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Multiparameter Analysis Of The Stress Field Around A Crack Tip

M. Guagliano[a], M. Sangirardi[a], A. Sciuccati[a,*], M. Zakeri[b]

[a]Department of Mechanical Engineering, Politecnico di Milano, Via La Masa 1, 20156 Milano, Italy
[b]Department of Aerospace Engineering, Khaje Nasir Toosi University of Technology, Vafadar St. East, Tehran 16569-83911, Iran

Abstract

The elastic stress field around a crack tip is fully defined through multiparameter equations such as the general relations of Westergaard proposed by Sanford or the most recent of Atluri and Kobayashi. Using the relations of Atluri and Kobayashi, a code was implemented to evaluate the characteristic parameters of the stress field around a crack tip by photoelastic analysis. The possibility to change the number of parameters makes it possible to adapt the study to different cases, increasing the extension of the analyzed area in order to have a correct modeling of the photoelastic fringes. The performed experimental tests allow emphasizing the importance of using multiparameter equations in the study of the stress field around the crack tip.

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1. Introduction

By means of photoelasticity [1,2,3] it is possible to obtain the distribution of the isochromatic fringe patterns and the stress field around the crack tip in order to evaluate the parameters K. Among the various methods used to evaluate those parameters from photoelastic observation, the study conducted by Sanford and Dally is one of the most used [1]. The computation is based on the three main parameters K_I, K_{II} and σ_{0x} for Mode I, Mode II and Mixed Mode. In this paper, using the method proposed by Sanford and Dally, implemented using multi-parameter equations of Atluri and Kobayashi [4], the study of field stress around a crack tip is developed utilizing a larger number of parameters in order to better approximate the field.

2. Photoelastic analysis using equations with reduced number of parameters

Based on Westergaard studies, Irwin introduced the concept of stress intensity factor and published the equations that allowed to evaluate the stress distribution in the vicinity of the crack tip. The expressions obtained were named classical equations of Westergaard.

* Corresponding author. Tel.: +39 339 80 36 978.
E-mail address: augusto.sciuccati@mecc.polimi.it.

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Nomenclature

a	crack length
f_σ	stress fringe value
h	specimen thickness
K	stress intensity factor
N	isochromatic fringe order
r	distance from crack tip
r_m	distance of the apogee of an isochromatic fringe loop near the crack tip
T-Stress	constant stress acting parallel to the crack
θ	polar coordinate around the crack tip
σ_{0x}	constant stress acting parallel to the crack
τ_m	maximum shear stress

Irwin noted that, under certain conditions, the stress distribution obtained with the previous relations, did not provide accurate results.

In this regard he proposed a correction, and introduced a constant term called σ_{0x} or T-Stress. The Westergaard equations were then modified as like Eqs. (1) for Mode I.

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\begin{Bmatrix}1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\\ 1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\\ \sin\frac{\theta}{2}\cos\frac{3\theta}{2}\end{Bmatrix}+\begin{Bmatrix}-\sigma_{0x}\\ 0\\ 0\end{Bmatrix} \quad (1)$$

Irwin also reported a method to evaluate K_I and σ_{0x} from isochromatic field knowing the value of the maximum shear stress τ_m of a fringe and the distance of the apogee of the same isochromatic fringe loop near the crack tip [1]. Etheridge and Dally [5] showed that the method is affected by an error of 5% as long as it is used for central cracks, $r_m<0.03a$ and $73°<\theta_m<139°$. Outside this range the error increases rapidly. In addition, to correctly determine the parameters K_I and σ_{0x} a single observation is not enough because of the significant errors involving reading data from the photoelastic field. It was then introduced, by Bradley and Kobayashi [6], a method that allows to consider more than an isochromatic fringe order. The solution is still valid only around the area previously defined but implies a more modest increase of the error away from this area. Because of these complications it is indispensable define the stress by equations that take into account terms of higher order.

3. Photoelastic analysis by multiparameter equations

Eqs. (2), that allow to obtain the stress distribution using a higher number of parameters in a situation that produces a load Mode I, Mode II, or a Mixed Mode, have been introduced by Atluri and Kobayashi [4]. The method to evaluate the stress field is proposed by Ramesh et al. [7].

