

Splines and index theorem

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Abstract In this article we want to present a meeting ground between rather different mathematical topics from numerical analysis to index theory and symplectic geometry. The unifying idea is that of linear representation of tori inside which the combinatorics, analysis and geometry is developed.

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1 Introduction

In 1968 appears the fundamental work of Atiyah and Singer on the index theorem of elliptic operators, a theorem formulated in successive steps of generality [3–7].

This theorem is the crowning point of a long sequence of ideas starting from the theory of algebraic functions of Riemann and passing through the theory of De Rham and Hodge on cohomology on manifolds. The index theorem generalizes the Riemann–Roch theorem of Hirzebruch, a cornerstone in modern algebraic geometry [25].

What is an index? Recall that the index $i(A)$ of a linear operator, when defined, is $i(A) := \dim(\ker A) - \dim(\ker A^*)$, where $\ker(A)$ is the space of solutions of the homogeneous equation $Au = 0$ and A^* denotes the adjoint.

- The linear operators considered by Atiyah and Singer are differential (or rather pseudodifferential) operators on manifolds.
- Precisely they operate not simply on functions but, as usual in global geometry, on *sections of vector bundles*.

The condition of being *elliptic* (like the Laplace operator) is needed to insure that the index is well defined and a finite integer. The index theorem produces a formula for the index through cohomological data associated to the operator, the manifold and the bundles involved (Chern character and Todd class).

The need for K -theory

An essential step in the index theory is given by the construction of a generalized cohomology theory, called K -theory.

Then the index theorem passes through several steps:

- to an operator one associates the *symbol* Σ , which is a (matrix valued) function on the cotangent bundle.
- to the symbol an element of a suitable group of K -theory, the index can be computed this way.

- Alternatively from K -theory one passes to cohomology using the Chern character $\text{ch}(\Sigma)$ of Σ .
- The final formula is proved by showing enough properties of all these steps which reduce the formula to some basic cases.

One general and useful setting is for operators on a manifold M which:

- (i) satisfy a symmetry with respect to a compact Lie group G
- (ii) are elliptic in directions transverse to the G -orbits.

In this case the values of the index are generalized functions on G . In fact in this case the kernel of A or of A^* is no more finite dimensional, but it is a particularly well behaved representation $\rho : G \rightarrow U(\mathbb{H})$, called of trace class, from the group of symmetries G to unitary operators on a Hilbert space. This means that we can integrate the C^∞ functions of G to operators $T_f := \int_G f(g)\rho(g)$ which have a trace and $f \mapsto \text{Tr}(T_f)$ is a distribution.

Being of trace class implies that both spaces of solutions are

- a (possibly infinite) direct sum of irreducible representations of G each with finite multiplicity
- the multiplicities have some moderate growth so that they can be interpreted as Fourier coefficients of generalized functions (distributions) on G .

The analytic index of the operator A is the virtual trace class representation of G , obtained as difference of the spaces of solutions of A and its adjoint A^* in an appropriate Sobolev space. This is equivalently described by an integral valued function on the set of irreducible representations, the *index multiplicity*.

The main topological ingredient of the theory is the group $K_{G,c}^0(T_G^*M)$ of equivariant K -theory with compact support, which is defined in a topological fashion.

If T^*M is the *cotangent bundle*, we denote by T_G^*M the closed subset of T^*M , union of the conormals to the G orbits. Using a Riemannian metric we may replace the cotangent bundle with the tangent bundle, and then conormal is replaced with orthogonal.

Example 1.1 $G = S^1$ acting by rotations on the plane \mathbb{R}^2 , the orbits are the circles centered at the origin, we identify $T^*M = \mathbb{R}^4$ with pairs of vectors.

A pair of vectors $(a, b) \in \mathbb{R}^2 \times \mathbb{R}^2$ is in $T_{S^1}^*M$ if either $a = 0$, in this case its orbit reduces to 0 and b arbitrary, or $a \neq 0$ its orbit is a circle and b is orthogonal to the orbit, that is it is proportional to a .

Assume first that M is compact. The condition of being transversally elliptic is expressed by the fact that the symbol Σ is invertible on T_G^*M minus the zero section. So one repeats the analysis for the index, substituting to K -theory *equivariant K -theory*.

This index depends only of the class defined by Σ in $K_{G,c}^0(T_G^*M)$, so that in the end the index defines a $R(G)$ module homomorphism from $K_{G,c}^0(T_G^*M)$ to virtual trace class representations of G .

Alternatively, from equivariant K -theory one passes to equivariant cohomology with compact supports using the equivariant Chern character $\text{ch}(\Sigma)$ of Σ . This is an element of a suitable G -equivariant cohomology group, with compact supports, of T_G^*M .

In his Lecture Note describing joint work with I.M. Singer, *Elliptic operators and compact groups*, [1], Atiyah explains how to reduce general computations to the case in which

- G is a torus, e.g. $G = (S^1)^s$, $\hat{G} = \mathbb{Z}^s$ its character group.
- The manifold M is a complex linear representation $M_X = \bigoplus_{a \in X} L_a$,
- $X \subset \hat{G} = \mathbb{Z}^s$ is a finite list of characters and L_a denotes the one dimensional complex line where G acts by the character $a \in X$.

He then computes explicitly in several cases and ends his introduction saying "...for a circle (with any action) the results are also quite explicit. However for the general case we give only a reduction process and one might hope for something explicit. This probably requires the development of an appropriate algebraic machinery, involving cohomology but going beyond it."

A solution to this question is given in the paper *Vector partition functions and index of transversally elliptic operators*, by C. De Concini, C. Procesi, M. Vergne [20].

In this paper I want to give an idea of the algebraic machinery which provides a complete solution to this question. This turns out to be a spinoff of the theory of splines, a development of the classic Theory of Schoenberg, [31] as in the classic book [15] by de Boor, Höllig, and Riemenschneider. For most of this material the reader can consult the recent book by De Concini, and Procesi *Topics in hyperplane arrangements, polytopes and box-splines* [23].

The main goal is to describe explicitly the map induced by the index, from $K_{G,c}^0(T_G^*M)$ to virtual trace class representations of G . This is described combinatorially.

Once we identify the character group of G with the free abelian group \mathbb{Z}^s we have, as values of the index, a space of integer valued functions on \mathbb{Z}^s . Our main result is for a linear representation M_X associated to a list of characters X in \mathbb{Z}^s :

- We identify the space of functions, image of the index, for $M = M_X$.
- We prove that the index is an isomorphism between $K_{G,c}^0(T_G^*M)$ and its image in the space of functions.

We need the basic notion of *Partition function* for a list X of vectors in \mathbb{Z}^s .

If there is a linear form ϕ so that $\langle \phi | a \rangle > 0$, $\forall a \in X$ we have a \mathbb{Z} valued function $P_X(b)$ on \mathbb{Z}^s defined as:

$$P_X(b) := \# \left\{ (t_1, \dots, t_m) \in \mathbb{N}^m \mid \sum_{i=1}^m t_i a_i = b \right\}.$$

The function $P_X(b)$ is best expressed via its generating series

$$\sum_{b \in \mathbb{Z}^s} P_X(b) e^b = \prod_{a \in X} \frac{1}{1 - e^a}.$$

If X is the list of characters of a linear representation M_X the partition function can be viewed as the trace of the representation of S^s in the symmetric algebra $S(M_X)$.

Atiyah-Singer constructed a “pushed” $\bar{\partial}$ operator on M_X , with index the partition function.

For a general list X consider a linear function ϕ which is non-zero on each element of X , then X is divided into two parts A where ϕ is positive, and B respectively where ϕ is negative. We may thus consider the partition function $P_{A,-B}$.

Lemma 1.2 *The function $P_{A,-B}$ depends only on the chamber F , of the hyperplane arrangement defined by X in the dual space, in which ϕ lies and will be denoted P_X^F .*

Each of these partition functions can be viewed as multiplicity index of a “pushed” $\bar{\partial}$ operator on M_X by changing the complex structure.

Integer valued functions on \mathbb{Z}^s are a module (using translation) over the group algebra $\mathbb{Z}[\mathbb{Z}^s]$ (also denoted by $R(T)$, if T is the torus with character group $\hat{T} = \mathbb{Z}^s$). We can thus consider the $\mathbb{Z}[\mathbb{Z}^s]$ module $\tilde{\mathcal{F}}(X)$ of functions on \mathbb{Z}^s generated by all the partition functions $P_{A,-B}$ for all the chambers.

The main Theorem 1 states that the index induces an isomorphism between $K_{G,c}^0(T_G^*M_X)$ and $\tilde{\mathcal{F}}(X)$. Moreover there is a very precise description of the group $\tilde{\mathcal{F}}(X)$.

1.2.1. The case S^1 . For $s = 1$, or $G = S^1$ we denote by t the basic character of $S^1 := \{t \mid |t| = 1\}$, so that $R(S^1) = \mathbb{Z}[t, t^{-1}]$. We write generalized functions by their generating series $\sum_{n \in \mathbb{Z}} a_n t^n$.

When S^1 acts by homotheties on \mathbb{C}^{k+1} we have $\mathbb{C}^{k+1} = M_X$ and $X = [t, t, \dots, t]$, $k + 1$ times. We start by describing Theorem 1 in this special case.

By the “pushed” $\bar{\partial}$ operator we get the generalized function

$$\Theta_X(t) := \sum_{n=0}^{\infty} \binom{n+k}{k} t^n$$

where

$$\binom{n+k}{k} = \frac{(n+k)(n+k-1)\dots(n+1)}{k!}, \quad n \geq 0$$

is the partition function for $X = \underbrace{\{1, \dots, 1\}}_{k+1}$.

In this case we only have two chambers and the other partition function

$$\binom{-n+k}{k}, \quad n \leq 0$$

for $X = \underbrace{\{-1, \dots, -1\}}_{k+1}$.

The map $n \mapsto \binom{n+k}{k}$ is a polynomial function on \mathbb{Z} which, for any n positive or negative, represents the dimension of a *virtual space*, the alternate sum of the cohomology spaces of the sheaf $\mathcal{O}(n)$ on k -dimensional projective space.

One easily sees that this function gives the Fourier coefficients of a generalized function $\theta_k(t) := \sum_{n=-\infty}^{\infty} \binom{n+k}{k} t^n$, on S^1 supported at $t = 1$. In fact for $k = 0$ the generalized function $\theta_0(t) := \sum_{n=-\infty}^{\infty} t^n$, is the Fourier expansion of the delta function δ_1 and we obtain $\theta_k(t)$ by applying suitable derivatives to $\theta_0(t)$.

One can prove that the tangential Cauchy–Riemann operator, on the unit sphere S_{2k+1} of \mathbb{C}^{k+1} , is a transversally elliptic operator with index $\theta_k(t)$.

Proposition 1.3 *The index map is an isomorphism from $K_{S^1,c}^0(T_{S^1}^*M_X)$ to the space $\tilde{\mathcal{F}}(X)$ of generalized functions on S^1 generated by Θ_X and $\theta_X = \theta_k(t)$ under multiplication by elements of $R(S^1) = \mathbb{Z}[t, t^{-1}]$.*

In fact the $R(S^1)$ module generated by Θ_X is free over $R(S^1)$.

The $R(S^1)$ module generated by θ_X is the torsion submodule.

- This submodule is the module of polynomial functions on \mathbb{Z} of degree at most k so it is a free \mathbb{Z} -module of rank $k + 1$.
- It corresponds to indices of operators on $\mathbb{C}^{k+1} - \{0\}$, the set where S^1 acts freely
- It is the space of solutions of the difference equation $\nabla^k f = 0$, where we define the operator ∇ by $(\nabla f)(n) = f(n) - f(n - 1)$.

1.3.1. Higher dimension. In higher dimensions, the single difference equation $\nabla^k f = 0$ must be replaced by a system of difference equations, discovered by Dahmen–Michelli as the natural generalization of a system of differential equations associated to splines in approximation theory.

We need a basic definition of combinatorial nature. Consider a sequence X of integral vectors (i.e. weights of a torus G with character group Λ).

Definition 1.4 We say that a sublist $Y \subset X$ is a *cocircuit*, if the elements in $X \setminus Y$ do not span V and Y is minimal with this property.

For an integral vector $a \in \Lambda$ (often identified with \mathbb{Z}^s) define the *translation operator* τ_a and the *difference operator* ∇_a by

$$\tau_a(f)(x) := f(x - a), \quad \nabla_a = 1 - \tau_a.$$

These operators act on functions on \mathbb{Z}^s . For a list Y of integral vectors we set $\nabla_Y := \prod_{a \in Y} \nabla_a$ and introduce thus the system of difference equations $\nabla_Y f = 0$ associated to cocircuits Y of $X \subset \mathbb{Z}^s$. The space of solutions is denoted by $DM(X)$. There is a beautiful Theory of this space which we will in part recall presently (cf. Definition 7.5).

Recall the $\mathbb{Z}[\mathbb{Z}^s]$ module $\tilde{\mathcal{F}}(X)$ of functions on \mathbb{Z}^s generated by all the partition functions $P_{A,-B}$ for all the chambers.

Theorem 1 *The index induces an isomorphism between $K_{G,c}^0(T_G^*M_X)$ and $\tilde{\mathcal{F}}(X)$.*

Moreover there is a very precise description of the group $\tilde{\mathcal{F}}(X)$.

The main result is about the open set M_X^f of M_X where G acts with finite stabilizers.

Theorem 2 *The index map induces an isomorphism between $K_{G,c}^0(T_G^*M_X^f)$ and the space $DM(X)$.*

For the general case we need some associated spaces of functions, generalizing $DM(X)$. We thus introduce the spaces $\tilde{\mathcal{F}}_i(X)$ associated to the orbit types in M_X .

Any linear subspace \underline{r} generated by vectors in X is called a *rational subspace*, we denote by S_X the set of all rational subspaces. We define:

$$\mathcal{F}(X) := \{f \mid \nabla_{X \setminus \underline{r}} f \text{ is supported on } \underline{r} \text{ for every rational subspace } \underline{r}\}.$$

Clearly $DM(X)$ is contained in $\mathcal{F}(X)$.

One can show that $\tilde{\mathcal{F}}(X)$ is the space of functions generated by $\mathcal{F}(X)$ under translations by integral vectors.

Denote by $S_X^{(i)}$ the subset of subspaces $\underline{r} \in S_X$ of dimension i . Define the spaces

$$\mathcal{F}_i(X) := \bigcap_{\underline{r} \in S_X^{(i-1)}} \ker \nabla_{X \setminus \underline{r}} \cap \mathcal{F}(X).$$

Denote by $\tilde{\mathcal{F}}_i(X)$ the space of functions generated by $\mathcal{F}_i(X)$ under translations by integral vectors. Set $M_{\geq i}$ as the open set of points in M_X with the property that the orbit has dimension $\geq i$.

Theorem 3 *For each $s \geq i \geq 0$, the index multiplicity map ind_m gives an isomorphism between $K_G^0(T_G^*M_{\geq i})$ and the space $\tilde{\mathcal{F}}_i(X)$.*

Finally one has very explicit descriptions of $DM(X)$ and of the spaces $\tilde{\mathcal{F}}_i(X)$ which are built from various spaces $DM(X_i)$.

We then investigate the formulas allowing us to pass from equivariant K -theory to equivariant cohomology.

In order to understand these explicit formulas in [18] we have introduced the *infinitesimal index*, called for short “infdex”, a map from the equivariant cohomology, with compact supports, of the zeroes of the moment map to distributions on \mathfrak{g}^* .

We have proved several properties for this map which, at least in the case of the space T_G^*M , in principle allow us to reduce the computations to the case in which G is a torus and the manifold is a complex linear representation M_X of G . Where X is a list of characters as before.

The equivariant cohomology of the open sets $M_{X,\geq i}$ can be computed from the structure of the algebra $S[\mathfrak{g}^*][(\prod_{a \in X} a)^{-1}]$ as a module over the Weyl algebra studied in [17].

The equivariant cohomology with compact supports $H_{G,c}^*(T_G^*M_X^{fin})$ of $T_G^*M_X^{fin}$ is isomorphic, as a $S[\mathfrak{g}^*]$ -module, to a remarkable finite dimensional space $D(X)$ of polynomial functions on \mathfrak{g}^* , where $S[\mathfrak{g}^*]$ acts by differentiation. The space $D(X)$ is the continuous analogue of $DM(X)$ and it is defined as the space of solutions of a set of linear partial differential equations combinatorially associated to X and has been of importance in approximation theory (see for example [13, 14]). In fact in a natural way $D(X)$ is a component of $DM(X)_{\mathbb{C}}$ and it coincides with $DM(X)_{\mathbb{C}}$ in the special case of X unimodular 7.2.

At this point the notion of infinitesimal index comes into play. One can show, Theorem 34, that the infinitesimal index gives an isomorphism between $H_{G,c}^*(T_G^*M_X^{fin})$ and $D(X)$. After this one shows that, for each i , the infinitesimal index establishes an isomorphism between $H_{G,c}^*(T_G^*M_{X,\geq i})$ and a space of splines $\tilde{\mathcal{G}}_i(X)$, introduced in [19] and analogous to the arithmetic spaces $\tilde{\mathcal{F}}_i(X)$, and generalizing $D(X)$.

Then Theorem 37 is a *deconvolution* formula which allows us to compute the function $\text{ind}_m(A)$ in function of the distribution $\text{infdex}(\text{ch}(\Sigma))$ by applying to it a Todd operator.

In order to develop this formula one needs results obtained by Dahmen–Micchelli in the purely combinatorial context of the semi-discrete convolution with the Box spline.

In this paper we do not give details for the proofs which can be found in the papers quoted.

Part 1. Polytopes and splines

2 Polytopes

Polytopes play a special role in the Theory. A convex polytope can be defined as the convex envelop of finitely many points or by finitely many linear inequalities. We take this second point of view and study polytopes which vary into families.

2.1 From a list of vectors

We start from a real, sometimes integer $n \times m$ matrix X . We always think of $X := (a_1, \dots, a_m)$ as a list of vectors in $V = \mathbb{R}^n$, its columns. We assume that 0 is NOT in the convex hull of its columns.

From A we make several constructions, algebraic, combinatorial, analytic etc.

Many people have contributed to the Theory from various branches of mathematics. *From numerical analysis:* A.A. Akopyan; Ben-Artzi, Asher; C.K. Chui, C. De Boor, W. Dahmen, H. Diamond, N. Dyn, K. Höllig, C. Micchelli, Jia, Rong Qing, A. Ron, A.A. Saakyan.

From geometry and combinatorics: In fact a lot of work originated from the seminal paper of Khovanskiĭ, Pukhlikov, who interpret the counting formulas as Riemann–Roch formulas for toric varieties. We should also mention Orlik and Solomon on cohomology, Baldoni, Brion, Szenes, Vergne, Jeffrey and Kirwan, on partition functions, polytopes etc.

From enumerative combinatorics: A.I. Barvinok, Matthias Beck, Sinai Robins, Richard Stanley.

From X we produce first a system of linear equations:

$$\sum_{i=1}^m a_i x_i = b, \quad \text{or} \quad Xx = b, \quad X := (a_1, \dots, a_m) \quad (1)$$

The columns a_i, b are vectors with n coordinates $a_{j,i}, b_j, j = 1, \dots, n$.

As in Linear Programming Theory we deduce and want to study the, convex and bounded, variable polytopes:

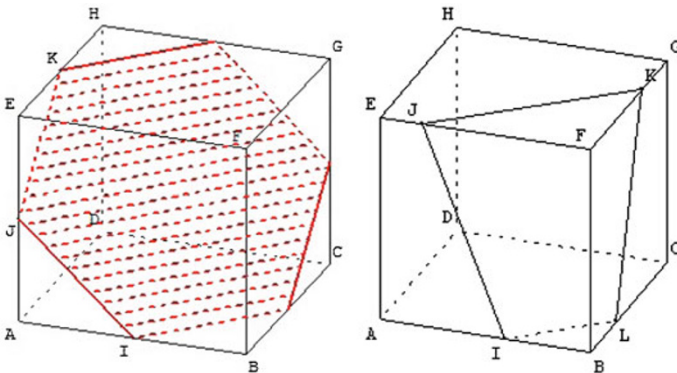
$$\Pi_X(b) := \{x \mid Xx = b, x_i \geq 0, \forall i\}, \quad \Pi_X^1(b) := \{x \mid Xx = b, 1 \geq x_i \geq 0, \forall i\}.$$

The variable polytope $\Pi_X(b)$ is empty unless b belongs to $C(X) := \{\sum_{i=1}^m x_i a_i, x_i \geq 0\}$, the cone of positive combinations of the a_i . The hypothesis that 0 is NOT in the convex hull of the a_i implies that $C(X)$ is pointed, i.e. there is a linear function ϕ with $\phi(a_i) > 0$ for all i so that $\phi(a_i) > c > 0$. In other words ϕ is strictly positive on all non zero points of $C(X)$.

The property of being bounded is trivial for $\Pi_X^1(b)$ while for $\Pi_X(b)$ depends on the fact that

$$\sum_i x_i a_i = b \implies \sum_i x_i \phi(a_i) = \phi(b) \implies \sum_i x_i < \phi(b)c^{-1}.$$

The polytopes $\Pi_X^1(b)$ are sections of a hypercube,



The object of study are two basic functions

- Set $T_X(x), B_X(x)$ to be the volume of $\Pi_X(x), \Pi_X^1(x)$.
- If X, b have integer coordinates set $P_X(x)$ to be the number of solutions of the system in which the coordinates x_i are non negative integers.
- In other words $P_X(x)$ is the number of integral points in the variable polytope $\Pi_X(x)$.

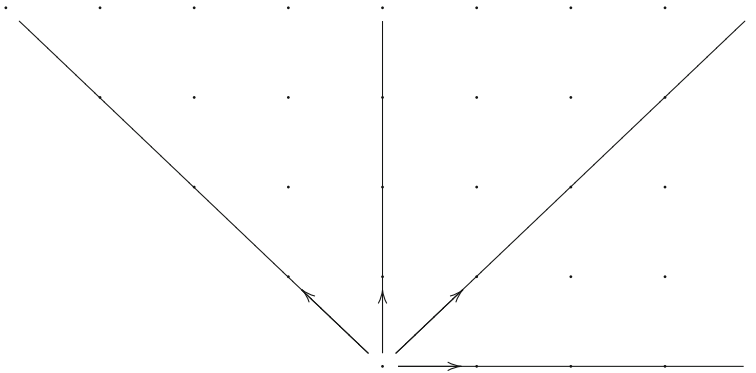
Up to a multiplicative normalization constant: $T_X(x)$ is the *Multivariate-spline*, $B_X(x)$ the *Box-spline*, $P_X(x)$ is called the *partition function*. We are interested in computing the three functions $T_X(x), B_X(x), P_X(x)$ and describe their qualitative properties.

We shall discuss applications of these functions to arithmetic, numerical analysis, Lie theory and Index Theory.

