Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2008, Article ID 164808, 24 pages
doi:10.1155/2008/164808

Research Article

Shannon Wavelets Theory

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Received 30 May 2008; Accepted 13 June 2008

Recommended by Cristian Toma

Shannon wavelets are studied together with their differential properties (known as connection coefficients). It is shown that the Shannon sampling theorem can be considered in a more general approach suitable for analyzing functions ranging in multifrequency bands. This generalization coincides with the Shannon wavelet reconstruction of $L_2(\mathbb{R})$ functions. The differential properties of Shannon wavelets are also studied through the connection coefficients. It is shown that Shannon wavelets are C^∞-functions and their any order derivatives can be analytically defined by some kind of a finite hypergeometric series. These coefficients make it possible to define the wavelet reconstruction of the derivatives of the C^ℓ-functions.

1. Introduction

Wavelets [1] are localized functions which are a very useful tool in many different applications: signal analysis, data compression, operator analysis, and PDE solving (see, e.g., [2] and references therein). The main feature of wavelets is their natural splitting of objects into different scale components [1, 3] according to the multiscale resolution analysis. For the $L_2(\mathbb{R})$ functions, that is, functions with decay to infinity, wavelets give the best approximation. When the function is localized in space, that is, the bottom length of the function is within a short interval (function with a compact support), such as pulses, any other reconstruction, but wavelets, leads towards undesirable problems such as the Gibbs phenomenon when the approximation is made in the Fourier basis. In this paper, it is shown that Shannon wavelets are the most expedient basis for the analysis of impulse functions (pulses) [4]. The approximation can be simply performed and the reconstruction by Shannon wavelets range in multifrequency bands. Comparing with the Shannon sampling theorem where the frequency band is only one, the reconstruction by Shannon wavelets can be done for functions ranging in different frequency bands. Shannon sampling theorem [5] plays a fundamental role in signal analysis and, in particular, for the reconstruction of a signal from a digital sampling. Under suitable hypotheses (on a given signal function) a few sets of values (samples) and a preliminary chosen basis (made by the sinc function) enable us to completely reconstruct the continuous signal. This reconstruction is alike the reconstruction of a function as a series expansion (such as polynomial, i.e., Taylor series, or trigonometric functions, i.e., Fourier series), but for the first time the reconstruction (in the sampling theorem) makes use of the sinc function, that is a localized function with decay to zero. Together with the Shannon sampling theorem (and reconstruction), also the wavelets series become very popular, as well as the bases with compact support. It has been recognized that on the sinc functions one can settle the family of Shannon wavelets. The main properties of these wavelets will be shown and discussed. Moreover, the connection coefficients [6–9] (also called refinable integrals) will be computed by giving some finite formulas for any order derivatives (see also some preliminary results in [2, 10–12]). These coefficients enable us to define any order derivatives of the Shannon scaling and wavelet basis and it is shown that also the derivatives are orthogonal.

2. Shannon Wavelets

Sinc function or Shannon scaling function is the starting point for the definition of the Shannon wavelet family [11]. It can be shown that the Shannon wavelets coincide with the real part of the harmonic wavelets [2, 10, 13, 14], which are the band-limited complex functions

$$\psi_k^n(x) \overset{\text{def}}{\equiv} 2^{n/2}\frac{e^{4\pi i(2^n x-k)} - e^{2\pi i(2^n x-k)}}{2\pi i(2^n x-k)}, \tag{2.1}$$

with $n, k \in \mathbb{Z}$. Harmonic wavelets form an orthonormal basis and give rise to a multiresolution analysis [1–3, 14, 15]. In the frequency domain, they are very well localized and defined on compact support intervals, but they have a very slow decay in the space variable. However, in dealing with real problems it is more expedient to make use of real basis. By focussing on the real part of the harmonic family, we can take advantage of the basic properties of harmonic wavelets together with a more direct physical interpretation of the basis.

Let us take, as scaling function $\varphi(x)$, the sinc function (Figure 1)

$$\varphi(x) = \operatorname{sinc} x \overset{\text{def}}{=} \frac{\sin \pi x}{\pi x} = \frac{e^{\pi i x} - e^{-\pi i x}}{2\pi i x} \tag{2.2}$$

and for the dilated and translated instances

$$\begin{aligned}\varphi_k^n(x) &= 2^{n/2}\varphi(2^n x - k) = 2^{n/2}\frac{\sin \pi(2^n x-k)}{\pi(2^n x-k)} \\ &= 2^{n/2}\frac{e^{\pi i(2^n x-k)} - e^{-\pi i(2^n x-k)}}{2\pi i(2^n x-k)}.\end{aligned} \tag{2.3}$$

The parameters n, k give, respectively, a compression (dilation) of the basic function (2.2) and a translation along the x-axis. The family of translated instances $\{\varphi(x-k)\}$ is an orthonormal basis for the banded frequency functions [5] (Shannon theorem). For this reason, they can be used to define the Shannon multiresolution analysis as follows. The scaling functions do not represent a basis, in a functional space, therefore we need to define a family of

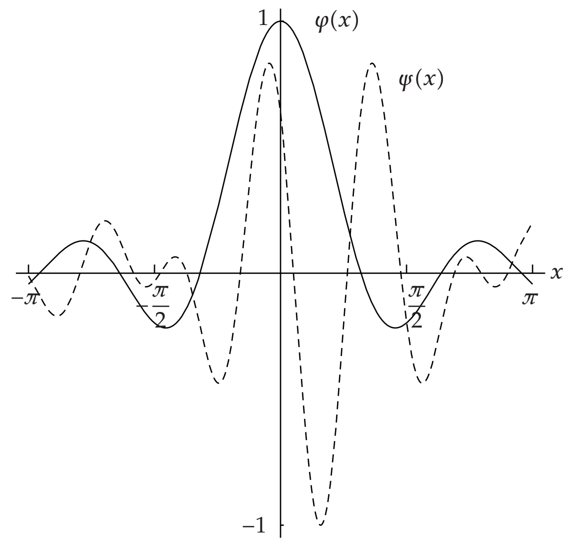

Figure 1: Shannon scaling function $\varphi(x)$ (thick line) and wavelet (dashed line) $\psi(x)$.

functions (based on scaling) which are a basis; they are called the wavelet functions and the corresponding analysis the multiresolution analysis.

Let

$$\widehat{f}(\omega) = \widehat{f(x)} \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \tag{2.4}$$

be the Fourier transform of the function $f(x) \in L_2(\mathbb{R})$ and

$$f(x) = 2\pi\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega x}\mathrm{d}\omega \tag{2.5}$$

its inverse transform. The Fourier transform of (2.2) gives us

$$\widehat{\varphi}(\omega) = \frac{1}{2\pi}\chi(\omega + 3\pi) = \begin{cases} \dfrac{1}{(2\pi)}, & -\pi \le \omega < \pi, \\ 0, & \text{elsewhere}, \end{cases} \tag{2.6}$$

with

$$\chi(\omega) = \begin{cases} 1, & 2\pi \le \omega < 4\pi, \\ 0, & \text{elsewhere}. \end{cases} \tag{2.7}$$

Analogously for the dilated and translated instances of scaling function it is

$$\widehat{\varphi}_k^n(\omega) = \frac{2^{-n/2}}{2\pi}e^{-i\omega(k+1)/2^n}\chi\left(\frac{\omega}{2^n} + 3\pi\right). \tag{2.8}$$

From the given scaling function, it is possible to define the corresponding wavelet function [1, 15] according to the following.

Theorem 2.1. *The Shannon wavelet, in the Fourier domain, is*

$$\widehat{\psi}(\omega) = \frac{1}{2\pi} e^{-i\omega} [\chi(2\omega) + \chi(-2\omega)]. \tag{2.9}$$

Proof. It can be easily shown that the scaling function (2.6) fulfills the condition

$$\widehat{\varphi}(\omega) = H\left(\frac{\omega}{2}\right)\widehat{\varphi}\left(\frac{\omega}{2}\right), \tag{2.10}$$

which characterizes the multiresolution analysis [1] with

$$H\left(\frac{\omega}{2}\right) = \chi(\omega + 3\pi). \tag{2.11}$$

Thus the corresponding wavelet function can be defined as [1, 15]

$$\widehat{\psi}(\omega) = e^{-i\omega}\overline{H\left(\frac{\omega}{2} \pm 2\pi\right)}\widehat{\varphi}\left(\frac{\omega}{2}\right). \tag{2.12}$$

With $H(\omega/2 - 2\pi)$ we have

$$\begin{aligned}\widehat{\psi}(\omega) &= e^{-i\omega}\overline{H\left(\frac{\omega}{2} - 2\pi\right)}\widehat{\varphi}\left(\frac{\omega}{2}\right)\\ &= e^{-i\omega}\chi(\omega + 3\pi - 2\pi)\frac{1}{2\pi}\chi\left(\frac{\omega}{2} + 3\pi\right)\\ &= \frac{1}{2\pi}e^{-i\omega}\chi(\omega + \pi)\chi\left(\frac{\omega}{2} + 3\pi\right)\\ &= \frac{1}{2\pi}e^{-i\omega}\chi(2\omega),\end{aligned} \tag{2.13}$$

then analogously with $H(\omega/2 + 2\pi)$ we obtain

$$\widehat{\psi}(\omega) = \frac{1}{2\pi}e^{-i\omega}\chi(-2\omega), \tag{2.14}$$

so that (2.9) follows. □

For the whole family of dilated-translated instances, it is

$$\widehat{\psi}^n_k(\omega) = \frac{2^{-n/2}}{2\pi}e^{-i\omega(k+1)/2^n}\left[\chi\left(\frac{\omega}{2^{n-1}}\right) + \chi\left(\frac{-\omega}{2^{n-1}}\right)\right]. \tag{2.15}$$

