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Point-free foundation of geometry looking at laboratory activities

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Abstract: Researches in "point-free geometry", aiming to found geometry without using points as primitive entities, have always paid attention only to the logical aspects. In this paper, we propose a point-free axiomatization of geometry taking into account not only the logical value of this approach but also, for the first time, its educational potentialities. We introduce primitive entities and axioms, as a sort of theoretical guise that is grafted onto intuition, looking at the educational value of the deriving theory. In our approach the notions of convexity and half-planes play a crucial role. Indeed, starting from the Boolean algebra of regular closed subsets of $\mathbb{R}^n$, representing, in an excellent natural way, the idea of region, we introduce an *n-dimensional prototype of point-free geometry* by using the primitive notion of convexity. This enable us to define *Re*-half-planes, *Re*-lines, *Re*-points, polygons, and to introduce axioms making not only meaningful all the given definitions but also providing adequate tools from a didactic point of view. The result is a theory, or a seed of theory, suitable to improve the teaching and the learning of geometry.

Subjects: 97E40; 97E50; 97G40; 97D40; 03B30

Keywords: point; point-free geometry; theory; axioms; region; convex; Re-half-planes; regular; laboratory activities

ABOUT THE AUTHOR

Point-free geometry represents a crucial topic in the research activity of the authors. Giangiacomo Gerla, Emeritus Professor at the Mathematics Department of the University of Salerno, is one of the greatest experts in this field. Annamaria Miranda is researcher at the same Department, her interests include point-free geometry and topology as well. Their papers, devoted to a point-free axiomatization of geometry, are part of a series of research that have always focused attention only to the logical aspects of the theory. In this paper, on the other hand, they propose an axiomatization of geometry taking into account not only the logical value of this approach but also its educational potentialities. It is only the first step in the development of a project aiming just to explore these potentialities. Primitive notions and axioms are simple and intuitive and could be experimented by building teaching-learning laboratory activities for students at all levels.

PUBLIC INTEREST STATEMENT

Research on the foundation of geometry shows that the historically dominant choice of assuming "points" as primitive entities is not the only possible one. An alternative choice is proposed by "Point-free geometry", a geometry in which the notion of "solid body", or, more in general, "region", is assumed as primitive whereas points, lines and plane surfaces are defined. The main motivation of these researches is ontological in nature since the existence of solid bodies in the space looks more acceptable than the existence of points. In this paper we propose an approach to Point-free geometry by introducing primitive entities and axioms, as a sort of theoretical guise that is grafted onto intuition, looking at the educational potentialities. The perspective to explore a deductive-hypothetical model, and, at a critical level, to compare different ways to propose geometry, should provide a cultural heritage useful not only for future teachers' training but also in everyone's life.

1. Introduction

The name "point-free geometry" denotes a series of researches on foundation of geometry in which the main primitive notion is the one of "solid body" (or, better, "region"), while "points", "lines", "planes" are defined. The main motivation of these researches is ontological in nature since the existence of solid bodies in the space looks more acceptable than the existence of points, lines, planes. In particular, an entity without extension as a point is not conceivable and it is inconceivable that three-dimensional entities are formed by entities without extension. Regardless of the validity of these motivations, the researches on point-free geometry show that the historically dominant choice of assuming points as primitive entities is only one of the possible choices (maybe the best one) and that alternatives are possible. In our opinion this fact alone is sufficient to show the importance of point-free geometry.

This paper starts from the observation that, as far as we know, the existing researchers in point-free geometry paid attention only to the logical aspects of the proposed theory (consistency, categoricity, independence and so on). However, in our opinion, it is interesting to search for point-free axiomatizations taking into account also their possible educational potentialities. We propose simple and intuitive primitive notions and axioms whose validity can be experimented through appropriate laboratory activities.

> It is thanks to a relationship between epistemological and didactic reflections on mathematics that we arrive at the debate on the first elements, to understand how there is no coincidence between primitive elements for a novice student and primitive terms in mathematics (D'Amore & Fandiño Pinilla, 2009).

In this paper, we are interested in the possibility of identifying a system of primitive notions and axioms for a point-free geometry expressing in a direct way the childhood spatial experiences. We refer to experiences inside and outside schools. In particular, we refer to activities devoted to develop the geometrical intuition. We share the following claim.

> Experimental and intuitional methods are not identical. … Take the equality of vertically opposite angles. If I measure the angles I am proceeding experimentally; if I open out two sticks crossed in the form of an X, and say that it is obvious to me that the amount of opening is equal on the two sides, then I am using intuition. (Godfrey & Siddons, 1931, p. 21)

The choice of the point-free approach is in accordance with the following critical remark.

> The tendency to reproduce the Euclidean approach in basic school continues today; many teachers introduce this discipline starting from concepts such as points, lines and planes which are important for a rational treatment, but distant from the student's experience or from definitions that should instead be considered as a point of arrival of a constructive, personal learning process (Arrigo & Sbaragli, 2004).

We propose a point-free theory that might be treated not only at University, in a course devoted to analyze the question of an axiomatic foundation of geometry, a course we consider essential for future teachers, but it could be also introduced in high school, as a rational arrangement of space exploration activities and, under teacher mediation, in primary and secondary school students. In the latter case, we are talking about a theory that should not be directly offered but it could be experimentally derived, even partially, by children during classroom activities. In choosing primitive notions we always look at possible building of teaching-learning laboratory activities. For instance, in plane geometry some interesting possibilities are related to folding and to equidecomposability activities and therefore to artifacts such as scissors and paper sheets.

Our hope is that in an experimental teaching-learning activity we can realize the social construction of a "germ-theory":

> It is still an initial theory built gradually, but with a potential for expansion and a tendency to develop into a complete theory. In other words, it contains some statements which, although they do not exhaust the axioms traditionally assumed in one of the possible theoretical arrangements allow the production of conjectures or construction methods and their proofs (Bartolini Bussi, 1999).

A germ of theory, therefore, although playing from a logical point of view the same role as a theory, does not aim to establish a branch of mathematical knowledge (Ferrari & Gerla, 2015). We hope to find a formal basis, a theory or a seed of theory, which provides tools that help the teaching and the learning of geometry.

Another reason leading us to consider the point-free approach to geometry in a teaching-learning perspective is the gap between the theoretical approach to geometry proposed at school and the geometric knowledge in the reality.

The importance of the study of the contrast between the learning of mathematics at school and the extra-scholastic or pre-scholastic mathematical knowledge, between capacity to work with mathematics "taught" at school and capacity to use mathematics in a spontaneous way in reality is emphasized in the following statement.

> A clear gap between the mathematically rich situations, mainly in the numerical field, that children experience in out-of-school and the classroom practice." (Bonotto, 2001).

We think that this contrast occurs also between knowledge and intuitions acquired in the activities and laboratories of the first years of life, during the pre-school, and the theoretical knowledge of the following years.

We believe that the existing gap between simple and intuitive primitive notions coming from experience and abstract entities proposed at school, such as points and lines, could be sought in the context of point-free geometry.

The researches in point-free geometry originate from the proposals by S. Leśniewski and the Polish logic school (Leśniewski, 1992) and from the analysis by A. N. Whitehead (Whitehead, 1919, 1920). Focusing on what Whitehead said, the notion of "event" and of "inclusion" of events are assumed as *primitive*. Instead, the points (and other "abstract" entities such as lines and surfaces) are defined by "abstraction processes". Several papers proposing various kinds of point-free axiomatic approaches to geometry have been produced in this last century, as we will emphasize in Section 2.

The novelty of our approach lies in the fact that in choosing primitive notions and axioms we always look at possible building of teaching-learning laboratory activities.

In our approach, inspired by Sniatycki (1968), the notions of convexity and half-planes play a crucial role. Indeed, in Section 3, we introduce an *n-dimensional prototype of point-free geometry* by using the primitive notion of convexity. The 2-dimensional prototype, briefly denoted by *PFP*, enable us to define the notion of *Re*-half-plane, fundamental to define a lot of other concepts, such as *Re*-lines, *Re*-points, polygons. Moreover, it gives us the possibility to face with success the important question to put an order on a Re-line.

The theory *T*, introduced in Section 5, able to capture the prototypical point-free model, has been deduced starting from the following statements.