An important example is when X is the list of positive roots of a root system, e.g. B_2 :

$$X = \begin{vmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix}$$

The associated cone $C(X)$ has three *big cells* (cf. Sect. 2.9):



In the literature of numerical analysis the Box spline associated to the root system B_2 is called the Zwart–Powell or ZP element, $2T_X$ is 0 outside the cone and on the three cells (Fig. 1):

The computation of the box-spline has some geometric, combinatorial and algebraic flavor. It appears as a piecewise polynomial function on a compact polyhedron. *From simple data we get soon a complicated picture!* (Fig. 2).

2.2. The partition function

When X, b have integer entries, it is natural to think of an expression like: $b = t_1 a_1 + \dots + t_m a_m$, with t_i not negative integers, as a *partition of b with the vectors a_i* ,

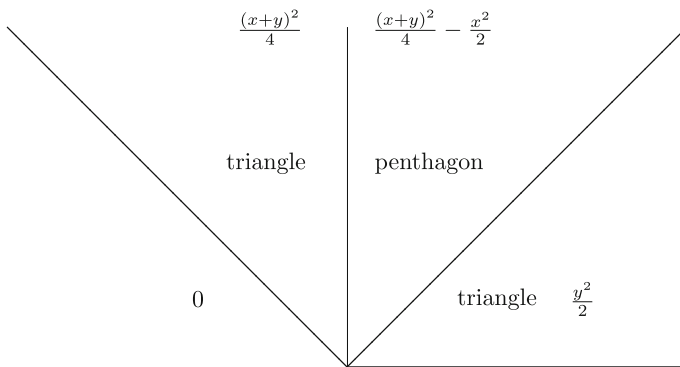


Fig. 1 The area and the shape as $b = (x, y)$ varies in the cone

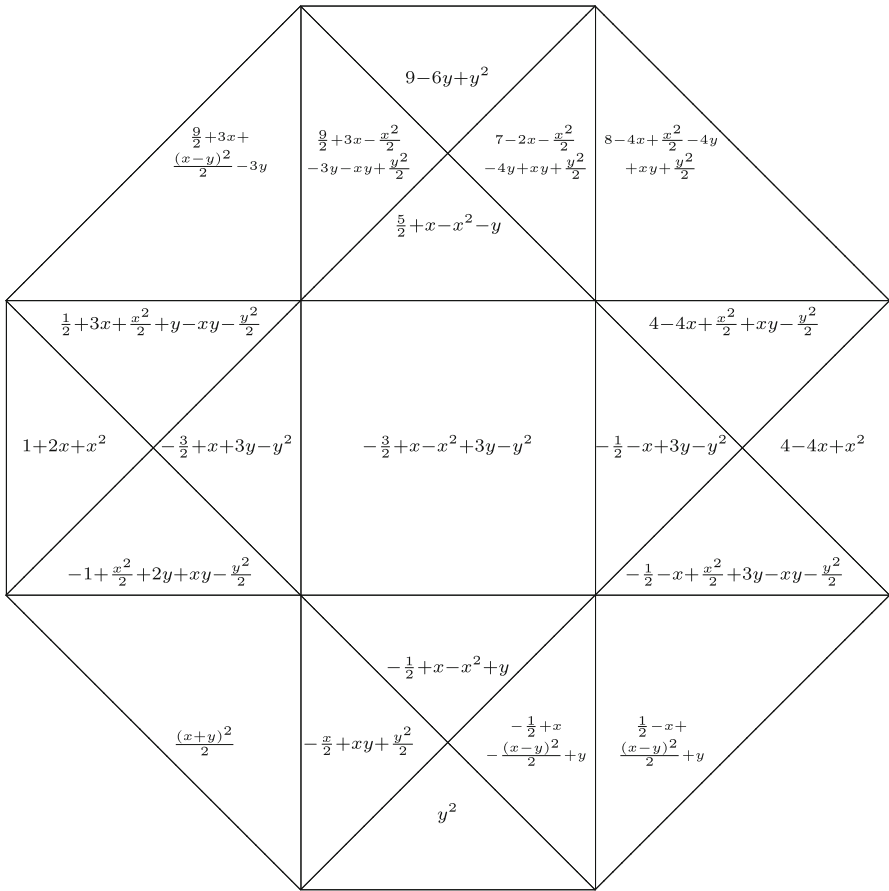


Fig. 2 The box spline $2B_X$, ZP element

in $t_1 + t_2 + \dots + t_m$ parts, hence the name *partition function* for the number $P_X(b)$, thought of as a function of the vector b .

Of special interest is the case when the a_i are numbers. Given positive integers $\underline{h} := (h_1, \dots, h_m)$, the problem of counting the number of ways in which a positive integer n can be written as a linear combination $\sum_{i=1}^n k_i h_i$, with the k_i again positive integers, is a basic question in arithmetic. Denote by $P_{\underline{h}}(n)$ this number.

Example 2.3 $m = 2, n = 1, X = \{2, 3\}$. In how many ways can you write a number b as: $b = 2x + 3y, \quad x, y \in \mathbb{N}$? The answer depends on the class of n modulo 6

$$\frac{n}{6} + 1, \frac{n}{6} - \frac{1}{6}, \frac{n}{6} + \frac{2}{3}, \frac{n}{6} + \frac{1}{2}, \frac{n}{6} + \frac{1}{3}, \frac{n}{6} + \frac{1}{6}$$

A function on a lattice Λ which is a polynomial on each coset of some sublattice M of finite index is called a *quasi-polynomial*.

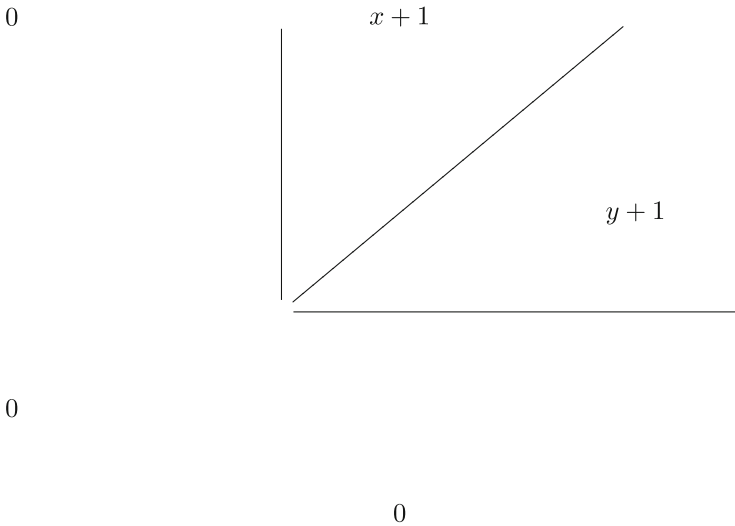


Fig. 3 The partition function for A_2

For the root system A_2 we may take $X = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$. The corresponding partition function P_X is piecewise polynomial with top degree coinciding with T_X (Fig. 3).

Example 2.4 In Lie theory the *Kostant partition function* counts in how many ways can you decompose a weight as a sum of positive roots.

This is used in many computations.

2.4.1. The case of integers. The number $P_{\underline{h}}(n)$ of ways in which n is written as combination of m numbers h_i is already studied by Euler and called by Sylvester a *denumerant*. The first results on denumerants are due to Cayley and Sylvester, who proved that such a denumerant is a polynomial in n of degree $m - 1$ plus a periodic polynomial of lower degree called an *undulant*. A different approach is also developed by Bell in (1943). One starts with

Lemma 2.5 $P_{\underline{h}}(n)$ is the coefficient of x^n in the power series expansion of:

$$S_{\underline{h}}(x) := \prod_{i=1}^m \frac{1}{1 - x^{h_i}} = \sum_{n=0}^{\infty} P_{\underline{h}}(n)x^n.$$

In order to compute for a given $n \geq 0$, the coefficient $P_{\underline{h}}(n)$ of x^n in $S_{\underline{h}}(x)$ we can use two essentially equivalent strategies:

1. Develop $S_{\underline{h}}(x)$ in partial fractions.
2. Compute the residue

$$\frac{1}{2\pi i} \oint \frac{x^{-n-1}}{\prod_i 1 - x^{h_i}} dx$$

around 0.

In both cases first we must expand in a suitable way the function $S_{\underline{h}}(x)$. Given k let us denote by $\zeta_k := e^{\frac{2\pi i}{k}}$, a primitive k th root of 1; we can write

$$S_{\underline{h}}(x) = \prod_{i=1}^m \prod_{j=0}^{h_i-1} \frac{1}{1 - \zeta_{h_i}^j x}. \tag{2}$$

Let μ be the least common multiple of the numbers h_i , and write $\mu = h_i k_i$. If $\zeta = e^{2\pi i/\mu}$, we have $\zeta_{h_i} = \zeta^{k_i}$ therefore we have

Lemma 2.6

$$S_{\underline{h}}(x) = \prod_i \frac{1}{1 - x^{h_i}} = \prod_{i=1}^m \prod_{j=0}^{h_i-1} \frac{1}{1 - \zeta^{k_i j} x} = \prod_{\ell=0}^{\mu-1} \frac{1}{(1 - \zeta^\ell x)^{b_\ell}}$$

where the integer b_ℓ is the number of k_i which are divisors of ℓ .

In particular the function $x^{-n-1} S_{\underline{h}}(x)$, $n \geq 0$ has poles at 0 and at the μ th roots of 1 (but not at ∞).

The classical method starts from the fact that there exist numbers c_i for which:

$$\prod_{i=0}^{\mu-1} \frac{1}{(1 - \zeta^i x)^{b_i}} = \sum_{i=0}^{\mu-1} \sum_{k=1}^{b_i} \frac{c_{i,k}}{(1 - \zeta^i x)^k}. \tag{3}$$

We then use the simple formula:

$$\frac{1}{(1 - t)^k} = \sum_{h=0}^{\infty} \binom{k-1+h}{h} t^h \tag{4}$$

to get:

$$\prod_{i=0}^{\mu-1} \frac{1}{(1 - \zeta^i x)^{b_i}} = \sum_{i=0}^{\mu-1} \sum_{k=1}^{b_i} c_{i,k} \left[\sum_{h=0}^{\infty} \binom{k-1+h}{h} (\zeta^i x)^h \right]. \tag{5}$$

We have thus obtained a formula for the coefficient

$$P_{\underline{h}}(n) = \sum_{i=0}^{\mu-1} \zeta^{in} \left[\sum_{k=1}^{b_i} c_{i,k} \binom{k-1+n}{n} \right].$$

Let us remark now that

$$\binom{k-1+n}{n} = \frac{(n+1)(n+2) \cdots (n+k-1)}{(k-1)!}$$

is a polynomial of degree $k - 1$ in n , while the numbers ζ^{in} depend only from the coset of n modulo μ . Given an $0 \leq a < \mu$ and restricting us to the numbers $n = \mu k + a$, we have:

Theorem 4 *The function $P_{\underline{h}}(\mu k + a)$ is a polynomial in the variable k of degree $\leq \max(b_i)$, that can be computed.*

The fact that $P_{\underline{h}}(n)$ is a polynomial on every coset means that it is a *quasi-polynomial* or a *periodic polynomial*.

Second method: computation of residues

Here the strategy is the following: shift the computation of the residue to the remaining poles, taking advantage of the fact that the sum of residues at all the poles of a rational function is 0.

From the theory of residues we have:

$$\frac{1}{2\pi i} \oint \prod_i \frac{x^{-n-1}}{1-x^{h_i}} dx = - \sum_{j=1}^{\mu-1} \frac{1}{2\pi i} \oint_{C_j} \prod_{t=1}^{\mu} \frac{x^{-n-1}}{(1-\zeta^t x)^{b_t}} dx$$

where C_j is a small circle around ζ^{-j} . In order to compute the term

$$\frac{1}{2\pi i} \oint_{C_j} \prod_{t=1}^{\mu} \frac{x^{-n-1}}{(1-\zeta^t x)^{b_t}} dx \tag{6}$$

we perform the change of coordinates $x = w + \zeta^{-j}$ obtaining:

$$\frac{1}{2\pi i} \oint_{C_j-\zeta^{-j}} \prod_{t=1}^{\mu} \frac{(w + \zeta^{-j})^{-n-1}}{(1-\zeta^{t-j} - \zeta^t w)^{b_t}} dw.$$

Finally we have that (6) equals:

$$\begin{aligned} & \frac{(-1)^{b_j}}{2\pi i} \oint_{C_j-\zeta^{-j}} \zeta^{j(n+1-b_j)} \left(\sum_{k=0}^{\infty} (-1)^k \binom{n+k}{k} \zeta^{jk} w^k \right) \left(\sum_{h=0}^{\infty} a_{j,h} w^h \right) w^{-b_j} dw \\ & = (-1)^{b_j} \zeta^{j(n+1-b_j)} \sum_{k+h=b_j-1} (-1)^k \zeta^{jk} \binom{n+k}{k} a_{j,h}. \end{aligned}$$

Summing over j we obtain an explicit formula for $P_{\underline{h}}(n)$, again as a *quasi-polynomial*.

Remark that, in order to develop these formulae it suffices to compute a finite number of coefficients $a_{j,h}$.

In these formulae roots of unity appear while the final partition functions are clearly integer valued. An algorithmic problem remains. We need to know how to manipulate these expressions involving roots on 1.

This is an elementary but computationally very complex problem on cyclotomic polynomials. Set $\zeta_k := e^{\frac{2\pi i}{k}}$ a primitive root of 1. Define:

$$\psi(k, x) := \text{tr}_{\mathbb{Q}}^{\mathbb{Q}(\zeta_k)}(\zeta_k^x), \quad x \in \mathbb{Z}.$$

This is a periodic integral valued function, computable through the Möbius function and the Euler ϕ function. One can use these functions in order to find explicit formulas for the partition function. For instance:

Example 2.7 ($X = 2, 3, 4, 4, 6, 9, 10, 40$).

- The formula is over all the roots of 1 of order a divisor of an element in X .
- The 10 Galois orbits of primitive roots correspond to the divisors 2, 3, 4, 5, 6, 8, 9, 10, 20, 40.
- The quasi-polynomial associated to the partition function, decomposes according to the divisors.

This has been computed with Mathematica and it is:

$$\begin{aligned} & \frac{1262093963}{2985984000} + \frac{252100403x}{2508226560} + \frac{81263x^2}{2457600} + \frac{3303691x^3}{995328000} + \frac{27443x^4}{199065600} + \frac{1639x^5}{597196800} + \frac{13x^6}{497664000} + \\ & \frac{x^7}{10450944000} + \frac{1173757\psi(2,x)}{4096000} + \frac{483129x\psi(2,x)}{4096000} + \frac{26273x^2\psi(2,x)}{2457600} + \frac{1621x^3\psi(2,x)}{4423680} + \frac{13x^4\psi(2,x)}{2457600} + \\ & \frac{x^5\psi(2,x)}{36864000} + \frac{1009\psi(3,x)}{5832} - \frac{11x\psi(3,x)}{4374} - \frac{x^2\psi(3,x)}{8748} - \frac{149\psi(3,1+x)}{972} - \frac{13x\psi(3,1+x)}{729} - \\ & \frac{x^2\psi(3,1+x)}{4374} + \frac{81\psi(4,x)}{1280} + \frac{39x\psi(4,x)}{10240} + \frac{x^2\psi(4,x)}{20480} + \frac{117\psi(4,1+x)}{10240} + \frac{3x\psi(4,1+x)}{10240} + \frac{89\psi(5,x)}{10000} + \\ & \frac{x\psi(5,x)}{5000} + \frac{123\psi(5,1+x)}{10000} + \frac{3x\psi(5,1+x)}{10000} + \frac{111\psi(5,2+x)}{10000} + \frac{3x\psi(5,2+x)}{10000} + \frac{67\psi(5,3+x)}{10000} + \\ & \frac{x\psi(5,3+x)}{5000} - \frac{\psi(6,x)}{648} + \frac{\psi(6,1+x)}{324} - \frac{\psi(8,1+x)}{640} - \frac{2\psi(9,x)}{81} + \frac{\psi(9,2+x)}{81} - \frac{\psi(9,3+x)}{81} - \frac{\psi(9,4+x)}{81} + \\ & \frac{13\psi(10,x)}{10000} + \frac{169\psi(10,1+x)}{10000} + \frac{x\psi(10,1+x)}{2000} - \frac{221\psi(10,2+x)}{10000} - \frac{x\psi(10,2+x)}{2000} + \frac{13\psi(10,3+x)}{10000} + \\ & \frac{\psi(20,1+x)}{200} - \frac{\psi(20,2+x)}{200} + \frac{\psi(20,3+x)}{400} + \frac{\psi(20,4+x)}{400} + \frac{\psi(20,5+x)}{400} - \frac{\psi(20,6+x)}{200} + \frac{\psi(20,7+x)}{200} - \\ & \frac{3\psi(40,3+x)}{80} - \frac{\psi(40,4+x)}{80} - \frac{\psi(40,5+x)}{80} - \frac{\psi(40,6+x)}{20} - \frac{\psi(40,7+x)}{40} - \frac{\psi(40,9+x)}{20} - \frac{\psi(40,11+x)}{40} - \\ & \frac{\psi(40,12+x)}{20} - \frac{\psi(40,13+x)}{80} - \frac{\psi(40,14+x)}{80} - \frac{3\psi(40,15+x)}{80} \end{aligned}$$

2.8 Main objects associated to X

We now go back to the general case of a list $X = \{a_1, \dots, a_m\}$ of vectors. We think of the elements a_i as linear equations defining hyperplanes in the dual space $U = V^*$. This is a central hyperplane arrangement.

One way to study this arrangement is to study the algebra of rational functions which have poles only on this arrangement. That is we consider the algebra

$$R_X := S[V] \left[\prod_{i=1}^m a_i^{-1} \right]$$

of polynomials $S[V]$ localized at $\prod_{i=1}^m a_i$.

The algebra R_X consists of those rational functions which have at the denominator a product of powers of the linear forms a_i . It is clearly a module under the Weyl algebra of differential operators with polynomial coefficients and we analyze in depth the module structure.

When the coordinates of the elements $a_i = (a_{i,1}, \dots, a_{i,s})$ are integers we take instead a multiplicative point of view and think of a_i as a character $\prod_{j=1}^s x_j^{a_{i,j}}$.

This defines a subgroup of the torus $(\mathbb{C}^*)^s$ the points in which the character is 1 or where $1 - \prod_{j=1}^s x_j^{a_{i,j}}$ vanishes.

This is call a *central toric arrangement* (by the exponential map this corresponds in fact to a *periodic hyperplane arrangement*).

One way to study this arrangement is to study the algebra of Laurent polynomials $S[x_1^{\pm 1}, \dots, x_s^{\pm 1}]$ localized at $\prod_{i=1}^m (1 - \prod_{j=1}^s x_j^{a_{i,j}})$ that is the algebra

$$S[x_1^{\pm 1}, \dots, x_s^{\pm 1}] \left[\prod_{i=1}^m \left(1 - \prod_{j=1}^s x_j^{a_{i,j}} \right)^{-1} \right].$$

We study also this as a module over the algebra of differential operators with coefficients Laurent polynomials.

2.9 The theorems

There are several general formulas to compute the previous functions which are obtained by a mixture of techniques.

A main geometric notion that plays a role is that of *big cells*. Assume that X spans V .

- The *singular points* $C^{sing}(X)$ are the points in the cone $C(X)$ lying in some cone $C(Y)$ for any sublist Y of X which does NOT span the ambient space.
- The other points are called *regular*.
- A *big cell* is a connected component of the set of regular points.

In other words, take *all* the hyperplanes H spanned by sublists of X and then the cones $C(H \cap X)$. The union of all these cones forms the set of singular vectors.

It is easy to see that the big cells are convex polyhedra.

Take all bases \underline{b} extracted from X , for each basis consider the cone $C(\underline{b})$ generated by \underline{b} . Its boundary is made of singular points.

A standard fact of polyhedra is that $C(X) = \cup_{\underline{b}} C(\underline{b})$ so if a point $p \in C(X)$ is regular it lies either in the interior $\overset{\circ}{C}(\underline{b})$ of outside each $C(\underline{b})$. It follows that the big cell c in which p lies is the intersection of all the $\overset{\circ}{C}(\underline{b})$ containing it: $c = \bigcap_{\underline{b} | p \in \overset{\circ}{C}(\underline{b})} \overset{\circ}{C}(\underline{b})$.

Example 2.10 The positive roots of type A_3 are $\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3$.

We want to decompose the cone $C(X)$ into *big cells* and see its singular and regular points. We do this on a transversal section (Fig. 4).

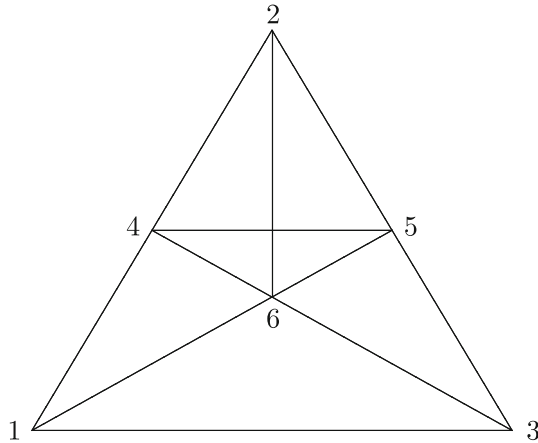


Fig. 4 The 7 big cells for type A_3 in cross section

2.10.1. *Towards the Jeffrey–Kirwan formula.* Assume X spans \mathbb{R}^n . Let $m(X)$ denote the minimum number of columns that one can remove from X so that the remaining columns do not span \mathbb{R}^n .

- (i) $T_X(x)$ has support on the cone $C(X)$.
- (ii) $T_X(x)$ is of class $m(X) - 2$.
- (iii) $T_X(x)$ coincides with a homogeneous polynomial of degree $m - n$ on each big cell.

In order to compute $T_X(x)$ we need to

- (i) Determine the decomposition of $C(X)$ into cells.
- (ii) Compute on each big cell the homogeneous polynomial of degree $m - n$ coinciding with $T_X(x)$.

One can find explicit polynomials $p_{\underline{b},X}(x)$, indexed by a combinatorial object called *unbroken bases* (see Sect. 3.1) and characterized by certain explicit differential equations so that, given a point x in the closure of a big cell c we have the (cf. Sect. 9.6.1)

$$\text{Jeffrey–Kirwan residue formula } T_X(x) = \sum_{\underline{b} \mid c \subset C(\underline{b})} |\det(\underline{b})|^{-1} p_{\underline{b},X}(-x).$$

From T_X one computes B_X . For a given subset S of X define $a_S := \sum_{a \in S} a$; the basic formula is:

$$B_X(x) = \sum_{S \subset X} (-1)^{|S|} T_X(x - a_S).$$

So T_X is the fundamental object.

