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Research Article

Linear Sobolev Type Equations with Relatively p-Sectorial Operators in Space of "Noises"

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The concept of "white noise," initially established in finite-dimensional spaces, is transferred to infinite-dimensional case. The goal of this transition is to develop the theory of stochastic Sobolev type equations and to elaborate applications of practical interest. To reach this goal the Nelson-Gliklikh derivative is introduced and the spaces of "noises" are developed. The Sobolev type equations with relatively sectorial operators are considered in the spaces of differentiable "noises." The existence and uniqueness of classical solutions are proved. The stochastic Dzektser equation in a bounded domain with homogeneous boundary condition and the weakened Showalter-Sidorov initial condition is considered as an application.

1. Introduction

Let U and F be Banach spaces, the operator $L \in \mathscr{L}(U;F)$ (linear and continuous), and the operator $M \in \mathrm{cl}(U;F)$ (linear, closed, and densely defined). Consider the equation

$$L\dot{u} = Mu, \quad \ker L \neq \{0\}. \tag{1}$$

Equations of the form (1) were firstly studied in the works of A. Poincare. Then they appeared in the works of S. V. Oseen, J. V. Boussinesq, S. G. Rossby, and other researchers that were dedicated to the investigation of some hydrodynamics problems. Their systematical study started in the middle of the XX century with the works of S. L. Sobolev. The first monograph [1] devoted to the study of equations of the form (1) appeared in 1999. Nowadays the number of works devoted to such equations is increasing extensively [1–3]. Sometimes such equations are called "equations that are not of Cauchy-Kovalevskaya type," "pseudoparabolic equations," "degenerate equations," or "equations unsolved with respect to the higher derivative." We call equations of the form (1) the Sobolev type equations. This term was firstly proposed in the works of Carroll and Showalter [4]. The Sobolev type equations constitute the vast area in nonclassical equations of mathematical physics [5]. The theory of degenerate semigroups of operators is a suitable mathematical tool for the study of such problems [2].

The right part of (1) can be subjected to random perturbations, such as white noise. Abstract stochastic equations are of great interest nowadays due to the large amount of applications. Linear stochastic differential equation in the simplest case can be represented in the form

$$d\eta = (S\eta + \psi)\,dt + A\,dw, \tag{2}$$

where S and A are some linear operators; $\psi = \psi(t)$ is a deterministic external influence and $w = w(t)$ is a stochastic external influence; $\eta = \eta(t)$ is unknown random process. Firstly dw was understood in the sense of differential of the Wiener process $w = W(t)$ and was traditionally treated as white noise. K. Ito was the first to study ordinary differential equations of the form (2); then R. L. Stratonovich and A. V. Skorokhod developed research. The Ito-Stratonovich-Skorokhod approach in the finite-dimensional case remains popular to this day [6, 7]. Moreover, it was successfully distributed to infinite-dimensional situation [8, 9], and even it was applied to studies of the Sobolev type equations [10, 11]. Another approach was presented in [12], where (2) was considered in the Schwartz spaces and the distributional derivative of the Wiener process makes sense.

A new approach to studying (2), where the noise is defined by the Nelson-Gliklikh derivative of the Wiener process, appeared recently and is actively developing [13, 14]. At first white noise was used in the theory of optimal measurements [15], where a special noises space was constructed [16]. In [17] the concept of white noise was also extended to the infinite-dimensional space $\mathbf{C}_K\mathbf{L}_2$ of K-random processes with a.s. continuous trajectories and the space $\mathbf{C}_K^l\mathbf{L}_2$ of K-random processes, whose trajectories are a.s. continuously differentiable in the Nelson-Gliklikh sense up to order $l \in \mathbf{N}$. The solvability of Showalter-Sidorov problem for linear stochastic Sobolev type equations with relatively bounded operators was studied in [17]. Our purpose is to study the solvability of weakened (in sense of S. G. Krein) Showalter-Sidorov problem for linear stochastic Sobolev type equation with relatively sectorial operator. The purpose of such extention is the development of the theory of stochastic Sobolev type equations and application of this theory to nonclassical models of mathematical physics of practical value.