The Shannon wavelet function in the real domain can be obtained from (2.9) by the inverse Fourier transform (Figure 1)

$$\begin{aligned}\psi(x) &= \frac{\sin\pi(x - 1/2) - \sin 2\pi(x - 1/2)}{\pi(x - 1/2)}\\ &= \frac{e^{-2i\pi x}(-i + e^{i\pi x} + e^{3i\pi x} + ie^{4i\pi x})}{(\pi - 2\pi x)},\end{aligned} \tag{2.16}$$

and by the space shift and compression we have the whole family of dilated and translated instances:

$$\psi^n_k(x) = 2^{n/2}\frac{\sin\pi(2^n x - k - 1/2) - \sin 2\pi(2^n x - k - 1/2)}{\pi(2^n x - k - 1/2)}. \tag{2.17}$$

By summarizing (2.3) and (2.17), the Shannon wavelet theory is based on the following functions [11]:

$$\varphi_k^n(x) = 2^{n/2}\frac{\sin \pi(2^n x - k)}{\pi(2^n x - k)},$$
$$\psi_k^n(x) = 2^{n/2}\frac{\sin \pi(2^n x - k - 1/2) - \sin 2\pi(2^n x - k - 1/2)}{\pi(2^n x - k - 1/2)} \tag{2.18}$$

in the space domain, and collecting (2.8) and (2.15), we have in the frequency domain

$$\widehat{\varphi}_k^n(\omega) = \frac{2^{-n/2}}{2\pi}e^{-i\omega k/2^n}\chi\left(\frac{\omega}{2^n}+3\pi\right),$$
$$\widehat{\psi}_k^n(\omega) = -\frac{2^{-n/2}}{2\pi}e^{-i\omega(k+1/2)/2^n}\left[\chi\left(\frac{\omega}{2^{n-1}}\right)+\chi\left(\frac{-\omega}{2^{n-1}}\right)\right]. \tag{2.19}$$

The inner product is defined as

$$\langle f,g\rangle \overset{\text{def}}{\equiv} \int_{-\infty}^{\infty} f(x)\overline{g(x)}\mathrm{d}x, \tag{2.20}$$

which, according to the Parseval equality, can be expressed also as

$$\langle f,g\rangle \overset{\text{def}}{\equiv} \int_{-\infty}^{\infty} f(x)\overline{g(x)}\mathrm{d}x = 2\pi\int_{-\infty}^{\infty}\widehat{f}(\omega)\overline{\widehat{g}(\omega)}\mathrm{d}\omega = 2\pi\langle \widehat{f},\widehat{g}\rangle, \tag{2.21}$$

where the bar stands for the complex conjugate.

With respect to the inner product (2.20), we can show the following theorem [11].

Theorem 2.2. *Shannon wavelets are orthonormal functions in the sense that*

$$\langle \psi_k^n(x), \psi_h^m(x)\rangle = \delta^{nm}\delta_{hk}, \tag{2.22}$$

with δ^{nm}, δ_{hk} being the Kroenecker symbols.

Proof.

$$\begin{aligned}
&\langle \psi_k^n(x), \psi_h^m(x)\rangle \\
&= 2\pi\langle \widehat{\psi}_k^n(\omega), \widehat{\psi}_h^m(\omega)\rangle \\
&= 2\pi\int_{-\infty}^{\infty}\frac{2^{-n/2}}{2\pi}e^{-i\omega(k+1/2)/2^n}\left[\chi\left(\frac{\omega}{2^{n-1}}\right)+\chi\left(\frac{-\omega}{2^{n-1}}\right)\right]\frac{2^{-m/2}}{2\pi}e^{i\omega(h+1/2)/2^m}\left[\chi\left(\frac{\omega}{2^{m-1}}\right)+\chi\left(\frac{-\omega}{2^{m-1}}\right)\right]\mathrm{d}\omega \\
&= \frac{2^{-(n+m)/2}}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega(k+1/2)/2^n+i\omega(h+1/2)/2^m}\left[\chi\left(\frac{\omega}{2^{n-1}}\right)+\chi\left(\frac{-\omega}{2^{n-1}}\right)\right]\left[\chi\left(\frac{\omega}{2^{m-1}}\right)+\chi\left(\frac{-\omega}{2^{m-1}}\right)\right]\mathrm{d}\omega
\end{aligned} \tag{2.23}$$

which is zero for $n \neq m$. For $n = m$ it is

$$\langle \psi_k^n(x), \psi_h^n(x)\rangle = \frac{2^{-n}}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega(h-k)/2^n}\left[\chi\left(\frac{\omega}{2^{n-1}}\right)+\chi\left(\frac{-\omega}{2^{n-1}}\right)\right]\mathrm{d}\omega \tag{2.24}$$

and, according to (2.7), by the change of variable $\xi = \omega/2^{n-1}$

$$\langle \varphi_k^n(x), \varphi_h^n(x) \rangle = \frac{1}{4\pi}\left[\int_{-4\pi}^{-2\pi} e^{-2i(h-k)\xi}\mathrm{d}\xi + \int_{2\pi}^{4\pi} e^{-2i(h-k)\xi}\mathrm{d}\xi\right]. \tag{2.25}$$

For $h = k$ (and $n = m$), it is trivially

$$\langle \varphi_k^n(x), \varphi_k^n(x) \rangle = 1. \tag{2.26}$$

For $h \neq k$, it is

$$\int_{2\pi}^{4\pi} e^{-2i(h-k)\xi}\mathrm{d}\xi = \frac{i}{2(h-k)}\left(e^{-4i\pi(h-k)} - e^{-8i\pi(h-k)}\right) = 0\,, \tag{2.27}$$

and analogously $\int_{-4\pi}^{-2\pi} e^{-2i(h-k)\xi}\mathrm{d}\xi = 0$. □

Moreover, we have the following theorem [11].

Theorem 2.3. *The translated instances of the Shannon scaling functions $\varphi_k^n(x)$, at the level $n = 0$, are orthogonal in the sense that*

$$\langle \varphi_k^0(x), \varphi_h^0(x) \rangle = \delta_{kh}\,, \tag{2.28}$$

being $\varphi_k^0(x) \stackrel{\text{def}}{=} \varphi(x-k)$.

Proof. It is

$$\begin{aligned}\langle \varphi_k^n(x), \varphi_h^m(x) \rangle &= 2\pi \langle \widehat{\varphi}_k^n(\omega), \widehat{\varphi}_h^m(\omega) \rangle \\ &= 2\pi \int_{-\infty}^{\infty} \frac{2^{-n/2}}{2\pi} e^{-i\omega k/2^n} \chi\left(\frac{\omega}{2^n} + 3\pi\right) \frac{2^{-m/2}}{2\pi} e^{i\omega h/2^m} \chi\left(\frac{\omega}{2^m} + 3\pi\right)\mathrm{d}\omega \\ &= \frac{2^{-(n+m)/2}}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(k/2^n - h/2^m)} \chi\left(\frac{\omega}{2^n} + 3\pi\right)\chi\left(\frac{\omega}{2^m} + 3\pi\right)\mathrm{d}\omega\,.\end{aligned} \tag{2.29}$$

When $m = n$, we have

$$\langle \varphi_k^n(x), \varphi_h^n(x) \rangle = \frac{2^{-n}}{2\pi}\int_{-2^n\pi}^{2^n\pi} e^{-i\omega(k-h)/2^n}\mathrm{d}\omega = 2^n \frac{\sin[(h-k)\pi]}{(h-k)\pi}\,. \tag{2.30}$$

Since $h, k \in \mathbb{Z}$, there follows that

$$\frac{\sin[(h-k)\pi]}{(h-k)\pi} = \begin{Bmatrix} 1, & h = k \\ 0, & h \neq k \end{Bmatrix} = \delta_{kh}, \tag{2.31}$$

that is,

$$\langle \varphi_k^n(x), \varphi_h^n(x) \rangle = \delta_{kh}. \tag{2.32}$$

When $m \neq n$, let's say $m < n$, we have

$$\langle \varphi_k^n(x), \varphi_h^m(x) \rangle = \frac{2^{-(n+m)/2}}{2\pi}\int_{-2^m\pi}^{2^m\pi} e^{-i\omega(k/2^n - h/2^m)}\mathrm{d}\omega, \tag{2.33}$$

that is,

$$\langle \varphi_k^n(x), \varphi_h^m(x) \rangle = 2^{(m+n)/2}\frac{\sin[(h-2^{m-n}k)\pi]}{(h-2^{m-n}k)\pi}. \tag{2.34}$$

When $m \neq n$, the last expression is always different from zero, in fact (since $m < n$)

$$\sin\left[\left(h - \frac{k}{2^{|m-n|}}\right)\pi\right] = 0 \Longrightarrow \left[h - \frac{k}{2^{|m-n|}}\right]\pi = s\pi, \quad s \in \mathbb{Z} \tag{2.35}$$

that is,

$$h = s + \frac{k}{2^{|m-n|}}, \quad h, k, s \in \mathbb{Z} \tag{2.36}$$

and $h \in \mathbb{Z}$ only if $m = n$. Therefore, in order to have the orthogonality it must be$m = n$, so that

$$\langle \varphi_k^n(x), \varphi_h^n(x) \rangle = 2^n \delta_{kh}. \tag{2.37}$$

and, in particular, when $n = 0$,

$$\langle \varphi_k^0(x), \varphi_h^0(x) \rangle = \delta_{kh}. \tag{2.38}$$

□

As a consequence of this proof we have that

$$\varphi_k^0(h) = \delta_{kh} \quad (h, k \in \mathbb{Z}). \tag{2.39}$$

The scalar product of the (Shannon) scaling functions with the corresponding wavelets is characterized by the following [11].