Notice that the local pieces of B_X are no more homogeneous polynomials.

3 Combinatorics

3.1 Unbroken bases

We now explain some ideas relating unbroken bases with cells.

From the theory of matroids to cells.

Let $\underline{c} := a_{i_1}, \dots, a_{i_k} \in X$, $i_1 < i_2 \cdots < i_k$, be a sublist of linearly independent elements.

Definition 3.2 We say that a_i *breaks* \underline{c} if there is an index $1 \leq e \leq k$ such that:

- $i \leq i_e$.
- a_i is linearly dependent on a_{i_e}, \dots, a_{i_k} .

Example 3.3 Take as X the list of positive roots for type A_3 .

$$X = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}.$$

We have 16 bases 10 broken and 6 unbroken, all contain necessarily α_1 :

$$\begin{aligned} &\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3. \\ &\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3. \\ &\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3. \\ &\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3. \\ &\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3. \\ &\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3. \end{aligned}$$

The overlapping theorem states that, by overlapping the cones generated by the unbroken bases one obtains the entire decomposition into big cells!!

PROBLEM: Describe the previous pictures for root systems. For type A_n the unbroken bases are known and can be indexed by certain binary graphs or by permutations of n elements. The decomposition into cells is unknown.

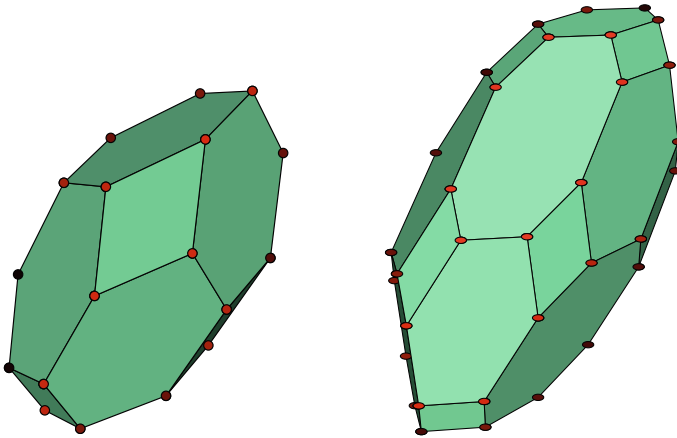
3.3.1. The zonotope. The box spline $B_X(x)$ is supported in the compact polytope: called the *zonotope*:

$$B(X) := \left\{ \sum_{i=1}^m t_i a_i \right\}, 0 \leq t_i \leq 1, \forall i.$$

When X is the set of positive roots of a root system and $\rho := 1/2 \sum_{a \in X} a$ we have:

Theorem 5 $B(X) - \rho$ is the convex envelop of the orbit of ρ under the action of the Weyl group.

The zonotopes associated to the root system A_3B_3



We now want to describe the faces of $B(X)$. Consider a sublist A of X spanning a hyperplane H_0 and such that $A = H_0 \cap X$.

Take a linear equation ϕ_{H_0} for H_0 and set

$$B = \{a \in X \mid \langle \phi_{H_0} \mid a \rangle > 0\}, \quad C = \{a \in X \mid \langle \phi_{H_0} \mid a \rangle < 0\}.$$

Let A, B be two disjoint sublists of X such that A does not span V and $B(A)$ its associated zonotope. Let

$$B(X)_{A,B} := \sum_{a_i \in A} t_i a_i + \sum_{b \in B} b, \quad 0 \leq t_i \leq 1 = B(A) + \lambda_B$$

with $\lambda_B := \sum_{b \in B} b$.

Proposition 3.4 • $B(X)_{A,B}$ and $B(X)_{A,C}$ are exactly the two codimension one faces in $B(X)$ parallel to H_0 .

- All codimension 1 faces occur in this way.

Let \mathcal{H}_X be the real hyperplane arrangement defined, in dual space, by the vectors of X thought of as linear equations.

Theorem 6 • There is an order reversing 1–1 correspondence between faces of $B(X)$ and faces of \mathcal{H}_X .

- To a face G of the hyperplane arrangement we associate the set of points in $B(X)$ where any element $f \in G$ takes its maximum value.
- This is the face $B(X)_{A,B}$ where $A := \{x \in X \mid \langle x \mid f \rangle = 0\}$ and $B := \{x \in X \mid \langle x \mid f \rangle > 0\}$.

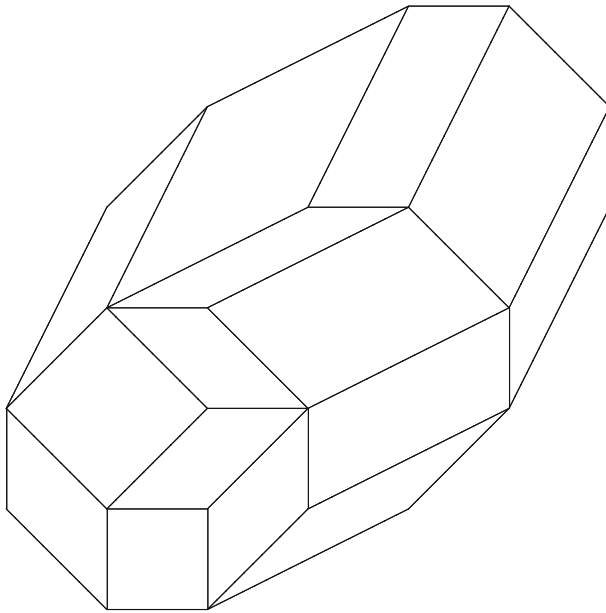
The zonotope $B(X)$ has a nice combinatorial structure, proved by Shephard, it can be paved by a set of parallelepipeds indexed by all the bases which one can extract from A !

Example 3.5 In the next example

$$X = \begin{vmatrix} 1 & 0 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{vmatrix}$$

we have 15 bases and 15 parallelograms.

$$X = \begin{vmatrix} 1 & 0 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{vmatrix}$$



This can be constructed stepwise adding a vector at a time.

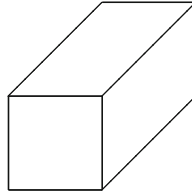
$$= \begin{vmatrix} 1 & 0 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{vmatrix}$$

START WITH

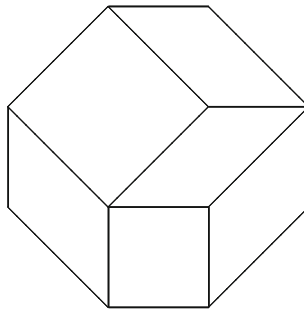
$$X = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$



Then $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$



$$\begin{vmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{vmatrix}$$



and so on.

4 Splines

By Fubini's Theorem the volume $V_X(x)$, of the variable polytope $\Pi_X(x)$, is $\sqrt{\det XX^t} T_X(x)$. The function $T_X(x)$ called *multivariate spline* is characterized by the formula:

$$\int_{\mathbb{R}^n} f(x) T_X(x) dx = \int_{\mathbb{R}_+^m} f\left(\sum_{i=1}^m t_i a_i\right) dt,$$

where $f(x)$ is any continuous function with compact support.

In general $T_X(x)$ is a tempered distribution, supported on the cone $C(X)$, only when X has maximal rank $T_X(x)$ is a function.

While the function $T_X(x)$ is the basic object, the more interesting object for numerical analysis is the *box spline*, that is the function $B_X(x)$ characterized by the formula:

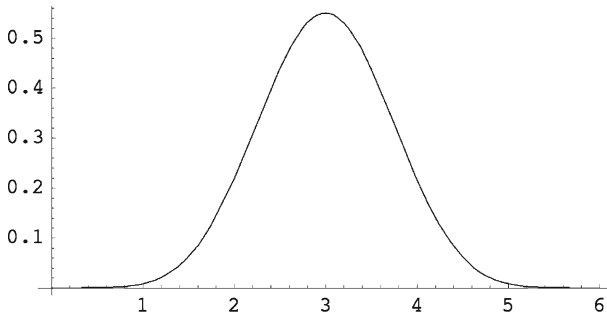


Fig. 5 The function $b_5(x)$

$$\int_{\mathbb{R}^n} f(x) B_X(x) dx = \int_{[0,1]^m} f\left(\sum_{i=1}^m t_i a_i\right) dt,$$

where $f(x)$ is any continuous function.

Example 4.1 The b_m -spline of Schoenberg $X = \{1, 1, \dots, 1\}$, $m+1$ -times is of class C^{m-1} , it is supported in the interval $[0, m+1]$, and given by the formula:

$$b_m(x) := \sum_{i=0}^k (-1)^i \binom{m+1}{i} \frac{(x-i)^m}{m!}, \quad \forall x \in [k, k+1].$$

4.1.1. Properties of box-splines.

$$\int_{\mathbb{R}^n} B_X(x) dx = 1, \quad B_{[X,v]}(x) = \int_0^1 B_X(x - tv) dt$$

In the case of integral vectors, we have that the translates $B_X(x - \lambda)$, as λ runs over the integral vectors form a partition of 1 (Fig. 5).

4.2 Approximation theory

The box spline, when the a_i are integral vectors, can be effectively used in the *finite element method* to approximate functions.

We define the *Cardinal spline space* as in [31].

$$\mathcal{S}_X := \left\{ \sum_{i \in \mathbb{Z}^s} B_X(x - i) a_i \right\}$$

The function a_i on \mathbb{Z}^s is called a *mesh function*.

Define $D(X)$ to be the space of polynomials contained in \mathcal{S}_X .

For a function f we have the *semi-discrete convolution*

$$f \mapsto B_X *' f := \sum_{i \in \mathbb{Z}^s} B_X(x - i) f(i)$$

The theory of Dahmen–Micchelli consists of several Theorems. The basic results are:

- Semi-discrete convolution maps $D(X)$ into $D(X)$.
- The dimension of $D(X)$ equals the number of bases which one can extract from X .
- $D(X)$ contains all polynomials of degree $< m(X)$ ($m(X)$ is the minimum length of a cocircuit).
- $D(X)$ is the space of solutions of the differential equations $D_Y f = 0$ as
 - (i) Y runs over the cocircuits of X .
 - (ii) $D_Y := \prod_{a \in Y} D_a$
 - (iii) D_a is the directional derivative relative to a .

Example 4.3 The cardinal spline space associated to $b_m(x)$ is the space of all functions of class C^{m-1} which, on each interval $[n, n + 1]$, $n \in \mathbb{Z}$ coincide with a polynomial of degree $\leq m$.

5 Algebra

How to compute T_X ? or the partition function P_X ? We use the Fourier–Laplace transform $Lf(u) := \int_V e^{-\langle u | v \rangle} f(v) dv$ which will change the analytic problem to one in algebra.

In coordinates $u = (y_1, \dots, y_s)$, $v = (x_1, \dots, x_s)$ we have

$$Lf(y_1, \dots, y_s) := \int_{\mathbb{R}^s} e^{-\sum_{i=1}^s y_i x_i} f(x_1, \dots, x_s) dx_1 \dots dx_s.$$

5.0.1. Basic properties. Given $p \in U, w \in V$, write p, D_w for the linear function $\langle p | v \rangle$ and the directional derivative on V . We then have:

$$L(D_w f)(u) = wLf(u), \quad L(pf)(u) = -D_p Lf(u), \quad L(e^p f)(u) = Lf(u - p), \\ L(f(v + w))(u) = e^w Lf(u).$$

An easy computation gives the Laplace transforms:

$$LB_X = \prod_{a \in X} \frac{1 - e^{-a}}{a}, \quad LT_X = \prod_{a \in X} \frac{1}{a}.$$

We need to rewrite $LT_X = \prod_{a \in X} a^{-1}$, for this we need to develop a theory of partial fractions in several variables, in this case for the algebra

$$S[V] \left[\prod_{a \in X} a^{-1} \right].$$

We do this using non commutative algebra.

Set $W(V)$, $W(U)$ be the two Weyl algebras of differential operators with polynomial coefficients on V and U . Both algebras are generated by $V \oplus U$ with the same relations so there is an algebraic Fourier isomorphism between these two algebras, so any $W(V)$ module M becomes a $W(U)$ module, denoted by \hat{M} .

5.0.2. D -modules in Fourier duality. The Laplace transform gives rise, using the previous construction, to an isomorphism of two modules under the two algebras $W(V)$, $W(U)$.

1. The D -module $\mathcal{D}_X := W(V)T_X$ generated, in the space of tempered distributions, by T_X under the action of the algebra $W(V)$ of differential operators on V with polynomial coefficients.
2. The algebra $R_X := S[V][\prod_{a \in X} a^{-1}] = W(U)d_X^{-1}$ obtained from the polynomials on U by inverting the element $d_X := \prod_{a \in X} a$. The algebra R_X is the coordinate ring of the open set \mathcal{A}_X complement of the union of the hyperplanes of U of equations $a = 0$, $a \in X$. It is a cyclic module under $W(U)$ generated by d_X^{-1} .

It is well known that, once we invert an element in a polynomial algebra, we get a holonomic module over the algebra of differential operators (cf. [12]).

In particular R_X is holonomic, cyclic and it has a finite composition series. In order to understand this composition series we observe that for each subspace W of U we have an irreducible module N_W (over $W(U)$) generated by the δ function of $W(f \mapsto \int_W f(w)dw)$.

To be explicit, take coordinates $x_1, \dots, x_n \in V$ so that $W = \{x_1 = x_2 = \dots = x_k = 0\}$. N_W is generated by an element δ_W satisfying: $x_i \delta_W = 0$, $i \leq k$, $\frac{\partial}{\partial x_i} \delta$, $i > k$. N_W is free of rank 1 generated by δ_W over: $\mathbb{C}[x_1, x_2, \dots, x_k, \frac{\partial}{\partial x_{k+1}}, \dots, \frac{\partial}{\partial x_n}]$.

In particular the composition factors of R_X are of the form N_W as W runs over the subspaces of the hyperplane arrangement, in U , given by the equations $a_i = 0$, $a_i \in X$. In order to explain this we exhibit an explicit filtration.

5.0.3. The filtration of R_X by polar order

Definition 5.1 We denote by $R_{X,k}$ the span of all the fractions $f \prod_{a \in X} a^{-h_a}$, $h_a \geq 0$ for which the set of vectors a , with $h_a > 0$, spans a space of dimension $\leq k$.

- R_X is filtered by the $W(U)$ -submodules $R_{X,k}$.
- We have $R_{X,s} = R_X$.
- For all k we have that $R_{X,k}/R_{X,k-1}$ is semisimple.
- The isotypic components of $R_{X,k}/R_{X,k-1}$ are of type N_W as W runs over the subspaces of the arrangement of codimension k .

- The space $R_{X,s}/R_{X,s-1}$ is a free module over

$$S[U] = \mathbb{C} \left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s} \right].$$

In fact a more precise statement holds. Denote by R_X^s the linear span of all the fractions $\prod_{i=1}^m a_i^{-h_i}$, $h_i \geq 0$ so that the a_i with $h_i > 0$ span the space.

R_X^s is a complement of $R_{X,s-1}$ in $R_{X,s}$ and it is a free module over

$$S[U] = \mathbb{C} \left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s} \right].$$

It is important to choose a basis.

Theorem 7 A basis for $R_X^s \equiv R_{X,s}/R_{X,s-1}$ over $S[U]$ is given by the elements $\prod_{a \in \underline{b}} a^{-1}$ as \underline{b} runs over the set of unbroken bases.

Denote by \mathcal{NB} the unbroken bases extracted from X .

We have in particular an expansion of $d_X^{-1} = \prod_{a \in X} a^{-1}$

$$d_X^{-1} = \prod_{a \in X} a^{-1} = \sum_{\underline{b} \in \mathcal{NB}} p_{\underline{b}} \prod_{a \in \underline{b}} a^{-1}, \quad p_{\underline{b}} \in S[U] = \mathbb{C} \left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s} \right].$$

Example 5.2 A_2 also called *Courant element* in the Theory of splines.

$$X = [x + y, x, y] = [x, y] \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$\frac{1}{(x + y)xy} = \frac{1}{x(x + y)^2} + \frac{1}{y(x + y)^2} = -\frac{\partial}{\partial y} \left(\frac{1}{x(x + y)} \right) - \frac{\partial}{\partial x} \left(\frac{1}{y(x + y)} \right)$$

Example 5.3 B_2 also called *ZP element* in the Theory of splines.

$$X = [x + y, x, y, -x + y] = [x, y] \begin{vmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{vmatrix}$$

$$\begin{aligned} \frac{1}{(x + y)xy(-x + y)} &= \frac{1}{(x + y)^3x} + \frac{4}{(x + y)^3(-x + y)} - \frac{1}{(x + y)^3y} \\ &= 1/2 \left[\frac{\partial^2}{\partial^2 y} \left(\frac{1}{(x + y)x} \right) + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 \left(\frac{1}{(x + y)(-x + y)} \right) - \frac{\partial^2}{\partial^2 x} \left(\frac{1}{(x + y)y} \right) \right] \end{aligned}$$

We now need the basic inversion. Let $X = \{a_1, \dots, a_n\}$ be a basis, $d := |\det(a_1, \dots, a_s)|$ and $\chi_{C(X)}$ the characteristic function of the positive quadrant $C(X)$ generated by X .

$$L(d^{-1}\chi_{C(X)}) = \prod_{i=1}^n a_i^{-1}$$

We want to invert

$$d_X^{-1} = \sum_{\underline{b} \in \mathcal{NB}} p_{\underline{b}, X} \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s} \right) \prod_{a \in \underline{b}} a^{-1}.$$

From the basic example and the properties we get

$$L^{-1}d_X^{-1} = \sum_{\underline{b} \in \mathcal{NB}} p_{\underline{b}, X}(-x_1, \dots, -x_s) d_{\underline{b}}^{-1} \chi_{C(\underline{b})}.$$

Inverting

$$1/2 \left[\frac{\partial^2}{\partial^2 y} \left(\frac{1}{x(x+y)} \right) + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 \left(\frac{1}{(x+y)(-x+y)} \right) - \frac{\partial^2}{\partial^2 x} \left(\frac{1}{y(x+y)} \right) \right]$$

we get

$$1/2 \left[y^2 \chi_{C((1,0),(1,1))} + \frac{(x+y)^2}{2} \chi_{C((1,1),(-1,1))} - x^2 \chi_{C((0,1),(1,1))} \right].$$

The same analysis for the ZP element leads to Fig. 1.

5.4 The theory of Dahmen–Micchelli

Given a vector a denote by D_a the directional derivative, a first order operator. For a list Y of vectors set $D_Y := \prod_{a \in Y} D_a$, a differential operator of order $|Y|$ with constant coefficients.

We now fix, as usual, a list X of vectors spanning V .

For a given unbroken basis \underline{b} , consider the element $D_{\underline{b}} := \prod_{a \notin \underline{b}} D_a$. The polynomials $p_{\underline{b}, X}$ are characterized by the differential equations

$$D_Y p = 0, \quad \forall Y, \quad \text{a cocircuit in } X, \quad D_{\underline{b}} p_{\underline{c}, X}(x_1, \dots, x_s) = \begin{cases} 1 & \text{if } \underline{b} = \underline{c} \\ 0 & \text{if } \underline{b} \neq \underline{c} \end{cases}.$$

5.4.1. A remarkable space of polynomials.

$$D(X) := \{p \mid D_Y p = 0, \forall Y, \text{ a cocircuit in } X\}$$

By a theorem of Dhamen Micchelli $\dim D(X)$ equals the total number of bases extracted from X , the polynomials $p_{\underline{b}, X}$ form a basis for the part of top degree $m - n$ of $D(X)$.

The graded dimension of $D(X)$ is given by $H_X(q) = \sum_{\underline{b} \in \mathcal{B}(X)} q^{m-n(\underline{b})}$. The number $n(\underline{b})$ (called external activity in the Theory of Matroids) is the number of elements of X breaking \underline{b} .

For A_3 the graded dimension is: $6q^3 + 6q^2 + 3q + 1$. Remark that for all polynomials in three variables the generating function for the dimension is: $\dots + 10q^3 + 6q^2 + 3q + 1$.

6 Approximation

The aim of this section is to explain some motivations, for the study of box-splines, coming from numerical analysis.

6.1 The Strang–Fix conditions

The interest of the space of polynomials $D(X)$ comes in approximation theory from the problem of studying the approximation of a function $f(x)$, on \mathbb{R}^s , by the *finite element method* (cf. Strang and Fix [33]). Using a box-spline we construct the approximations:

$$B_X *' f := f(x) \mapsto \sum_{\underline{i} \in \mathbb{Z}^s} B_X(x - \underline{i}) f(\underline{i}), \text{ semi-discrete convolution.}$$

Or at order $n \in \mathbb{N}$:

$$f(x) \mapsto \sum_{\underline{i} \in \mathbb{Z}^s} B_X(nx - \underline{i}/n) f(\underline{i}/n).$$

In general, given a function $M(x)$ with compact support (as B_X) we want to find optimal *weights* $c_{\underline{i}}(n)$ in order to approximate f by $f(x) \mapsto \sum_{\underline{i} \in \mathbb{Z}^s} M(nx - \underline{i}/n) c_{\underline{i}}(n)$ and determine a constant $k \in \mathbb{N}$ so that on any given bounded region there is a constant C , independent of n with:

$$\left| f(x) - \sum_{\underline{i} \in \mathbb{Z}^s} M(nx - \underline{i}/n) c_{\underline{i}}(n) \right| \leq Cn^{-k}, \quad \forall n.$$

The maximum k is the *approximation power* of $M(x)$.

Given a spline $M(x)$ on \mathbb{R}^s , with compact support one may define the *cardinal spline space* to be the space \mathcal{S}_M of all (infinite) linear combinations:

$$\mathcal{S}_M := \left\{ \sum_{\underline{i} \in \mathbb{Z}^s} M(x - \underline{i})c_{\underline{i}} \right\}.$$

The approximation power of $M(x)$ is related to two questions:

1. For which polynomials $f(x)$ we have that $\sum_{\underline{i} \in \mathbb{Z}^s} M(x - \underline{i})f(\underline{i})$ is a polynomial?
2. Which polynomials lie in the cardinal spline space?

Theorem 8 *A polynomial $f(x)$ is in the cardinal spline space of B_X if and only if*

$$D_Y f = 0, \forall Y \subset X \mid \text{the span of } X \setminus Y \text{ is not } V$$

In other words $D_Y f = 0, \forall Y \subset X$ which are cocircuits.

The Strang–Fix conditions is a general statement:

The approximation power of a function M is the maximum r such that the space of all polynomials of degree $\leq r$ is contained in the cardinal space \mathcal{S}_M .

For the cardinal spline space in the case $B_X(x)$ with X integral we have:

1. $D(X)$ is characterized as the space of polynomials $f(x)$ which *reproduce*, i.e. map to polynomials under the *semi-discrete convolution*.
2. $D(X)$ is also characterized as the space of polynomials lying in the cardinal spline space.