(a) *T* must at least make meaningful the definitions

(b) The axioms in *T* must be satisfied in *PFP* so that every theorem of *T* is valid in this model.

(c) *T* must be categorical.

(d) The axioms in *T* must refer to properties of regions and ovals as directly as possible (evident and intuitive)

(e) The axioms have to be satisfactory from a didactic and not only from a logical point o

Some open problems related to this point of view emerged. The paper is only the first step in a research project that will develop in different directions.

2. From seminal books by A. N. Whitehead to formal theories

Thanks to the omnipresence of analytic geometry, in the present time the role of the points is absolute. Indeed, lines, planes, and all the geometric elements are defined as sets of points satisfying certain algebraic conditions. The role is also absolute in the metrical approaches to geometry and to the ones based on the notions of betweenness or equidistance. Nevertheless, in general, the axiomatization of geometry, in accordance with Euclid and Hilbert, requires also to assume as a primitive the notions of line and plane. Traces of this point of view are also in frequent expressions as "the point *P lies* on the line *r*" or "the line *r passes* through the point *P*" (instead of "the point *P belongs* to the line *r*"), "the line *r lies* on the plane *σ*" (instead of "the line *r is included* in the plane *σ*"), the two circles *meet* at a point *P*" (instead of "there is a point *P* that *belongs* to both the two circles").

As we have already said, in point-free geometry one proceeds in a totally different way since *the only primitive notion is that of region*[1] while points are defined as suitable sets of regions. In other words, with a radical reversal of the usual point of view, instead of considering a region as a set of points, one defines a point as a set of regions. Then, according to the language of mathematical logic, a region is a first-order object and a point a second-order object.

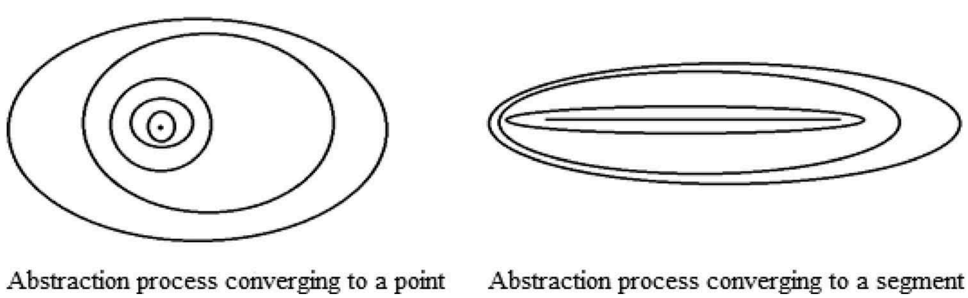
The researches in point-free geometry originate from the proposals by S. Leśniewski and the Polish logic school (Leśniewski, 1992) and from the analysis by A. N. Whitehead (Whitehead, 1919, 1920). Focusing on what Whitehead said, the notion of "event" (understood as a region in a four-dimensional space) and of "inclusion" of events are assumed as *primitive*. Instead, the points (and other "abstract" entities such as lines and surfaces) are defined by "abstraction processes".

Let us assume the notion of *region* and the *inclusion* relation as primitive.

Definition 2.1. An *abstraction process* is an order-reversing sequence $(r_n)_{n \in N}$ of regions such that there is no region r such that $r \le r_n$ for all $n \in N$ (see Figure 1).

We denote by *AP* the class of abstraction processes.

Figure 1. Some abstraction processes.



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In some sense, an abstraction process represents the "abstract limit" to which the process converges. In Figure 1 we illustrate an abstraction process that "converges" to a point and an abstraction process that "converges" to a segment. As a matter of fact, Whitehead does not deal with sequences but with classes of regions. We refer to sequences because this most closely resembles the method of constructing real numbers through Cauchy sequences of rational numbers. Both of them rely on identifying an endless approximation process with the object you want to approximate.

Of course, as in the case of Cauchy sequences, two different processes may characterize the same abstract entity. For example, in Figure 2 the sequence of squares and one of the circles "converges" to the same point. Then, Whitehead proposed the following definition of a pre-order and, consequently, of an equivalence relation. This definition is probably inspired by the fact that if $(X_n)_{n\square N}$ and $(Y_n)_{n\square N}$ are two decreasing sequences of subsets of a given set and if for every Y_n there exists X_m such that $X_m \leq Y_n$, then $\cap_{n\square N} X_{n\square N} \subseteq \cap_{n\square N} Y_n$.

Definition 2.2. Given two abstraction processes $(x_n)_{n\boxtimes N}$ and $(y_n)_{n\boxtimes N}$, we define the relation $\leq^*$ by setting $(x_n)_{n\boxtimes N} \leq^* (y_n)_{n\boxtimes N}$ if for every y_n there exists x_m such that $x_m \leq y_n$. We define the relation $\equiv^*$ by putting $(x_n)_{n\boxtimes N} \equiv^* (y_n)_{n\boxtimes N}$ if $(x_n)_{n\boxtimes N} \leq^* (y_n)_{n\boxtimes N}$ and $(y_n)_{n\boxtimes N} \leq^* (x_n)_{n\boxtimes N}$. In this case we say that the two sequences are *equiconvergent*.

Since $\leq^*$ is a pre-order, $\equiv^*$ is an equivalence relation in the class *AP* of abstraction processes. Also, in the quotient $AP/\equiv^*$ an order relation $\leq^{**}$, is defined putting $[(x_n)_{n\square N}] \leq^{**} [(y_n)_{n\square N}]$ if and only if $(x_n)_{n\square N} \leq^* (y_n)_{n\square N}$. In this case, we say that $[(x_n)_{n\square N}]$ is a *part of* $[(y_n)_{n\square N}]$.

Definition 2.3. The elements of the quotient $AP/\boxtimes^*$ are named *"abstract geometric entities"*. We call *point* an abstract geometric entity minimal with respect to $\leq^{**}$.

It should be noted that the request that no region is contained in all the regions of the process stems from the fact that we want to avoid abstract geometric figures containing a region, since we want the abstract geometric figures to be smaller than the regions. Now, this way of proceeding has some technical difficulties as this example shows.

In the Euclidean plane denote by G_{-O}, G_O and G_{+O} the abstractive sequences of open balls with radius $1/n$ and centre in $(-1/n,0)$, $(0,0)$ and $(1/n,0)$, respectively (see Figure 3). From an intuitive point of view, G_O represents the point $O \equiv (0,0)$. Unfortunately, $[G_O]$ is not a point since G_O covers both G_{-O} and G_{+O} but these abstractive sets are not equivalent with G_0.

Figure 2. Two equivalent abstraction processes.

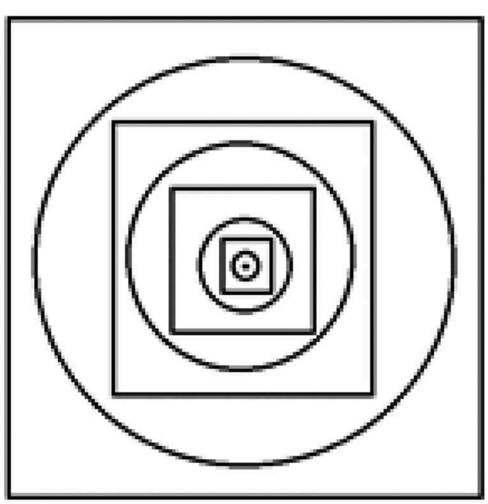
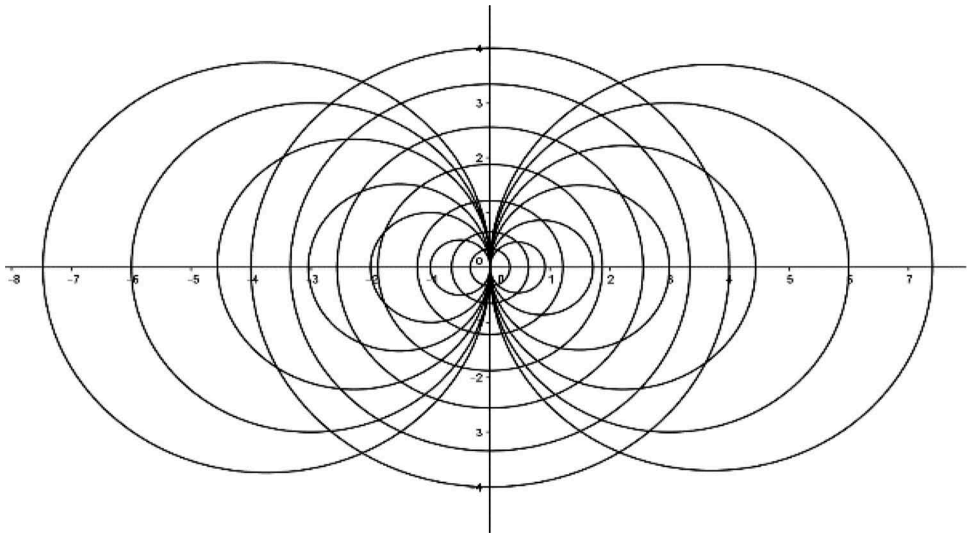