Theorem 2.4. *The translated instances of the Shannon scaling functions $\varphi_k^n(x)$, at the level $n = 0$, are orthogonal to the Shannon wavelets in the sense that*

$$\langle \varphi_k^0(x), \psi_h^m(x) \rangle = 0, \quad m \geq 0, \tag{2.40}$$

being $\varphi_k^0(x) \overset{\text{def}}{=} \varphi(x - k)$.

Proof. It is

$$\begin{aligned}
&\langle \varphi_k^n(x), \psi_h^m(x) \rangle \\
&\quad = 2\pi \langle \widehat{\varphi}_k^n(\omega), \widehat{\psi}_h^m(\omega) \rangle \\
&\quad = 2\pi \int_{-\infty}^{\infty} 2^{-n/2} e^{-i\omega k/2^n} \chi\left(\frac{\omega}{2^n} + 3\pi\right) \frac{2^{-m/2}}{2\pi} e^{i\omega(h+1/2)/2^m} \left[\chi\left(\frac{\omega}{2^{m-1}}\right) + \chi\left(\frac{-\omega}{2^{m-1}}\right)\right] d\omega \\
&\quad = 2^{-(n+m)/2} \int_{-\infty}^{\infty} e^{-i\omega k/2^n + i\omega(h+1/2)/2^m} \chi\left(\frac{\omega}{2^n} + 3\pi\right) \left[\chi\left(\frac{\omega}{2^{m-1}}\right) + \chi\left(\frac{-\omega}{2^{m-1}}\right)\right] d\omega
\end{aligned} \tag{2.41}$$

which is zero for $m \geq n \geq 0$ (since, according to (2.7), the compact support of the characteristic functions do not intersect).

On the contrary, it can be easily seen that, for $m < n$, it is

$$\begin{aligned}
\langle \varphi_k^n(x), \psi_h^m(x) \rangle &= 2^{-(n+m)/2} \int_{2^m\pi}^{2^n\pi} e^{-i\omega k/2^n + i\omega(h+1/2)/2^m} d\omega \\
&= -\frac{2^{1+(m+n)/2}\left(ie^{i\pi[2^{-m+n-1}(1+2h)-k]} + e^{i\pi(h-2^{m-n}k)}\right)}{2^n(1+2h) - 2^{1+m}k}
\end{aligned} \tag{2.42}$$

and this product, in general, does not vanish. □

3. Reconstruction of a Function by Shannon Wavelets

Let $f(x) \in L_2(\mathbb{R})$ be a function such that for any value of the parameters $n, k \in \mathbb{Z}$, it is

$$\left| \int_{-\infty}^{\infty} f(x) \varphi_k^0(x) \mathrm{d}x \right| \le A_k^n < \infty, \qquad \left| \int_{0}^{\infty} f(x) \psi_k^n(x) \mathrm{d}x \right| \le B_k^n < \infty, \tag{3.1}$$

and $B \subset L_2(\mathbb{R})$ the Paley-Wiener space, that is, the space of band-limited functions such that,

$$\operatorname{supp} \widehat{f} \subset [-b, b], \quad b < \infty. \tag{3.2}$$

For the representation with respect to the basis (2.18), it is $b = \pi$. According to the sampling theorem (see, e.g., [5]) we have the the following.

Theorem 3.1 (Shannon). *If* $f(x) \in L_2(\mathbb{R})$ *and* $\operatorname{supp} \widehat{f} \subset [-\pi, \pi]$, *the series*

$$f(x) = \sum_{k=-\infty}^{\infty} \alpha_k \varphi_k^0(x) \tag{3.3}$$

uniformly converges to $f(x)$, *and*

$$\alpha_k = f(k). \tag{3.4}$$

Proof. In order to compute the values of the coefficients, we have to evaluate the series in correspondence of the integer:

$$f(h) = \sum_{k=-\infty}^{\infty} \alpha_k \varphi_k^0(h) = \sum_{k=-\infty}^{\infty} \alpha_k \delta_{kh} = \alpha_h, \tag{3.5}$$

having taken into account (2.39).

The convergence follows from the hypotheses on $f(x)$. In particular, the importance of the band-limited frequency can be easily seen by applying the Fourier transform to (3.3):

$$\begin{aligned} \widehat{f}(\omega) &= \sum_{k=-\infty}^{\infty} f(k) \widehat{\varphi}_k^0(x) \\ &\overset{(2.8)}{=} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} f(k) e^{-i\omega k} \chi(\omega + 3\pi) \\ &= \frac{1}{2\pi} \chi(\omega + 3\pi) \sum_{k=-\infty}^{\infty} f(k) e^{-i\omega k} \end{aligned} \tag{3.6}$$

so that

$$\widehat{f}(\omega) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\sum_{k=-\infty}^{\infty} f(k) e^{-i\omega k}, & \omega \in [-\pi, \pi], \\ 0, & \omega \notin [-\pi, \pi]. \end{cases} \tag{3.7}$$

In other words, if the function is band limited (i.e., with compact support in the frequency domain), it can be completely reconstructed by a discrete Fourier series. The Fourier coefficients are the values of the function $f(x)$ sampled at the integers. □

As a generalization of the Paley-Wiener space, and in order to generalize the Shannon theorem, we define the space $\mathcal{B}_{\psi} \supseteq \mathcal{B}$ of functions $f(x)$ such that the integrals

$$\alpha_k = \langle f(x), \varphi_k^0(x) \rangle = \int_{-\infty}^{\infty} f(x)\varphi_k^0(x)\mathrm{d}x,$$
$$\beta_k^n = \langle f(x), \psi_k^n(x) \rangle = \int_{-\infty}^{\infty} f(x)\psi_k^n(x)\mathrm{d}x \tag{3.8}$$

exist and are finite. According to (2.20) and (2.21), it is in the Fourier domain that

$$\alpha_k = 2\pi \langle \widehat{f(x)}, \widehat{\varphi_k^0(x)} \rangle = \int_{-\infty}^{\infty} \widehat{f}(\omega)\widehat{\varphi}_k^0(\omega)\mathrm{d}\omega = \int_0^{2\pi} \widehat{f}(\omega)e^{i\omega k}\mathrm{d}\omega,$$
$$\beta_k^n = 2\pi \langle \widehat{f(x)}, \widehat{\psi_k^n(x)} \rangle = \cdots = 2^{-n/2}\int_{2^{n+1}\pi}^{2^{n+2}\pi} \widehat{f}(\omega)e^{i\omega k/2^n}\mathrm{d}\omega. \tag{3.9}$$

Let us prove the following.

Theorem 3.2 (Shannon generalization). *If* $f(x) \in B_{\psi} \subset L_2(\mathbb{R})$ *and* $\operatorname{supp} \widehat{f} \subseteq \mathbb{R}$, *the series*

$$f(x) = \sum_{h=-\infty}^{\infty} \alpha_h \varphi_h^0(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \beta_k^n \psi_k^n(x) \tag{3.10}$$

converges to $f(x)$, *with* α_h *and* β_k^n *given by* (3.8) *and* (3.9). *In particular, when* $\operatorname{supp} \widehat{f} \subseteq [-2^N\pi, 2^N\pi]$, *it is*

$$f(x) = \sum_{h=-\infty}^{\infty} \alpha_h \varphi_h^0(x) + \sum_{n=0}^{N}\sum_{k=-\infty}^{\infty} \beta_k^n \psi_k^n(x). \tag{3.11}$$

Proof. The representation (3.10) follows from the orthogonality of the scaling and Shannon wavelets (Theorems 2.2, 2.3, 2.4). The coefficients, which exist and are finite, are given by (3.8). The convergence of the series is a consequence of the wavelet axioms.

It should be noticed that

$$\operatorname{supp}\widehat{f} = [-\pi, \pi] \bigcup_{n=0,\ldots,\infty} [-2^{n+1}\pi, -2^n\pi] \cup [2^n\pi, 2^{n+1}\pi] \tag{3.12}$$

so that for a band-limited frequency signal, that is, for a signal whose frequency belongs to the first band $[-\pi, \pi]$, this theorem reduces to the Shannon. But, more in general, one has to deal with a signal whose frequency range in different bands, even if practically banded, since it is $N < \infty$. In this case, we have some nontrivial contributions to the series coefficients from all the bands, ranging from $[-2^N\pi, 2^N\pi]$:

$$\operatorname{supp}\widehat{f} = [-\pi, \pi] \bigcup_{n=0,\ldots,N} [-2^{n+1}\pi, -2^n\pi] \cup [2^n\pi, 2^{n+1}\pi]. \tag{3.13}$$

In the frequency domain, (3.10) gives

$$f(x) = \sum_{h=-\infty}^{\infty} \alpha_h \varphi_h^0(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \beta_k^n \psi_k^n(x),$$

$$\widehat{f}(\omega) = \sum_{h=-\infty}^{\infty} \alpha_h \widehat{\varphi}_h^0(\omega) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \beta_k^n \widehat{\psi}_k^n(\omega),$$

$$\widehat{f}(\omega) \overset{(2.19)}{=} \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} \alpha_h e^{-i\omega h}\chi(\omega+3\pi) + \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} 2^{-n/2}\beta_k^n e^{-i\omega(k+1)/2^n}\left[\chi\left(\frac{\omega}{2^{n-1}}\right)+\chi\left(\frac{-\omega}{2^{n-1}}\right)\right], \tag{3.14}$$

that is,

$$\begin{aligned}\widehat{f}(\omega) &= \frac{1}{2\pi}\chi(\omega+3\pi)\sum_{h=-\infty}^{\infty}\alpha_h e^{-i\omega h}\\ &+ \frac{1}{2\pi}\chi\left(\frac{\omega}{2^{n-1}}\right)\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} 2^{-n/2}\beta_k^n e^{-i\omega(k+1)/2^n}\\ &+ \frac{1}{2\pi}\chi\left(\frac{-\omega}{2^{n-1}}\right)\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} 2^{-n/2}\beta_k^n e^{-i\omega(k+1)/2^n}.\end{aligned} \tag{3.15}$$