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\sum_{n=1}^{\infty}\frac{n}{2}A_{In}r^{\frac{n-2}{2}}\begin{Bmatrix}\left\{2+(-1)^n+\frac{n}{2}\right\}\cos\left(\frac{n}{2}-1\right)\theta-\left(\frac{n}{2}-1\right)\cos\left(\frac{n}{2}-3\right)\theta\\ \left\{2-(-1)^n-\frac{n}{2}\right\}\cos\left(\frac{n}{2}-1\right)\theta+\left(\frac{n}{2}-1\right)\cos\left(\frac{n}{2}-3\right)\theta\\ -\left\{(-1)^n+\frac{n}{2}\right\}\sin\left(\frac{n}{2}-1\right)\theta+\left(\frac{n}{2}-1\right)\sin\left(\frac{n}{2}-3\right)\theta\end{Bmatrix}-\sum_{n=1}^{\infty}\frac{n}{2}A_{IIn}r^{\frac{n-2}{2}}\begin{Bmatrix}\left\{2-(-1)^n+\frac{n}{2}\right\}\sin\left(\frac{n}{2}-1\right)\theta-\left(\frac{n}{2}-1\right)\sin\left(\frac{n}{2}-3\right)\theta\\ \left\{2+(-1)^n-\frac{n}{2}\right\}\sin\left(\frac{n}{2}-1\right)\theta+\left(\frac{n}{2}-1\right)\sin\left(\frac{n}{2}-3\right)\theta\\ -\left\{(-1)^n-\frac{n}{2}\right\}\cos\left(\frac{n}{2}-1\right)\theta-\left(\frac{n}{2}-1\right)\cos\left(\frac{n}{2}-3\right)\theta\end{Bmatrix} \quad (2)$$

Where $A_{I1}=K_I/(2\pi)^{0.5}$, $A_{II1}=-K_{II}/(2\pi)^{0.5}$ and $4A_{I2}=-\sigma_{0x}$. In order to solve the system of nonlinear equations in the unknown coefficients, it is necessary to apply the following method. From the fundamental relationship of the photoelasticity, and for a plane principal stress problem, Eqs. (3),

$$\frac{N\cdot f_\sigma}{t}=\sigma_1-\sigma_2 \qquad \sigma_1,\sigma_2=\frac{\sigma_x+\sigma_y}{2}\pm\sqrt{\frac{(\sigma_x-\sigma_y)^2}{4}+(\tau_{xy})^2} \tag{3}$$

it is possible to define Eq. (4) depending on where the difference between the main stresses are evaluated.

$$g_m=\left\{\frac{\sigma_x-\sigma_y}{2}\right\}_m^2+(\tau_{xy})_m^2-\left\{\frac{N_m\cdot f_\sigma}{2\cdot t}\right\}^2 \tag{4}$$

Substituting equation that describes the evolution of stresses around the crack tip, nonlinear equations in the unknown coefficients $A_{I1}, A_{I2},..., A_{Ik}$ and $A_{II1}, A_{II2},..., A_{IIl}$ is obtained; k is the number of parameters representing the Mode I while the number of parameters that characterize the Mode II is defined by the subscript l. In order to solve the system it is necessary to impose an initial value of the coefficients; if this value is correct, g_m would be zero. Of course it is impossible to find a priori the correct parameters values, therefore, an iterative process to approximate the solution is required. In this regard, the value of these coefficients is estimated by means of a Taylor series reported in Eq. (5),

$$(g_m)_{i+1}=(g_m)_i+\frac{\partial g_m}{\partial A_{I1}}(\Delta A_{I1})_i+\frac{\partial g_m}{\partial A_{I2}}(\Delta A_{I2})_i+...+\frac{\partial g_m}{\partial A_{Ik}}(\Delta A_{Ik})_i+\frac{\partial g_m}{\partial A_{II1}}(\Delta A_{II1})_i+\frac{\partial g_m}{\partial A_{II2}}(\Delta A_{II2})_i+...+\frac{\partial g_m}{\partial A_{IIl}}(\Delta A_{IIl})_i \tag{5}$$

where subscript i refers to the number of iterations reached and $\Delta A_{I1}, \Delta A_{I2},..., \Delta A_{Ik}$ and $\Delta A_{II1}, \Delta A_{II2},..., \Delta A_{IIl}$ are the corrections of the previous parameters estimations $A_{I1}, A_{I2},..., A_{Ik}$ and $A_{II1}, A_{II2},..., A_{IIl}$. The corrections are determined by imposing perfect correlation between the terms of the fundamental relationship of photoelasticity, therefore, $(g_m)_{i+1} = 0$. The derivations for every m point respect to the coefficients A_{In} and A_{IIn} are reported in Eqs. (6).