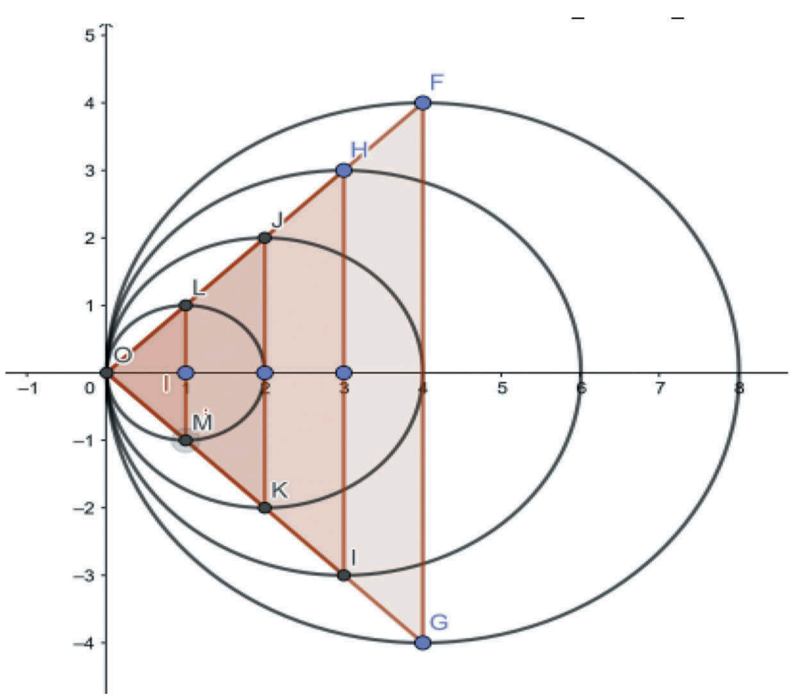
Figure 3. Non-minimal abstract geometric entities.



On the other hand, neither $[G_{-O}]$ nor $[G_{+O}]$ are points. In fact, denote by OA_nB_n the triangle defined by the points $O = (0,0)$, $A_n = (1/n,1/n)$, $B_n = (1/n, -1/n)$ (see Figure 4), then $<OA_nB_n>_{n\in N}$ is an abstractive sequence which is covered by G_{+0} but is not equivalent to G_{+0}. This means that the geometrical element $[G_{+0}]$ is not a point since it is not minimal in the class of geometrical elements. In a similar way, one proves that $[G_{-O}]$ is not a point. Again, there is no difficulty to prove that $[<OA_nB_n>_{n\in N}]$ is not a point, and one begins to suspect that Whitehead's definition of point is empty.

These difficulties led Whitehead to formulate in Process and Reality (Whitehead, 1929) a different notion of abstraction process. Indeed, he considers as a primitive the topological relation

Figure 4. An abstractive class of triangles covered by an abstractive class of tangent balls.



of connection (i.e. to be either overlapping or in contact) as a tool to define the non-tangential inclusion.[2] This will allow a definition of abstraction process in which the inclusion is non-tangential.

It should be kept in mind that Whitehead's research does not want to have a mathematical character but only to be a qualitative analysis of how points, lines and all "non-concrete" entities can be defined through the regions (namely through the "events", i.e. objects in 4-dimension space). However, this analysis has inspired a long series of researches, mathematical in nature, on point-free geometry.

The first rigorous mathematical paper is due to the famous Polish logician Alfred Tarski who in a few pages proposes a point-free theory based only on the notions of inclusion and sphere (Tarski, 1929). Then, Tarski studies structures as $(Re, \leq, S)$ where

- Re is intended as the class of "regions",
- the relation $\leq$ as "inclusion"
- the subset S of Re as the class of "spheres".

Obviously, suitable axioms are imposed to this class of structures. For example, one requires that $(Re, \leq)$ is an order relation. The main step of Tarski's paper is the definition of various types of tangency between spheres in order to define in S the relation of "being concentric" we denote by $\equiv$. This relation is an equivalence (Gruszczyński & Pietruszczak, 2008) and this enables us to define a point as an element of the quotient of S modulo $\equiv$. In other words, a point is a complete class of concentric spheres. Successively, Tarski defines in the set of points the ternary relation "X and Y are equidistant from Z". This allows him to "cannibalize" the proposal for a foundation of geometry made by M. Pieri based on this relationship (Pieri, 1908). Indeed, by the following postulate he adds the whole Pieri's system of axioms to the axioms referring directly to spheres and regions properties.

Postulate 1. (Cannibalization axiom) The defined notions of point and equidistance of two points from a third point satisfy Pieri's postulates of three-dimensional Euclidean geometry.[3]

This postulate has been named by us "cannibalization axiom" since Tarski constructs his theory starting from Pieri's axioms transforming them in axioms of his own language, and therefore in formulas expressing properties of the structures $(Re, \leq, S)$.

After Tarski's paper, research focused on *point-free topology* in which a topology could be viewed from a lattice theoretical point of view. Indeed, the class of open sets is a complete lattice satisfying an infinitely distributive property. However, it is evident that topological notions alone are not sufficient to establish Euclidean geometry. A point-free metric space theory was proposed by F. Previale (1966a). In this case, the lattice is equipped with a diameter function and this allows elegant definitions of point and of a metric in the set of points. A further step (Gerla, 1990) was done considering both diameter and distance between regions as undefined notions (see also Di Concilio & Gerla, 2006). Every model of the proposed theories is associated with a metric space. This makes possible to obtain an implicit point-free axiomatization of Euclidean geometry by cannibalizing, for example, the metric-based foundation of geometry proposed in (Blumenthal, 1970).

The literature on point-free geometry is rather large and it is not possible to cite all the papers on this subject. A survey is contained in two papers (Gerla, 2019; Gerla & Miranda, 2008). Various routes to define points have been crossed (see for example, Coppola et al., 2010; Gerla, 2001; Gerla & Miranda, 2004; Gerla & Tortora, 1992). This paper is partially inspired by the approach proposed

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by Sniatycki (1968) and by a successive paper (Gerla & Gruszczyński, 2017) in which both the notions of half-plane and convex region play a crucial role.

3. A possible mathematical models of the notion of region

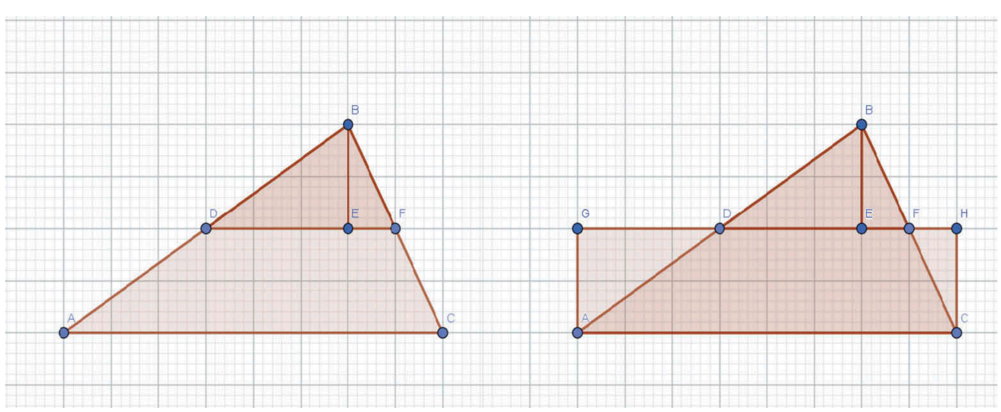
Two options are possible to identify an adequate system of axioms for a “geometry of regions”. The first one consists in referring to an intuitive idea of region and in proposing axioms that reflect this intuition. It is the way followed by Euclid to construct his theory. The second option, we adopt in this paper, is to lean on the existing analytical geometry and therefore on the linear structure of $\mathbb{R}^n$ (where $\mathbb{R}$ is the ordered field of real numbers). Specifically, the tools provided by □ ^{n} will help us to construct a structure whose domain is a particular class of subsets of $\mathbb{R}^n$ which we will call *regions*. This structure will then be used as a guiding structure for the formulation of an axiomatic theory whose axioms will be chosen among the properties valid in it. Notice that it is no longer shameful to lean on a structure for constructing an alternative to the structure itself. Non-Euclidean geometries developed in a similar way by constructing their own non-Euclidean models within the classical Euclidean plane.