There follows that the Fourier transform is made by the composition of coefficients at different frequency bands. When $\beta_k^n = 0$, for all $n, k \in \mathbb{Z}$, we obtain the Shannon sampling theorem as a special case. □

Of course, if we limit the dilation factor $n \leq N < \infty$, for a truncated series, we have the approximation of $f(x)$, given by

$$f(x) \cong \sum_{h=-S}^{S} \alpha_h \varphi(x-h) + \sum_{n=0}^{N}\sum_{k=-M}^{M} \beta_k^n \psi_k^n(x). \tag{3.16}$$

By rearranging the many terms of the series with respect to the different scales, for a fixed N we have

$$\begin{aligned} f(x) &\cong \sum_{h=-S}^{S} \alpha_h \varphi(x-h) + \sum_{n=0}^{N} f_n(x),\\ f_n(x) &= \sum_{k=-M}^{M} \beta_k^n \psi_k^n(x),\end{aligned} \tag{3.17}$$

where $f_n(x)$ represent the component of the function $f(x)$ at the scale $0 \leq n \leq N$ (i.e., in the band $[2^n\pi, 2^{n+1}\pi]$), and $f(x)$ results from a multiscale approximation or better from the multiband reconstruction.

3.1. Examples

Let us first compute the approximate wavelet representation of the even function

$$f(x) = e^{-4x^2} \cos 2\pi x. \tag{3.18}$$

The bottom length (i.e., the main part) of the function $f(x)$ is concentrated in the interval $[-0.2, 0.2]$. With a low scale $n = 3$, we can have a good approximation (Figures 2, 4) of the function even with a small number k of translation. In fact, with $|k| \le 3$ the absolute value of the approximation error is less than 7% (see Figure 4). The higher number of the translation parameter k improves the approximation of the function on its "tails," in the sense that by increasing the number of translation parameters k the oscillation on "tails" is reduced. We can see that with $|k| \le 10$ the approximation error is reduced up to 3%. Moreover, the approximation error tends to zero with $|x| \to \infty$.

The multiscale representation is given by

$$\begin{aligned} f(x) &\cong \alpha_0 \varphi(x) + \sum_{n=0}^{3} f_n(x), \\ f_n(x) &= \sum_{k=-3}^{3} \beta_k^n \psi_k^n(x), \end{aligned} \tag{3.19}$$

so that at the higher scales there are the higher frequency oscillations (see Figure 2). It should be also noticed that the lower scale approximations $f_0(x)$, $f_1(x)$, $f_2(x)$ represent the major content of the amplitude. In other words, $f_0(x) + f_1(x) + f_2(x)$ gives a good representation of (3.18) in the origin, while $f_3(x)$, with its higher oscillations, makes a good approximation of the tails of (3.18). Therefore, if we are interested in the evolution of the peak in the origin, we can restrict ourselves to the analysis of the lower scales. If we are interested in the evolution either of the tails or the high frequency, we must take into consideration the higher scales (in our case $f_3(x)$).

If we compare the Shannon wavelet reconstruction with the Fourier integral approach, in the Fourier method the following hold.

(1) It is impossible to have a series expansion except for the periodic functions.

(2) It is impossible to focus, as it is done with the Shannon series, on the contribution of each basis to the function. In other words, the projection of $f(x)$ on each term $\{\cos \xi x, \sin \xi x\}$ of the Fourier basis is not evident. There follows that it is impossible to decompose the profile with the components at different scales.

(3) The integral transform performs an integral over the whole real axis for a function which is substantially zero (over $\mathbb{R}$), except in the "small" interval $(-\varepsilon, \varepsilon)$.

As a second example, let us consider the approximate wavelet representation of the odd function

$$f(x) = e^{-(16x)^2/2} + e^{-4x^2} \sin 2\pi x. \tag{3.20}$$

The bottom length (i.e., the main part) of the function $f(x)$ is concentrated in the interval $[-0.2, 0.2]$. Also in this case, for a localized function, with a low scale $n = 3$ we can have a good approximation (Figures 3, 4) of the function even with a small number k of translation. However, in this case, the error can be reduced (around the origin) by adding some translated instances, but it remains nearly constant far from the origin. In fact, with

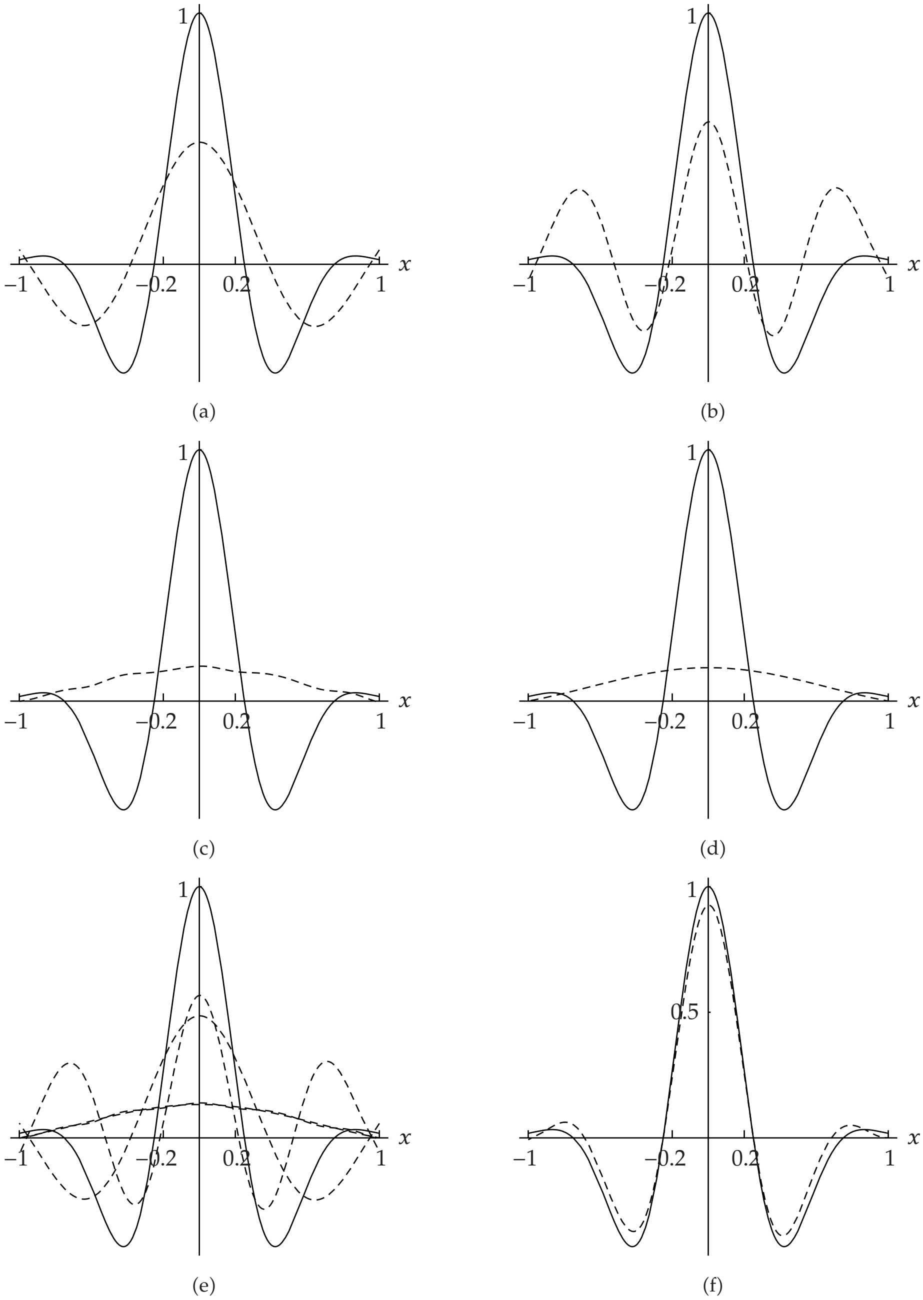

Figure 2: Shannon wavelet reconstruction (dashed) of the even function $f(x) = e^{-4x^2} \cos 2\pi x$, with $n_{\max} = 3$, $-3 \le k \le 3$ (bottom right). Scale approximation with (a) $n = 0$, $-3 \le k \le 3$, (b) $n = 1$, $-3 \le k \le 3$, (c) $n = 2$, $-3 \le k \le 3$, (d) $n = 3$, $-3 \le k \le 3$, (e) $0 \le n \le 3$, $-3 \le k \le 3$, (f) $n = 3$, $-5 \le k \le 5$.