$$\frac{\partial g_m}{\partial A_{In}}=\frac{1}{2}(\sigma_x-\sigma_y)_m\left(\frac{\partial\sigma_x}{\partial A_{In}}-\frac{\partial\sigma_y}{\partial A_{In}}\right)_m+2\left(\tau_{xy}\frac{\partial\tau_{xy}}{\partial A_{In}}\right)_m \qquad \frac{\partial g_m}{\partial A_{IIn}}=\frac{1}{2}(\sigma_x-\sigma_y)_m\left(\frac{\partial\sigma_x}{\partial A_{IIn}}-\frac{\partial\sigma_y}{\partial A_{IIn}}\right)_m+2\left(\tau_{xy}\frac{\partial\tau_{xy}}{\partial A_{IIn}}\right)_m \tag{6}$$

At this point it is possible to calculate the system $\{g\}_i=-[b]_i\{\Delta A\}_i$ and the vector of corrections $\{\Delta A\}_i=-[c]_i^{-1}\{d\}_i$, where $[c]_i=[b]_i^T[b]_i$ and $\{d\}_i=[b]_i^T\{g\}_i$. In the second iteration the new correction $\{A\}_{i+1}=\{A\}_i+\{\Delta A\}_i$ is used. Deriving the equations of Atluri and Kobayashi, in relation to the unknown coefficients, Eqs. (7) can be obtained.

$$\begin{Bmatrix}\frac{\partial\sigma_x}{\partial A_{In}}\\ \frac{\partial\sigma_y}{\partial A_{In}}\\ \frac{\partial\tau_{xy}}{\partial A_{In}}\end{Bmatrix}=\frac{n}{2}r^{\frac{n-2}{2}}\begin{Bmatrix}\left\{2+(-1)^n+\frac{n}{2}\right\}\cos\left(\frac{n}{2}-1\right)\theta-\left(\frac{n}{2}-1\right)\cos\left(\frac{n}{2}-3\right)\theta\\ \left\{2-(-1)^n-\frac{n}{2}\right\}\cos\left(\frac{n}{2}-1\right)\theta+\left(\frac{n}{2}-1\right)\cos\left(\frac{n}{2}-3\right)\theta\\ -\left\{(-1)^n+\frac{n}{2}\right\}\sin\left(\frac{n}{2}-1\right)\theta+\left(\frac{n}{2}-1\right)\sin\left(\frac{n}{2}-3\right)\theta\end{Bmatrix} \quad \begin{Bmatrix}\frac{\partial\sigma_x}{\partial A_{IIn}}\\ \frac{\partial\sigma_y}{\partial A_{IIn}}\\ \frac{\partial\tau_{xy}}{\partial A_{IIn}}\end{Bmatrix}=-\frac{n}{2}r^{\frac{n-2}{2}}\begin{Bmatrix}\left\{2-(-1)^n+\frac{n}{2}\right\}\sin\left(\frac{n}{2}-1\right)\theta-\left(\frac{n}{2}-1\right)\sin\left(\frac{n}{2}-3\right)\theta\\ \left\{2+(-1)^n-\frac{n}{2}\right\}\sin\left(\frac{n}{2}-1\right)\theta+\left(\frac{n}{2}-1\right)\sin\left(\frac{n}{2}-3\right)\theta\\ -\left\{(-1)^n-\frac{n}{2}\right\}\cos\left(\frac{n}{2}-1\right)\theta-\left(\frac{n}{2}-1\right)\cos\left(\frac{n}{2}-3\right)\theta\end{Bmatrix} \tag{7}$$

It is possible to observe that stresses derivatives are independent of the unknown coefficients and do not vary with the evolution of iterations. This is not true for the derivatives of the function g_m which vary at each iteration. The implementation is done as follows. Initially, vectors of the type $\partial\sigma_x/\partial A_{In}$ are computed for each A_I and A_{II}. Then stress values σ_x, σ_y and τ_{xy} are evaluated by mean of terms A_{In} and A_{IIn}. Now it is possible to build the matrix [b] and the corrections values $\{\Delta A\}_i$ and the parameters to be used in the next iteration. Particular attention must be made in adopting a right convergence method. It is possible to used two methods. An error criterion based on the difference between parameters of two consecutive steps, therefore a stop of the iterative process is done if the difference $|\{\Delta A\}_{i+1} - \{\Delta A\}_i|$ is lower than a determinate value and an approach based on the error between the difference of the fringes distribution obtained experimentally and analytically. The iterative process is stopped when the error calculated is 0.1 [7]. It was noted that, while the error based on the difference between parameters give a not justified stop of the iteration process without a correct result, fringe error leads to a robust and accurate result.

