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Persistent currents and magnetic susceptibility of two-junction quantum interferometers

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ABSTRACT

Persistent currents and magnetic susceptibility of two-junction quantum interferometers are calculated by means of perturbation analysis by solving, to second order in the SQUID parameter β, the coupled non-linear differential equations governing the dynamics of this superconducting device in the absence of bias current. Comparison is made with results obtained to first order in β.

1. Introduction

The properties of Superconducting QUantum Interference Devices (SQUIDs) have been thoroughly studied in the literature by means of the following well-known non-linear model [1]:

$$\frac{d\phi}{d\tau} + \cos \pi\psi \sin \phi = \frac{i_B}{2}. \tag{1a}$$

$$\pi\frac{d\psi}{d\tau} + \cos \phi \sin \pi\psi + \frac{\psi}{2\beta} = \frac{\psi_{ex}}{2\beta}. \tag{1b}$$

In (1b), the parameter β is defined, for a symmetric SQUID, as LI_{J0}/Φ_0, L being the inductance of a single current branch of the device, I_{J0} the maximum Josephson current of both junctions, and Φ_0 the elementary flux quantum. The normalized time τ is linked to the laboratory time t by the following $\tau = 2\pi RI_{J0}t/\Phi_0$, R being resistive parameter of both junctions. The variable ϕ is the average value between the two superconducting phase differences, ϕ_1 and ϕ_2, across the two identical junctions, and ψ is the flux number variable, which represents the flux linked to the SQUID loop normalized to Φ_0. The forcing terms in the ordinary differential Eqs. (1a) and (1b) are the normalized bias current i_B and the normalized applied flux ψ_{ex}.

Most descriptions of the characteristic features of the model rely either on a fully numerical analysis [2] or, by assuming the characteristic parameter β could be taken to be approximately equal to zero [3,4], on a simplified – though fully analytic – approach. In the latter case, however, one should postulate absence of the persistent current i_S. In fact, we define, for finite β values, the persistent current as

$$i_S = \frac{\psi - \psi_{ex}}{\beta}. \tag{2}$$

In the case $\beta = 0$, one can show that $\psi = \psi_{ex}$, so that Eq. (2) does not result to be a well-defined ratio. Therefore, at least a perturbed first-order solution to Eq. (1b) is needed, in order to correctly define i_S. Semi-analytical description of SQUIDs and more complex systems in the case of non-zero inductances has already been given in the literature [5–7], given the usefulness of these devices in the present days. In fact, apart from the extensive use of SQUIDs in the realm of applied science research and related fields [8], SQUID-based systems have been proposed as qubits in quantum computing [9–13]. Therefore, due to the necessity of describing the basic properties of the non-linear dynamical model for finite values of the parameter β, De Luca and Romeo [6] have considered a single-junction effective model for SQUID systems.

In the present work a second-order perturbation analysis of the dynamic model (1a) and (1b) is proposed as an extension of the analysis done in Ref. [6]. In the following section we derive a single-junction effective model (SJEM) by which the dimensionality of the system of coupled non-linear ordinary differential Eqs. (1a) and (1b) is reduced from two to one by eliminating the flux number variable ψ. In Section 3, the proposed second-order perturbation solution to the problem is used to derive the persistent current expression to first order one in β and the magnetic susceptibility to second order in β. It is concluded that, though the present

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analysis is similar to what is done in Ref. [6], it allows knowledge of the expression of i_S and of the magnetic susceptibility χ to higher order in β. It is noted that, by comparing the analytic expressions of i_S and χ with respect to the corresponding numerically evaluated expressions, obtained by solving Eqs. (1a) and (1b) with computer assisted means, the second-order approximation, as expected, fits the numerical values better than the first-order solution obtained in Ref. [6].

2. Single-junction effective model

Let us consider the second ordinary differential equation in (1). By setting

$$\psi = \psi_{ex} + \beta\psi_1 + \beta^2\psi_2 + O(\beta^3), \tag{3}$$

after having multiplied both sides of (1b) by β, we can collect terms of equal power of β, obtaining the following:

$$\psi_1 = -2\sin\pi\psi_{ex}\cos\phi, \tag{4a}$$

$$\psi_2 = -2\frac{d\psi_1}{dt} - 2\psi_1\cos\pi\psi_{ex}\cos\phi. \tag{4b}$$

Therefore, the perturbed solution for ψ, to second order in the parameter β, is the following:

$$\psi = \psi_{ex} - 2\beta\sin\pi\psi_{ex}\cos\phi + 2\pi\beta^2[\sin 2\pi\psi_{ex} - i_B\sin\pi\psi_{ex} \times \sin\phi]. \tag{5}$$

We can now substitute Eq. (5) into Eq. (1a), obtaining

$$\frac{d\phi}{d\tau} + f(\psi_{ex},\phi)\sin\phi = \frac{i_B}{2}, \tag{6}$$

where

$$f(\psi_{ex},\phi) = \cos\pi\psi_{ex} + 2\pi\beta\sin^2\pi\psi_{ex}\cos\phi - 2\pi^2\beta^2 \times \sin^2\pi\psi_{ex}[\cos\pi\psi_{ex}(2+\cos^2\phi) - i_B\sin\phi]. \tag{7}$$