Now to give a rigorous definition of region in $\mathbb{R}^2$ it is necessary to first establish what topological properties are satisfied by the regions. For instance, is a triangle, intended as a set of points, is an open set? To address this question, assume that T is a set of point in $\mathbb{R}^n$ candidate to be considered *a triangle*, then its interior $i(T)$ and its closure $c(T)$ have the same sides as T. It follows that, by the congruence principle, there is an isometry between $i(T)$ and $c(T)$. But this is an absurdity since an isometry is a homeomorphism. This forces us to assume that either every triangle is open or every triangle is closed. More in general that either all the regions are open sets or all the regions are closed sets. Unfortunately, this creates difficulties in defining the notion of equidecomposability. Indeed, let *ABC* be a triangle where *AC* is a base (see Figure 5). In order to prove the well-known formula for the area, one proceeds as follows:

(1) *“Cut into pieces”* the triangle: first making a horizontal cut *DF* at half the height and then a vertical cut *BE* in the obtained triangle *DBF*. In this way, we obtain two triangles *DBE* and *EBF* and the trapezoid *ADFC*.
(2) *“Rotate”* the triangle DBE around the point *D* and the triangle *EBF* around point *F* so that we obtain the rectangle *AGHC* that has the same base as the triangle *ABC* and half the height of the triangle *ABC*.

The famous formula for the area of a triangle follows. Now, if we assume that all the regions are closed set of points, then the regions *DBE, EBF* and *ADFC* are not a partition since they have common sides.[4] If we assume that these regions are open, then they do not define a partition since their union is strictly contained in *ABC*.

Figure 5. An equidecomposability for the triangle.



Of course, there are various tricks to solve this kind of problems. For example, it is possible to change the notion of partition by calling *partition* of a shape *F* a class P of shapes whose union is *F* and such that the interiors of two different elements in P are disjoints. However, such a solution and other "ad hoc" solutions involve topological notions which are out of proportion with the immediacy of *"cutting shapes into several parts"*.

Notice that if we interpret an isometry as the result of a movement, it is easy to find further paradoxes. For example, in the rotation around the points *D* and *F* the point *E* splits into two points *G* and *H*. Since rectangles are closed sets, no hole is admitted in *GH*, and this means that the point *E* remains in its position. Therefore, a point becomes three points, in a sense. It should be noted that this kind of paradoxes, which can be easily extended in three-dimensional spaces, has always interested the philosophers. Aristotle posed an analogous problem (Metaphysics 3.5, 1002a28—b11):

> For as soon as bodies come into contact or are divided, the boundaries simultaneously become one if they touch and two if they are divided. Hence, when the bodies have been put together, one boundary does not exist, but has ceased to exist, and when they have been divided, the boundaries exist which they did not exist before …

It is the case to recall the famous citation related with the contact of two surfaces.

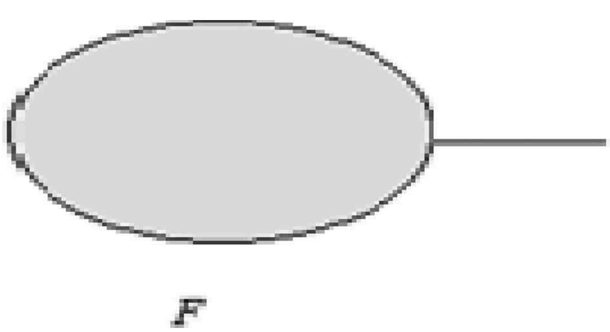
> What does it separate air from water? Is it air or water?
> (Leonardo da Vinci: Codex Atlanticus, UTET 1966: 546)

Now we go forward in searching for a class *CR* of subsets of $\mathbb{R}^n$ we can consider acceptable candidates to represent the notion of region.

Firstly, in accordance with the previous observation, it is possible to assume, for example, that all the subsets in *CR* are closed.[5] Obviously, in the case of the space $\mathbb{R}^3$ this class must contain the cubes, the spheres and all the three-dimensional solids usually considered in geometry. Also, since we imagine a region as a part of the space that can be occupied by a solid body, we have to require that points, lines and surfaces are not in *CR*. Moreover, we have to exclude "mixed" figures such as a sphere with a one-dimensional "tail" as the set in Figure 6. Indeed, it is difficult to imagine a part of a solid body able to fill the segment. Obviously, it should be useful that *CR* is the domain of a suitable algebraic structure. Similar considerations hold true for the regions in the plane $\mathbb{R}^2$. A possible definition of region in accordance with these conditions is furnished by the notion of closed regular subset (Tarski's proposal).

Definition 3.1. We call *regulator* the operator $r: P(\mathbb{R}^n) \to P(\mathbb{R}^n)$ defined by setting $r(X) = c(i(X))$ where *i* and *c* are the interior and the closure operators in the topological space $\mathbb{R}^n$, respectively. We call

Figure 6. A non-regular set.



closed regular (in brief, *regular*) a fixed point of *r*, and we denote by CR_n the class of the closed regular subsets of $\mathbb{R}^n$.

Then, a subset *F* of $\mathbb{R}^n$ is closed regular if it coincides with the closure of its interior. For every figure *F* we have that *r*(*r*(*F*)) = *r*(*F*) and therefore *r*(*F*) is regular (see Figure 7).

Notice that the notion of *regulator* operator can be potentially applicable to the study of confining regions extending the previous version of linear spaces (see for example, Shang, 2017).

Moreover, it should also be noted that the choice of representing the notion of region by the closed regular subsets is not shared by some authors (see for example, Lando & Scott., 2019).

The first important property of CR_n is given by the following theorem.

Theorem 3.2. Let $B_n = (CR_n, \subseteq)$ be the ordered class of regular closed subsets of a topological space $\mathbb{R}^n$. Then B_n is an atomless complete Boolean algebra in which

- $\emptyset$ and $\mathbb{R}^n$ are the minimum and the maximum, respectively
- the Boolean operations ·, +, ~ are defined by setting,

$$X \cdot Y = r(X \cap Y);\ X + Y = r(X \cup Y); \sim(X) = r(-X)$$

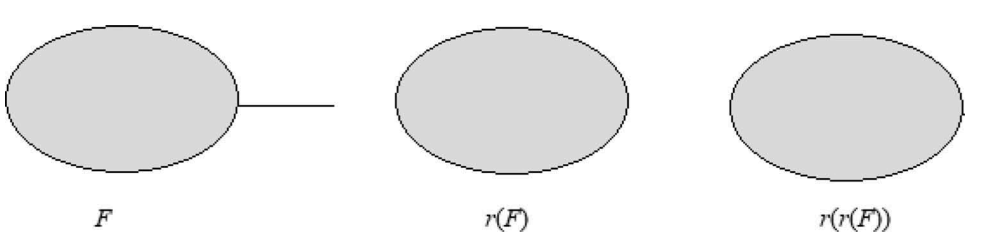
In other words, the Boolean operations are defined by regularizing the set-theoretical operations.[6] In point-free geometry, regular closed sets are frequently assumed as a reference model for the idea of a region. Indeed, it is immediate that all the desired properties are satisfied by the Boolean algebra $(CR_n, \subseteq)$. Moreover, the decomposition of the triangle given in Section 3 is actually a "partition" when we define this notion by the Boolean operations in Theorem 3.2 instead of the set-theoretic operations. Indeed, the product of two different pieces is the empty set since it is the closure of the interior of a segment. Moreover, the sum of all pieces coincides with the whole triangle. This is in complete accordance with the intuition for which by cutting a figure by a pair of scissors, we obtain a partition of this figure.