4. Implementation of the code for the analysis of the photoelastic stress field

In order to check the correct implementation of the program a second code that allows the plotting of the distribution of the isochromatic field around a crack tip is developed. The required input data are the crack length, the number of parameters to characterize the stress field around the crack tip, the thickness of the photoelastic model and the stress fringe value of the material in order to combine the stress intensity to the isochromatic fringes. Assuming an hypothesized set of parameters it is possible to evaluate the corresponding isochromatic field around a crack tip. The parameters chosen are given in Table 1.

Table 1. Parameters chosen for the hypothesized isochromatic field and relative stress intensity factors and T-Stress

Mode I Parameters	Mode II Parameters	SIF of hypothesized field
$A_{I1} = 2\ MPa\sqrt{mm}$	$A_{II1} = 3\ MPa\sqrt{mm}$	$K_{I_{8par}} = 5.0132\ MPa\sqrt{mm}$
$A_{I2} = 0.8\ MPa$	$A_{II2} = 1\ MPa$	$K_{II_{8par}} = -7.5199\ MPa\sqrt{mm}$
$A_{I3} = 0.3\ MPa\ mm^{-1/2}$	$A_{II3} = 0.4\ MPa\ mm^{-1/2}$	$\sigma_{0x_{8par}} = -3.2\ MPa$
$A_{I4} = 0.06\ MPa\ mm^{-1}$	$A_{II4} = 0.09\ MPa\ mm^{-1}$	

The code allows to extract an accurate estimation of the stress around the crack tip, in fact, as shown in Fig. 1, the analytically obtained fringes, completely overlap the hypothesized pattern close and also far from the defect.

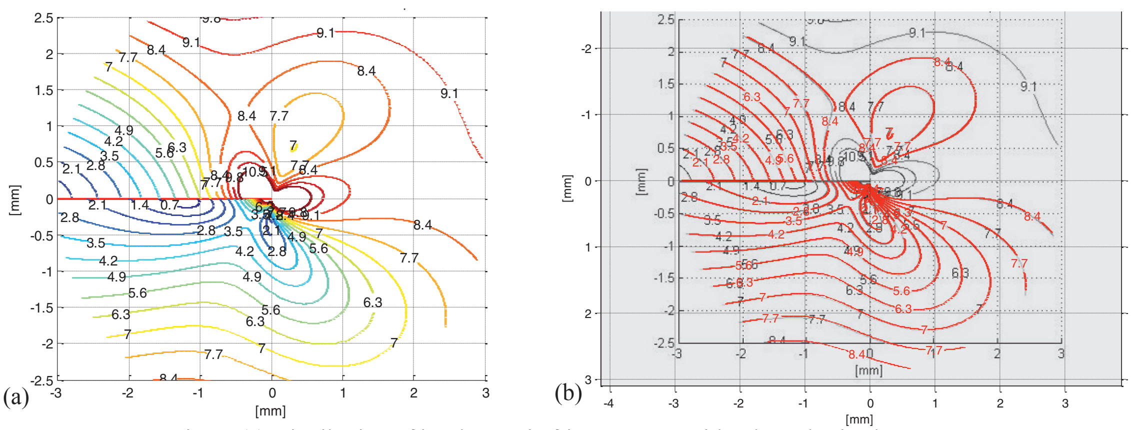

Fig. 1. (a) Distribution of isochromatic fringe pattern with 8 hypothesized parameters
(b) overlap of the analytical evaluation of the pattern on the hypothesized pattern.

The computed parameters and errors at the end of the iterative process are reported in Table 2.

Table 2. Parameters evaluated from the iterative process and relative stress intensity factors, T-Stress and errors

Mode I Parameters	Mode II Parameters	SIF from Code	Errors
$A_{I1} = 1.9588\ MPa\sqrt{mm}$	$A_{II1} = 2.9519\ MPa\sqrt{mm}$	$K_{I_{anl}} = 4.9100\ MPa\sqrt{mm}$	$K_I = 2.1\ \%$
$A_{I2} = 0.8062\ MPa$	$A_{II2} = 1\ MPa$	$K_{II_{anl}} = -7.3993\ MPa\sqrt{mm}$	$K_{II} = 1.6\ \%$
$A_{I3} = 0.3062\ MPa\ mm^{-1/2}$	$A_{II3} = 0.4027\ MPa\ mm^{-1/2}$	$\sigma_{0x_{anl}} = -3.2250\ MPa$	$\sigma_{0x} = 0.8\ \%$
$A_{I4} = 0.0615\ MPa\ mm^{-1}$	$A_{II4} = 0.0934\ MPa\ mm^{-1}$		