$|k| \le 3$ the absolute value of the approximation error is less than 10% (8% in the origin, Figure 4). The higher number of the translation parameter k improves the approximation of the function on its "tails," in the sense that by increasing the number of translation

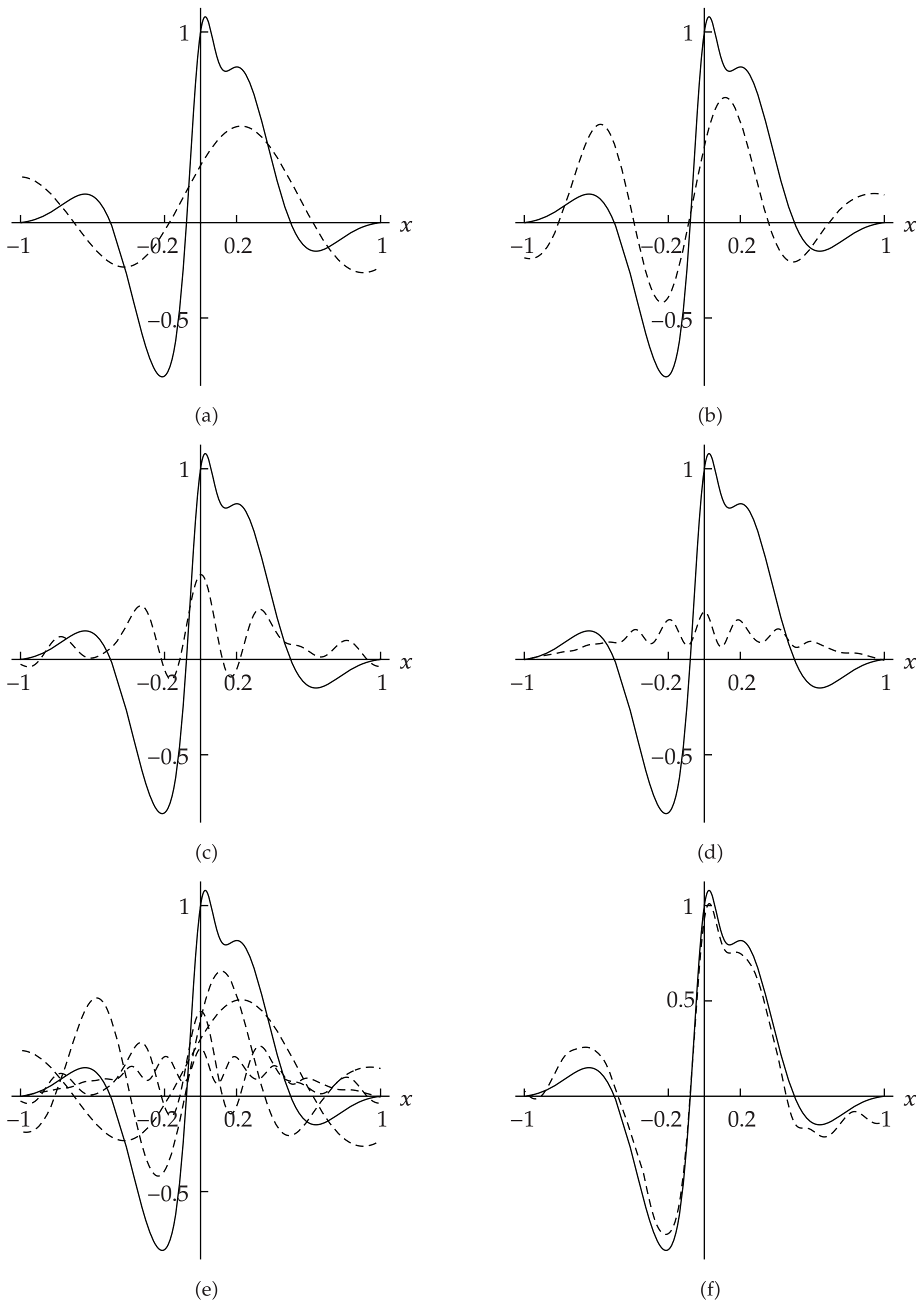

Figure 3: Shannon wavelet reconstruction (dashed) of the odd function $f(x) = e^{-(16x)^2/2} + e^{-4x^2} \sin 2\pi x$, with $N = n_{\max} = 3$, $-3 \le k \le 3$ (bottom right). Scale approximation with (a) $n = 0$, $-3 \le k \le 3$, (b) $n = 1$, $-3 \le k \le 3$, (c) $n = 2$, $-3 \le k \le 3$, (d) $n = 3$, $-3 \le k \le 3$, (e) $0 \le n \le 3$, $-3 \le k \le 3$, (f) $n = 3$, $-5 \le k \le 5$.

parameters$k$ the oscillation on "tails" is reduced and becomes constant (around 10%). But we can see that with $|k| \le 10$ the approximation error in the origin is reduced up to 3%.

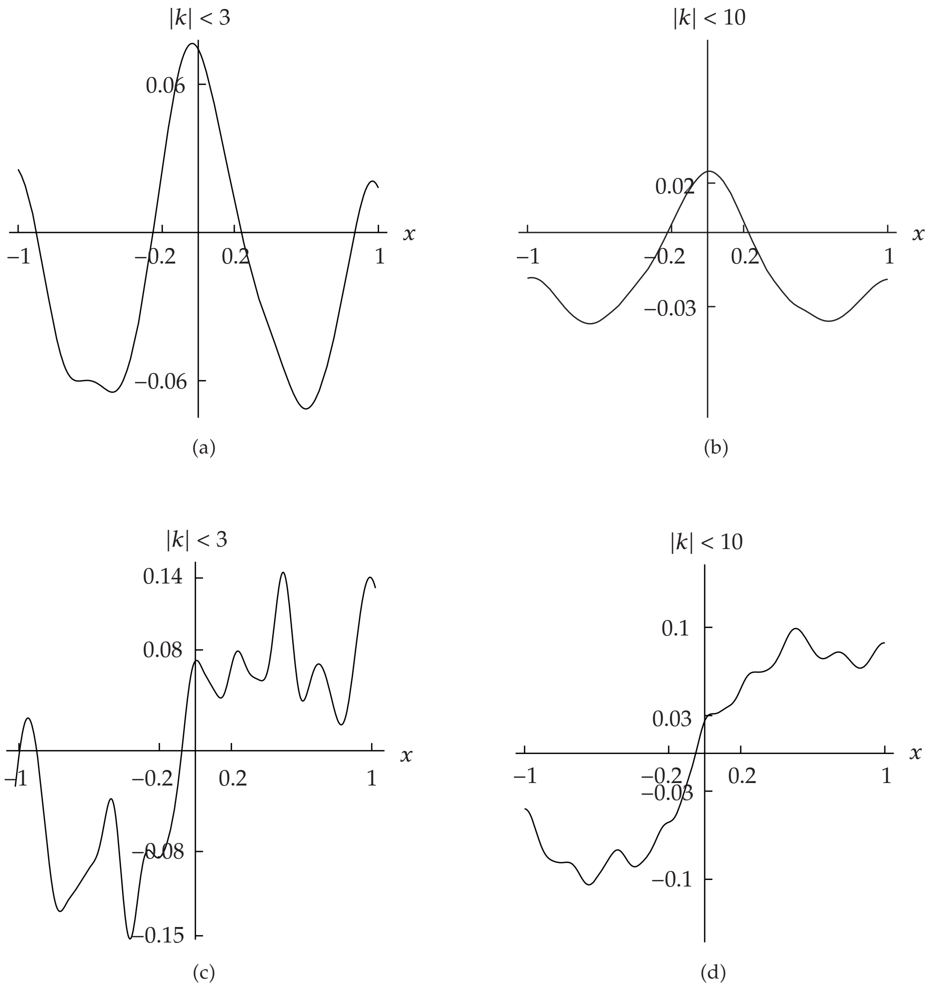

Figure 4: Error of the Shannon wavelet reconstruction of the even function (top) $f(x) = e^{-4x^2} \cos 2\pi x$, with $N = n_{\max} = 3$ and the odd function $f(x) = e^{-(16x)^2/2} + e^{-4x^2} \sin 2\pi x$, with $N = n_{\max} = 3$ (bottom right) with different values of $k_{\max}$.

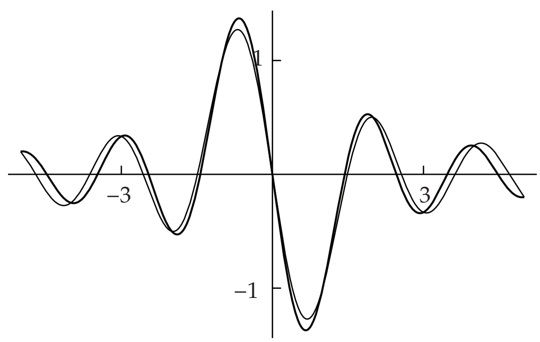

Figure 5: Approximation (plain) of the first derivative of the function $\varphi_0^0(x)$ (bold) by using the connection coefficients.

4. Reconstruction of the Derivatives

Let $f(x) \in L_2(\mathbb{R})$ and let $f(x)$ be a differentiable function $f(x) \in C^p$ with p sufficiently high. The reconstruction of a function $f(x)$ given by (3.10) enables us to compute also its derivatives in terms of the wavelet decomposition

$$\frac{d^\ell}{dx^\ell} f(x) = \sum_{h=-\infty}^{\infty} \alpha_h \frac{d^\ell}{dx^\ell} \varphi_h^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n \frac{d^\ell}{dx^\ell} \psi_k^n(x), \tag{4.1}$$

so that, according to (3.10), the derivatives of $f(x)$ are known when the derivatives

$$\frac{d^\ell}{dx^\ell} \varphi_h^0(x), \qquad \frac{d^\ell}{dx^\ell} \psi_k^n(x) \tag{4.2}$$

are given.