In this way, the effective CPR of the single-junction effective model of the two-junction quantum interferometer is given by the quantity $f(\psi_{ex},\phi)\sin\phi$. One should here consider that this unconventional CPR is not due to intrinsic properties of the Josephson junction, but only to the particular form of the electromagnetic coupling between the junctions dynamics, governed by Eq. (1a), and of the flux time evolution coming from Eq. (1b). In the limit of small values of the parameter β, the flux dynamics can be approximated, to various orders in β, as shown in this section. In this way, the superconducting phase dynamics can be described by the single-junction effective model given in Eq. (6).

3. Persistent currents and susceptibility at zero bias

Let us consider the persistent current for $i_B = 0$. In this case the expression for i_S is the following:

$$i_S = -2\sin\pi\psi_{ex}\cos\phi + 2\pi\beta\sin 2\pi\psi_{ex}. \tag{8}$$

On the other hand, keeping in mind that the magnetic susceptibility of the system can be defined as

$$\chi = \frac{\psi - \psi_{ex}}{\psi_{ex}} = \frac{\beta i_S}{\psi_{ex}}, \tag{9}$$

we find

$$\chi = -2\pi\beta\frac{\sin\pi\psi_{ex}}{\pi\psi_{ex}}\cos\phi + (2\pi\beta)^2\frac{\sin 2\pi\psi_{ex}}{2\pi\psi_{ex}}. \tag{10}$$

In Eqs. (8) and (10) we need to decide the value to give to ϕ. Therefore, we need to consider the fixed-point solutions of Eq. (6). Because of (7), we only need to consider solutions to first order in β. In this way, by setting

$$\phi = \phi_0 + \beta\phi_1 + O(\beta^2), \tag{11}$$

we may obtain the following relations for ϕ_0 and ϕ_1:

$$\frac{d\phi_0}{d\tau} + \cos\pi\psi_{ex}\sin\phi_0 = \frac{i_B}{2}, \tag{12a}$$

$$\frac{d\phi_1}{d\tau} + \phi_1\cos\pi\psi_{ex}\cos\phi_0 = -\pi\sin^2\pi\psi_{ex}\sin 2\phi_0. \tag{12b}$$

Eq. (12a) can be solved exactly for ϕ_0 [3]. By knowledge of ϕ_0, a solution for ϕ_1 in Eq. (12b) can be found by standard application of the theory of ordinary differential equations. We here omit details on the closed solution to the problem, but make the following important considerations. First, we notice that, being ϕ_1 defined by means of ϕ_0, we only need to define the stable fixed points for ϕ_0 in (12a). Second, for $i_B = 0$, we see that fixed-point solutions for ϕ_0 can be either $\phi_0 = 0$ or $\phi_0 = \pi$. We therefore need to choose between these two solutions on the basis of stability, by means of the phase-plane analysis shown in Fig. 1. We therefore argue that, for $\cos\pi\psi_{ex} > 0$, we have a stable equilibrium point at $\phi_0 = 0$. On the other hand, for $\cos\pi\psi_{ex} < 0$, we have a stable equilibrium point at $\phi_0 = \pi$. Therefore, we have the following final expression for i_S:

$$i_S = \begin{cases} -2\sin\pi\psi_{ex} + 2\pi\beta\sin 2\pi\psi_{ex} & \text{for } \cos\pi\psi_{ex} > 0 \\ +2\sin\pi\psi_{ex} + 2\pi\beta\sin 2\pi\psi_{ex} & \text{for } \cos\pi\psi_{ex} < 0 \end{cases}. \tag{13}$$

For the magnetic susceptibility χ, on the other hand, we may write:

$$\chi = \begin{cases} -2\pi\beta\frac{\sin\pi\psi_{ex}}{\pi\psi_{ex}} + (2\pi\beta)^2\frac{\sin 2\pi\psi_{ex}}{2\pi\psi_{ex}} & \text{for } \cos\pi\psi_{ex} > 0 \\ +2\pi\beta\frac{\sin\pi\psi_{ex}}{\pi\psi_{ex}} + (2\pi\beta)^2\frac{\sin 2\pi\psi_{ex}}{2\pi\psi_{ex}} & \text{for } \cos\pi\psi_{ex} < 0 \end{cases}. \tag{14}$$

In Fig. 2 and in Fig. 3, we show the curves obtained for the persistent current and the magnetic susceptibility, respectively, as a function of ψ_{ex} for $\beta = 0.02$. In particular, in Fig. 2 the red and green curves represent the persistent currents i_S calculated by means of the first-order and second-order approximation of the solution of ψ, respectively; the numerical integration is represented by empty blue circles. It can be seen that, as expected, the second-order approximation fits better the numerical solution. In Fig. 3, on the other hand, the red and green curves represent the magnetic susceptibility χ calculated by means of the first-order and second-order approximation of the solution of ψ, respectively. As in Fig. 2,