The power of approximation by discrete convolution is measured by the maximum degree of the space of polynomials which reproduce under discrete convolution.

6.1.1. Superfunctions. Consider the following algorithm applied to a function g :

$$g_h := \sum_{\underline{i} \in \Lambda} F(x/h - \underline{i})g(h\underline{i}), \quad h = 1, 2, \dots$$

There are functions F in the cardinal spline space such that this transformation is the identity on polynomials of degree $< m(X)$, these are the super-functions. For such functions the previous algorithm satisfies the requirements of the Strang–Fix approximation

Theorem 9 *We have, under the explicit algorithm previously constructed that, for any domain G :*

$$\|f_h - f\|_{L^\infty(G)} = O(h^{m(X)}).$$

For every multi-index $\alpha \in \mathbb{N}^s$ with $|\alpha| \leq m(X) - 1$, we have:

$$\|\partial^\alpha f_h - \partial^\alpha f\|_{L^\infty(G)} = \|\partial^\alpha (f_h - f)\|_{L^\infty(G)} = O(h^{m(X)-|\alpha|}).$$

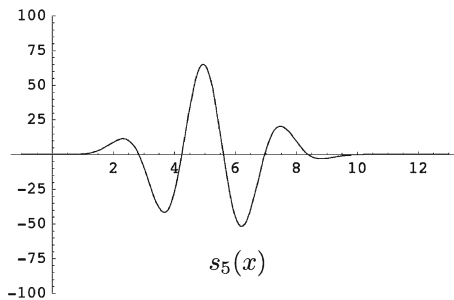
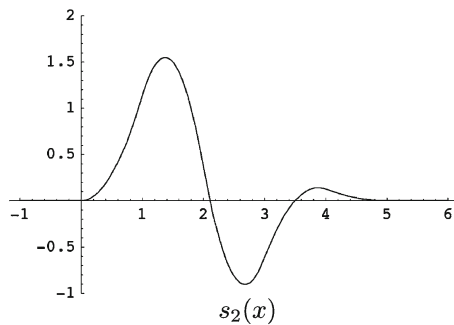
Theorem 10 When $p \in D(X)$ we have that also $B_X *' p \in D(X)$.

This defines a linear isomorphism F of $D(X)$ to itself, given explicitly by the invertible differential operator $\prod_{a \in X} \frac{1 - e^{-D_a}}{D_a}$.

- If R is any difference operator inverting $B_X *' -$ on $D(X)$ then RB_X is a superfunction.
- We can then take as R a truncation of

$$Q := \prod_{a \in X} \left(\sum_{i=0}^{\infty} \frac{\nabla_a^i}{i+1} \right).$$

Let us give as example some cases of the function $s_m(x)$ obtained by this procedure from $b_m(x)$.



Part 2. Arithmetic and combinatorics

7 Arithmetic

7.1 The partition function

The partition function is a quasi polynomial on each big cell, in fact on the larger neighborhood $c - B(X)$ of the big cell c .

We think of the partition function $P_X(b) = \#\{t_1, \dots, t_m \in \mathbb{N} \mid \sum_{i=1}^m t_i a_i = b\}$ as a distribution $\sum_{\lambda \in \Lambda} P_X(\lambda) \delta_\lambda$ with Laplace transform $\sum_{\lambda \in \Lambda} P_X(\lambda) e^{-\lambda} = \prod_{a \in X} \frac{1}{(1 - e^{-a})}$.

There are explicit formulas relating the partition function P_X with the multivariate splines T_A , $A \subset X$. One could derive the Formula of Example 2.7 in this way.

We used hyperplane arrangements in order to study multivariate splines, for partition functions we use *toric arrangements*.

7.2 The toric arrangement

Let T be the torus of character group Λ , X a list of characters $u_X = \prod_{a \in X} (1 - e^a)$ and $S_X = \mathbb{C}[\Lambda][u_X^{-1}]$ is the coordinate ring of the open set $P_X \subset T$ complement of the union of the subgroups of T of equations $e^a = 1$, $a \in X$. If $a \in \mathbb{Z}^s$ is a vector of coordinates (n_1, \dots, n_s) we also write $e^a = \prod_{i=1}^s x_i^{n_i}$.

The toric arrangement is the finite set consisting of all the connected components of the subvarieties obtained by intersecting the subgroups of T of equations $e^a = 1$, $a \in X$.

EXAMPLE $s = 1$, $T = \mathbb{C}^*$, $X = \{5, 3\}$

The arrangement consists of the connected components of the variety $x^5 = 1$ or $x^3 = 1$, i.e. of the five, fifth roots of 1 and the three third roots of 1.

The elements of the toric arrangement are ordered by reverse inclusion, particular importance is given to the *points of the arrangement*, $P(X)$ which are the zero-dimensional, i.e. points, elements of the arrangement.

A very special case is when $P(X)$ reduces to the point 1, this is the *unimodular case*. For root systems the only unimodular case is A_n .

Each point $p \in P(X)$ determines a sublist:

$$X_p := \{a \in X \mid e^a(p) = 1\}.$$

EXAMPLE ZP

$$X = \left| \begin{array}{cccc} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \end{array} \right|$$

The subgroups are

$$xy = 1, \quad x = 1, \quad y = 1, \quad x^{-1}y = 1.$$

We have *two points in $P(X)$*

$$(1, 1), \quad (-1, -1).$$

$$X_{(1,1)} = X, \quad X_{(-1,-1)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}.$$

7.3 The filtration

The formulas we are aiming at are built from basic functions. These are analogues, in the discrete case, of the Laplace transforms of the characteristic functions of the cones generated by bases extracted from X . Given a basis \underline{b} of $V = \Lambda \otimes \mathbb{R}$, contained in Λ , we introduce the set

$$R_{\underline{b}} = \left\{ \lambda \in \Lambda \mid \lambda = \sum_{b \in \underline{b}} p_b b, \text{ with } 0 \leq p_b < 1 \right\}. \tag{7}$$

Notice that this is a set of coset representatives of $\Lambda \cap \langle \underline{b} \rangle / \Lambda_{\underline{b}}$. Given a character $\phi : \Lambda \rightarrow \mathbb{C}^*$ which is 1 on \underline{b} set

$$\xi_{\phi} := \sum_{v \in \Lambda \cap C(\underline{b})} e^{\langle \phi \mid v \rangle} \delta_v.$$

ξ_{ϕ} is a tempered distribution supported on the cone $C(\underline{b})$ and its Laplace transform has an analytic meaning and we get:

Proposition 7.4 *The Laplace transform of ξ_{ϕ} equals*

$$|\det(\underline{b})| \frac{e(\phi)}{\prod_{a \in \underline{b}} (1 - e^{-a})}.$$

$$e(\phi) := \frac{\sum_{\lambda \in R_{\underline{b}}} e^{\langle \phi \mid \lambda \rangle} e^{-\lambda}}{|\det(\underline{b})|} \tag{8}$$

We filter S_X by *polar order* that is $S_{X,k}$ is formed by all fractions which have at the denominator a product $\prod_{a \in X} (1 - e^a)^{h_a}$ so that the a with $h_a > 0$ span a subspace of dimension $\leq k$.

A connected component of the toric arrangement is of the form pP for some subtorus P . For each connected component pP we have an irreducible module *generated by the delta function*.

Concretely, if pP has codimension k we construct a copy $N_{pP, \underline{b}}$ in $S_{X,k}/S_{X,k-1}$ as generated by the class of an element $\boxed{|\det(\underline{b})| \frac{e(\phi)}{\prod_{a \in \underline{b}} (1 - e^{-a})}}$ as \underline{b} is a basis for the span of the sublist X_{pP} of the elements in X which are 1 on pP .

Theorem 11 $S_{X,k}/S_{X,k-1}$, decomposes as direct sum of the modules $N_{pP, \underline{b}}$ as pP runs over all components of the arrangement of codimension k and \underline{b} over the unbroken bases on pP .

For each pP , F_{pP} is the isotypic component of type N_{pP} .

Take the space of polar parts $SP_X := S_X/S_{X,s-1}$, its isotypic components are F_ϕ indexed by points of the arrangement.

$$SP_X := S_X/S_{X,s-1} = \bigoplus_{\phi \in \tilde{P}(X)} F_\phi, \quad F_\phi = \bigoplus_{\underline{b} \in \mathcal{NB}_{X_\phi}} N_{\underline{b}, e^\phi}.$$

Let us consider the element v_X , in SP_X , class of the function $\prod_{a \in X} (1 - e^{-a})^{-1}$. Decompose it uniquely as a sum of elements v_{X_ϕ} in F_ϕ , each one of these elements is uniquely expressed as a sum

$$v_{X_\phi} = \sum_{\underline{b} \in \mathcal{NB}_{X_\phi}} q_{\underline{b}, \phi} \omega_{\underline{b}, \phi}$$

for suitable polynomials $q_{\underline{b}, \phi} \in S[U_{\mathbb{C}}]$. Thus:

$$v_X = \sum_{\phi \in \tilde{P}(X)} \sum_{\underline{b} \in \mathcal{NB}_{X_\phi}} q_{\underline{b}, \phi} \omega_{\underline{b}, \phi}. \tag{9}$$

The main formula that one can effectively use for computing the partition function P_X :

Theorem 12 Let Ω be a big cell, $B(X)$ the box associated to X . For every $x \in \Omega - B(X)$,

$$P_X(x) = \sum_{\phi \in \tilde{P}(X)} e^{(\phi | x)} \sum_{\underline{b} \in \mathcal{NB}_{X_\phi} \mid \Omega \subset C(\underline{b})} |\det(\underline{b})|^{-1} q_{\underline{b}, \phi}(-x). \tag{10}$$

Finally we compute the contributions by residues.

The isotypic components appearing in grade k correspond to the connected components of the toric arrangement of codimension k .

For the top part $S_{X,n}/S_{X,n-1}$ we have a sum over the points of the arrangement $P(X)$.

The isotypic component associated to a point e^ϕ decomposes as direct sum of irreducibles indexed by the unbroken bases in $X_\phi := \{a \in X \mid e^{(a | \phi)} = 1\}$.

7.4.1. *Local structure of P_X .* The previous formula shows in particular, that the partition function is on each cell a quasi polynomial, as for the case of the multivariate spline:

The quasi polynomials appearing in the formula for P_X satisfy special difference equations.

Denote by $\mathcal{C}[\Lambda]$ the abelian group of \mathbb{Z} -valued functions on Λ .

If $a \in \Lambda$, the *difference operator* $\nabla_a(f)(x) := f(x) - f(x - a)$ acts on $\mathcal{C}[\Lambda]$. Given X a list in Λ spanning the ambient space we define

Definition 7.5

$$DM(X) := \left\{ f \in \mathcal{C}[\Lambda] \mid \nabla_Y f = \prod_{a \in Y} \nabla_a f = 0 \right\}$$

as Y runs over all cocircuits of X .

7.5.1. *Second Theorem of Dahmen–Micchelli.* The dimension of $DM(X)$ is the *Volume* $\delta(X)$ of the box $B(X)$:

$$\delta(X) = \sum_{b \in B(X)} |\det(\underline{b})|. \tag{11}$$

This follows from a more precise statement.

Theorem 13 *If r is a regular vector $\delta(r \mid X) := r - B(X) \cap \Lambda$ has cardinality $\delta(X)$. The restriction of $DM(X)$ to \mathbb{Z} valued functions on $\delta(r \mid X)$ is an isomorphism.*

In other words the values on $\delta(r \mid X)$ of a function in $DM(X)$ determine uniquely this function and in particular $DM(X)$ is a free abelian group of rank $\delta(X)$.

Example 7.6 (The binomial coefficients) $X = \underbrace{\{1, 1, \dots, 1\}}_{m+1}$ the zonotope $B(X) = [0, m + 1]$ is the interval.

In this case we only have one equation

$$\nabla_1 f(x) = f(x) - f(x - 1), \quad \nabla_1^{m+1} f = 0.$$

The space of solutions has as integral basis the elements

$$\binom{x+k}{m} = \frac{(x+k)(x+k-1)\dots(x+k-m+1)}{m!}, \quad k = 0, \dots, m.$$

Take a regular point close to 0, $0 < a < 1$ then

$$\delta(a \mid X) = a - [0, m + 1] \cap \mathbb{Z} = [-m - 1 + a, a] \cap \mathbb{Z} = [-m, \dots, 0].$$

$$\binom{i+k}{m} = 0, \quad i \in [-k, \dots, m - 1 - k], \quad \binom{m-k+k}{m} = 1.$$

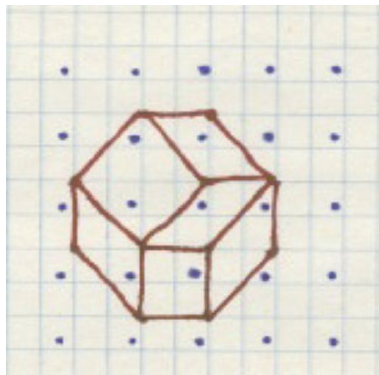
The partition function is $\binom{x+m}{m}$. The matrix of the values of $\binom{x+i}{m}$ for $i \in [0, m]$ at the points $0, -1, \dots, -m$ are given by the matrix, invertible over \mathbb{Z} and establishes the isomorphism of Theorem 13:

$$m = 10, \begin{pmatrix} 92378 & 43758 & 19448 & 8008 & 3003 & 1001 & 286 & 66 & 11 & 1 & 0 \\ 43758 & 19448 & 8008 & 3003 & 1001 & 286 & 66 & 11 & 1 & 0 & 0 \\ 19448 & 8008 & 3003 & 1001 & 286 & 66 & 11 & 1 & 0 & 0 & 0 \\ 8008 & 3003 & 1001 & 286 & 66 & 11 & 1 & 0 & 0 & 0 & 0 \\ 3003 & 1001 & 286 & 66 & 11 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1001 & 286 & 66 & 11 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 286 & 66 & 11 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 66 & 11 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Formula (11) has a strict connection with the paving of the box.

Example 7.7 Let us take

$$X = \begin{vmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \end{vmatrix}$$



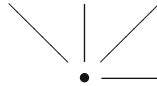
See that $\delta(X) = 1 + 1 + 1 + 1 + 1 + 2 = 7$ is the number of points in which the box $B(X)$, shifted generically a little, intersects the lattice!

7.8 Combinatorics

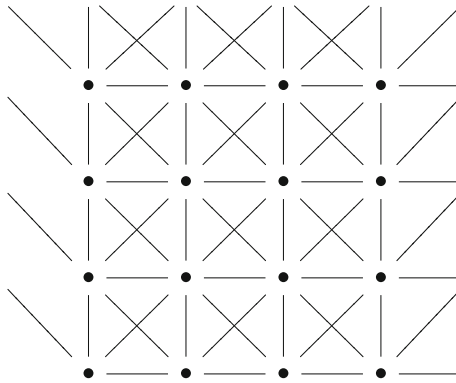
Λ is a lattice in a real vector space V and X a list of vectors in Λ . In coordinates $V = \mathbb{R}^s$, $\Lambda = \mathbb{Z}^s$. Assume that X generates a pointed cone $C(X)$. We have already seen in Sect. 2.9 the notions of singular, regular points and big cells. We now extend these notions as follows:

- (i) The translates $C^{sing}(X) + \Lambda$ give a periodic hyperplane arrangement.
- (ii) The union of the hyperplanes of this periodic arrangement is called *the cut locus* and each connected component of its complement *a chamber*.
- (iii) Each chamber of this arrangement is the interior of a bounded polytope.
- (iv) The function B_X is a polynomial on each chamber.
- (v) The set of chambers is invariant under translation by elements of Λ .

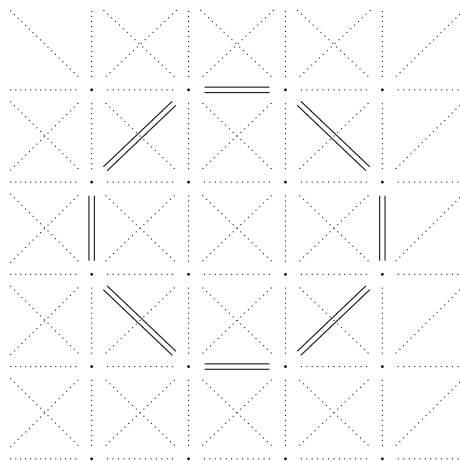
Example 7.9 $X = \begin{vmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \end{vmatrix},$



When we translate we have:



We draw the zonotope and the cut-locus



If X is a basis of the lattice Λ we have that the chambers are open parallelepipeds.

8 Difference theorem

One of the main results of the Theory is the fact that a partition function P_X is a quasi-polynomial on the regions $\Omega - B(X)$, for each big cell Ω .

The fact that P_X is a quasi-polynomial not just on the big cells but in fact in the larger regions $\Omega - B(X)$ may be considered as a discrete analogue of the differentiability properties of the multivariate spline T_X . In this section we discuss some properties of the quasi-polynomials describing P_X .

- P_X on $\Omega - B(X)$ coincides with an element of $DM(X)$.
- The element is uniquely determined by its *initial values*, that is the values it takes on $\delta(u | X)$ where $u \in \Omega$ is close to zero.
- In this case $\delta(u | X) \cap C(X) = \{0\}$ so that, in order to coincide with P_X on $\Omega - B(X)$ we must have that $f(0) = 1$ while $f(a) = 0$ for all non-zero elements of $\delta(u | X)$.
- These are the defining initial conditions.

An idea of a possible proof will be given in Sect. 9.20.

Let us start to explain why the system of difference equations gives a system of recursion which allow to determine the values of a function $f \in DM(X)$ by the values that it takes in a set $\delta(u | X)$. To begin notice that

$$\tau_a \tau_b = \tau_{a+b} \implies \prod_{i=1}^m \nabla_{a_i} = \sum_{S \subset \{1, \dots, m\}} (-1)^{|S|} \tau_{(\sum_{i \in S} a_i)}.$$

Thus the recursive formula for a function $f(x)$ satisfying $\prod_{i=1}^m \nabla_{a_i} f = 0$.

$$f(x) = \sum_{S \subset \{1, \dots, m\}, S \neq \emptyset} (-1)^{|S|+1} f\left(x - \sum_{i \in S} a_i\right). \tag{12}$$

Example 8.1 $X = \{2, 3\}$, $\nabla_2 = 1 - \tau_2$, $\nabla_3 = 1 - \tau_3$, so

$$\nabla_2 \nabla_3 f(x) = f(x) - f(x - 2) - f(x - 3) + f(x - 5)$$

easily implies that a function satisfying $\nabla_2 \nabla_3 f(x) = 0$ is determined by the values that it takes on the interval $[-4, 0]$.

The partition function $P_{2,3}$ for the list 2, 3 is characterized by the fact that: for every number $i > -5$, the function $P_{2,3}$ coincides with the unique function $f(x)$ which satisfies the difference equation $\nabla_2 \nabla_3 f(x) = 0$ and which takes values 0, 0, 0, 0, 1 on $-4, -3, -2, -1, 0$.

A function f satisfying $\nabla_Y f = 0$ for all cocircuits Y extracted from X is determined by the values on a set $\delta(c_0 | X)$ where c_0 is a given chamber.

- First $\Lambda = \cup_{\mathfrak{c}} \delta(\mathfrak{c} \mid X)$ as \mathfrak{c} runs over all chambers.
- Next we want to prove that if f vanishes on a given $\delta(\mathfrak{c} \mid X)$ for a chamber \mathfrak{c} then it also vanishes on $\delta(\mathfrak{g} \mid X)$ for a chamber \mathfrak{g} which is adjacent to \mathfrak{c} .
- Then we go by recursion.

The fact that the values of a function $f \in DM(X)$ are determined by the values it takes on a single set $\delta(\mathfrak{c} \mid X)$ comes from a *wall crossing formula* when we pass from $\delta(\mathfrak{c} \mid X)$ to $\delta(\mathfrak{g} \mid X)$ for a chamber \mathfrak{g} which is adjacent to \mathfrak{c} .

8.2 Arithmetic

8.2.1. Connection between $DM(X)$ and $D(X)$. There is a formal machinery which allows us to interpret, locally around a point, difference equations as restriction to the lattice of differential equations, we call it the logarithm isomorphism. We have this for any module over the periodic Weyl algebra $\mathbb{C}[\frac{\partial}{\partial x_i}, e^{x_i}]$ as soon as for algebraic reasons (nilpotency) we can deduce from the action of e^{x_i} also an action of x_i .

The main point is that the ideal J_X generated by the elements $\prod_{a \in Y} (1 - e^a)$ as Y runs over the cocircuits defines (as a scheme) the set $P(X)$ of points of the arrangement.

Thus

$$\mathbb{C}[\Lambda]/J_X = \bigoplus_{p \in P(X)} \mathbb{C}[\Lambda]/J_X(p).$$

Where $\mathbb{C}[\Lambda]/J_X(p)$ is a local algebra supported at p .

This implies that the dual of $\mathbb{C}[\Lambda]/J_X$ which is the complexified form

$$DM_{\mathbb{C}}(X) = DM(X) \otimes \mathbb{C},$$

decomposes as direct sum of contributions at the various points in $P(X)$.

Choose representatives $\tilde{P}(X)$ for $P(X)$:

$$DM_{\mathbb{C}}(X) = \bigoplus_{\phi \in \tilde{P}(X)} e^{(\phi \mid v)} D_{\phi} \tag{13}$$

where D_{ϕ} is the space of polynomials f such that $e^{(\phi \mid v)} f \in DM_{\mathbb{C}}(X)$.

Proposition 8.3 *The space D_{ϕ} is the differentiable Dahmen–Micchelli space $D(X_p)$, $p = e^{\phi}$.*

This explains why $DM(X)$ is formed by quasi-polynomials.

- if $v \in V$, the difference operator ∇_v and the differential operator ∂_v satisfy $\nabla_v = T_v \partial_v$ with $T_v = \frac{1 - e^{-\partial_v}}{\partial_v}$.
- T_v is invertible on the space of polynomials and commutes with all the ∇_a .
- Hence $\nabla_Y = A D_Y$ where A is invertible on polynomials.

- The equations $\nabla_Y f = 0$ are equivalent to $D_Y f = 0$ for polynomials.
- Thus we obtain that $e^{\langle \phi | v \rangle} f \in DM_{\mathbb{C}}(X)$ if and only if $f \in D(X_p)$.

$$DM_{\mathbb{C}}(X) = \bigoplus_{\phi \in \tilde{P}(X)} e^{\langle \phi | v \rangle} D(X_{e\phi})$$

- In particular the contribution of the point 1 to $DM_{\mathbb{C}}(X)$ is the space of polynomials $D(X)$.
- In the unimodular case $DM_{\mathbb{C}}(X) = D(X)$.
- In general we also have the contribution of the other points, that will be denoted by

$$E(X) := \bigoplus_{p \in P(X), p \neq 1} e^{\langle \phi | v \rangle} D(X_p). \tag{14}$$

8.3.1. *Supports.* We want now to explain the nature of $DM_{\mathbb{C}}(X)$ as Fourier coefficients of distributions on the compact torus T , supported at the points of the arrangement.