The paper is organized as follows. In the second section we introduce the definition of a strongly relatively p-sectorial operator and construct semigroups of the resolving operators. In the third section the Nelson-Gliklikh derivative of K-random process with values in real separable Hilbert spaces is considered. In particular the K-Wiener process is studied. Then the space of such processes, containing the K-Wiener process and its Nelson-Gliklikh derivative (i.e., white noise), is constructed. In the fourth section the theory of stochastic Sobolev type equations with relatively p-sectorial operators is developed; namely, the stochastic Sobolev type equation

$$L \overset{\circ}{\eta} = M\eta + Nw \tag{3}$$

is considered. Here $\eta = \eta(t)$ is the unknown random process, $\overset{\circ}{\eta}$ is its Nelson-Gliklikh derivative, $w = w(t)$ is a random process, responsible for external influence; the operators $L, M, N \in \mathscr{L}(U; F)$; moreover the operator M is (L, p)-sectorial, $p \in \{0\} \cup \mathbf{N}$. Add to (3) with a weakened Showalter-Sidorov condition

$$\lim_{t \to 0+} \left[R_\alpha^L(M)\right]^{p+1} (\eta(t) - \xi_0) = 0, \tag{4}$$

where $R_\alpha^L(M) = (\alpha L - M)^{-1}L$, $\alpha \in \rho^L(M)$. Condition (4) is a natural generalization of condition

$$L(u(0) - u_0) = 0 \tag{5}$$

which is in its turn the generalization of the Cauchy condition

$$u(0) = u_0. \tag{6}$$

Note that condition (4) is more natural for the Leontieff type system and for the Sobolev type equations [5] than the traditional Cauchy condition (6). Problem (5) for the deterministic Sobolev type equation was firstly studied in [1]. This investigation formed the basis of the study of problem (5) for linear stochastic Sobolev type equation (3) [11]. The existence and the uniqueness of classical solution for problem (3), (4) are proved in the fourth section of our paper. In the fifth section we apply the abstract scheme to the investigation of the Dzektser model [18], describing free surface evolution of filtered liquid.

2. Holomorphic Degenerate Semigroups of Operators

Let U and F be Banach spaces, and let the operators $L \in \mathscr{L}(U; F)$, $M \in \mathrm{cl}(U; F)$. Consider the L-resolvent set of M, $\rho^L(M) = \{\mu \in \mathbf{C} : (\mu L - M)^{-1} \in \mathscr{L}(F; U)\}$, the L-spectrum $\sigma^L(M) = \mathbf{C} \setminus \rho^L(M)$ of the operator M, and the right and the left L-resolvents of the operator M, $R_\mu^L(M) = (\mu L - M)^{-1}L$, $L_\mu^L(M) = L(\mu L - M)^{-1}$, respectively. Let $\mu_k \in \rho^L(M)$, $k = 0, 1, \ldots, p$. The operator-functions

$$\begin{gathered} R^L_{(\mu,p)} = \prod_{k=0}^{p} R^L_{\mu_k}(M), \\ L^L_{(\mu,p)}(M) = \prod_{k=0}^{p} L^L_{\mu_k}(M) \end{gathered} \tag{7}$$

are called *the right* and *the left* (L, p)-resolvents of the operator M.

Definition 1. Operator M is said to be *p-sectorial*, $p \in \{0\} \cup \mathbf{N}$ with respect to the operator L (or shortly (L, p)-*sectorial*), if

(i) $\exists a \in \mathbf{R}$, $\theta \in (\pi/2, \pi)$ such that the sector

$$S^L_{a,\theta} = \{\mu \in \mathbf{C} : |\arg(\mu - a)| < \theta,\ \mu \neq a\} \subset \rho^L(M); \tag{8}$$

(ii) $\exists K > 0$ such that

$$\begin{aligned} \max &\left\{\left\|R^L_{(\mu,p)}(M)\right\|_{\mathscr{L}(U)}, \left\|L^L_{(\mu,p)}(M)\right\|_{\mathscr{L}(F)}\right\} \\ &\leq \frac{K}{\prod_{q=0}^{p}\left|\mu_q - a\right|}, \end{aligned} \tag{9}$$

for all $\mu_q \in S^L_{a,\theta}$, $q = 0, 1, \ldots, p$.

Remark 2. Without loss of generality we can put $a = 0$ in Definition 1. Indeed, if we find a resolving semigroup of (1) $\{\mathbf{V}^t \mid t \in \mathbf{R}_+\}$ for $a = 0$, then the semigroup $\{e^{at}\mathbf{V}^t \mid t \in \mathbf{R}_+\}$ will be resolving when $a \neq 0$.

Let $\alpha \in \rho^L(M)$. Consider two equivalent forms of the linear homogeneous Sobolev type equation (1)

$$R^L_\alpha(M)\,\dot{u} = (\alpha L - M)^{-1} Mu, \tag{10}$$

$$L^L_\alpha(M)\,\dot{f} = M(\alpha L - M)^{-1} f \tag{11}$$

as concrete interpretations of the equation

$$A\dot{v} = Bv, \tag{12}$$

defined on a Banach space V, where the operators $A, B \in \mathscr{L}(V)$. Operator $(\alpha L - M)^{-1}M = -\mathbf{I} + \alpha(\alpha L - M)^{-1}L$ is linear bounded on a dense set in U and it can be uniquely continued to a bounded operator $-\mathbf{I} + \alpha(\alpha L - M)^{-1}L$ defined on U. For (10) the space $V = U$ and for (11) $V = F$.