In a real case it is impossible to know the number of parameters that allows a correct approximation of the isochromatic field because, this value, depends also on the distance from the crack tip at which the field is characterized. Furthermore, using a low number of parameters, for example only the three basic parameters A_{I1}, A_{I2} and A_{II1}, the area for the data collection is particularly small. The complications increase further if the load applied on the photoelastic model is so low to lead the appearance of few isochromatic fringes. That's why multiparameter analysis is fundamental in such studies.

5. Study of the stress field in a central cracked disk

The study of the stress field distribution in a diametrically compressed disc with a central crack is widely used to study photoelastic fracture stress fields [8]. The main advantage is the simplicity in the application of a Mode I, a Mode II or a Mixed Mode load only by changing the orientation of the crack in relation to the load application direction. Fig. 2(a) shows a comparison between the diametrically compressed disk, observed at the polariscope in monochromatic circularly polarized light, with the axis of the crack perpendicular to the direction of load application and the analysis with finite elements and Fig. 2(b) shows the distribution of the computed isochromatic pattern, in red, that the code plots on the stress field image gotten from polariscope.

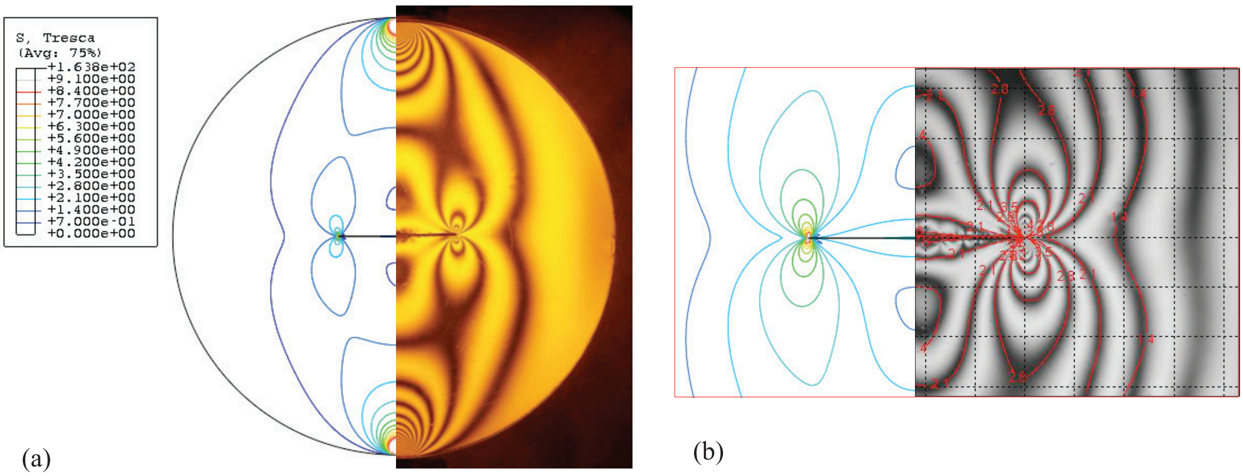

Fig. 2. (a) Comparison between stress field distribution obtained by photoelastic observation and FE analysis and (b) Comparison between computed, photoelastic and numerical stress field distribution

As it is possible to see, there is a good correlation between photoelastic observation and FE analysis. By FEM it is possible to evaluate K_I, K_{II}, and σ_{0x} for the analyzed situation. Using the code it is possible to derive the parameters K_I, K_{II}, and σ_{0x} from the photoelastic observation.
In order to define this field, 20 parameters have been used. Stress intensity factors evaluated both from FE analysis and from the photoelastic observation and errors are shown in Table 3.