Denote by L a language containing symbols ≤, +, ·, ~, 0, *U* (the usual ones for Boolean algebra theory), a monadic predicate symbol *CO* (to represent the convexity property) and the relation symbol ≡ (to represent the congruence). The intended interpretation is that objects of which L speaks are regions, ≤ is the inclusion, +, ·, ~ correspond to the union, intersection, complement, 0 and *U* are interpreted by the empty region and the universe, respectively and, finally, *CO* as the convexity property.

Definition 3.3. Given $n \in N$, we call *n-dimensional prototype of point-free geometry* the interpretation of L defined by the structure $(B_n, CO_n, \equiv)$ where:

- B_n denotes the Boolean algebra of regular closed subsets of $\mathbb{R}^n$,
- CO_n is the set of convex elements of B_n,
- ≡ is the usual congruence relation.

Figure 7. The fixed point r(F).



We denote by *PFP* the 2-dimensional prototype.

When we refer to this interpretation we use terms as "*Re-union*", "*Re-intersection*", "*Re-complement*" for the interpretations of +, ·, ~, respectively. Prefix *Re* reminds us that the elements of the structure are intended to be regular subsets and that the operations are not necessarily set-theoretical in nature.

By these structures, we obtain an elegant solution to the difficulties discussed in Section 2.

Definition 3.4. We say that two regions x and y are *Re-disjoint* if $x{\cdot}y = 0$. Given a region x, a set $\{x_1, \ldots, x_n\}$ of non-empty regions is a *Re-partition of x* if the elements of this set are mutually *Re*-disjoint and x is their *Re*-union, that is $x = x_1+ \ldots +x_n$. A *Re*-partition is said to be *convex* if every element in the partition is convex.

Hence, two regions x and y are *Re*-disjoint if the only regular closed set contained in both them is the empty region. In $(B_n, CO_n, \equiv)$ this means that two regions sharing only a common boundary (therefore not disjoint) are *Re*-disjoint.

Definition 3.5. Two regions x and x are called *Re-equidecomposable* if there exists a *Re*-partition $\{x_1, \ldots, x_n\}$ of x and a *Re*-partition $\{x_1, \ldots, x_n\}$ of x with $x_i \equiv x_i$ for $i = 1, \ldots, n$.

The triangle and the rectangle mentioned in this Section are *Re*-equidecomposable according to this definition.

4. Defining half-planes, lines, points and polygonal regions

In what follows we will examine the expressive power[7] of L with respect to the interpretation provided by *PFP*, our two-dimensional prototype. It is evident that, in an analogous way, we can examine the expressive power in higher dimensions.

A *region* is a regular closed subset of $\mathbb{R}^2$.

Definition 4.1. A *Re-half-plane* is a nonempty convex region h whose complement $\sim h$ is a convex nonempty region. We denote by *HP* the class of *Re*-half-planes. A *Re-line* is a convex *Re*-partition of the universe $\mathbb{R}^2$ consisting of a *Re*-half-plane and its complement $\{h, \sim h\}$. In this case h and $\sim h$ are called *sides of l*.[8]

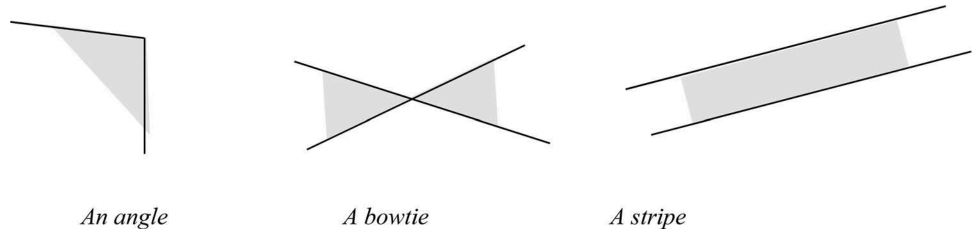
One proves that in $\mathbb{R}^2$ the *Re*-half planes coincide with the closed half-planes (Gerla & Miranda, 2017). Then a *Re*-line is a pair consisting of a closed half-plane and the closure of its set-theoretical complement. This means that a *Re*-line does not directly denote a line of the Euclidean plane but the pair $\{h, \sim h\}$ which is able to identify a line. This since there is a bijection between the set of *Re*-lines and the set of usually defined lines of $\mathbb{R}^2$. Obviously there are no difficulties in extending these notions to higher dimension, using an appropriate terminology.

Proposition 4.2. Two *Re*-lines $l = \{h, \sim h\}$ and $t = \{k, \sim k\}$ define a convex *Re*-partition of the universe U whose elements are the nonempty regions of the set $\{h{\cdot}k, h{\cdot}(\sim k), (\sim h){\cdot}k, (\sim h){\cdot}(\sim k)\}$. Namely, the following cases can occur:

- the *Re*-partition has 4 elements (see Figure 8).
- the *Re*-partition has 3 elements consisting of two disjoint *Re*-half-planes, for example, h and k, and a non-empty convex region $\sim h{\cdot}\sim k$ which we call *stripe* (see Figure 8).
- the *Re*-partition has 2 elements and this happens only in the case $l = t$.

Proof. It is immediate that two elements of the set $\{h{\cdot}k, h{\cdot}(\sim k), (\sim h){\cdot}k, (\sim h){\cdot}(\sim k)\}$ are *Re*-disjoint. Moreover, $h{\cdot}k + h{\cdot}(\sim k)+(\sim h){\cdot}k+(\sim h){\cdot}(\sim k) = h{\cdot}(k+\sim k)+(\sim h){\cdot}(k+(\sim k)) = h+(\sim h) = U$ and this proves that the set of nonempty regions in $\{h{\cdot}k, h{\cdot}(\sim k), (\sim h){\cdot}k, (\sim h){\cdot}(\sim k)\}$ is a *Re*-partition.

Figure 8. Re-partitions.



Assume that one of the products is 0, for example, $h{\cdot}k = 0$, then $h \sim\sim k$ and $k \sim\sim h$, and therefore $h{\cdot}k + h{\cdot}(\sim k) + (\sim h){\cdot}k + (\sim h){\cdot}(\sim k) = h + k + \sim h{\cdot}\sim k = U$. Then, in the case $\sim h{\cdot}\sim k \neq 0$ we have a 3 elements partition of U. In the case $\sim h{\cdot}\sim k = 0$, $\{h, k\}$ is a two element *Re*-partition and therefore $\sim h = k$, and $l = \{h, \sim h\} = \{k, \sim k\} = t$.

Definition 4.3. We say that two *Re*-lines $l = \{h, \sim h\}$ and $t = \{k, \sim k\}$ are *incident* if the associated *Re*-partition has 4 elements. We call *parallel* two *Re*-lines which are not incident. (see Figure 8)

Definition 4.4. Given two incident *Re*-lines $l = \{h, \sim h\}$ and $t = \{k, \sim k\}$, we call *angle* each element of the associated partition. In the angle $h{\cdot}k$, replacing h with its complement or k with its complement, we get two angles $\sim h{\cdot}k$ and $h{\cdot}(\sim k)$ called *adjacent to* $h{\cdot}k$. Replacing both h and k with their complements we obtain an angle $(\sim h){\cdot}(\sim k)$ we call *opposite* to $h{\cdot}k$. A *bowtie* is the union of an angle and its opposite.

Then, two intersecting lines define a four elements partition consisting in angles. These angles define two complementary bowties, $h{\cdot}k+(\sim h){\cdot}(\sim k)$ and $(\sim h){\cdot}k + h{\cdot}(\sim k)$. Two parallel lines define a three element partition consisting in a stripe and two *Re*-half-planes.[9]

Paradoxically the definition of a point and its belonging to a line appear a little more complex. The following proposal is partially inspired by Sniatycki (1968) where a point arises from the intersection between two lines or, equivalently, as the vertex of an angle.

Definition 4.5. Let us call a *pseudo-point*, a pair $P = \{r, t\}$ of incident *Re*-lines and indicate by *PP* the set of *pseudo*-points.