For this we consider the space $D(X)$ (or $D(X_p)$) as a space of polynomial differential operators on U and hence also on T with constant coefficients.

The Haar measure on T of total mass 1. This allows us to identify generalized functions on T and distributions on T .

Call $\widehat{DM}(X)$ the space of distributions on T of which $DM_{\mathbb{C}}(X)$ gives the Fourier coefficients.

We deduce

Proposition 8.4 $\widehat{DM}(X)$ is the direct sum of the spaces of distributions $D(X_p)\delta_p$ as $p \in P(X)$.

In particular we see that $\widehat{DM}(X)$ is supported at the finite set $P(X)$.

9 Residues

9.1 Wonderful models

There is an approach to compute the partition function based on residues (as for numbers in Sect. 2.4.1), also in higher dimension.

In order to define multidimensional residues we need *divisors with normal crossings*. That is we consider functions which, in some coordinates x_i , have poles only on the hyperplanes $x_i = 0$, $i = 1, \dots, s$. The residue is the coefficient of $\prod_i x_i^{-1}$.

The construction of these divisors with normal crossings follows [16] and in our case starts with a

Definition 9.2 Given a subset $A \subset X := \{x_1, \dots, x_s\}$ the list $\overline{A} := X \cap \langle A \rangle$ will be called the completion of A . In particular A is called complete if $A = \overline{A}$.

The space of vectors $\phi \in U$ such that $\langle a | \phi \rangle = 0$ for every $a \in A$ will be denoted by A^\perp . Notice that clearly \overline{A} equals to the list of vectors $a \in X$ which vanish on A^\perp .

From this we see that we get a bijection between the complete subsets of X and subspaces of the arrangement defined by X .

A central notion in what follows is given by

Definition 9.3 Given a complete set $A \subset X$, a **decomposition** is a decomposition $A = A_1 \cup A_2$ in non empty sets, such that:

$$\langle A \rangle = \langle A_1 \rangle \oplus \langle A_2 \rangle.$$

Clearly the two sets A_1, A_2 are necessarily complete.

We shall say that: a complete set A is irreducible if it does not have a non trivial decomposition.

Theorem 14 Every set A can be decomposed as $A = A_1 \cup A_2 \cup \dots \cup A_k$ with the A_i irreducible and:

$$\langle A \rangle = \langle A_1 \rangle \oplus \langle A_2 \rangle \oplus \dots \oplus \langle A_k \rangle.$$

This decomposition is unique up to order.

$A = A_1 \cup A_2 \cup \dots \cup A_k$ is called the *decomposition into irreducibles* of A .

Example 9.4 An interesting example is that of the configuration space of s -ples of point in a line (or the root system A_{s-1}). In this case $X = \{z_i - z_j | 1 \leq i < j \leq s\}$.

In this case, irreducible sets are in bijection with subsets of $\{1, \dots, s\}$ with least 2 elements. If S is such a subset the corresponding irreducible is $I_S = \{z_j - z_j | \{i, j\} \subset S\}$.

Given a complete set C , the irreducible decomposition of C corresponds to a family of disjoint subsets S_1, \dots, S_k of $\{1, \dots, s\}$ each with at least 2 elements.

Definition 9.5 A family \mathcal{S} of irreducibles A_i is called *nested* if, given elements $A_{i_1}, \dots, A_{i_h} \in \mathcal{S}$ mutually incomparable we have that $C := A_1 \cup A_2 \cup \dots \cup A_i$ is complete and $C := A_1 \cup A_2 \cup \dots \cup A_i$ is its decomposition into irreducibles.

Consider the hyperplane arrangement \mathcal{H}_X and the open set

$$\mathcal{A}_X = U / (\cup_{H \in \mathcal{H}_X} H)$$

complement of the union of the given hyperplanes.

Let us denote by \mathcal{I} the family of irreducible subsets in X .

We construct a minimal smooth variety Z_X containing \mathcal{A}_X as an open set with complement a normal crossings divisor, plus a proper map $\pi : Z_X \rightarrow U$ extending the identity of \mathcal{A}_X .

For any irreducible subset $A \in \mathcal{I}$ take the vector space V/A^\perp and the projective space $\mathbb{P}(V/A^\perp)$.

Notice that, since $A^\perp \cap \mathcal{A}_X = \emptyset$ we have a natural projection $\pi_A : \mathcal{A}_X \rightarrow \mathbb{P}(V/A^\perp)$. If we denote by $j : \mathcal{A}_X \rightarrow U$ the inclusion we get a map The model

$$i := j \times (\times_{a \in \mathcal{I}} \pi_A) : \mathcal{A}_X \rightarrow U \times (\times_{a \in \mathcal{I}} \mathbb{P}(U/A^\perp))$$

Definition 9.6 The model Z_X is the closure of $i(\mathcal{A}_X)$ in $U \times (\times_{a \in \mathcal{I}} \mathbb{P}(U/A^\perp))$.

There is a very efficient approach to computations by residue at points at infinity in the wonderful compactification of the associated hyperplane arrangement.

Points at infinity correspond to maximal nested sets. Around each such point one can consider a s -dimensional torus and its class in homology

A basis of the homology or of the corresponding residues corresponds to the tori around special points indexed by unbroken bases.

9.6.1. The non linear coordinates. We now apply the previous Theory to the multivariate spline or the partition function associated to a list X . One can find explicit polynomials $p_{\underline{b}, X}(x)$ (given in Formula (16)), indexed by the points at infinity associated to the maximal nested set generated by *unbroken bases* so that, given a point x in the closure of a big cell \mathfrak{c} we have *Jeffrey–Kirwan residue formula* [26].

$$T_X(x) = \sum_{\underline{b} \mid \mathfrak{c} \subset C(\underline{b})} |\det(\underline{b})|^{-1} p_{\underline{b}, X}(-x).$$

There is a parallel theory for the partition function, as a result we can compute a set of polynomials $q_{\underline{b}, \phi}(-x)$ (given in Formula (17)) indexed by pairs, a character ϕ of finite order and a unbroken basis in $X_\phi = \{a \in X \mid \phi(e^a) = 1\}$.

The analogue of the Jeffrey–Kirwan formula is:

Theorem 15 *Given a point x in the closure of a big cell \mathfrak{c} we have a Residue formula for partition function*

$$P_X(x) = \sum_{\phi \in P(X)} e^\phi \sum_{\underline{b} \in \mathcal{NB}_{X_\phi} \mid \mathfrak{c} \subset C(\underline{b})} q_{\underline{b}, \phi}(-x)$$

Finally one can deduce the partition function from some combinatorics and multivariate splines (with parameters):

Theorem 16 *For the points x in the interior of big cells \mathfrak{c} we have*

$$P_X(x) = \sum_{\phi \in P(X)} \widehat{Q}_\phi T_{X_\phi, \phi}(x)$$

$$Q_\phi = \prod_{a \notin X_\phi} \frac{1}{1 - e^{-a}} \prod_{a \in X_\phi} \frac{a - \langle \phi \mid a \rangle}{1 - e^{-a + \langle \phi \mid a \rangle}}$$

9.6.2. Residues. The polynomials building the multivariate spline and the partition functions are *residues*.

Given a MNS \mathcal{S} choose a basis $\underline{b} := b_1, \dots, b_s$ from X so that if X_i is the minimal element of \mathcal{S} containing b_i we have all the X_i distinct.

Construct new coordinates $z_A, A \in \mathcal{S}$ using the monomial expressions:

$$b_A := \prod_{B \in \mathcal{S}, A \subseteq B} z_B. \tag{15}$$

The residue at the point 0 for these coordinates is denoted by $res_{\underline{b}}$.

$$p_{\underline{b}, X}(-y) = det(\underline{b})res_{\underline{b}}\left(\frac{e^{(y|x)}}{\prod_{a \in X} \langle x | a \rangle}\right) \text{ Spline.} \tag{16}$$

$$q_{\underline{b}, X_\phi}(-y) = det(\underline{b})res_{\underline{b}, \phi}\left(\frac{e^{(y|z)}}{\prod_{a \in X} (1 - e^{-a(z) - \langle \phi | a \rangle})}\right) \text{ Partition function.} \tag{17}$$

Theorem 17 (Brion–Vergne [10]) *For every $h = \sum_{\underline{b} \in \mathcal{NB}(X)} q_{\underline{b}, X} d_{\underline{b}}^{-1}$ with $q_{\underline{b}, X} \in S[U]$. We have:*

$$q_{\underline{b}, X}(-y) = det(\underline{b})res_{\underline{b}}(e^{(y|x)} h(x)). \tag{18}$$

The proof is easy applying formal properties of the residue. This gives a *simple algorithm* to compute the polynomials $q_{\underline{b}, X}$.

- (i) First make the non-linear change of coordinates $b_i = \prod_{j=1}^i z_j$ in the function h .
- (ii) You get a Laurent series in the variables z_i with coefficients polynomials in the variables y .
- (iii) The polynomial $q_{\underline{b}, X}(-y)$ appears now as the coefficient of $\prod_{i=1}^s z_i^{-1}$.

9.7 The abelian group $\mathcal{F}(X)$.

We denote by $\mathcal{C}[\Lambda, \underline{r}]$ the subgroup in $\mathcal{C}[\Lambda]$ consisting of the elements supported in $\Lambda \cap \underline{r}$. The following abelian group will play a key role in what follows

Definition 9.8

$$\mathcal{F}(X) := \{f \in \mathcal{C}[\Lambda] \mid \nabla_{X \setminus \underline{r}} f \in \mathcal{C}[\Lambda, \underline{r}], \text{ for all } \underline{r} \in S_X\}. \tag{19}$$

Notice that if $f \in \mathcal{F}(X)$ then f must in particular satisfy the relation corresponding to the space $\underline{r} = \{0\}$ that is

$$\nabla_X f = c\delta_0$$

or the equivalent relation

$$\prod_{a \in X} (1 - e^a) Lf = c.$$

with $c \in \mathbb{Z}$. In particular $\mathcal{F}(X) \subset \mathcal{R}$ (the rational elements) and $L_r(\mathcal{F}(X)) \subset C(\Lambda)$ is the one dimensional abelian group spanned by

$$\prod_{a \in X} \frac{1}{1 - e^{-a}}.$$

Example 9.9 Let us give a simple example. Let $\Lambda = \mathbb{Z}$, and $X = [2, -1]$. Then it is easy to see that $\mathcal{F}(X)$ has as integral basis

$$\begin{aligned} \theta_1 &= \sum_{n \in \mathbb{Z}} \delta_n, & \theta_2 &= \sum_{n \in \mathbb{Z}} n \delta_n, \\ \theta_3 &= \sum_{n \in \mathbb{Z}} \left(\frac{n}{2} + \frac{1 - (-1)^n}{4} \right) \delta_n, & \theta_4 &= \sum_{n \geq 0} \left(\frac{n}{2} + \frac{1 - (-1)^n}{4} \right) \delta_n. \end{aligned}$$

Here $\theta_1, \theta_2, \theta_3$ is an integral basis of $DM(X)$.

Proposition 9.10 $\mathcal{F}(X)$ is a free abelian group whose rank equals the number of integral points in the Zonotope $B(X)$.

Recall that, in Lemma 1.2 we have defined the partition functions P_X^F associated to the faces of the hyperplane arrangement given by X .

The first important fact on this abelian group is the following:

- (i) If F is a regular face for X , then P_X^F lies in $\mathcal{F}(X)$.
- (ii) The abelian group $DM(X)$ is contained in $\mathcal{F}(X)$.

Proof (i) Indeed, $\nabla_{X \setminus \underline{r}} P_X^F = P_{X \cap \underline{r}}^F \in C[\Lambda, \underline{r}]$.

- (ii) is clear from the definitions. □

Each P_X^F is a partition function. In particular, if X generates a pointed cone, then the partition function P_X equals P_X^F for the face F which is positive on X .

9.10.1. Some properties of $\mathcal{F}(X)$. Let \underline{r} be a rational subspace and $F_{\underline{r}}$ be a regular face for $X \setminus \underline{r}$.

Proposition 9.11 (i) The map $g \mapsto P_{X \setminus \underline{r}}^{F_{\underline{r}}} * g$ gives an injection from $\mathcal{F}(X \cap \underline{r})$ to $\mathcal{F}(X)$. Moreover

$$\nabla_{X \setminus \underline{r}} (P_{X \setminus \underline{r}}^{F_{\underline{r}}} * g) = g, \quad \forall g \in \mathcal{F}(X \cap \underline{r}). \tag{20}$$

- (ii) $\nabla_{X \setminus \underline{r}}$ maps $\mathcal{F}(X)$ surjectively to $\mathcal{F}(X \cap \underline{r})$.
- (iii) If $g \in DM(X \cap \underline{r})$, then $\nabla_{X \setminus \underline{t}} (P_{X \setminus \underline{t}}^{F_{\underline{t}}} * g) = 0$ for any rational subspace \underline{t} such that $\underline{t} \cap \underline{r} \neq \underline{r}$.

Recall that $S_X^{(i)}$ the set of rational subspaces of dimension i . Define the abelian groups

$$\mathcal{F}(X)_i := \bigcap_{\underline{r} \in S_X^{(i-1)}} \ker(\nabla_{X \setminus \underline{r}}) \cap \mathcal{F}(X).$$

Notice that by definition $\mathcal{F}(X)_{\{0\}} = \mathcal{F}(X)$, that $\mathcal{F}(X)_{\dim V}$ is the abelian group $DM(X)$ and that $\mathcal{F}(X)_{i+1} \subseteq \mathcal{F}(X)_i$.

Choose, for every rational space \underline{r} , a regular face $F_{\underline{r}}$ for $X \setminus \underline{r}$.

Let $\underline{r} \in S_X^{(i)}$.

- (i) The image of $\nabla_{X \setminus \underline{r}}$ restricted to $\mathcal{F}(X)_i$ is contained in the abelian group $DM(X \cap \underline{r})$.
- (ii) If f is in $DM(X \cap \underline{r})$, then $P_{X \setminus \underline{r}}^{F_{\underline{r}}} * f \in \mathcal{F}(X)_i$.

Theorem 18 *With the previous choices, we have:*

$$\mathcal{F}(X) = \bigoplus_{\underline{r} \in S_X} P_{X \setminus \underline{r}}^{F_{\underline{r}}} * DM(X \cap \underline{r}). \tag{21}$$

Definition 9.12 A collection $\mathbf{F} = \{F_{\underline{r}}\}$ of faces $F_{\underline{r}} \subset \underline{r}^\perp$ regular for $X \setminus \underline{r}$, indexed by the rational subspaces $\underline{r} \in S_X$ will be called a *X-regular collection*.

Given a X-regular collection \mathbf{F} , we can write, using Theorem 18, an element $f \in \mathcal{F}(X)$ as

$$f = \sum_{\underline{r} \in S_X} f_{\underline{r}} \quad \text{with} \quad f_{\underline{r}} \in P_{X \setminus \underline{r}}^{F_{\underline{r}}} * DM(X \cap \underline{r}).$$

This expression for f will be called the \mathbf{F} decomposition of f . In this decomposition, we always have $F_V = \{0\}$, $P_{X \setminus V}^{F_V} = \delta_0$ and the component f_V is in $DM(X)$.

9.13 Localization theorem

There is a more restricted notion than that of big cell, it is the notion of *tope* by this we mean a connected component of the complement of the union of all proper rational subspaces generated by subsets of X .

In this section we are going to discuss the fact that every element $f \in \mathcal{F}(X)$ coincides with a quasi-polynomial on the sets $(\tau - B(X)) \cap \Lambda$ as τ varies over all topes (we simply say f is a quasi-polynomial on $\tau - B(X)$).

Definition 9.14 Let τ be a tope and \underline{r} be a proper rational subspace. We say that a regular face $F_{\underline{r}}$ for $X \setminus \underline{r}$ is non-positive on τ if there exists $u_{\underline{r}} \in F_{\underline{r}}$ and $x_0 \in \tau$ such that $\langle u_{\underline{r}}, x_0 \rangle < 0$.

Given $x_0 \in \tau$, it is always possible to choose a regular face $F_{\underline{r}} \subset \underline{r}^\perp$ for $X \setminus \underline{r}$ such that x_0 is negative on some vector $u_{\underline{r}} \in F_{\underline{r}}$, since the projection of x_0 on V/\underline{r} is not zero.

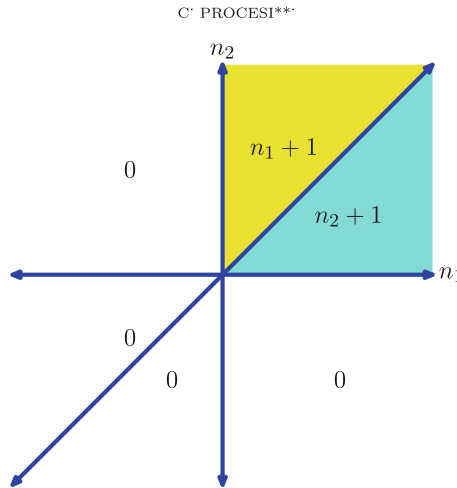


Fig. 6 The partition function of $X := (a, b, c)$

Definition 9.15 Let $\mathbf{F} = \{F_{\underline{r}}\}$ be a X -regular collection. We shall say that \mathbf{F} is non-positive on τ if each $F_{\underline{r}}$ is non-positive on τ .

Let $f \in \mathcal{F}(X)$ and let $f = \sum f_{\underline{r}}$ be the \mathbf{F} decomposition of f .

The following result is quite similar to Paradan’s localization theorem [29].

Theorem 19 (Localization theorem) *Let τ be a tope. Let $\mathbf{F} = \{F_{\underline{r}}\}$ be a X -regular collection non-positive on τ .*

The component f_V of the \mathbf{F} decomposition $f = \sum_{r \in \mathcal{S}_X} f_{\underline{r}}$ is a quasi-polynomial function in $DM(X)$ such that $f = f_V$ on $(\tau - B(X)) \cap \Lambda$.

Example 9.16 Figure 6 describes the partition function P_X for $X := (a, b, c)$ with $a := \omega_1, b := \omega_2, c := \omega_1 + \omega_2$ in the lattice $\Lambda := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. Figure 7 describes the \mathbf{F} decomposition relative to the tope containing x_0 . Notice how the choice of \mathbf{F} has the effect of pushing the supports of the elements $f_{\underline{r}} (\underline{r} \neq V)$ away from τ .

A quasi-polynomial is completely determined by the values that it takes on $(\tau - B(X)) \cap \Lambda$. Thus f_V is independent on the construction so:

Definition 9.17 We shall denote by f^τ the quasi-polynomial coinciding with f on $(\tau - B(X)) \cap \Lambda$.

The open subsets $\tau - B(X)$ cover V , when τ runs over the topes of V (with possible overlapping). Thus the element $f \in \mathcal{F}(X)$ is entirely determined by the quasi-polynomials f^τ .

If $f \in \mathcal{F}(X)$, the element $\nabla_{X \setminus \underline{r}} f$ is in $\mathcal{F}(X \cap \underline{r})$ and coincides with a quasi-polynomial $(\nabla_{X \setminus \underline{r}} f)^\tau \in DM(X \cap \underline{r})$ on each tope τ for the system $X \cap \underline{r}$.

Theorem 20 *Let $\beta \in V$ be generic with respect to all the rational subspaces \underline{r} . Let $F_{\underline{r}}^\beta$ be the unique regular face for $X \setminus \underline{r}$ containing $p_{\underline{r}^\perp} \beta$.*

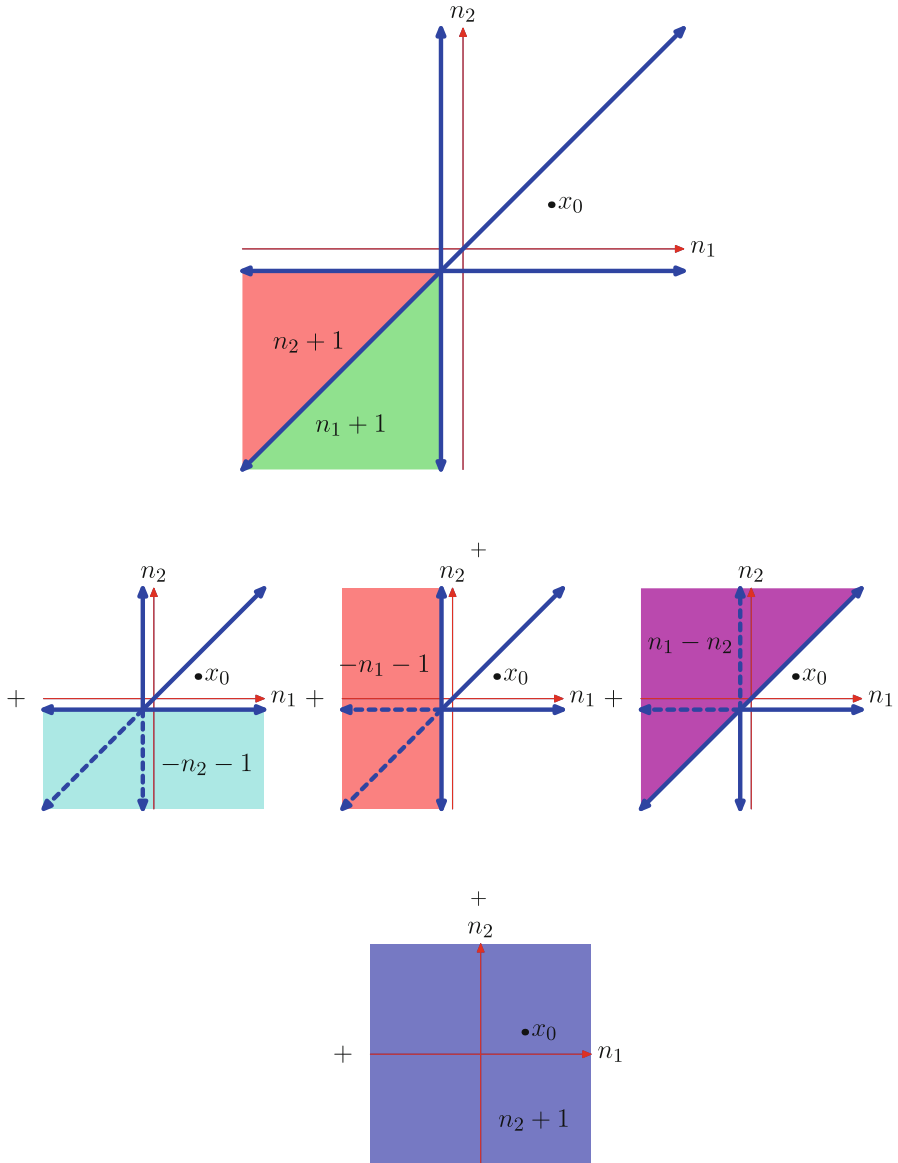


Fig. 7 F decomposition of the partition function of $X := (a, b, c)$ for \mathbf{F} non-positive on τ

Then

$$f = \sum_{\underline{r} \in S_X} P_{X \setminus \underline{r}}^{-F_{\underline{r}}^\beta} * (\nabla_{X \setminus \underline{r}} f)^{\tau(p_{\underline{r}} \beta)}.$$

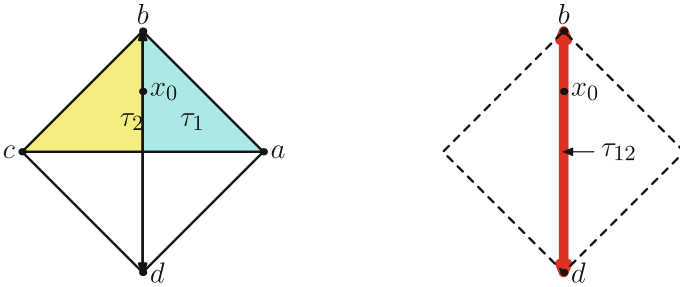


Fig. 8 Two adjacent topes of $X := (a, b, c, d)$

9.17.1. Wall crossing formula. We first describe a general formula of how the functions f^τ change when crossing a wall. This formula implies that the partition function P_X is a quasi-polynomial on $\Omega - B(X)$, where Ω is a big cell.