Table 3. Stress intensity factor, T-Stress and Errors between photoelastic and FE analysis

SIF from FEM	Parameters from Code	SIF from Code	Errors
$K_{I_{fem}} = -15.711\ MPa\sqrt{mm}$	$A_{I1} = -6.2539\ MPa\sqrt{mm}$	$K_{I_{anl}} = -15.6763\ MPa\sqrt{mm}$	$K_I = 0.2\ \%$
$K_{II_{fem}} = 0\ MPa\sqrt{mm}$	$A_{II1} = 0.0117\ MPa\sqrt{mm}$	$K_{II_{anl}} = -0.0292\ MPa\sqrt{mm}$	-
$\sigma_{0x_{fem}} = 2.3145\ MPa$	$A_{I2} = -0.5813\ MPa$	$\sigma_{0x_{anl}} = 2.3253\ MPa$	$\sigma_{0x} = 0.5\ \%$

As it is possible to see, errors between terms suggested from finite element analysis and those extracted from photoelastic observation are small. The negative sign in the value of K_I has no physical meaning since it would involve an overlap of the edges of the crack under compressive loading. Again, if a number of parameters less than 20 would be used to define the stress field in a so large area of investigation, a non-correct stress pattern approximation will be calculated and the parameters will not be correct. Fig. 3 shows the trend of the isochromatic pattern using a number of parameters less than 20.

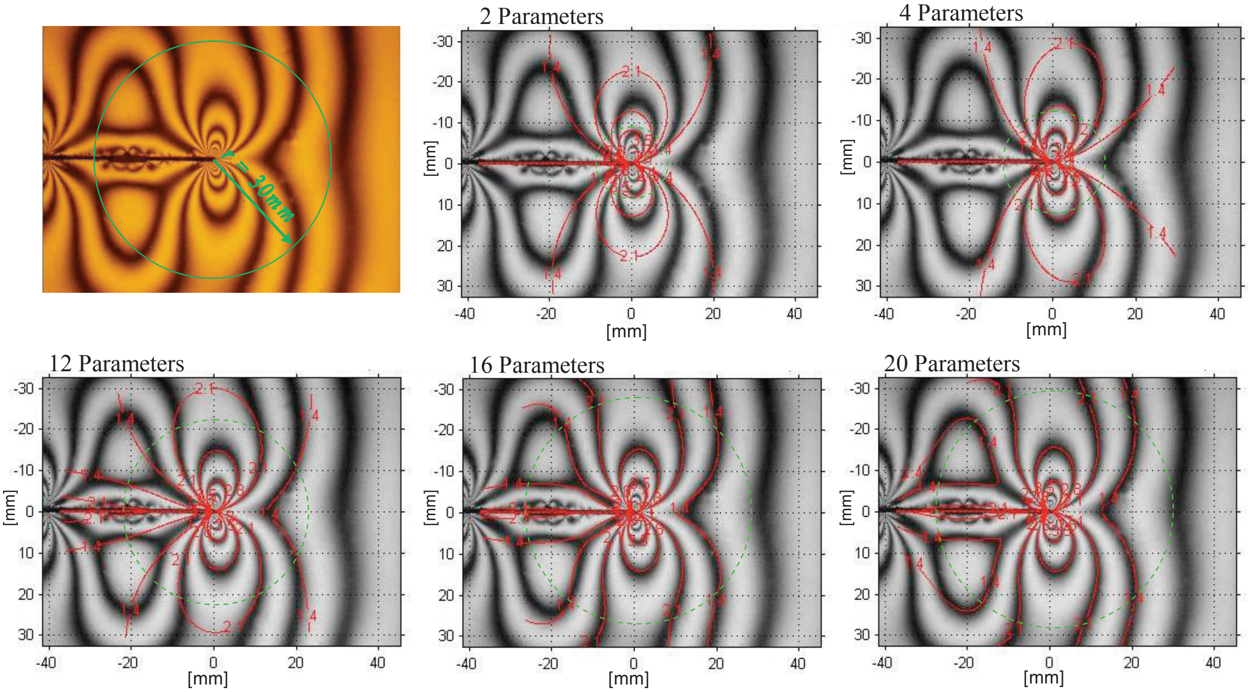

Fig. 3. Comparison between photoelastic observation and analytical field distribution computed with different number of parameters

In the image regarding the estimation with 2 parameters, it is possible to observe the total absence of the term associated to the T-Stress; and it is possible to visualize the effect of this parameter from the image obtained based on the estimation with 4 parameters. From the figure, it is clear that as the number of parameters increase better is the approximation of the isochromatic field; such that for this case, perfect correlation for the fringes around the crack tip is achieved by using 16 parameters. However, 20 or more parameters are needed to simulate the fringe patterns far from the tip.

CONCLUSIONS

In present work, problems regarding the study of the stress field around a crack tip are discussed with the aim of photoelastic techniques. The significant advantages using multiparameter equations in the analysis of the stress field are shown and the errors that a study with a limited number of terms produce is demonstrated. The comparison with finite element analysis highlighted the importance and precision of the photoelastic observation for the evaluation of the fracture mechanics parameters.

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