Definition 4.6. We say that a pseudo-point *P lies in a convex* region x if x overlaps all four angles determined by *P*. *P lies in a region* x if it lies in a convex region which is part of x. *P lies* in a line $l = \{h, \sim h\}$ or that *l passes through P*, if *P* does not lie neither in h nor in $\sim h$.

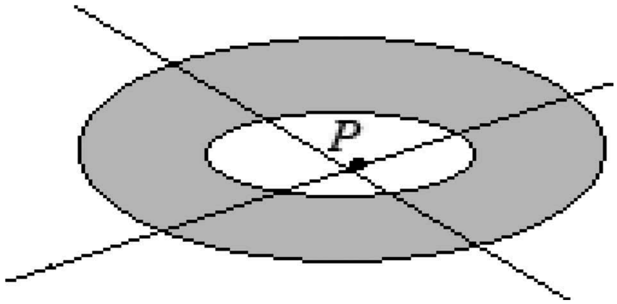
Figure 9 shows the reason of the two steps definition of lying. Indeed, all the angles defined by *P* overlap the circular crown in spite of the fact that *P* is not in the crown.

Definition 4.7. Two *pseudo*-points *P* and *Q* are said to be *separable* if there are two disjoint convex regions x and y such that *P* lies in x and *Q* in y. We write $P \cong Q$ if *P* and *Q* are not separable.

In other words, $P \cong Q$ if *P* and *Q* lie on the same convex regions and this entails that ≅ is an equivalence relation. Since two disjoint convex regions can be separated by a line, "separable" is equivalent to "separable by a line".[10]

Definition 4.8. We call *Re-point* an element of the quotient $PP/\cong$, i.e. an equivalence class modulo ≅ of *pseudo*-points. We say that a *Re*-point [*P*] *lies on a Re-line r* if every *pseudo*-point in [*P*] lies in *r*. We indicate by *Points*(*r*) the set of points lying on *r*.

Figure 9. An overlapping non-convex region.



Trivially, if P lies in $r = \{h, \sim h\}$ and $Q \cong P$, then Q lies in r, too. Moreover, there is a large list of properties compatible with $\cong$. We leave to the readers the obvious interpretation of further geometrical notions, for example, the one of set X of *aligned Re-points* or of *Re-vertex of an angle* and so on.

Observe that there are no difficulties in extending these notions to higher dimension.

We can define all main polygonal regions as a finite intersection of *Re*-half-planes.

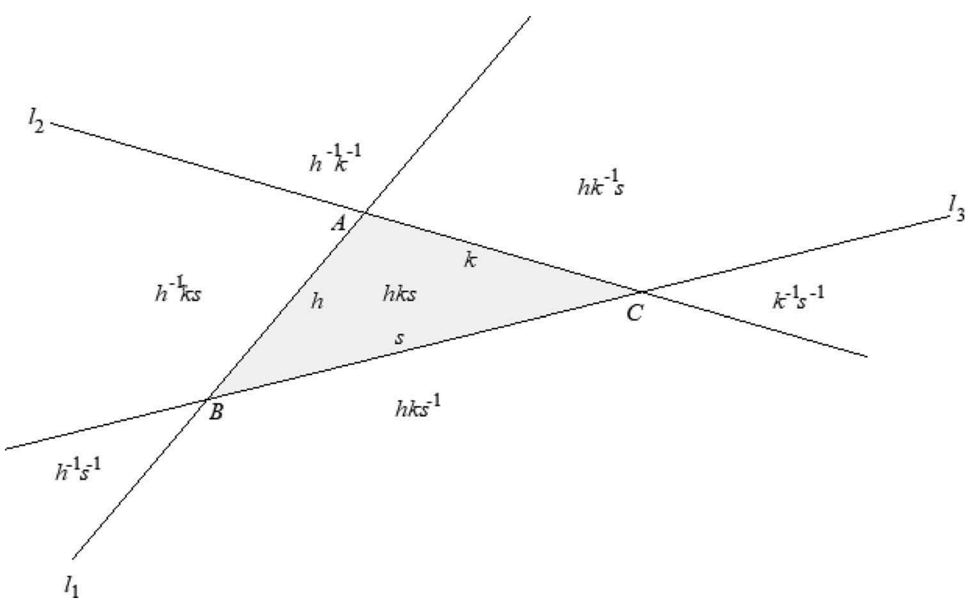
Definition 4.9. Let l_1, l_2, l_3 be three pairwise non-parallel lines defining three different *pseudo*-points $A = \{l_1, l_2\}$, $B = \{l_1, l_3\}$, $C = \{l_2, l_3\}$ and assume that these points are not aligned (there is no line in which A, B, C lie). Denote

- by h the side of l_1 containing C,
- by k the side of l_2 containing B,
- by s the side of l_3 containing A,

then we *call triangle of vertices A, B, C* the region $h \cdot k \cdot s$ (see Figure 10)., and we say that $h \cdot s$, $h \cdot k$ and $k \cdot s$ are the *inner angles* of the triangle[11] . We call *bounded* a region contained in a triangle.

Definition 4.10. Given two strips obtained by the lines l_1, t_1 and l_2, t_2, respectively, we say that they are *parallel* if the lines are parallel. A *parallelogram* is the *Re*-intersection of two non-parallel strips.

Figure 10. The triangle.



So far, the introduced notions do not involve congruence which is instead crucial in what follows.

Definition 4.11. We say that two incident *Re*-lines are *perpendicular* if they form four congruent angles.[12] In this case the angles are said to be *right angles*. Two non-parallel strips are called *perpendicular* if they determine a parallelogram with four right angles. The *Re*-intersection (i.e. the product) of two perpendicular strips is called *rectangle*.

Definition 4.12 Let us call *Re-bisector* of an angle $h{\cdot}k$ an *Re*-line that divides $h{\cdot}k$ into two congruent parts. A *Re-diagonal* of a parallelogram is a *Re*-line that divides it into two congruent triangles. Let us call *square* a rectangle that is divided by its diagonals into four congruent triangles.

We stop here since we prefer to face the basic question of the orderings in the set of points lying in a line. We will do this by observing that these ordering can be inherited by the inclusion in the following way. In the set *HP* of *Re*-half-planes, we denote by $\equiv_c$ the relation of *comparability* and therefore we write $h \equiv_c k$ if $h \geq k$ or $h \sim k$. Trivially, this relation is reflexive and symmetric. To prove that $\equiv_c$ is transitive we use the fact that in our prototype the set of *Re*-half-planes contained in a given *Re*-half-plane is totally ordered (see the proof of Theorem 5.2).

Definition 4.13. An *ordered bundle of Re-half-planes*, in brief a *bundle*, is an ordered structure $(F, \sim)$ where F is a class in the quotient of *HP* modulo $\equiv_c$ and ~ is the inclusion relation.

Then, by definition, F is an ordered structure of *Re*-half-planes if a *Re*-half-plane k exists such that

$$F = [k] = \{k' \in HP\colon k' \text{ is comparable with } k\} = \{k' \in HP\colon k' \sim k\} \cup \{k' \in HP\colon k' \geq k\}.$$

Moreover, since two *Re*-half-planes are comparable if an only if the associated *Re*-lines are parallel, the set $\{\{k', \sim k'\}\colon k' \in F\}$ is the bundle of lines parallel to $\{k, \sim k\}$.

Definition 4.14. Given a *Re*-line $r = \{h, \sim h\}$, let $\{k, \sim k\}$ be a line perpendicular to r and let $F = [k]$. Then we call *projection* of F in r the function $p\colon F \to Points(r)$ defined by setting $p(k') = [(r, \{k', \sim k'\})]$ for $k' \in F$.

We can prove that p is a one-to-one function and this allows us to define an order relation in *Point*(r).

Definition 4.15. In *Points*(r) we define the relation $\sim_k$ by placing

$$[P] \sim_k [Q] \Leftrightarrow p^{-1}([P]) \sim p^{-1}([Q]).$$

By definition, p is an isomorphism from $(F, \leq)$ to $(Points(r), \leq_k)$ and this proves that $\leq_k$ is a total order. Also, if k^* is comparable with k, then $\leq_{k^*}$ coincides with $\leq_k$.

