1/6 & -\sqrt{3}/6 & 0 & 0 \\ \sqrt{3}/6 & 1/6 & 0 & 0 \\ 0 & 0 & 1/6 & -\sqrt{3}/6 \\ 0 & 0 & \sqrt{3}/6 & 1/6 \end{pmatrix}. \tag{6.16}$$

Finally, let us apply the transformation which first rotates $w_1([A, B])$ of an angle $\theta = 120°$ around the fixed point A, and then it translates the rotated simplex by the vector $\mathbf{v} = (2/3,\ 0)$. Accordingly, it is

$$w_4 : \begin{pmatrix} x_A \\ y_A \\ x_B \\ y_B \end{pmatrix} \Longrightarrow M'' \cdot \begin{pmatrix} x_A \\ y_A \\ x_B \\ y_B \end{pmatrix} + \begin{pmatrix} 2/3 \\ 0 \\ 2/3 \\ 0 \end{pmatrix}, \tag{6.17}$$

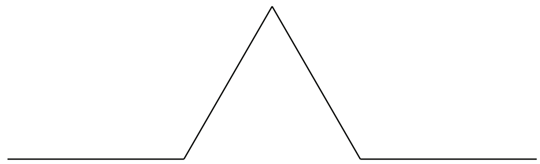

Figure 4: Image of $w([\sigma^1]) = \bigcup_{i=1,\ldots,4} w_i([\sigma^1])$, where the 1-simplex σ^1 is the unitary interval.

where M'' is the matrix

$$M'' = \begin{pmatrix} -1/6 & -\sqrt{3}/6 & 0 & 0 \\ \sqrt{3}/6 & -1/6 & 0 & 0 \\ 0 & 0 & -1/6 & -\sqrt{3}/6 \\ 0 & 0 & \sqrt{3}/6 & -1/6 \end{pmatrix}. \quad (6.18)$$

Since, as previously shown, rotation and translation are isometries, for each $w_i([\sigma^1])$, $i = 1, 2, 3, 4$, we obtain

$$\widehat{\mathbf{L}}_{ij} = \frac{1}{3}\mathbf{L}_{ij}, \qquad \widehat{l}_{ij} = \frac{1}{3}l_{ij}, \qquad \widehat{\Omega} = \frac{1}{3}\Omega. \quad (6.19)$$

In order to visualize the von Koch pattern, let us consider the 1-simplex $\{A, B\} = \{\{0,\ 0\}, \{1,\ 0\}\}$; since the point A has been chosen as the origin of the reference system, and w_3 and w_4 are obtained as rotation leaving fixed the origin, the transformed instances can be easily computed so that, at the first step, the IFS maps on 1-simplexes can be drawn (Figure 4).

Reiterating this process for each remaining segment, at the step k we will obtain the compact set T_k made of 2^{2k} segments whose sides are contracted by a factor $(1/3)^k$. The IFS map $(\mathbb{R}^1, d, w_1, w_2, w_3, w_4)$ gives us the Koch curve $L = \bigcap_k L_k$ with similarity dimension equal to

$$s = \frac{\log 4}{\log 1/(1/3)} = \frac{\log 4}{\log 3}. \quad (6.20)$$

This is also the similarity dimension of the Koch snowflake [27, 28]. So, the Koch curve (Figure 5) is obtained as a combination of IFS simplicial maps generating the known fractal-shaped curve.

6.3.2. Koch Flake

According to (5.16) and to the examples previously given, Koch flake (snowflake) can be constructed in a non-classical approach as IFS of maps on a 2-simplex. Koch snowflake can be seen as the image of a suitable system of iterated homotheties acting on a 2-simplex, given by suitable translations of the boundary 1-simplexes.

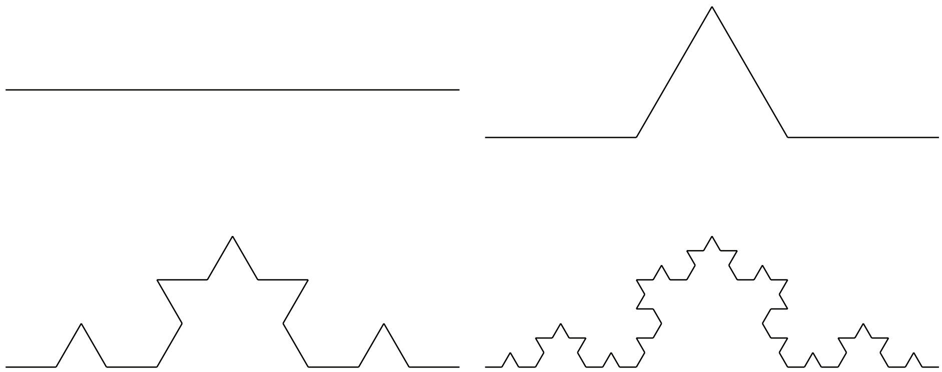

Figure 5: Koch curve.

In this process, the total length of each side of a triangle increases by one-third, and thus, the total length at the kth step will be $(4/3)^k$ of the original triangle perimeter.

7. Conclusion

In this paper, a nonclassical approach to fractal generation based on IFS of maps on simplexes has been given. Some of the most popular fractals, as the Sierpinski gasket and the von Koch flake, were obtained by iterative maps on simplexes. All maps were also intrinsically defined by using the affine (barycentric) coordinates and some basic measures on simplexes. The method proposed in this paper could be used to generate some new classes of fractals in any dimension, by simply defining suitable IFS on simplexes, thus opening new perspectives in fractal lattice geometry.

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