By a direct computation, we can easily evaluate the first and second derivatives of the scaling function

$$\begin{aligned} \frac{d}{dx} \varphi_k^n(x) &= \frac{2^n \left[-1 + (2^n x - k)\pi \cot((2^n x - k)\pi) \right]}{2^n x - k} \varphi_k^n(x), \\ \frac{d^2}{dx^2} \varphi_k^n(x) &= \frac{2^{2n} \left\{ 2 - [\pi(2^n x - k)]^2 - 2\pi(2^n x - k) \cot((2^n x - k)\pi) \right\}}{(2^n x - k)^2} \varphi_k^n(x), \end{aligned} \tag{4.3}$$

respectively. However, on this way, higher-order derivatives cannot be easily expressed. Indeed, according to (3.10), we have to compute the wavelet decomposition of the derivatives:

$$\begin{aligned} \frac{d^\ell}{dx^\ell} \varphi_h^0(x) &= \sum_{k=-\infty}^{\infty} \lambda_{hk}^{(\ell)} \varphi_k^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \Lambda_{hk}^{(\ell)n} \psi_k^n(x), \\ \frac{d^\ell}{dx^\ell} \psi_h^m(x) &= \sum_{k=-\infty}^{\infty} \Gamma_{hk}^{(\ell)m} \varphi_k^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \gamma_{hk}^{(\ell)mn} \psi_k^n(x), \end{aligned} \tag{4.4}$$

with

$$\lambda_{kh}^{(\ell)} \stackrel{\text{def}}{\equiv} \left\langle \frac{d^\ell}{dx^\ell} \varphi_k^0(x), \varphi_h^0(x) \right\rangle, \qquad \gamma_{kh}^{(\ell)nm} \stackrel{\text{def}}{\equiv} \left\langle \frac{d^\ell}{dx^\ell} \psi_k^n(x), \psi_h^m(x) \right\rangle, \tag{4.5}$$

$$\Lambda_{kh}^{(\ell)n} \stackrel{\text{def}}{\equiv} \left\langle \frac{d^\ell}{dx^\ell} \varphi_k^0(x), \psi_h^n(x) \right\rangle, \qquad \Gamma_{hk}^{(\ell)m} \stackrel{\text{def}}{\equiv} \left\langle \frac{d^\ell}{dx^\ell} \psi_h^n(x), \varphi_h^0(x) \right\rangle, \tag{4.6}$$

being the connection coefficients [6–9, 11] (or refinable integrals).

Their computation can be easily performed in the Fourier domain, thanks to equality (2.21). In fact, in the Fourier domain the ℓ-order derivatives of the (scaling) wavelet functions are

$$\widehat{\frac{d^\ell}{dx^\ell} \varphi_k^n(x)} = (i\omega)^\ell \widehat{\varphi}_k^n(\omega), \qquad \widehat{\frac{d^\ell}{dx^\ell} \psi_k^n(x)} = (i\omega)^\ell \widehat{\psi}_k^n(\omega) \tag{4.7}$$

and according to (2.19),

$$\widehat{\frac{d^{\ell}}{dx^{\ell}}\varphi_k^n(x)} = (i\omega)^{\ell}\frac{2^{-n/2}}{2\pi}e^{-i\omega k/2^n}\chi\left(\frac{\omega}{2^n}+3\pi\right),$$
$$\widehat{\frac{d^{\ell}}{dx^{\ell}}\psi_k^n(x)} = -(i\omega)^{\ell}\frac{2^{-n/2}}{2\pi}e^{-i\omega(k+1/2)/2^n}\left[\chi\left(\frac{\omega}{2^{n-1}}\right)+\chi\left(\frac{-\omega}{2^{n-1}}\right)\right]. \tag{4.8}$$

Taking into account (2.21), we can easily compute the connection coefficients in the frequency domain

$$\lambda_{kh}^{(\ell)} = 2\pi\left\langle \widehat{\frac{d^{\ell}}{dx^{\ell}}\varphi_k^0(x)}, \widehat{\varphi_h^0(x)}\right\rangle, \qquad \gamma_{kh}^{(\ell)nm} = 2\pi\left\langle \widehat{\frac{d^{\ell}}{dx^{\ell}}\psi_k^n(x)}, \widehat{\psi_h^m(x)}\right\rangle, \tag{4.9}$$

with the derivatives given by (4.8).

For the explicit computation, we need some preliminary theorems (for a sketch of the proof see also [11]).

Theorem 4.1. *For given $m \in \mathbb{Z}$, $\ell \in \mathbb{N}$, it is*

$$\int x^{\ell}e^{mx}dx = (1-|\mu(m)|)\frac{x^{\ell+1}}{\ell+1} + \mu(m)\frac{e^{mx}}{|m|^{\ell+1}}\sum_{s=1}^{\ell+1}(-1)^{[1+\mu(m)](2\ell-s+1)/2}\frac{\ell!(|m|x)^{\ell-s+1}}{(\ell-s+1)!} + Const, \tag{4.10}$$

where

$$\mu(m) = \operatorname{sign}(m) = \begin{cases} 1, & m > 0, \\ -1, & m < 0, \\ 0, & m = 0. \end{cases} \tag{4.11}$$

Proof. When $m = 0$, (4.10) trivially follows. When $m \neq 0$, by a partial integration we get the iterative formula

$$\int x^{\ell}e^{mx}dx = \begin{cases} \mu(m)\dfrac{1}{|m|}e^{mx}, & \ell = 0, \\ \mu(m)\dfrac{1}{|m|}\left[x^{\ell}e^{mx} - \ell\displaystyle\int x^{\ell-1}e^{mx}dx\right], & \ell > 0, \end{cases} \tag{4.12}$$

from where by the explicit computation of iterative terms and rearranging the many terms, (4.10) holds. □

The following corollary follows. From Theorem 4.1, after a substitution $x \to i\xi$, we have the following corollary.

Corollary 4.2. *For given $m \in \mathbb{Z}$, $\ell \in \mathbb{N}$, it is*

$$\int (i\xi)^{\ell}e^{im\xi}d\xi = i^{\ell}(1-|\mu(m)|)\frac{\xi^{\ell+1}}{\ell+1} - i\mu(m)e^{im\xi}\sum_{s=1}^{\ell+1}(-1)^{[1+\mu(m)](2\ell-s+1)/2}\frac{\ell!(i\xi)^{\ell-s+1}}{(\ell-s+1)!|m|^{s}} + Const. \tag{4.13}$$

In particular, taking into account that

$$e^{ik\pi} = (-1)^k = \begin{cases} 1, & k = \pm 2s, \\ -1, & k = \pm(2s+1), \quad s \in \mathbb{N}, \end{cases} \tag{4.14}$$

we have the following corollary.

Corollary 4.3. *For given $m \in \mathbb{Z} \cup \{0\}$, $\ell \in \mathbb{N}$, and $n \in \mathbb{N}$, it is*

$$\begin{aligned}\int_{-n\pi}^{n\pi} (i\xi)^\ell e^{im\xi} d\xi &= i^\ell (1 - |\mu(m)|) \frac{(n\pi)^{\ell+1}[1 + (-1)^\ell]}{\ell + 1} \\ &\quad + i\mu(m)(-1)^{mn+1} \sum_{s=1}^{\ell+1} (-1)^{[1+\mu(m)](2\ell-s+1)/2} \frac{\ell!(in\pi)^{\ell-s+1}}{(\ell - s + 1)!|m|^s} \left[1 - (-1)^{\ell-s+1}\right].\end{aligned} \tag{4.15}$$

More in general, the following corollary holds.

Corollary 4.4. *For given $m \in \mathbb{Z}$, $\ell \in \mathbb{N}$, and $a, b \in \mathbb{Z}$ $(a < b)$, it is*

$$\begin{aligned}\int_{a\pi}^{b\pi} (i\xi)^\ell e^{im\xi} d\xi &= i^\ell (1 - |\mu(m)|) \frac{\pi^{\ell+1}(b^{\ell+1} - a^{\ell+1})}{\ell + 1} \\ &\quad - i\mu(m) \sum_{s=1}^{\ell+1} (-1)^{[1+\mu(m)](2\ell-s+1)/2} \frac{\ell!(i\pi)^{\ell-s+1}}{(\ell - s + 1)!|m|^s} \left[(-1)^{mb} b^{\ell-s+1} - (-1)^{ma} a^{\ell-s+1}\right].\end{aligned} \tag{4.16}$$

As a particular case, the following corollaries hold.

Corollary 4.5. *For given $m \in \mathbb{Z}$, $\ell \in \mathbb{N}$, and $b \in \mathbb{Z}$ $(0 < b)$, it is*

$$\begin{aligned}\int_{0}^{b\pi} (i\xi)^\ell e^{im\xi} d\xi &= i^\ell (1 - |\mu(m)|) \frac{\pi^{\ell+1} b^{\ell+1}}{\ell + 1} \\ &\quad - i\mu(m) \left[\sum_{s=1}^{\ell} (-1)^{[1+\mu(m)](2\ell-s+1)/2} \frac{\ell!(i\pi)^{\ell-s+1}(-1)^{mb} b^{\ell-s+1}}{(\ell - s + 1)!|m|^s} \right. \\ &\quad \left. + \frac{(-1)^{(1+\mu(m))\ell/2} \ell! [(-1)^{mb} - 1]}{|m|^{\ell+1}} \right].\end{aligned} \tag{4.17}$$

Corollary 4.6. *For given $m \in \mathbb{Z}$, $\ell \in \mathbb{N}$, it is*

$$\int_{0}^{2\pi} (i\xi)^\ell e^{im\xi} d\xi = i^\ell (1 - |\mu(m)|) \frac{(2\pi)^{\ell+1}}{\ell + 1} - i\mu(m) \sum_{s=1}^{\ell} (-1)^{[1+\mu(m)](2\ell-s+1)/2} \frac{\ell!(2i\pi)^{\ell-s+1}}{(\ell - s + 1)!|m|^s}. \tag{4.18}$$

Thus we can show that the following theorem holds.