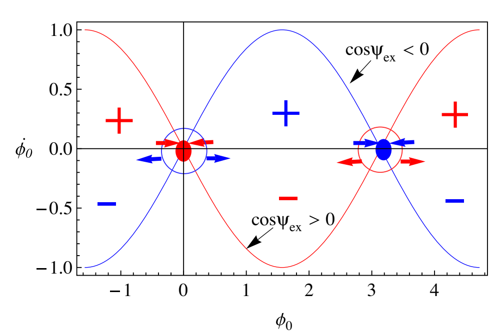

Fig. 1. Phase space representation of the superconducting phase dynamics to leading order in the parameter β. A graphical representation of the system behavior is given for $\cos\pi\psi_{ex} < 0$ (full-line curve) and for $\cos\pi\psi_{ex} > 0$ (dotted curve). By this analysis the stable fixed points are found to be $\phi_0 = 0$ for $\cos\pi\psi_{ex} > 0$, and $\phi_0 = \pi$ for $\cos\pi\psi_{ex} < 0$.

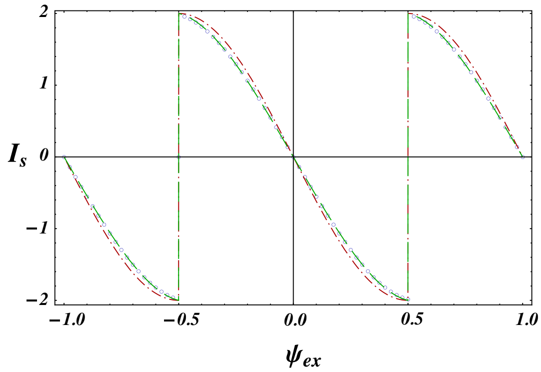

Fig. 2. Persistent current i_S as a function of the applied normalized flux ψ_{ex} for $\beta = 0.02$. The red dashed-dotted line and the green dashed line represent the persistent currents calculated by means of first-order and second-order approximation of the solution ψ, respectively; the numerical integration is represented by a collection of empty blue circles. It can be noticed that the green curve fits the numerical solution better than the red curve.

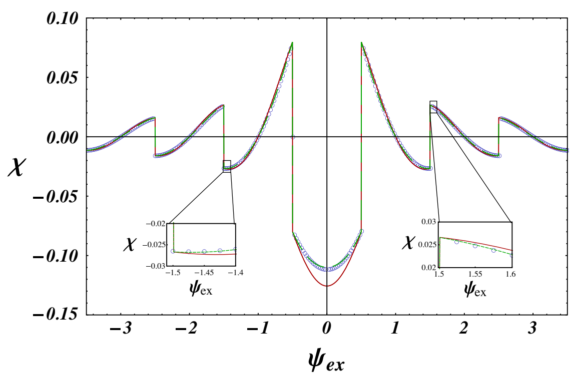

Fig. 3. Magnetic susceptibility χ as a function of the applied normalized flux ψ_{ex} for $\beta = 0.02$. The red full line and the green dashed line represent the magnetic susceptibility calculated by means of first-order and second-order approximation of the solution ψ, respectively; the numerical integration is represented by a collection of empty blue circles. It can be noticed that, in the ψ_{ex}-interval $(-1/2,1/2)$ the green curve fits the numerical solution much better than the red curve. A similar behavior is shown in the insets for values of ψ_{ex} outside the interval $(-1/2,1/2)$.

the empty blue circles give the outcome of the numerical integration of Eqs. (1a) and (1b). In Fig. 3 we finally notice that the evident disagreement of the value of χ calculated by means of the first-order approximation of the solution of ψ in the ψ_{ex} -interval $(-1/2,1/2)$ disappears when the expression for χ given in Eq. (12) is used. The insets in Fig. 3 show that, as in Fig. 2, the agreement between the numerically calculated expression for χ and the analytic expressions given in (12) is greater for the second-order approximation for χ than for the first-order expression, as one would expect.

4. Conclusions

Considering two-junction quantum interferometers, we have found an approximate analytic expression for persistent currents i_S and magnetic susceptibility χ as a function of the externally applied normalized magnetic flux ψ_{ex}. We have shown that the second-order approximation in β of the dynamical variable ψ gives a better fit, with respect to the first-order approximation, of the numerically calculated values of i_S and χ. The advantage of finding an analytic expression for these physical quantities resides on the actual and future potential use that can be made of these superconducting systems both in applied science and quantum computing. Furthermore, the model itself presents very challenging mathematical aspects and deserves attention for its very interesting intrinsic properties. In fact, these systems, though being made of superconducting parts, can show alternating diamagnetic and paramagnetic behavior, depending on the value of the normalized applied flux. This characteristic response can also shed some light on the controversial behavior denoted as "Paramagnetic Meissner Effect" [14] in high-T_c granular superconductors. Finally, as further developments of the present analysis, the quantum properties of a single-junction effective model of a two-junction quantum interferometer built in nanoscale dimensions will be sought in the future.

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