Let H be a rational hyperplane and $u \in H^\perp$ a non zero element. Then the two open faces in H^\perp are half-lines $F_H = \mathbf{R}_{>0}u$ and $-F_H$. If $q \in DM(X \cap H)$, then $w := (P_{X \setminus H}^{F_H} - P_{X \setminus H}^{-F_H}) * q$ is an element of $DM(X)$.

Remark 9.18 In [11], a one-dimensional residue formula is given for w allowing us to compute it.

Assume that τ_1, τ_2 are two adjacent topes, namely $\bar{\tau}_1 \cap \bar{\tau}_2$ spans a hyperplane H . The hyperplane H is a rational subspace. Let τ_{12} be the unique tope for $X \cap H$ such that $\bar{\tau}_1 \cap \bar{\tau}_2 \subset \bar{\tau}_{12}$.

Example 9.19 Let C be the cone generated by the vectors $a := \omega_3 + \omega_1, b := \omega_3 + \omega_2, c := \omega_3 - \omega_1, d := \omega_3 - \omega_2$ in a 3-dimensional space $V := \mathbf{R}\omega_1 \oplus \mathbf{R}\omega_2 \oplus \mathbf{R}\omega_3$. Figure 8 represents the section of C cut by the affine hyperplane containing a, b, c, d . We consider $X := (a, b, c, d)$.

On the left of the picture we show the intersection of C with the two topes τ_1, τ_2 adjacent along the hyperplane H generated by b, d and, on the right, that with the tope τ_{12} . The list $X \cap H$ is $[b, d]$. The closure of the tope τ_{12} is “twice bigger ” than $\bar{\tau}_1 \cap \bar{\tau}_2$.

Let $f \in \mathcal{F}(X)$. The function $\nabla_{X \setminus H} f$ is an element of $\mathcal{F}(H \cap X)$, thus there exists a quasi-polynomial $(\nabla_{X \setminus H} f)^{\tau_{12}}$ on H such that $\nabla_{X \setminus H} f$ agrees with $(\nabla_{X \setminus H} f)^{\tau_{12}}$ on τ_{12} .

Theorem 21 Let $\tau_1, \tau_2, H, \tau_{12}$ be as before and $f \in \mathcal{F}(X)$. Let F_H be the half line in H^\perp positive on τ_1 . Then

$$f^{\tau_1} - f^{\tau_2} = (P_{X \setminus H}^{F_H} - P_{X \setminus H}^{-F_H}) * (\nabla_{X \setminus H} f)^{\tau_{12}}. \tag{22}$$

In the case in which $f = P_X$ we deduce Paradan’s formula [28, Theorem 5.2]

$$P_X^{\tau_1} - P_X^{\tau_2} = (P_{X \setminus H}^{F_H} - P_{X \setminus H}^{-F_H}) * P_{X \cap H}^{\tau_{12}}. \tag{23}$$

9.20 The partition function

We assume that $C(X)$ is a pointed cone. Let us now consider a big cell Ω . Given a big cell Ω , let τ_1, \dots, τ_k be all the topes contained in Ω . Then:

$$\Omega - B(X) = \cup_{i=1}^k (\tau_i - B(X)).$$

Now in order to prove the statement for big cells, we need to see what happens when we cross a wall between two adjacent topes by the previous formulas one has

Theorem 22 *On $(\Omega - B(X)) \cap \Lambda$, the partition function P_X agrees with a quasi-polynomial $P_X^\Omega \in DM(X)$.*

This theorem was proven [14] by Dahmen–Micchelli for topes, and by Szenes–Vergne [34] for cells. In many cases, the sets $\Omega - B(X)$ are the maximal domains of quasi-polynomiality for \mathcal{T}_X .

There is an important point of the theory of Dahmen–Micchelli that we point out.

Theorem 23 *Let Ω be a big cell contained in $C(X)$. Then P_X^Ω is the unique element $f \in DM(X)$ such that $f(0) = 1$ and $f(a) = 0, \forall a \in \delta(c|X), a \neq 0$.*

Part 3. Index theory

10 The Atiyah–Singer index theorem

10.1 Equivariant K -theory

We briefly review the notations for K -theory that we will use, for a systematic treatment see Atiyah [2] and Segal [32]. Let G be a compact Lie group acting on a locally compact space N .

- One has the notion of the equivariant topological K -theory group $K_G^0(N)$.
- If N is compact $K_G^0(N)$ is the Grothendieck ring of equivariant vector bundles.
- For a point p it is the character ring $R[G]$ of G .
For N locally compact set $\hat{N} = N \cup \infty$ be the one point compactification.
- One sets $K_G^0(N)$ to be the kernel of the restriction of $K_G^0(\hat{N})$ to $K_G^0(\infty)$
- $K_G^0(N)$ is then a contravariant functor for proper maps and covariant for open embeddings.

Remark 10.2 Notice that, when G is abelian, the character ring $R[G]$ equals the group ring $\mathbb{Z}[\hat{G}]$ of its character group \hat{G} .

Representatives of the K -theory group $K_G^0(N)$ can be described in the following way.

Given two G -equivariant complex vector bundles E^0, E^1 on N and a G -equivariant bundle map $f : E^0 \rightarrow E^1$, the support $supp(f)$ of f is the set of points where $f_x : E_x^0 \rightarrow E_x^1$ is not an isomorphism.

A G -equivariant bundle map f with compact support defines an element $[f]$ of $K_G^0(N)$. All elements can be described this way.

Let $f : E^0 \rightarrow E^1$ and $g : F^0 \rightarrow F^1$ be two G -equivariant bundle maps. Using G -invariant Hermitian metrics on the bundles E^i, F^i we can define:

$$f \odot g : E^0 \otimes F^0 \oplus E^1 \otimes F^1 \rightarrow E^1 \otimes F^0 \oplus E^0 \otimes F^1$$

by

$$f \odot g := \begin{pmatrix} f \otimes 1 & -1 \otimes g^* \\ 1 \otimes g & f^* \otimes 1 \end{pmatrix}.$$

The support of $f \odot g$ is the intersection of the supports of f, g thus $f \odot g$ induces an element in $K_G^0(N)$ as soon as one of the two f, g has compact support.

In particular this defines a product $[f][g] := [f \odot g]$ on $K_G^0(N)$.

If $N = \text{pt}$ is a point, $K_G^0(\text{pt})$ is isomorphic to the Grothendieck ring $R(G)$ of finite dimensional representations of G .

In general take the projection $\pi : N \rightarrow \text{pt}$, given $\tau \in R(G)$ and $\sigma \in K_G^0(N)$, we have that $[\pi^*(\tau) \odot \sigma] \in K_G^0(N)$ and this gives a $R(G)$ module structure to $K_G^0(N)$.

10.2.1. Clifford action. Let W be a Hermitian vector space and let $E = \bigwedge W$.

For $w \in W$, consider the exterior multiplication $m(w) : E \rightarrow E$ and the Clifford action

$$c(w) = m(w) - m(w)^*, \quad m(w)(\omega) := w \wedge \omega \tag{24}$$

of W on $\bigwedge W$.

We have $c(w)^2 = -\|w\|^2$, so that $c(w)$ is an isomorphism, if $w \neq 0$. We shall use this as follows:

10.2.2. Bott symbol and Thom isomorphism. If $p : W \rightarrow M$ is a G -equivariant complex vector bundle over a G -space M , the fiberwise Clifford action $c(w_x) : \bigwedge^{\text{even}} W_x \rightarrow \bigwedge^{\text{odd}} W_x$ defines a morphism $\mathbf{c}_W : p^* \bigwedge^{\text{even}} W \rightarrow p^* \bigwedge^{\text{odd}} W$ of vector bundles over W , called the *Bott symbol*.

Take a bundle map $f : E \rightarrow F$ of complex equivariant vector bundles on M which is an isomorphism outside a compact set, and denote still by f its pull back $f : p^*E \rightarrow p^*F$. Then $f \odot \mathbf{c}_W$ is a bundle map of bundles over W , which is an isomorphism outside the support of f embedded in W via the zero section. We set

$$\text{Thom isomorphism} \quad C_W : K_G^0(M) \rightarrow K_G^0(W), \quad C_W([f]) = [f \odot \mathbf{c}_W]. \tag{25}$$

The map $[f] \mapsto C_W([f]) = [f \odot \mathbf{c}_W]$ is the Thom isomorphism between $K_G^0 M$ and $K_G^0 W$.

10.2.3. *Bott periodicity.* The Bott periodicity is the Thom isomorphism when $W = M \times \mathbb{C} = M \times \mathbb{R}^2$. We then define inductively the groups

$$K_G^{i+1}(N) := K_G^i(N \times \mathbb{R}).$$

One has $K_G^0(\mathbb{R}) = K_G^1(\text{pt}) = 0$.

There is a natural isomorphism $K_G^i(N) \rightarrow K_G^{i+2}(N) = K_G^i(N \times \mathbb{R}^2)$ given by Bott periodicity.

10.2.4. *A long exact sequence.* If F is a G -invariant closed subset of N , denote by $i : F \rightarrow N$ the closed embedding and $j : N \setminus F \rightarrow N$ the open embedding. There is a long exact sequence of $R(G)$ modules:

$$\dots \rightarrow K_G^i(N \setminus F) \xrightarrow{j^*} K_G^i(N) \xrightarrow{i^*} K_G^i(F) \xrightarrow{\delta} K_G^{i+1}(N \setminus F) \rightarrow \dots \tag{26}$$

10.3 Transversally elliptic operators

Consider a manifold M with an action of a compact Lie group G . We then have the notion of *transversally elliptic operator* between two equivariant complex vector bundles E, F on M .

Such an operator is a pseudo-differential operator $A : \Lambda(M, E) \rightarrow \Lambda(M, F)$ from the space $\Lambda(M, E)$ of smooth sections of E to the space $\Lambda(M, F)$ of smooth sections of F , which commutes with the action of G , is *elliptic* in the directions transversal to the orbits of G and is “trivial” at infinity.

Let T^*M denote the cotangent bundle of M and $p : T^*M \rightarrow M$ the canonical projection.

Definition 10.4 By a *symbol*, one means a smooth section on T^*M of the bundle $\text{hom}(p^*(E), p^*(F))$: in other words, for each point $(x, \xi), x \in M, \xi \in T_x^*M$, we have a linear map $\sigma(x, \xi) : E_x \rightarrow F_x$.

Inside T^*M , there is a special closed subset denoted by T_G^*M (the zero set of the moment map cf. Sect. 13.3). Its fiber over a point $x \in M$ is formed by all the cotangent vectors $\xi \in T_x^*M$ which vanish on the tangent space to the orbit of x under G , in the point x . Thus each fiber $(T_G^*M)_x$ is a linear subspace of T_x^*M . In general the dimension of $(T_G^*M)_x$ is not constant and this space is not a vector bundle.

Assume first that M is a compact manifold. To the pseudo-differential operator A , one associates its principal *symbol* σ_p which is defined outside the zero section of T^*M . The operator A is said to be G -transversally elliptic if its principal symbol $\sigma_p(x, \xi)$ is invertible for all $(x, \xi) \in T_G^*M$ such that $\xi \neq 0$.

Using a G -invariant function χ on T^*M identically equal to 1 in a neighborhood of M and compactly supported, then $\sigma(x, \xi) := (1 - \chi(x, \xi))\sigma_p(x, \xi)$ is defined on the whole space T^*M . Furthermore $\sigma(x, \xi)$ restricted to T_G^*M is an isomorphism outside a compact G -invariant subset of T_G^*M .

Thus, by restriction to T_G^*M , the symbol σ defines a K -theory class $[\sigma]$ in the topological equivariant K -theory group $K_G^0(T_G^*M)$.

This class does not depend of the choice of χ . We still say that this class σ is the symbol of A .

Let \hat{G} be the set of equivalence classes of finite dimensional irreducible representations of G , and let $\mathcal{C}[\hat{G}]$ be the group of \mathbb{Z} -valued functions on \hat{G} . Let $\chi_\tau(g) = \text{Tr}(\tau(g))$ be the character of the representation $\tau \in \hat{G}$ of G . We associate to an element $f \in \mathcal{C}[\hat{G}]$ a formal (virtual) character $\Theta(f) = \sum_\tau f(\tau)\chi_\tau$, that is a formal combination of the characters χ_τ with multiplicities $f(\tau) \in \mathbb{Z}$. When $f(\tau)$ satisfies certain moderate growth conditions, then the series $\Theta(f)(g) = \sum_\tau f(\tau)\chi_\tau(g)$ converges, in the distributional sense, to a generalized function on G .

The *index map* associates to a transversally elliptic operator A an element of $\mathcal{C}[\hat{G}]$ constructed as follows. For every $\tau \in \hat{G}$, the space $\text{hom}_G(\tau, \ker(A))$ is finite dimensional of dimension $m(\tau, A)$. Thus $m(\tau, A)$ is the multiplicity of τ in the space $\ker(A)$ of smooth solutions of A . We choose a G -invariant metric on M and G -invariant Hermitian structures on E, F . Then $A^* : \Lambda(M, F) \rightarrow \Lambda(M, E)$ is also transversally elliptic.

Definition 10.5 The index multiplicity of the pseudo-differential operator A is the function $\text{ind}_m(A) \in \mathcal{C}[\hat{G}]$ defined by

$$\text{ind}_m(A)(\tau) := m(\tau, A) - m(\tau, A^*).$$

It follows also from Atiyah-Singer [1] that the series $\sum_\tau m(\tau, A)\chi_\tau(g)$ defines a generalized function on G . Thus we may also associate to A the generalized function

$$\text{ind}(A)(g) = \sum_\tau \text{ind}_m(A)(\tau)\chi_\tau(g)$$

on G with integral Fourier coefficients. One of the main points in the index theory consists in showing that the index factors through the symbols and defines a homomorphism of $R(G)$ modules from $K_{G,c}^0(T_G^*M)$ to $\mathcal{C}[\hat{G}]$.

If $j : U \rightarrow M$ is an open G -invariant set of a compact G manifold M , we still denote by j the corresponding open embedding from T_G^*U to T_G^*M . Then j_* defines a map from $K_G^0(T_G^*U)$ to $K_{G,c}^0(T_G^*M)$. The index of $\sigma \in K_G^0(T_G^*U)$ is defined to be the index of $j_*(\sigma)$. The excision property of the index shows that this is independent of the choice of the open embedding j and thus allows us to define the index map also for manifolds which can be embedded as open sets of compact ones.

In particular, if V is a vector space with a linear action of a compact group G , then V is diffeomorphic to the sphere, minus a point. Thus we can define the index of any $\sigma \in K_G^0(T_G^*V)$. More generally, if U is an open G -invariant subset of a vector bundle on a compact manifold, we can define the index of $\sigma \in K_G^0(T_G^*U)$.

The problem of computing the index can be reduced, at least theoretically, to the case in which G is a torus. For a given compact manifold M , one embeds M into a linear representation and then is reduced to perform the computations in the representation.

10.5.1. Some properties of the index map

- (i) Any element $\sigma \in K_G^0(T_G^*M)$ arises from the restriction to T_G^*M of a G - bundle morphism $\sigma(x, \xi) : E_x \rightarrow F_x$, such that $supp(\sigma) \cap T_G^*M$ is a compact set. Here E, F are G -equivariant complex vector bundles over M .
- (ii) If $j : \mathcal{U} \rightarrow M$ is an open embedding, the map $j_* : K_G^0(T_G^*\mathcal{U}) \rightarrow K_G^0(T_G^*M)$ is compatible with the index.
- (iii) Let W be a real vector space with a linear representation of G and W' be the dual vector space. We identify $TW = W \times W$ with $W_{\mathbb{C}}$ by $(v, w) \rightarrow v + iw$. Furthermore, we identify $TW = W \times W$ with $T^*W = W \times W'$ using an Euclidean structure on W . Thus the Bott symbol $c_{W_{\mathbb{C}}}(v + i\xi)$ acting on $\bigwedge W_{\mathbb{C}}$ defines a G -equivariant elliptic symbol on W . Its G -equivariant index is identically equal to 1.
 Let $i : N \rightarrow M$ be an injection of G -manifolds, this factors through a tubular neighborhood $j : \mathcal{U} \rightarrow M$. Using a similar construction we obtain a map $i_! : K_G^0(T_G^*N) \rightarrow K_G^0(T_G^*(\mathcal{U}))$ given at the level of symbols by $\sigma \mapsto \sigma \odot c_{W_{\mathbb{C}}}$. The index of σ is equal to the index of $i_!\sigma$.
- (iv) In case $M = N \times \mathbb{R}$ with the trivial action on \mathbb{R} , $T_G^*(N \times \mathbb{R}) = T_G^*N \times T^*\mathbb{R}$ and thus $i_!$ is an isomorphism by Bott periodicity.
- (v) Let H be a closed subgroup of G . Then there is a surjective map $\hat{G} \rightarrow \hat{H}$ induced by the restriction of characters. The dual map induces an injection $Ind_H^G : \mathcal{C}[\hat{H}] \rightarrow \mathcal{C}[\hat{G}]$.

Let M be a space with H action (open subset a compact H -manifold), and let $N := G \times_H M$ be the G space with typical fiber M over G/H . It is easy to see that there is an isomorphism

$$i_H^G : K_H^i(T_H^*M) \rightarrow K_G^i(T_G^*N) \tag{27}$$

and, by [1, Theorem 4.1], for any $\sigma \in K_H^0(T_H^*M)$,

$$ind_m(i_H^G(\sigma)) = Ind_H^G(ind_m(\sigma)). \tag{28}$$

We shall also need the following simple consequence of the previous facts:

Lemma 10.6 *Let G be a compact Lie group and $\chi : G \rightarrow S^1$ be a surjective character. Set $H := \ker \chi$ be the kernel of χ .*

Take a manifold M over which G acts and consider the product $\mathbb{C}^ \times M$, with the action of G on the first factor induced by χ .*

There is an isomorphism

$$k : K_H^i(T_H^*M) \cong K_G^i(T_G^*(\mathbb{C}^* \times M)). \tag{29}$$

*Moreover, if $\sigma \in K_H^0(T_H^*M)$, we have $ind_m(k(\sigma)) = Ind_H^G(ind_m(\sigma))$.*

Since $\mathbb{C}^* = S^1 \times \mathbb{R}^+$, we get by iv) that the inclusion $i : S^1 \times M \rightarrow \mathbb{C}^* \times M$ induces the isomorphism

$$i_! : K_G^i(T_G^*(S^1 \times M)) \rightarrow K_G^i(T_G^*(\mathbb{C}^* \times M))$$

which at the level of K^0 is compatible with the index.

On the other hand, the space $G \times_H M$ identifies with $S^1 \times M$ via the map $[g, m] \mapsto [\chi(g), g \cdot m]$. So (27) gives us the isomorphism

$$i_H^G : K_H^i(T_H^*(M)) \rightarrow K_G^i(T_G^*(S^1 \times M)).$$

If $\sigma \in K_H^0(T_H^*M)$, then $ind_m(i_H^G(\sigma)) = Ind_H^G(ind_m(\sigma))$ by Formula (28).

Thus we can take $k := i_!i_H^G$.

• **The tangential Cauchy–Riemann operator.**

Assume that the group G is an abelian compact Lie group. An irreducible representation a of G is a one dimensional complex vector space L_a , where G acts via a character $\chi_a : G \rightarrow S^1$, so that \hat{G} is identified with the abelian group of characters, denoted by Λ .

Definition 10.7 Let X be a finite list of elements of Λ . Define the complex vector space

$$M_X := \bigoplus_{a \in X} L_a. \tag{30}$$

The space M_X is a G -manifold and our goal is the determination of $K_G^0(T_G^*M_X)$. The basic tool that we shall use is the space of functions $DM^{(G)}(X)$ on Λ . This is defined as $DM(X)$ (and often denoted also $DM(X)$) but in this more general case of not necessarily connected G . The extension to non connected abelian groups is necessary in order to perform induction. In this case Λ has a torsion subgroup Λ_t and Theorem 13 gives the statement that $DM^{(G)}(X)$ is a free module over the group ring of Λ_t or rank $\delta(X)$.

With a given G -invariant Hermitian structure on M let S be the unit sphere of M . Let $\mathbb{P}(M)$ be the complex projective space of M . Consider on S the differential operator δ acting on the pull back of the Dolbeault complex on the associated projective space $\mathbb{P}(M)$ using $\bar{\partial} + \bar{\partial}^* : \sum \Omega^{0,2p} \rightarrow \sum \Omega^{0,2p+1}$. Then δ is a G -transversally elliptic differential operator *the tangential Cauchy–Riemann operator* on S .

Indeed, using the Hermitian structure, identify T^*S with its tangent bundle $TS \subset TM$, the subspace H_p of T_p^*S orthogonal to the line $\mathbb{R}J_u p$ is then identified to the complex subspace of M , orthogonal under the Hermitian form to p . We call it the horizontal cotangent space. The symbol of δ is $\sigma(p, \xi) = c(\xi^1)$ where ξ^1 is the projection of ξ on H_p , and c the Clifford action of H_p on $\wedge H_p$. This morphism is invertible if $\xi^1 \neq 0$. We have also $H_p \oplus \mathbb{R}\rho(u)p = T_p^*S$, as the eigenvalues of $-iu$ on M are all positive. Thus we see that $\sigma(p, \xi)$ restricted to T_G^*S is invertible outside the zero section.