The existence of two orders on a line is emphasized by the following proposition.

Proposition 4.16. We have that $[\sim k] = \{\sim k' \in HP\colon k' \in [k]\}$ and that $\leq_{\sim k}$ is the dual of $\leq_k$.

Proof. Indeed,

$$\{\sim k' \in HP\colon k' \in [k]\} = \{\sim k' \in HP\colon k' \geq k\} \cup \{\sim k' \in HP\colon k' \sim k\} = \{\sim k' \in HP\colon \sim k' \sim \sim k\} \cup \{\sim k' \in HP\colon \sim k' \geq \sim k\} = \{h \in HP\colon h \leq \sim k\} \cup \{h \in HP\colon h \geq \sim k\} = [\sim k].$$ □

5. The existence of a system of axioms adequate from a logical and didactic point of view

So far, we have only considered the interpretation of L given by the point-free prototype *PFP* and we have observed that in this prototype there is no difficulty in defining the primitive notions usually considered in a point-based approach. The next step is to show that a (suitable) system of axioms able to characterize *PFP* exists and therefore that a point-free axiomatization of Euclidean geometry is possible.

Now, let T_H be a system of axioms for Euclidean geometry, for example, a foundation of plane geometry in Hilbert's style whose primitives are points, lines together with the relations "lies in", "congruence" and "orderings". Also, denote by L_H the related language and consider a classical (point-based) model of T_H. Since T_H is categorical, it is not restrictive to assume that this model is the analytic model *AM* of T_H whose domain is $\mathbb{R}^2$ and such that the remaining symbols of L_H are interpreted as usual. Then, we have two mathematical structures both defined from the ordered field of real numbers:

- the two-dimensional prototype *PFP* in Definition 3.3 (interpretation of L),

- the classical analytic model *AM* (interpretation of L_H).

Now, these structures are, in some sense, equivalent.[13] Indeed, we defined *PFP* by starting from *AM*, and we defined *AM* by starting from *PFM*.[14] This means that every system of axioms able to capture *PFP* is also able to capture *AM* and vice versa. Then, it is possible to extend Tarski's cannibalization method and to rewrite T_H as a theory T_H* in the language L. This is obtained by replacing in any axiom α in T_H each L_H-symbol with the related definition in L. For example,

- "a line" has to be replaced with "a pair {*h*, ~*h*} with *h* and ~*h* convex",
- "two parallel lines" has to be replaced with "two sets {*h*, ~*h*} e {*k*, ~*k*} where *h*, ~*h, k*, ~*k* are convex and *h* is disjoint from *k*",
- "a point" can be replaced with "two convex regions *h* and *k* such that *h* and *k* generate a four elements partition of the universe".

Proceeding in this way it is possible to absorb all the axioms in T_H. For example, consider the axiom "Given two distinct points *A* and *B* there is a line passing through *A* and *B*". A first step of the proposed translation should be:

- "Given two separable *pseudo*-points *A* and *B*, a convex region *h* exists whose complement ~*h* is convex and such that *A* and *B* lie on {*h*, ~*h*}."

Moreover, we have to replace the expressions "*pseudo*-point", "separable", "lies" with the respective definitions written in L and so on.

The theory T_H* works well in the sense that in every (point-free) model of T_H* we can define a (point-based) model of T_H and therefore, in account of the categoricity of T_H, the classical analytic model *AM*.[15] Nevertheless, T_H* is not adequate from our point of view. On the other hand, also Tarski claims its dissatisfaction regarding the method used in its paper.

> The postulate system given above is far from being simple and elegant; it seems very likely that this postulate system can be essentially simplified by using intrinsic properties of the geometry of solids.

In accordance with this claim, in our project we have to search for a system of axioms expressing intrinsic properties of the geometry of two-dimensional regions.

In other words, to realize our project, we have to find a theory *T* in the point-free language L satisfying the following conditions.

(a) *T* must at least make meaningful the definitions given in Section 4 with reference to the prototype *PFP*. Now, there are definitions in which no problem exists. For example, if we indicate by (*A*, *CO*, ≡)

a generic interpretation of L (A denotes the algebraic part), then there is no difficulty in defining a half-plane as a region $h \in CO$ such that $\sim h \in CO$. The definition of line is also immediate. Some difficulties arise in defining an ordering on the set *Points*(*r*) of points of a line *r* since this definition requires that the projection from the bundle perpendicular to *r* to *P*(*r*) *r* is one-to-one.

(b) The axioms in *T* must be satisfied in *PFP* so that every theorem of *T* is valid in this model.

(c) *T* must be categorical.

(d) The axioms in *T* must refer to properties of regions and ovals as directly as possible (in contrast with cannibalization strategy).

(e) The axioms have to be satisfactory from a didactic and not only from a logical point of view.

In the following, we propose some axioms.

A1. Boolean algebra Axiom. The structure A is a complete atomless Boolean algebra.

This means that one assumes that in the class of regions operations are defined which correspond to the ones of union, intersection, complement. Moreover, no point exists. A further axiom concerns convexity.

A2. Closure system Axiom. The set *CO* of convex regions is an algebraic closure system in the Boolean algebra A.

Where the general definition of (abstract) algebraic closure system is given, for example, in Gerla (2001). Then, A2 means that

(1) *U* belongs to *CO*,

(2) the least upper bound of a class of convex regions is also a convex region,

(3) the greatest lower bound of a chain of regions is also a convex region.

Trivially, *PFP* satisfies these axioms.

Once we admit these axioms, we can test them with respect to some basic questions. For example, the ones related with the parallelism relation.

Theorem 5.1. While the fact that parallelism is a reflexive and symmetric relation is a trivial consequence of the proposed definition, transitivity cannot be proved and therefore this relation is not necessarily an equivalence.

Proof. It is not surprising that we have to refer to a model of non-Euclidean geometry, for example, Poincaré 's model. Indeed, denoted by *S* the set $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ of points and, given two points *P* and *Q* in *S*, denote by *PQ* the Poincaré segment connecting *P* ad *Q*. Then, we obtain a model (A, *CO*, ≡) of A1 and A2 by putting,

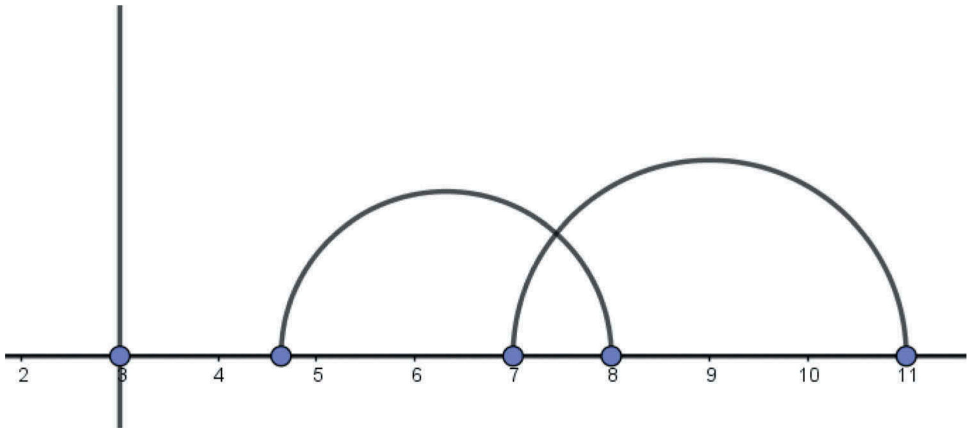
- A equal to the Boolean algebra of the closed regular subsets of *S*

- *C* equal to the class {*X: X* is a regular subset of *S* such that *PQ* □*X* for every *P, Q* in *X*}

- ≡ the congruence as defined in Poincaré's model.

In other words, the definition of (A, *CO*, ≡) is analogous to the one of the prototypical model *PFP*. Now, consider the lines in Figure 11 (a vertical half-line and two semicircles) which are right lines in accordance with Definition 4.1, i.e. a two element of a convex partition. Then, the lines defined by

Figure 11. Two different lines parallel to a third line which are not parallel.



the semicircle are not parallel in spite of the fact that are both parallel to the vertical line. This proves that to be parallel is not a transitive relation.

This theorem suggests the necessity of adding transitivity as an axiom. Obviously, should be possible to cannibalize the transitivity as it is expresses in its classical form. We obtain the following axiom in L.