Theorem 4.7. *The any order connection coefficients* $(4.5)_1$*of the scaling functions* $\varphi_k^0(x)$ *are*

$$\lambda_{kh}^{(\ell)} = \begin{cases} (-1)^{k-h} \dfrac{i^\ell}{2\pi} \displaystyle\sum_{s=1}^{\ell} \dfrac{\ell! \pi^s}{s! [i(k-h)]^{\ell-s+1}} [(-1)^s - 1], & k \neq h, \\ \dfrac{i^\ell \pi^{\ell+1}}{2\pi(\ell+1)} [1 + (-1)^\ell], & k = h, \end{cases} \tag{4.19}$$

or, shortly,

$$\lambda_{kh}^{(\ell)} = \frac{i^\ell \pi^\ell}{2(\ell+1)} [1 + (-1)^\ell](1 - |\mu(k-h)|) + (-1)^{k-h} |\mu(k-h)| \frac{i^\ell}{2\pi} \sum_{s=1}^{\ell} \frac{\ell! \pi^s}{s! [i(k-h)]^{\ell-s+1}} [(-1)^s - 1]. \tag{4.20}$$

Proof. From (4.9), (4.8), (4.7), (2.21), (2.19), it is

$$\lambda_{kh}^{(\ell)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\omega)^\ell e^{-i(k-h)\omega} \chi(\omega + 3\pi) \chi(\omega + 3\pi) \mathrm{d}\omega, \tag{4.21}$$

that is,

$$\begin{aligned} \lambda_{kh}^{(\ell)} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\omega)^\ell e^{-i(k-h)\omega} \chi(\omega + 3\pi) \chi(\omega + 3\pi) \mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (i\omega)^\ell e^{-i(k-h)\omega} \mathrm{d}\omega = \frac{i^\ell}{2\pi} \int_{-\pi}^{\pi} \omega^\ell e^{-i(k-h)\omega} \mathrm{d}\omega. \end{aligned} \tag{4.22}$$

The last integral, according to (4.15) (with $n = 1$), gives (4.20). □

Thus we have at the lower-order derivatives $\ell \leq 5$

$$\begin{aligned} \lambda_{kh}^{(1)} &= -\frac{(-1)^{k-h}}{k-h}, & \lambda_{00}^{(1)} &= 0, \\ \lambda_{kh}^{(2)} &= -\frac{2(-1)^{k-h}}{(k-h)^2}, & \lambda_{00}^{(2)} &= -\frac{\pi^2}{3}, \\ \lambda_{kh}^{(3)} &= (-1)^{k-h} \frac{(k-h)^2 \pi^2 - 6}{(k-h)^3}, & \lambda_{00}^{(3)} &= 0, \\ \lambda_{kh}^{(4)} &= 4(-1)^{k-h} \frac{(k-h)^2 \pi^2 - 6}{(k-h)^4}, & \lambda_{00}^{(4)} &= \frac{\pi^4}{5}, \\ \lambda_{kh}^{(5)} &= (-1)^{k-h} \frac{(k-h)^4 \pi^4 - 20(k-h)^2 \pi^2 + 120}{(k-h)^5}, & \lambda_{00}^{(5)} &= 0. \end{aligned} \tag{4.23}$$

Analogously for the connection coefficients $(4.5)_2$, we have the following theorem.

Theorem 4.8. *The any order connection coefficients* $(4.5)_2$ *of the Shannon wavelets* $(2.18)_2$ *are*

$$\begin{aligned}\gamma_{kh}^{(\ell)nm} = \delta^{nm}\Bigg\{ & i^{\ell}(1-|\mu(h-k)|)\frac{\pi^{\ell}2^{n\ell-1}}{\ell+1}(2^{\ell+1}-1)(1+(-1)^{\ell}) \\ & + \mu(h-k)\sum_{s=1}^{\ell+1}(-1)^{[1+\mu(h-k)](2\ell-s+1)/2}\frac{\ell! i^{\ell-s}\pi^{\ell-s}}{(\ell-s+1)!|h-k|^{s}}(-1)^{-s-2(h+k)}2^{n\ell-s-1} \\ & \times\left\{2^{\ell+1}\left[(-1)^{4h+s}+(-1)^{4k+\ell}\right]-2^{s}\left[(-1)^{3k+h+\ell}+(-1)^{3h+k+s}\right]\right\}\Bigg\},\end{aligned} \tag{4.24}$$

respectively, for $\ell \geq 1$, *and* $\gamma_{kh}^{(0)nm} = \delta_{kh}\delta^{nm}$.

Proof. From (4.9), (4.8), (4.7), (2.21), (2.19), it is

$$\begin{aligned}\gamma_{kh}^{(\ell)nm} & \overset{\text{def}}{=} \left\langle \frac{d^{\ell}}{dx^{\ell}}\psi_k^n(x), \psi_h^m(x)\right\rangle \\ & \overset{(4.9)}{=} 2\pi\left\langle \widehat{\frac{d^{\ell}}{dx^{\ell}}\psi_k^n(x)}, \widehat{\psi_h^m(x)}\right\rangle \\ & \overset{(4.7)}{=} 2\pi\left\langle (i\omega)^{\ell}\widehat{\psi}_k^n(\omega), \widehat{\psi}_h^m(\omega)\right\rangle \\ & \overset{(2.21)}{=} 2\pi\int_{-\infty}^{\infty}(i\omega)^{\ell}\psi_k^n(\omega)\overline{\widehat{\psi}_h^m(\omega)}d\omega \\ & \overset{(2.19)}{=} 2\pi\int_{-\infty}^{\infty}(i\omega)^{\ell}\frac{2^{-n/2}}{2\pi}e^{-i\omega(k+1/2)/2^n}\left[\chi\left(\frac{\omega}{2^{n-1}}\right)+\chi\left(\frac{-\omega}{2^{n-1}}\right)\right] \\ & \qquad \times\frac{2^{-m/2}}{2\pi}e^{i\omega(h+1/2)/2^m}\left[\chi\left(\frac{\omega}{2^{m-1}}\right)+\chi\left(\frac{-\omega}{2^{m-1}}\right)\right]d\omega,\end{aligned} \tag{4.25}$$

from where, according to the definition (2.7), it is

$$\gamma_{kh}^{(\ell)nm} = 0, \quad n \neq m, \tag{4.26}$$

and (for $n = m$)

$$\begin{aligned}\gamma_{kh}^{(\ell)nn} & = \frac{2^{-n}}{2\pi}\int_{-\infty}^{\infty}(i\omega)^{\ell}e^{-i\omega(k-h)/2^n}\left[\chi\left(\frac{\omega}{2^{n-1}}\right)+\chi\left(\frac{-\omega}{2^{n-1}}\right)\right]d\omega \\ & = \frac{2^{-n}}{2\pi}\left[\int_{-2^{n+1}\pi}^{-2^n\pi}(i\omega)^{\ell}e^{-i\omega(k-h)/2^n}d\omega+\int_{2^n\pi}^{2^{n+1}\pi}(i\omega)^{\ell}e^{-i\omega(k-h)/2^n}d\omega\right].\end{aligned} \tag{4.27}$$

By taking into account (4.16), (4.24) is proven. □

Theorem 4.9. *The connection coefficients are recursively given by the matrix at the lowest scale level:*

$$\gamma_{kh}^{(\ell)nn} = 2^{\ell(n-1)}\gamma_{kh}^{(\ell)11}. \tag{4.28}$$

Moreover, it is

$$\gamma_{kh}^{(2\ell+1)nn} = -\gamma_{hk}^{(2\ell+1)nn}, \qquad \gamma_{kh}^{(2\ell)nn} = \gamma_{hk}^{(2\ell)nn}. \tag{4.29}$$

Let us now prove that the mixed connection coefficients (4.6) are zero. It is enough to show the following theorem.

Theorem 4.10. *The mixed coefficients* $(4.6)_1$ *of the Shannon wavelets are*

$$\Lambda_{kh}^{(\ell)n} = 0. \tag{4.30}$$

Proof. From (4.9), (4.8), (4.7), (2.21), (2.19), it is

$$\begin{aligned}
\Lambda_{kh}^{(\ell)n} &\stackrel{\text{def}}{=} \left\langle \frac{d^{\ell}}{dx^{\ell}}\varphi_k^0(x), \psi_h^m(x) \right\rangle = 2\pi \left\langle \widehat{\frac{d^{\ell}}{dx^{\ell}}\varphi_k^0(x)}, \widehat{\psi_h^m(x)} \right\rangle \\
&\stackrel{(4.7)}{=} 2\pi \langle (i\omega)^{\ell}\widehat{\varphi}_k^0(\omega), \widehat{\psi}_h^m(\omega) \rangle \stackrel{(2.21)}{=} 2\pi \int_{-\infty}^{\infty} (i\omega)^{\ell} \varphi_k^0(\omega) \overline{\widehat{\psi}_h^m(\omega)} d\omega \\
&\stackrel{(2.19)}{=} 2\pi \int_{-\infty}^{\infty} (i\omega)^{\ell} \frac{2^{-n/2}}{2\pi} e^{-i\omega k/2^n} \chi\left(\frac{\omega}{2^n} + 3\pi\right) \\
&\quad \times \frac{2^{-m/2}}{2\pi} e^{i\omega(h+1/2)/2^m} \left[\chi\left(\frac{\omega}{2^{m-1}}\right) + \chi\left(\frac{-\omega}{2^{m-1}}\right)\right] d\omega,
\end{aligned} \tag{4.31}$$

from where, since

$$\chi\left(\frac{\omega}{2^n} + 3\pi\right)\left[\chi\left(\frac{\omega}{2^{m-1}}\right) + \chi\left(\frac{-\omega}{2^{m-1}}\right)\right] = 0, \tag{4.32}$$

the theorem is proven. □

As a consequence, we have that the ℓ-order derivatives of the Shannon scaling and wavelets are

$$\begin{aligned}
\frac{d^{\ell}}{dx^{\ell}}\varphi_h^0(x) &= \sum_{k=-\infty}^{\infty} \lambda_{hk}^{(\ell)} \varphi_k^0(x), \\
\frac{d^{\ell}}{dx^{\ell}}\psi_h^m(x) &= \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \gamma_{hk}^{(\ell)mn} \psi_k^n(x).
\end{aligned} \tag{4.33}$$

In other words, the following theorem holds.