The following formula is proven in [1] (Proposition 5.4).

Theorem 24 *Let M be provided with the complex structure J_u and let δ be the tangential Cauchy–Riemann operator, on the unit sphere of M . Then*

$$index(\delta)(g) = (-1)^{|X|} g^{a_X} (\Theta_X^{-F}(g) - \Theta_X^F(g))$$

where $a_X = \sum_{a \in X} a$.

We recall briefly the proof. Let S^1 be the circle group acting by homotheties on M . We decompose solution spaces with respect to characters $t \mapsto t^n$ of S^1 . The group G acts on $\mathbb{P}(M)$ and on every line bundle $\mathcal{O}(n)$ on $\mathbb{P}(M)$. Thus the index as an index of $G \times S^1$ is the sum of the index of G in the cohomology on $\mathbb{P}(M)$ of the line bundles $\mathcal{O}(n)$. Define $\chi_n(g)$ as the virtual character (as a representation of G) in the virtual finite dimensional vector space $\sum (-1)^i H^{0,i}(\mathbb{P}(M), \mathcal{O}(n))$. Then

$$index(\delta)(g) = \sum_{n \in \mathbb{Z}} \chi_n(g).$$

Let us show that

$$\sum_{n \geq 0} \chi_n(g) = (-1)^{|X|} g^{ax} \Theta_X^{-F}(g), \tag{31}$$

$$\sum_{n < 0} \chi_n(g) = (-1)^{|X|+1} g^{ax} \Theta_X^F(g). \tag{32}$$

For $n \geq 0$, $\mathcal{O}(n)$ has only 0-cohomology and $H^{0,0}(\mathbb{P}(M), \mathcal{O}(n))$ is just the space of homogeneous polynomials on M of degree n . So $\sum_{n=0}^\infty \chi_n$ is the character of the symmetric algebra $S[M^*] = \prod_{a \in X} S[L_{-a}]$. The function $\sum_{k=0}^\infty g^{-ka}$ is the character of the action of G in $S[L_{-a}]$. The function Θ_X^{-F} is the product of the functions $-g^{-1} \sum_{k=0}^\infty g^{-ka}$. Thus we obtain Formula (31). On the other hand, if $n < 0$, we have two cases. If $-|X| - 1 < n \leq -1$, then $H^{0,i}(\mathbb{P}(M), \mathcal{O}(n)) = 0$ for every i . Otherwise we apply Serre's duality and we obtain the second equality (32).

• **Atiyah-Singer pushed symbol.** We identify T^*M with $M \times M$, using the Hermitian metric on M . Let $c(v) : \bigwedge^{even} M \rightarrow \bigwedge^{odd} M$ be the Clifford action (24) of M on $\bigwedge M$. Given, as before, a regular element u in the Lie algebra of G and letting $\rho(u)$ denote its infinitesimal action on M , we define

Definition 10.8

$$At_u(v, \xi) = c(\xi + \rho(u)v).$$

The morphism $At_u(v, \xi)$ is invertible except if $\xi + \rho(u)v = 0$. If furthermore ξ is in T_G^*M , ξ is orthogonal to the tangent vector $\rho(u)v$. Thus the support of $At_u(v, \xi)$ restricted to T_G^*M is the unique point $v = 0, \xi = 0$ and At_u determines an element of $K_G^0(T_G^*M)$, which depends only of the open face F of u in the hyperplane arrangement dual to X . We denote it by At_F . The index of At_F is computed in [1] (Theorem 8.1). In more detail, in the Appendix of [9], it is constructed an explicit G -transversally elliptic pseudo-differential operator A on the product of the projective lines $\mathbb{P}(L_a \oplus \mathbb{C})$.

If $j : M_X \rightarrow \prod_{a \in X} \mathbb{P}(L_a \oplus \mathbb{C})$ is the natural open embedding, it is shown that $j_*(At_F)$ is homotopic to the symbol of A . By definition, the index of At_F is that of A and one has the explicit formula:

Theorem 25 *Let M be provided with the complex structure J_u and let $At_F \in K_G^0(T_G^*M)$ be the “pushed” $\bar{\partial}$ symbol. Then*

$$index(At_F)(g) = (-1)^{|X|} g^{a_X} \Theta_X^F(g). \tag{33}$$

11 The index for M_X

This section contains the main results on the index for the space M_X , that is Theorems 28 and 29.

11.1 K -theory

Let G be, as before, a compact abelian Lie group of dimension s and $M_X := \bigoplus_{a \in X} L_a$ as in (30). We assume that X has rank s .

Given a vector $v \in M_X$, its *support* is the sublist of elements $a \in X$ such that v has a non zero coordinate in the summand L_a .

If Y is the support of v , an element t of G stabilizes v if and only if $t^a = 1$ for all $a \in Y$. If Y spans a rational subspace of dimension k , the G -orbit of v has dimension k .

For any rational subspace \underline{r} , we may consider the subspace $M_{\underline{r}} := \bigoplus_{a \in X \cap \underline{r}} L_a$ of M_X . We set

$$M_{\leq i} := \bigcup_{\underline{r} \in S_X^{(i)}} M_{\underline{r}}, \quad M_{\geq i} := M_X \setminus M_{\leq i-1},$$

$$F_i := M_{\leq i} \setminus M_{\leq i-1} = M_{\geq i} \setminus M_{\geq i+1} = M_{\leq i} \cap M_{\geq i}. \tag{34}$$

Notice that

$$M_X = M_{\geq 0} \supset M_{\geq 1} \supset M_{\geq 2} \supset \dots \supset M_{\geq s} := M_X^f.$$

The set $M_{\leq i}$ is the closed set of points in M with the property that the orbit has dimension $\leq i$ while $M_{\geq i}$ is the open set of points in M with the property that the orbit has dimension $\geq i$. The set F_i is open in $M_{\leq i}$ and closed in $M_{\geq i}$ and it is the set of points in M whose orbit under G has dimension exactly i . In particular, $F_s = M_{\geq s} = M_X^f$ is the open set of points in M with finite stabilizer under the action of G , which plays a particular role.

Definition 11.2 Given a rational subspace \underline{r} , we denote by $G_{\underline{r}}$ the subgroup of G joint kernel of the elements in $\Lambda \cap \underline{r}$. The group $G_{\underline{r}}$ is a torus and acts trivially on $M_{\underline{r}} := \bigoplus_{a \in X \cap \underline{r}} L_a$ inducing an action of $G/G_{\underline{r}}$.

We define the set $F(\underline{r})$ to be the open set of $M_{\underline{r}}$ where $G/G_{\underline{r}}$ acts with finite stabilizers.

In other words, the connected component of the stabilizer of an element of $F(\underline{r})$ is exactly the group $G_{\underline{r}}$.

Remark 11.3 By definition of $G_{\underline{r}}$, the set $F(\underline{r})$ is non-empty. The set F_i is the disjoint union of the sets $F(\underline{r})$ as \underline{r} runs over all rational subspaces of dimension i . Thus the space M_X is the disjoint union of the locally closed strata $F(\underline{r})$.

We now analyze the equivariant K theory of $T_G^*M_X^f$.

Let $a \in X$ be an element of infinite order so that the homomorphism $g^a : G \rightarrow S^1$ is surjective. Set $Z := X \setminus \{a\}$ and $G_a := \ker g^a$. Denote by \tilde{Z} the list of the restrictions to G_a of the elements of Z . For $v \in M_X$, denote by $v_a \in \mathbb{C}$ its coordinate in L_a with respect to a choice of a basis of the one dimensional vector space L_a .

The set $M_Z^f := \{v \in M_X^f \mid v_a = 0\}$ is closed in M_X^f . Denote by $i : M_Z^f \rightarrow M_X^f$ the closed embedding and by $j : M_X^f \setminus M_Z^f \rightarrow M_X^f$ the open embedding of the complement.

Lemma 11.4 *There exists an isomorphism*

$$k : K_{G_a}^i(T_{G_a}^*M_Z^f) \rightarrow K_G^i(T_G^*(M_X^f \setminus M_Z^f)).$$

If $\sigma \in K_{G_a}^0(T_{G_a}^*M_Z^f)$, we have $ind_m(k(\sigma)) = Ind_{G_a}^G(ind_m(\sigma))$.

Take an element $(v_a, w) \in L_a \times M_Z$ with $v_a \neq 0$, its stabilizer in G is the subgroup of G_a stabilizing w , therefore the space $M_X^f \setminus M_Z^f$ is isomorphic to $\mathbb{C}^* \times M_Z^f$. Thus we are in the setting of Lemma 10.6.

For a real vector space W , we shall denote by W' its dual. Consider the projection $p : T_G^*M_X^f \rightarrow M_X^f$. Then $p^{-1}M_Z^f$ is a closed subset of $T_G^*M_X^f \subset M_X^f \times M_X'$ and $T_G^*M_X^f \setminus p^{-1}M_Z^f$ is equal to $T_G^*(M_X^f \setminus M_Z^f)$. We use the same notations i, j also in this setting for the closed and open embedding associated. Remark the following fact.

- Lemma 11.5** (i) *We have $p^{-1}M_Z^f = T_G^*M_Z^f \times L'_a$.*
 (ii) *We have an isomorphism $C_a : K_G^i(T_G^*M_Z^f) \rightarrow K_G^i(p^{-1}M_Z^f)$.*

The first assertion is immediate to verify. The second follows from the first and Thom isomorphism. The first theorem (see [20] for a proof) is:

- Theorem 26** (i) $K_G^1(T_G^*M_X^f) = 0$.
 (ii) *If $a \in X$ has infinite order, there is a short exact sequence:*

$$0 \rightarrow K_{G_a}^0(T_{G_a}^*M_Z^f) \xrightarrow{j_*k} K_G^0(T_G^*M_X^f) \xrightarrow{C_a^{-1}i^*} K_G^0(T_G^*M_Z^f) \rightarrow 0. \tag{35}$$

Choose $0 \leq i \leq s$. We pass now to study the G -invariant open subspace $M_{\geq i}$ of M . The set $M_{\geq i+1}$ is open in $M_{\geq i}$ with complement the space F_i disjoint union of the spaces $F(\underline{r})$ with $\underline{r} \in S_X^{(i)}$. Denote by $\tilde{T}_G^*F_i$ the restriction of T_G^*M to F_i , disjoint union of the spaces $\tilde{T}_G^*F(\underline{r})$. Denote $j : M_{\geq i+1} \rightarrow M_{\geq i}$ the open inclusion and $e : \tilde{T}_G^*F_i \rightarrow T_G^*M_{\geq i}$ the closed embedding. Let C_i be the Thom isomorphism from $K_G^0(T_G^*F_i)$ to $K_G^0(\tilde{T}_G^*F_i)$ direct sum of the Thom isomorphisms $C_{\underline{r}}$.

Theorem 27 For each $0 \leq i \leq s - 1$,

- (i) $K_G^1(T_G^*M_{\geq i}) = 0$.
- (ii) The following sequence is exact

$$0 \rightarrow K_G^0(T_G^*M_{\geq i+1}) \xrightarrow{j^*} K_G^0(T_G^*M_{\geq i}) \xrightarrow{C_i^{-1}e^*} K_G^0(T_G^*F_i) \rightarrow 0. \tag{36}$$

Since $M_{\geq s} = M_X^f$, we can assume by induction on $s - i$ that (i) holds for each $j > i$. Also by Theorem 26 (i) we get that $K_G^1(T_G^*F_i) = 0$ for each $0 \leq i \leq s - 1$. Using this both statements follow immediately from the long exact sequence of equivariant K -theory.

Remark 11.6 The fact that the sequence (36) is exact is proved in [1] using a splitting.

11.6.1. Two commutative diagrams Let N be a complex representation space for G . Recalling the structure of $R(G)$ module of the equivariant K -theory, the multiplication by the difference $\bigwedge^{even} N - \bigwedge^{odd} N$ will be denoted by $\bigwedge_{-1} N \otimes -$. This is by definition the action of the element $\det_N(1 - g) \in R(G)$ on the equivariant K -theory.

Lemma 11.7 Take a sublist Y in X and decompose $M_X = M_{X \setminus Y} \oplus M_Y$. Let U be an open G -invariant set contained in $M_{X \setminus Y} \times (M_Y \setminus \{0\})$. Then if $\sigma \in K_G^i(U)$ or $\sigma \in K_G^i(T_G^*U)$, we have $\bigwedge_{-1} M_Y \otimes \sigma = 0$.

We give the proof for U , the case of T_G^*U being identical.

Take $v \in U$ and decompose it as $v = v_{X \setminus Y} + v_Y$ with $v_{X \setminus Y} \in M_{X \setminus Y}$ and $v_Y \in M_Y$. The component v_Y is not zero by assumption. Consider the complex G -equivariant vector bundle $V_Y = U \times M_Y$ on U . Set now $E^+ := U \times \bigwedge^{even} M_Y$, $E^- = U \times \bigwedge^{odd} M_Y$. Choosing an Hermitian metric on M_Y , for every $u \in M_Y$, we get the Clifford action $c(u) : \bigwedge^{even} M_Y \rightarrow \bigwedge^{odd} M_Y$ of M_Y on $\bigwedge M_Y$, which is an isomorphism as soon as $u \neq 0$.

Going back to our bundles E^+, E^- , for every $\epsilon \in [0, 1]$, define the bundle map $\mathbf{c}_\epsilon : E^+ \rightarrow E^-$ by

$$\mathbf{c}_\epsilon(v, \omega) = (v, \epsilon c(v_Y)\omega).$$

If $\sigma \in K_G^0(U)$, the element $\bigwedge_{-1} M_Y \otimes \sigma \in K_G^0(U)$ is represented by the morphism $\mathbf{c}_0 \odot \sigma$, homotopic to $\mathbf{c}_1 \odot \sigma$. This last bundle map is an isomorphism since $v_Y \neq 0$ on U . This implies that $\bigwedge_{-1} M_Y \otimes \sigma = 0$.

We apply this to the open set M_X^f where G acts with finite stabilizers. If Y is a cocircuit, $M_{X \setminus Y} \cap M_X^f = \emptyset$, thus $\bigwedge_{-1} M_Y \otimes \sigma = 0$ for all $\sigma \in K^0(T_G^*M_X^f)$. As the index map is a $R(G)$ module map, this implies that for cocircuit Y , the generalized function $index(\sigma)(g)$ on G satisfies the equation $\prod_{a \in Y} (1 - g^a) index(\sigma)(g) = 0$. The function $ind_m(\sigma)$ on \hat{G} is the Fourier transform of the function $index(\sigma)$. It follows that $\nabla_Y ind_m(\sigma) = 0$.

Thus we obtain

Corollary 11.8 *The multiplicity index map ind_m maps $K_G^0(T_G^*M_X^f)$ to the space $DM(X)$.*

More generally the same argument shows that

Corollary 11.9 *Choose $0 \leq i \leq s$. If $\sigma \in K_G^0(T_G^*M_{\geq i})$ and \underline{t} is a rational subspace of dimension strictly less than i , then $\bigwedge_{-1} M_{X \setminus \underline{t}} \otimes \sigma = 0$.*

Let us now split $X = A \cup B$ and $M_X = M_A \oplus M_B$. Let $p : T_G^*M_X \rightarrow M_X$ be the projection and consider $\tilde{T}_G^*M_A := p^{-1}M_A$. We have $\tilde{T}_G^*M_A = T_G^*M_A \times M'_B$. In particular, we get a Thom isomorphism

$$C_{M'_B} : K_G^0(T_G^*M_A) \rightarrow K_G^0(\tilde{T}_G^*M_A) \cong K_G^0(T_G^*M_A \times M'_B).$$

Denote by i the closed inclusion $M_A \rightarrow M_X$, and, by abuse of notation, also the inclusion $\tilde{T}_G^*M_A \rightarrow T_G^*M_X$ above i . Then i induces the morphisms $i^* : K_G^0(T_G^*M_X) \rightarrow K_G^0(\tilde{T}_G^*M_A)$ and $i_! : K_G^0(T_G^*M_A) \rightarrow K_G^0(T_G^*M_X)$. Combining these 3 maps, we claim that

Lemma 11.10 *Take $\sigma \in K_G^0(T_G^*M_X)$, then $i_! C_{M'_B}^{-1} i^*(\sigma) = \bigwedge_{-1} M_B \otimes \sigma$.*

Since we are working on vector spaces, we can assume that all vector bundles are topologically trivial. Thus we can represent σ as given by a variable linear map $\sigma(v, w, \xi, \eta) : E \rightarrow F$ where E, F are complex representation spaces, $v \in M_A, w \in M_B, \xi \in M'_A, \eta \in M'_B$. Now σ restricts to an element $\tilde{\sigma} := i^*\sigma$ in $K_G^0(T_G^*M_X|M_A)$ which is represented by the map $\sigma(v, 0, \xi, \eta)$. Since $T_G^*M_X|M_A = T_G^*M_A \times M'_B$, the element $\tilde{\sigma}$ is equivalent to $\mathbf{c}_{M'_B} \odot q^*\tau$, where $q : T_G^*M_A \times M'_B \rightarrow T_G^*M_A$ is the projection, τ a transversally elliptic symbol on M_A and $\mathbf{c}_{M'_B}$ the Bott symbol with support the zero section of the bundle $T_G^*M_A \times M'_B$ on $T_G^*M_A$. Thus we have to show that $\bigwedge_{-1} M_B \otimes \sigma$ and $i_!(\tau)$ are homotopic.

By definition, a representative of the symbol $i_!(\tau)$ on $M_A \times M_B$ is the product of the symbol $\mathbf{c}_{M_B \otimes_{\mathbb{R}} \mathbb{C}}$ by the symbol $q^*\tau$. As $\mathbf{c}_{M_B \otimes_{\mathbb{R}} \mathbb{C}} = \mathbf{c}_{M_B} \odot \mathbf{c}_{M'_B}$, we see that $i_!(\tau) = q^*\tau \odot \mathbf{c}_{M_B} \odot \mathbf{c}_{M'_B} = \tilde{\sigma} \odot \mathbf{c}_{M_B}$.

As we have seen before, the symbol defined as $\bigwedge_{-1} M_B \otimes \sigma$ on the manifold $M_A \times M_B$ is homotopic to the element $\mathbf{c}_{M_B} \odot \sigma$. Now consider the symbol $\sigma(t)(v, w, \xi, \eta) = \sigma(v, tw, \xi, \eta)$ on T^*M . The intersection of the support of $\mathbf{c}_{M_B} \odot \sigma(t)$ with T_G^*M stays compactly supported for all t . Indeed its support remains constant: this is the intersection of the support of σ with $T_G^*M_A$. So we obtain the desired homotopy between $\bigwedge_{-1} M_B \otimes \sigma = \mathbf{c}_{M_B} \odot \sigma(1)$ and $i_!(\tau) = \mathbf{c}_{M_B} \odot \sigma(0)$ and the claim follows.

With the previous notations, $X = A \cup B, i : T_G^*M_A \times M'_B \rightarrow T_G^*M_X$.

Corollary 11.11 *Take $\sigma \in K_G^0(T_G^*M_X)$. Let $\sigma_0 = C_{M'_B}^{-1} i^*(\sigma) \in K_G^0(T_G^*M_A)$. Then, we have the equality of generalized functions on G :*

$$det_{M_B}(1 - g) index(\sigma)(g) = index(\sigma_0)(g).$$

We are now ready to compare the exact sequence (35) with a combinatorial exact sequence using the index. Notice that \tilde{Z} is a list of elements in $\Lambda/\mathbb{Z}a$ and functions on

$\Lambda/\mathbb{Z}a$ can be identified to functions on Λ which are invariant under translation by a . We get

Theorem 28 *The diagram*

$$\begin{CD}
 0 @>>> K_{G_a}^0(T_{G_a}^*M_Z^f) @>j_*>> K_G^0(T_G^*M_X^f) @>C_a^{-1}i^*>> K_G^0(T_G^*M_Z^f) @>>> 0 \\
 @. @V ind_m VV @V ind_m VV @V ind_m VV @. \\
 0 @>>> DM(\tilde{Z}) @>i_a>> DM(X) @>\nabla_a>> DM(Z) @>>> 0
 \end{CD} \tag{37}$$

is commutative. Its vertical arrows are isomorphisms.

In particular, the index multiplicity map gives an isomorphism between $K_G^0(T_G^*M_X^f)$ and $DM(X)$.

We start by remarking that, by Corollary 11.8, all the vertical maps in our diagram are indeed taking values in the corresponding Dahmen–Micchelli spaces.

So we need to show commutativity. To prove the commutativity of the square on the right hand side, using Fourier transform, we need to prove that $(1 - g^a) index(\sigma)(g) = index(C_a^{-1}i^*(\sigma))(g)$.

From the symbol σ on M_X^f , an open set in M_X , we deduce a symbol on M_X with same index, by the excision property of the index. Thus the commutativity follows from Corollary 11.11 applied to $A = Z, B = \{a\}$. As for the square on the left hand side, since j_* is an open embedding, it preserves indices. The statement thus follows from Proposition 11.4.

By induction we can then assume that the two external vertical arrows are isomorphism so, by the five Lemma, also the central one is and everything follows.

Summarizing we have isomorphisms

$$\boxed{K_G^{s+1}(M_X^f) \cong K_G^1(T_G^*M_X^f) = 0, \quad K_G^s(M_X^f) \cong K_G^0(T_G^*M_X^f) \cong DM(X).}$$

Let us make two obvious remarks on the isomorphism $K_G^0(T_G^*M_X^f) \cong DM(X)$.

Remark 11.12 The space $K_G^0(T_G^*M_X^f)$ depends only of the manifold M_X considered as a real manifold. The space $DM(X)$ depends only of the list X up to change of signs.

Remark 11.13 If U is a G manifold, the index of an element $\sigma \in K_G^0(T_G^*U)$ is a generalized function supported on the set of points $g \in G$ such that g has a fixed point in U .

We have seen in Sect. 8.3.1 that Fourier transforms of elements in $DM(X)$ are supported on the finite set of points $P(X)$. This is in agreement with the fixed point philosophy that we just recalled. In fact, an element $g \in G$ has a fixed point v in M_X^f if and only if $g \in P(X)$. Indeed if $g \in P(X)$, there exists a basis \underline{b} of V extracted from X with $g^{b_i} = 1$, for all $b_i \in \underline{b}$. Thus any element $v \in M_X$ with non zero coordinates on each L_{b_i} is fixed by g , and is in M_X^f .

We now come to our next commutative diagram.

Lemma 11.14 *For each $s \geq i \geq 0$, the index multiplicity map ind_m sends $K_G^0(T_G^*M_{\geq i})$ to the space $\tilde{\mathcal{F}}_i(X)$.*

Recall that $\tilde{\mathcal{F}}_i(X)$ is the subspace in $\tilde{\mathcal{F}}(X)$ such that $\nabla_{X \setminus \underline{t}} f = 0$ for all $\underline{t} \in S_X^{(i-1)}$. Denote by $\ell : \tilde{\mathcal{F}}_i(X) \rightarrow \tilde{\mathcal{F}}(X)$ the inclusion.