Axiom. Assume that h_1, h_2, h_3 are half-planes such that $h_1{\cdot}h_2 = 0$ and $h_2{\cdot}h_3 = 0$, then either $h_1{\cdot}h_3 = 0$ or $h_1{\cdot}({\sim}h_3) = 0$ or $({\sim}h_1){\cdot}h_3 = 0$ or $({\sim}h_1){\cdot}({\sim}h_3) = 0$.

Nevertheless, in accordance with our ideas, it should be preferable the following formulation referring directly to the half-planes.

A3. Two half-plane contained in the same half-plane are comparable.

In other words, the set of half-planes contained in a given half-plane is totally ordered with respect to the inclusion ≤.

Theorem 5.2. Assume A1-A2, A3, then the following claims holds true.

(a) The comparability relation $\equiv_c$ is an equivalence relation in the class of half-planes.

(b) The parallelism is an equivalence relation in the set of lines.

(c) A bundle is totally ordered with respect to the inclusion.

Proof. a) Suppose that h, k, s are half-planes such that $h \equiv_c k$, for example, that $h \sim k$, and that $k \equiv_c s$. Then, in the case $s \sim k$, by Axiom 3, $h \equiv_c s$, in the case $k \sim s$, since ~ is transitive, $h \sim s$. This proves that $h \equiv_c s$. □

(b) Assume that $r_1 = \{h_1, {\sim}h_1\}$ is parallel to $r_2 = \{h_2, {\sim}h_2\}$, for example, that $h_1{\cdot}h_2 = 0$ and that $r_2 = \{h_2, {\sim}h_2\}$ is parallel to $r_3 = \{h_3, {\sim}h_3\}$, for example, that $h_2{\cdot}h_3 = 0$. Then, $h_1 \leq {\sim}h_2$ and $h_3 \leq {\sim}h_2$ and, by (a), this proves that h_1 is comparable with h_3 and therefore that r_1 is parallel with r_3.

(c) Immediate.

Recall that in Euclidean geometry the existence of a parallel to a straight line passing through a given point is a theorem but the uniqueness is not demonstrable. This entails the necessity of the famous parallel axiom. Trivially in our proposal this axiom is a theorem. As a matter of fact, axiom A3 is a new formulation of parallel axiom in which the notion of point is not necessary.

Theorem 5.3. Parallel axiom is a consequence of A1, A2, A3.

Proof. An immediate consequence of Theorem 5.2. □

Further axiom are those about congruence and order. For the congruence relation, the following properties look to be reasonable.

A4. Congruence Axiom. The relation ≡ satisfies the following properties.

(a) ≡ is an equivalence relation in the set of regions

(b) *x'* ≡*x* and *x* convex □*x'* is convex

(c) *x'* ≡ *x* □ ~*x'* ≡ ~*x*

(d) *h* and *k* half-planes □ ~*h* ≡ ~*k*.

It is not clear whether these properties are sufficient or not to characterize the notion of congruence. From didactic point of view, perhaps it should be preferable to assume as a primitive notion the one of *"movement"* and, successively, to define the congruence by putting $x \equiv y$ if there is a movement τ such that $\tau(x) = y$. Indeed, the notion of movement is surely more adequate to laboratory activities as the "folding and cutting" ones. In this case, we have to define a group *G* of transformations of the Boolean algebra A satisfying suitable axioms. Notice that in (Pambuccian, 2005) one proposes a system of axiom for the group of movements in the plane.

Regarding the order in *Points*(*r*) of a line *r*, we recall that this relation is defined by the projection of the bundle orthogonal to *r* into *Points*(*r*). This requires the following axiom.

A5. Projection Axiom. The projection defined in Section 4 is a one-to-one map.

It is also a theorem the fundamental axiom of continuity that can easily be deduced from the completeness axiom of the Boolean algebra A.

Theorem 5.4. Axiom of continuity is a consequence of A1-A5.

Proof. Let *F* be a class of half-planes contained in a half plane *h*. Then Axiom 1 entails that the least upper bound sup(*F*) of *F* exists. Since *CO* is an algebraic closure system, *sup*(*F*) is a convex region. To prove that *sup*(*F*) is a half-plane we recall that in a complete Boole algebra the De Morgan law holds and therefore $\sim sup(F) = \sim (+\{k: k \leq h\}) = \cdot\{\sim k: k \leq h\} = inf\{\sim k: k \leq h\}$. Then, since *CO* is a closure system and ~*sup*(*F*) the greatest lower bound of a chain of convex regions, ~*sup*(*F*) is a convex region. So *sup*(*F*) is a half-plane. □

6. Open questions and future works

This paper is only the first fundamental step in the development of a project concerning the role of point-free geometry in education. It is an exploration of a possibility and not a series of definitive results. So, the list of open questions is very large.

The main question is to add to the proposed list of axioms in Sections 5 further axioms to obtain a theory *T* able to capture the prototypical point-free model. Obviously, these axioms have to express in a direct way evident and intuitive properties of the regions. For example, it should be interesting to add an axiom expressing the *Separation theorem*.

Separation axiom. For every pair of non-empty, disjoint ovals *x* and *y* a line *r* = {*h*, ~*h*} exists such that *x* ≤ *h* and *y* ≤~*h*.

The second fundamental step will be just to explore the educational value of our choices. The argument is suitable to build teaching-learning activities for students at all levels, even at university-level thinking to mathematics education courses. Due to the difficulties that may arise, in general, we will deal with plane geometry which is easier to handle. For instance, in plane geometry, some interesting possibilities are related to folding and to equidecomposability activities and therefore to artifacts such as scissors and paper sheets. More precisely, the simple notion of half-plane could allow, by appropriate laboratory activities, to verify the validity of the axioms and theorems usually studied at school, such as the Pythagoras's theorem or the equidecomposability theorems for the calculus of the areas of the main figures. It should be important to find simple proofs of these theorems. However, many of the things in the paper are valid for spaces of whatever dimension and we hope that a successive enlargement from plane to solid geometry is possible.

Obviously, it should be also interesting to analyze the possibility of extending the approach proposed in this paper to projective, and non-Euclidean geometry.

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Notes

1. In point-free geometry, the term “solid body” is used frequently as synonymous of “region”. This is rather questionable since the solid bodies are objects considered in physics and not in geometry. The same holds true for the difference between material points and geometrical points. While in physics we can move a material point, in geometry, in accordance with the fact that a point is a precise position in the space, it is impossible to move a point. As a matter of fact, the term “region” must be interpreted as a part of the space that a body can occupy in a given instant.
2. The intended meaning is that *X* is not tangentially contained in *Y* if *X* is contained in *Y* but the boundaries of these regions do not meet.
3. Further postulates are added by Tarski to obtain the categoricity of the theory.
4. Things would not have been better if these components were represented by open sets since in this case their union would not coincide with *ABC*.
5. As a consequence of a duality principle, the choice of the open subsets is equivalent.
6. As a matter of fact, since the union of two regular closed sets is a regular closed set, *X* + *Y* = *X*□*Y*; moreover, since the complement of a regular closed set is open, ~*X* = *c*(-*X*).
7. That is, the quantity and the relevance of the notions definable directly or “captured” by a definable notion. As we will see in what follows, examples of the first type are half-planes and angles, examples of the second type are lines and points.
8. In a laboratory activity we obtain a line by folding a sheet and an angle by two non-parallel folds. We can address the equidecomposability by suitable folds and successive cuts along the folds.
9. The folding-cutting related activities are evident.
10. Then, in terms of folding-cutting activities, this means that two *pseudo*-points are separable if one can separate them by a folding (and therefore by a scissors cut).
11. It is possible to read this definition as a way of constructing a triangle in a laboratory activity.
12. Then, once a folding enables us to obtain a line *l*, to obtain a line *t* orthogonal to *l*, it is sufficient a second folding leading *l* in *l*.
13. Obviously, we cannot compare *PFP* and *CM* with respect the notion of isomorphism since the primitives of these structures are different.
14. It is important to notice that in both the cases first-order logic is not sufficient for expressing the definitions.
15. A very interesting series of papers related to the comparison of theories in different languages provide the tools to define in a precise way this procedure (see, for example, Andréka & Németi, 2014).

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