Theorem 4.11. *The derivatives of the Shannon scaling function are orthogonal to the derivatives of the Shannon wavelets*

$$\left\langle \frac{d^{\ell}}{dx^{\ell}}\varphi_h^0(x), \frac{d^p}{dx^p}\psi_h^m(x) \right\rangle = 0. \tag{4.34}$$

Proof. It follows directly from (4.33) and the orthogonality of the Shannon functions according to Theorem 2.4. □

4.1. First- and Second-Order Connection Coefficients

For the first and second derivatives of the Shannon wavelets, we have (see [11])

$$\frac{d}{dx}\psi_k^n(x) = \sum_{h=-\infty}^{\infty} \gamma_{kh}^{\prime nn}\psi_h^n(x),$$
$$\frac{d^2}{dx^2}\psi_k^n(x) = \sum_{h=-\infty}^{\infty} \gamma_{kh}^{\prime\prime nn}\psi_h^n(x), \tag{4.35}$$

with (4.24)

$$\begin{aligned}\gamma_{kh}^{\prime nn} &= \mu(h-k)\sum_{s=1}^{2}(-1)^{[1+\mu(h-k)](2-s+1)/2}\frac{i^{1-s}\pi^{1-s}}{(2-s)!|h-k|^s}(-1)^{-s-2(h+k)}2^{n-s-1}\\&\quad\times\left\{4\left[(-1)^{4h+s}+(-1)^{4k+1}\right]-2^s\left[(-1)^{3k+h+1}+(-1)^{3h+k+s}\right]\right\},\\\gamma_{kh}^{\prime\prime nn} &= -(1-|\mu(h-k)|)\pi^2 2^{2n}\\&\quad+\mu(h-k)\sum_{s=1}^{3}(-1)^{[1+\mu(h-k)](5-s)/2}\frac{2i^{2-s}\pi^{2-s}}{(3-s)!|h-k|^s}(-1)^{-s-2(h+k)}2^{2n-s-1}\\&\quad\times\left\{8\left[(-1)^{4h+s}+(-1)^{4k+2}\right]-2^s\left[(-1)^{3k+h+2}+(-1)^{3h+k+s}\right]\right\},\end{aligned} \tag{4.36}$$

respectively.

A disadvantage in (4.33) is that derivatives are expressed as infinite sum. However, since the wavelets are mainly localized in a short range interval, a good approximation can be obtained with a very few terms of the series. The main advantage of (4.33) is that the derivatives are expressed in terms of the wavelet basis.

Analogously, we obtain for the first and second derivative of the scaling function

$$\frac{d}{dx}\varphi_k^0(x) = \sum_{h=-\infty}^{\infty} \lambda'_{kh}\varphi_h^0(x),$$
$$\frac{d^2}{dx^2}\varphi_k^0(x) = \sum_{h=-\infty}^{\infty} \lambda''_{kh}\varphi_h^0(x), \tag{4.37}$$

with (4.20)

$$\begin{aligned}\lambda'_{kh} &= (-1)^{h-k}\mu(h-k)\sum_{s=1}^{2}(-1)^{[1+\mu(h-k)](3-s)/2}\frac{i^{1-s}\pi^{1-s}}{2(2-s)!|h-k|^s}\left[1+(-1)^{1-s}\right],\\\lambda''_{kh} &= -(1-|\mu(h-k)|)\frac{\pi^2}{2}+(-1)^{h-k}\mu(h-k)\sum_{s=1}^{3}(-1)^{[1+\mu(h-k)](5-s)/2}\frac{i^{2-s}\pi^{2-s}}{(3-s)!|h-k|^s}\left[1+(-1)^{2-s}\right].\end{aligned} \tag{4.38}$$

The coefficients of derivatives are real values as can be shown by a direct computation

$$
\begin{array}{c|ccccc}
\gamma'^{11}_{kh} & k=-2 & k=-1 & k=0 & k=1 & k=2 \\
\hline
h=-2 & 0 & -\dfrac{1}{2} & -\dfrac{1}{4} & -\dfrac{1}{6} & -\dfrac{1}{8} \\[2ex]
h=-1 & \dfrac{1}{2} & 0 & -\dfrac{1}{2} & -\dfrac{1}{4} & -\dfrac{1}{6} \\[2ex]
h=0 & \dfrac{1}{4} & \dfrac{1}{2} & 0 & -\dfrac{1}{2} & -\dfrac{1}{4} \\[2ex]
h=1 & \dfrac{1}{6} & \dfrac{1}{4} & \dfrac{1}{2} & 0 & -\dfrac{1}{2} \\[2ex]
h=2 & \dfrac{1}{8} & \dfrac{1}{6} & \dfrac{1}{4} & \dfrac{1}{2} & 0
\end{array}
\tag{4.39}
$$

$$
\begin{array}{c|ccccc}
\gamma'^{22}_{kh} & k=-2 & k=-1 & k=0 & k=1 & k=2 \\
\hline
h=-2 & 0 & -1 & -\dfrac{1}{2} & -\dfrac{1}{3} & -\dfrac{1}{4} \\[2ex]
h=-1 & 1 & 0 & -1 & -\dfrac{1}{2} & -\dfrac{1}{3} \\[2ex]
h=0 & \dfrac{1}{2} & 1 & 0 & -1 & -\dfrac{1}{2} \\[2ex]
h=1 & \dfrac{1}{3} & \dfrac{1}{2} & 1 & 0 & -1 \\[2ex]
h=2 & \dfrac{1}{4} & \dfrac{1}{3} & \dfrac{1}{2} & 1 & 0
\end{array}
\tag{4.40}
$$

If we consider a dyadic discretization of the x-axis such that

$$x_k = 2^{-n}\left(k+\frac{1}{2}\right), \quad k \in \mathbb{Z}, \tag{4.41}$$

that is,

(4.42)

	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$
$n=0$	−1.5	−0.5	0.5	1.5	2.5
$n=1$	−0.75	−0.25	0.25	0.75	1.25
$n=2$	−0.375	−0.125	0.125	0.375	0.625

there results

$$\varphi_k^n\left(2^{-n}\left(k+\frac{1}{2}\right)\right) = -2^{n/2}, \quad k \in \mathbb{Z}. \tag{4.43}$$

Thus (4.33) at dyadic points $x_k = 2^{-n}(k+1/2)$becomes

$$\begin{aligned}\left[\frac{d}{dx}\varphi_k^n(x)\right]_{x=x_k} &= -2^{n/2}\sum_{h=-\infty}^{\infty}\gamma_{kh}^{nn},\\ \left[\frac{d^2}{dx^2}\varphi_k^n(x)\right]_{x=x_k} &= -2^{n/2}\sum_{h=-\infty}^{\infty}\Gamma_{kh}^{nn}.\end{aligned} \tag{4.44}$$

For instance, (see the above tables) in $x_1 = 2^{-1}(1+1/2)$,

$$\left[\frac{d}{dx}\varphi_1^1(x)\right]_{x=x_1=3/4} = -2^{1/2}\sum_{h=-\infty}^{\infty}\gamma_{1h}^{11} \cong -2^{1/2}\sum_{h=-2}^{2}\gamma_{1h}^{11} = -2^{1/2}\left(\frac{1}{6}+\frac{1}{4}\right) = -\frac{5\sqrt{2}}{12}. \tag{4.45}$$

Analogously, it is

$$\varphi_k^n\left(2^{-n}\left(k+\frac{1}{2}\right)\right) = \frac{2^{1+n/2}}{\pi}, \quad k \in \mathbb{Z}, \tag{4.46}$$

from where, in $x_k = (k+1/2)$, it is

$$\begin{aligned}\left[\frac{d}{dx}\varphi_k^0(x)\right]_{x=x_k} &= \frac{2}{\pi}\sum_{h=-\infty}^{\infty}\lambda_{kh},\\ \left[\frac{d^2}{dx^2}\varphi_k^0(x)\right]_{x=x_k} &= \frac{2}{\pi}\sum_{h=-\infty}^{\infty}\Lambda_{kh}.\end{aligned} \tag{4.47}$$

Outside the dyadic points, the approximation is quite good even with low values of the parameters n, k. For instance, we have (Figure 5) the approximation

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi_0^0(x) = \frac{\cos \pi x}{x} - \frac{\sin \pi x}{\pi x^2} \cong \sum_{h=-5}^{5}\lambda_{0h}\varphi_h^0(x). \tag{4.48}$$

5. Conclusion

In this paper, the theory of Shannon wavelets has been analyzed showing the main properties of these functions sharply localized in frequency. The reconstruction formula for the $L_2(\mathbb{R})$ functions has been given not only for the function but also for its derivatives. The derivative of the Shannon wavelets has been computed by a finite formula (both for the scaling and for the wavelet) for any order derivative. Indeed, to achieve this task, it was enough to compute connection coefficients, that is, the wavelet coefficients of the basis derivatives. These coefficients were obtained as a finite series (for any order derivatives). In Latto's method [6, 8, 9], instead, these coefficients were obtained only (for the Daubechies wavelets) by using the inclusion axiom but in approximated form and only for the first two order derivatives. The knowledge of the derivatives of the basis enables us to approximate a function and its derivatives and it is an expedient tool for the projection of differential operators in the numerical computation of the solution of both partial and ordinary differential equations [2, 3, 10, 13].

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