By Corollary 11.9, if $\sigma \in K_G^0(T_G^*M_{\geq i})$ and \underline{t} is a rational subspace of dimension strictly less than i , we have $\bigwedge_{-1} M_{X \setminus \underline{t}} \otimes \sigma = 0$. Thus $\nabla_{X \setminus \underline{t}} ind_m(\sigma) = 0$. It follows that the only thing we have to show is that, if $\sigma \in K_G^0(T_G^*M_X)$, then $ind_m(\sigma)$ lies in $\tilde{\mathcal{F}}(X)$. Take a rational subspace \underline{r} . By Lemma 11.10, the index of $\bigwedge_{-1} M_{X \setminus \underline{r}} \otimes \sigma$ equals the index of an element $\sigma_0 \in K_G^0(T_G^*M_{X \cap \underline{r}})$. But the action of G on $M_{X \cap \underline{r}}$ factors through the quotient $G/G_{\underline{r}}$ whose character group is $\Lambda_{\underline{r}}$. Thus $K_G^0(T_G^*M_{X \cap \underline{r}}) \cong R(G) \otimes_{R(G/G_{\underline{r}})} K_{G/G_{\underline{r}}}^0(T_{G/G_{\underline{r}}}^*M_{X \cap \underline{r}})$, hence $\nabla_{X \setminus \underline{r}} ind_m(\sigma) = ind_m(\sigma_0)$ lies in $R(G) \otimes_{R(G/G_{\underline{r}})} \mathcal{C}[\Lambda_{\underline{r}}]$ as desired.

Our second commutative diagram and main theorem characterizes the values of the index on the entire M_X . This time, we use the notations and the exact sequences contained in Theorem 27 and Corollary 11.11.

Theorem 29 *For each $0 \leq i \leq s$,*

- *the diagram*

$$\begin{array}{ccccccc}
 0 \rightarrow & K_G^0(T_G^*M_{\geq i+1}) & \xrightarrow{j_*} & K_G^0(T_G^*M_{\geq i}) & \xrightarrow{C_i^{-1}e^*} & K_G^0(T_G^*F_i) & \rightarrow 0 \\
 & \text{ind}_m \downarrow & & \text{ind}_m \downarrow & & \text{ind}_m \downarrow & \\
 0 \rightarrow & \tilde{\mathcal{F}}_{i+1}(X) & \xrightarrow{\ell} & \tilde{\mathcal{F}}_i(X) & \xrightarrow{\mu_i} & \bigoplus_{\underline{r} \in S_X^{(i)}} DM^{(G)}(X \cap \underline{r}) & \rightarrow 0
 \end{array}$$

commutes.

- *Its vertical arrows are isomorphisms.*
- *In particular, the index gives an isomorphism between $K_G^0(T_G^*M_X)$ and $\tilde{\mathcal{F}}(X)$.*

Lemma 11.14 tells us that the diagram is well defined. We need to prove commutativity.

We prove that the square on the right hand side is commutative using Corollary 11.11. The square on the left hand side is commutative since j_* is compatible with the index and ℓ is the inclusion.

Recall that $K_G^0(T_G^*M_{X \cap \underline{r}}) \cong R(G) \otimes_{R(G/G_{\underline{r}})} K_{G/G_{\underline{r}}}^0(T_{G/G_{\underline{r}}}^*M_{X \cap \underline{r}})$ and that $DM^{(G)}(X \cap \underline{r}) \cong R(G) \otimes_{R(G/G_{\underline{r}})} DM(X \cap \underline{r})$. Using Theorem 28, this implies that the right vertical arrow is always an isomorphism.

We then apply descending induction on i . When $i + 1 = s$, since $M_{\geq s} = M_X^f$ and $\tilde{\mathcal{F}}_{s-1}(X) = DM(X)$, Theorem 28 gives that the left vertical arrow is an isomorphism. So assume that the left vertical arrow is an isomorphism. We then deduce by the five Lemma that the central vertical arrow is an isomorphism and conclude by induction.

12 Generators

In this section, we show that the generators of $\tilde{\mathcal{F}}(X)$ correspond via the index map to the generators of $K_G^0(T_G^*M_X)$ constructed by Atiyah-Singer in [1].

Recall that, given a manifold M , a way to construct elements of $K_G^0(T_G^*M)$ is to take a closed G -manifold N embedded by $i : N \rightarrow M$. Then we have a map $i_! : K_G^0(T_G^*N) \rightarrow K_{G,c}^0(T_G^*M)$.

For our case $M = M_X$, we shall take the following manifolds. Take a flag ϕ of rational subspaces $0 = r_0 \subset r_1 \subset r_2 \subset \dots \subset r_s$ with $\dim(r_i) = i$ (and $s = \dim G$). Consider then the spaces $E_i := \bigoplus_{a \in (X \cap r_i) \setminus r_{i-1}} L_a$. We choose an orientation for each r_i and divide the set of characters $Z_i := (X \cap r_i) \setminus r_{i-1}$ into positive and negative elements A_i, B_i . Accordingly, we change the complex structure on each L_b for which b is negative into its conjugate structure. Let A be the union of the sets A_i and B the union of the sets B_i . Choose a G -invariant Hermitian metric h_i on E_i and consider the unit sphere $S_i(h_i)$ on E_i . The product $S_\phi(h) = \prod_{i=1}^s S_i(h_i)$ is a closed submanifold of M_X^f .

The tangent space at a point p of $S_\phi(h)$ decomposes as a vertical space generated by the rotations on each factor E_i and the horizontal space H_p , a lift of the tangent space to the corresponding product of projective spaces. The horizontal space is a Hermitian vector space. The tangential Cauchy–Riemann operator δ_ϕ is a differential operator on $S_\phi(h)$, the product of the operators δ_i described in Subsection 10.3. The index of δ_ϕ is the product of the indices of δ_i .

Let $c_\phi(p, \xi) = c(\xi^1)$ be the Clifford action on $\bigwedge H_p$ of the projection ξ^1 of ξ on the horizontal tangent space H_p .

We then have [1] the following theorem.

Theorem 30

$$index(\delta_\phi)(g) = (-1)^s (-1)^{|B|} g^{\sum_{a \in A} a} \theta_\phi^X(g).$$

In fact $c_\phi = c_1 \odot c_2 \odot \dots \odot c_s$ is the external product of the symbols c_i of the operators δ_i .

Let i_ϕ be the closed embedding of $S_\phi(h)$ in M_X^f . We can then give an “easy” proof of the following theorem of Atiyah-Singer (Theorem 7.9 of [1]).

Theorem 31 *Let i_ϕ be the closed injection of $S_\phi(h)$ to M_X . Then the elements $(i_\phi)_! c_\phi$ generate $K_G^0(T_G^*M_X^f)$.*

Consider M_X as a real representation of G , each connected component F of the space of regular elements in $Lie(G)$ gives us a complex structure J_F on M_X and a corresponding “pushed” symbol At_F with $index(At_F)(g) = (-1)^{|B|} g^{\sum_{a \in A} a} \Theta_X^F$ (Theorem 25).

Theorem 32 *The symbols $At_F \in K_G^0(T_G^*M_X)$, where F varies over all open faces of the arrangement \mathcal{H}_X , give us a set of generators for $K_G^0(T_G^*M_X)$.*

Remark 12.1 After checking naturality axioms, Theorem 32 reduces the proof of the cohomological index formula given by [9] or [30] to the case of the symbols At_F .

So Theorem 32 is crucial in establishing a cohomological formula valid for any transversally elliptic operator.

13 Equivariant cohomology and index

The purpose of this section is to establish *cohomological formulas* for the index of transversally elliptic operators (cf. [22] and [21]). We do this through a new invariant, the *infinitesimal index*.

This construction is motivated by taking the Fourier transform of the formula of Berline–Vergne for the equivariant index of a transversally elliptic operator [8, 9, 27] where one can also find the various notations and definitions. As we shall see the combinatorial spaces $DM(X)$ and $\mathcal{F}(X)$ appearing in K -theory and index Theory will be replaced by spaces of distributions completely analogues to the ones appearing in the Theory of splines and giving rise to precise analogues.

13.1 Equivariant de Rham cohomology

Let M be a C^∞ manifold with a C^∞ action of a compact Lie group G , we are going to define its equivariant cohomology with compact support following Cartan (see [24]).

We define the space of compactly supported equivariant forms as

$$\mathcal{A}_{G,c}(M) = (S(\mathfrak{g}^*) \otimes \mathcal{A}_c(M))^G$$

with the grading given setting \mathfrak{g}^* in degree 2. Here $\mathcal{A}_c(M)$ is the algebra of differential forms on M with compact support.

Each element $x \in \mathfrak{g}$ of the Lie algebra of G induces a vector field v_x on M , the *infinitesimal generator* of the action: here the sign convention is that $v_x = \frac{d}{d\epsilon} \exp(-\epsilon x) \cdot m$ in order that the map $x \rightarrow v_x$ be a Lie algebra homomorphism. A vector field V on M induces a derivation ι_V on forms, such that $\iota_V(df) = V(f)$ and for simplicity we denote by $\iota_x = \iota_{v_x}$.

One defines the differential as follows. Given $\alpha \in \mathcal{A}_{G,c}(M)$, we think of α as an equivariant polynomial map on \mathfrak{g} with values in $\mathcal{A}_c(M)$, thus for any $x \in \mathfrak{g}$ we set

$$D\alpha(x) := d(\alpha(x)) - \iota_x(\alpha(x)) \tag{38}$$

where d is the usual de Rham differential.

It is easy to see that D increases the degree by one and that $D^2 = 0$. Thus we can take cohomology and we get the G -equivariant cohomology of M with compact support.

Now take a G -stable closed set Z in a manifold M . Consider the open set $U = M \setminus Z$. Then U is a manifold and we have an inclusion of complexes $\mathcal{A}_{G,c}(U) \subset \mathcal{A}_{G,c}(M)$ given by extension by zero. We set

$$\mathcal{A}_{G,c}(Z, M) := \mathcal{A}_{G,c}(M) / \mathcal{A}_{G,c}(U).$$

Definition 13.2 The equivariant de Rham cohomology with compact support $H_{G,c}^*(Z)$ is the cohomology of the complex $\mathcal{A}_{G,c}(Z, M)$.

Notice that $\mathcal{A}_{G,c}(U)$ is an ideal in $\mathcal{A}_{G,c}(M)$ so $\mathcal{A}_{G,c}(Z, M)$ is a differential graded algebra and $H_{G,c}^*(Z)$ is a graded algebra (without 1 if Z is not compact).

In this model, a representative of a class in $H_{G,c}^*(Z)$ is an equivariant form $\alpha(x)$ with **compact support** on M . The form α is not necessary equivariantly closed on M , but there exists a neighborhood of Z such that the restriction of $\alpha(x)$ to this neighborhood is equivariantly closed.

13.3 Action form and the moment map

Let G be a Lie group and M a G -manifold.

Definition 13.4 An *action form* is a G -invariant real one form σ on M .

The prime examples of this setting are when M is even dimensional and $d\sigma$ is non degenerate. In this case $d\sigma$ defines a symplectic structure on M .

Example 13.5 For every manifold N , we may take its cotangent bundle $M := T^*N$ with projection $\pi : T^*N \rightarrow N$. The canonical action form σ on a tangent vector v at a point (n, ϕ) , $n \in N, \phi \in T_n^*N$ is given by

$$\langle \sigma | v \rangle := \langle \phi | d\pi(v) \rangle.$$

In this setting, $d\sigma$ is a canonical symplectic structure on T^*N and, if $r = \dim(N)$, the form $\frac{d\sigma^r}{r!}$ determines an orientation and a measure, the *Liouville measure* on T^*N . If a group G acts on N , then it acts also on T^*N preserving the canonical action form and hence the symplectic structure and the Liouville measure.

Remark 13.6 If M is a manifold with a G -invariant Riemannian structure, we can consider an invariant vector field instead of a 1-form.

Let v_x be the vector field on M associated to $x \in \mathfrak{g}$ and i_x the derivation on forms induced by contraction with v_x .

Definition 13.7 Given an action form σ we define the *moment map* $\mu_\sigma : M \rightarrow \mathfrak{g}^*$ associated to σ by:

$$\mu_\sigma(m)(x) := -\langle \sigma | v_x \rangle(m) = -i_x(\sigma)(m). \tag{39}$$

for $m \in M, x \in \mathfrak{g}$.

The moment map is related to the equivariant differential of σ as defined in the Cartan model (see Formula (38)).

$$\Omega(x) = D\sigma(x) = \mu(x) + d\sigma.$$

13.8 Infinitesimal index

We fix a translation invariant Lebesgue measure $d\xi$ on \mathfrak{g}^* . We choose a square root i of -1 and define the Fourier transform:

$$\hat{f}(x) := \int_{\mathfrak{g}^*} e^{-i\langle \xi | x \rangle} f(\xi) d\xi.$$

We normalize dx on \mathfrak{g} so that the inverse Fourier transform is

$$f(\xi) = \int_{\mathfrak{g}} e^{i\langle \xi | x \rangle} \hat{f}(x) dx. \tag{40}$$

The measure $dx d\xi$ is independent of the choice of $d\xi$.

If $f(\xi)$ is a C^∞ function on \mathfrak{g}^* with compact support in a ball B_R of radius R in \mathfrak{g}^* (for a choice of Euclidean structure on \mathfrak{g}^*) its Fourier transform $\hat{f}(x)$ is a rapidly decreasing function on \mathfrak{g} .

Theorem 33 *Let σ be an action form with moment map μ . Let $M^0 = \mu^{-1}(0)$. Then we can define a map*

$$\text{infdex}_G^\sigma : H_{G,c}^*(M^0) \rightarrow \mathcal{D}'(\mathfrak{g}^*)^G$$

setting for any $[\alpha] \in H_{G,c}^*(M^0)$ and for any smooth function with compact support f on \mathfrak{g}^*

$$\langle \text{infdex}_G^\sigma([\alpha]), f \rangle := \lim_{s \rightarrow \infty} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx.$$

The map infdex_G^σ is a well defined homomorphism of $S[\mathfrak{g}^*]^G$ modules.

If the one form σ moves along a smooth curve σ_t with moment map μ_t such that $\mu_t^{-1}(0)$ remains equal to M^0 , then

$$\text{infdex}_G^{\sigma_t} = \text{infdex}_G^\sigma.$$

13.8.1. Applications. We apply the Theory to a linear representation M_X of a torus, with list of characters X . Then we have, for the infinitesimal index, cohomological analogues of the K -theoretical results 28 and 29. on the index discussed in Sect. 11.

Theorem 34 *The map infdex is a graded isomorphism as $S[\mathfrak{g}^*]$ -modules of $H_{G,c}^*(T_G^* M_X^{\text{fin}})$ onto $D(X)$.*

Finally as for K -theory and the index we have

Theorem 35 For each $0 \leq i \leq s - 1$,

- (a) For each $h \geq 0$, $H_{G,c}^{2h+1}(T_G^*M_{X,\geq i}) = 0$.
- (b) For each $h \geq 1$, the following sequence is exact

$$0 \rightarrow H_{G,c}^{2h}(T_G^*M_{X,\geq i+1}) \xrightarrow{j_*} H_{G,c}^{2h}(T_G^*M_{X,\geq i}) \xrightarrow{C_i^{-1}e^*} \bigoplus_{\underline{s} \in S_X(i)} H_{G,c}^{2h-2|\underline{s}|}(T_G^*M_{\underline{s}}^f) \rightarrow 0. \tag{41}$$

Let \underline{r} be a vector subspace in \mathfrak{g}^* . We have an embedding $j : \mathcal{S}'(\underline{r}) \rightarrow \mathcal{S}'(\mathfrak{g}^*)$ by $j(\phi)(f) = \phi(f|_{\underline{r}})$ for any $\phi \in \mathcal{S}'(\underline{r})$, f a Schwartz function on \mathfrak{g}^* . We denote the image $j(\mathcal{S}'(\underline{r}))$ by $\mathcal{S}'(\mathfrak{g}^*, \underline{r})$ (sometimes we even identify $\mathcal{S}'(\underline{r})$ with $\mathcal{S}'(\mathfrak{g}^*, \underline{r})$ if there is no ambiguity). We next define the vector space:

Definition 13.9

$$\mathcal{G}(X) := \{f \in \mathcal{S}'(\mathfrak{g}^*) \mid \partial_{X \setminus \underline{r}} f \in \mathcal{S}'(\mathfrak{g}^*, \underline{r}), \text{ for all } \underline{r} \in S_X\}. \tag{42}$$

The following theorem characterizes the values of the infinitesimal index on the entire M_X . This time, we use the notations and the exact sequences contained in Theorem 27 and Corollary 11.11.

Theorem 36 For each $0 \leq i \leq s$,

- the diagram

$$\begin{array}{ccccccc} 0 \rightarrow & H_{G,c}^*(T_G^*M_{X,\geq i+1}) & \xrightarrow{j_*} & H_{G,c}^*(T_G^*M_{X,\geq i}) & \xrightarrow{C_i^{-1}e^*} & H_{G,c}^*(T_G^*M_{=i}) & \rightarrow 0 \\ & \text{infdex} \downarrow & & \text{infdex} \downarrow & & \text{infdex} \downarrow & \\ 0 \rightarrow & \tilde{\mathcal{G}}_{i+1}(X) & \xrightarrow{\ell} & \tilde{\mathcal{G}}_i(X) & \xrightarrow{\mu_i} & \bigoplus_{\underline{s} \in S_X(i)} D^{\mathfrak{g}}(X \cap \underline{s}) & \rightarrow 0 \end{array}$$

commutes.

- Its vertical arrows are isomorphisms.
- In particular, the infinitesimal index gives an isomorphism between $H_{G,c}^*(T_G^*M_X)$ and $\tilde{\mathcal{G}}(X)$.

13.9.1. Index and infinitesimal index. Our final task is to relate all commutative diagrams, of index and K -theory on one hand, infinitesimal index and cohomology on the other, in a unique commutative diagram involving the Chern character on one hand and Poisson summation formula on the other. The box spline now appears in Fourier transform since the following result is well known.

$$\text{ch}(\text{Bott}(M_X))(x) = (2i\pi)^{|\underline{X}|} \prod_{a \in X} \frac{e^{i\langle a, x \rangle} - 1}{i\langle a, x \rangle} \text{Thom}(M_X)(x)$$

in the cohomology group of smooth equivariant differential forms.

We denote by X_R the sequence $X \cup -X$ of characters. Remark that the zonotope associated to X_R contains 0 in its closure.

Theorem 37 (Convolution theorem) *Let $X \subset \Lambda$ be a system of characters of G . Let*

$$X_R = X \cup -X.$$

Let Σ be a G -invariant transversally elliptic symbol on M . Let $\text{ind}_m(\Sigma) \in \mathcal{C}_{\mathbb{Z}}[\Lambda]$ be its multiplicity index. Let $\text{infdex}_G^{-\mu}(\text{ch}(\Sigma))$ be the infinitesimal index of its Chern character. Then

$$B_{X_R} *_d \text{ind}_m(\Sigma) = (2i\pi)^{-2|X|} \text{infdex}_G^{-\mu} \text{ch}(\Sigma).$$

We need thus a *Deconvolution theorem* which we state in the unimodular case (for the general case see [22] and [21]).

Theorem 38 (Deconvolution theorem) *Assume that X is unimodular. Let c be an alcove in V containing 0 in its closure and contained in $Z(X)$. Then*

- (i) $\lim_c(\text{Todd}(X)_{pw} B_X) = \delta_0$.
- (ii) For any $K \in \mathcal{C}[\Lambda]$,

$$K = \lim_c(\text{Todd}(X)_{pw} (B_X *_d K)).$$

Using the deconvolution theorem in the unimodular case, Theorem 37 leads to the following theorem, which is strongly reminiscent of the Riemann–Roch theorem. Remark that as X_R contains 0 in its interior, we may use any alcove containing 0 in its closure in the limiting procedure.

We denote by $\text{Todd}(X_R)$ the Todd operator associated to X_R . It acts on the space of piecewise polynomial functions for the system (X, Λ) .

Theorem 39 *Let $X \subset \Lambda$ be a unimodular system of characters of G . Let*

$$\text{Todd}(X_R) = \prod_{a \in X \cup -X} \frac{\partial_a}{1 - e^{-\partial_a}}$$

be the Todd operator.

Let Σ be a G -invariant transversally elliptic symbol on M , $\text{ind}_m(\Sigma) \in \mathcal{C}_{\mathbb{Z}}[\Lambda]$ be its multiplicity index and $\text{infdex}_G^{-\mu}(\text{ch}(\Sigma))$ be the infinitesimal index of its Chern character. Then

- $\text{infdex}_G^{-\mu}(\text{ch}(\Sigma))$ is a piecewise polynomial measure on \mathfrak{g}^* .
- Let c be an alcove having 0 in its closure. We have

$$\text{ind}_m(\Sigma) = (2i\pi)^{-2|X|} \lim_c \text{Todd}(X_R)_{pw} \text{infdex}_G^{-\mu}(\text{ch}(\Sigma)).$$

Remark 13.10 It is possible to show in this unimodular case that the piecewise polynomial function $(2i\pi)^{-2|X|} \text{Todd}(X_R)_{pw} \text{infdex}_G^{-\mu}(\text{ch}(\Sigma))$ extends to a continuous function on \mathfrak{g}^* . Thus its restriction to Λ gives the index multiplicity.

We formulate now the general index theorem. We denote by $P(X) \subset G$ the points of the arrangement.

Theorem 40 *Let X be a sequence of elements in Λ and let $M := M_X$. Let*

$$X_R = X \cup -X.$$

Let Σ be a G -invariant transversally elliptic symbol on M and $\text{ind}_m(\Sigma) \in \mathcal{C}_{\mathbb{Z}}[\Lambda]$ be its multiplicity index.

For any $g \in P(X)$, let $\text{index}_G^{-\mu}(\text{ch}^g(\Sigma))$ be the distribution on \mathfrak{g}^ associated to the cohomology class $\text{ch}^g(\Sigma) \in \mathcal{H}_{G,c}^{\infty,m}(T_G^*M^g)$ by the infinitesimal index. Then*

- $\text{index}_G^{-\mu}(\text{ch}^g(\Sigma))$ is a piecewise polynomial measure on \mathfrak{g}^* .
- Let c be an alcove having 0 in its closure. We have

$$\text{ind}_m(\Sigma) = \sum_{g \in P(X)} (2i\pi)^{-2|X^g|} \hat{g} \lim_c D(X_R \setminus X_R^g, g)^{-1} \quad (43)$$

$$\text{Todd}(X_R^g) *_{pw} \text{index}_G^{-\mu}(\text{ch}^{g^{-1}}(\Sigma)).$$

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