Definition 3. The vector function $v \in C^{\infty}(\mathbf{R}_+; V)$ satisfying (12) on $\mathbf{R}_+$ $(\equiv (0, +\infty))$ is called *a solution* of (12).

Definition 4. The mapping $\mathscr{V}^{\bullet} \in C^{\infty}(\mathbf{R}_+; \mathscr{L}(V))$ is called *a semigroup of the resolving operators* (*a resolving semigroup*) of (12), if

(i) $\mathscr{V}^s\mathscr{V}^t v = \mathscr{V}^{s+t}v$ for all $s, t > 0$ and any $v \in V$;

(ii) $v(t) = \mathscr{V}^t v$ is a solution of (12) for any v from a dense set in V.

The semigroup is called *uniformly bounded*, if

$$\exists C > 0$$
$$\left\|\mathscr{V}^t\right\|_{\mathscr{L}(V)} \leq C \quad \forall t \in \mathbf{R}_+. \tag{13}$$

The semigroup is called *analytic*, if it can be extended to some sector containing the ray $\mathbf{R}_+$ with fulfillment of properties (i), (ii) in Definition 4.

Theorem 5 (see [2, p. 60]). *Let the operator M be (L, p)-sectorial, $p \in \{0\} \cup \mathbf{N}$. Then there exists a uniformly bounded and analytic resolving semigroup of (10) and (11) and it is represented by*

$$\begin{aligned} \mathscr{U}^t &= \frac{1}{2\pi i}\int_{\Gamma} R^L_{\mu}(M)\, e^{\mu t}\, d\mu, \\ \mathscr{F}^t &= \frac{1}{2\pi i}\int_{\Gamma} L^L_{\mu}(M)\, e^{\mu t}\, d\mu, \end{aligned} \tag{14}$$

where $t \in \mathbf{R}_+$ and contour $\Gamma \subset \rho^L(M)$ is such that $|\arg \mu| \to \theta$ for $\mu \to \infty$, $\mu \in \Gamma$.

For example, contour $\Gamma = \{\mu \in \mathbf{C} : |\arg \mu| = \theta\}$, where $\theta \in (\pi/2, \pi)$ is taken from Definition 1.

Let U^1 (F^1) be the closure of $\operatorname{im} R^L_{(\mu,p)}(M)$ $(\operatorname{im} L^L_{(\mu,p)}(M))$ in the norm of the space U (F). The set

$$\ker \mathscr{V}^{\bullet} = \left\{v \in V : \mathscr{V}^t v = 0 \ \exists t \in \mathbf{R}_+\right\} \tag{15}$$

is called *a kernel* [2, p. 61] of the semigrop $\mathscr{V}^{\bullet}$ and the set

$$\operatorname{im} \mathscr{V}^{\bullet} = \left\{v \in V : \lim_{t \to 0+} \mathscr{V}^t v = v\right\} \tag{16}$$

is called *an image* [2, p. 61] of the semigrop $\mathscr{V}^{\bullet}$.

Theorem 6 (see [2, p. 62]). *Let the operator M be (L, p)-sectorial. Then* $\operatorname{im} \mathscr{U}^{\bullet} = U^1$, $\operatorname{im} \mathscr{F}^{\bullet} = F^1$.

Further we assume that the operator M is (L, p)-sectorial. Set $U^0 = \ker \mathscr{U}^{\bullet}$, $F^0 = \ker \mathscr{F}^{\bullet}$. By L_0 (M_0) denote the restriction of the operator L (M) on U^0 $(U^0 \cap \operatorname{dom} M)$.

Theorem 7 (see [2, pp. 63, 64]). *Let the operator M be (L, p)-sectorial. Then*

(i) *the operator $L_0 \in \mathscr{L}(U^0; F^0)$, and the operator $M_0 : U^0 \cap \operatorname{dom} M \to F^0$;*

(ii) *there exists the operator $M_0^{-1} \in \mathscr{L}(F^0; U^0)$;*

(iii) *the operator $H = M_0^{-1}L_0 \in \mathscr{L}(U^0)$ is nilpotent with degree less or equal to p.*

By L_1 (M_1) denote the restriction of the operator L (M) on U^1 $(U^1 \cap \operatorname{dom} M)$.

Consider the following conditions:

$$\begin{aligned} U^0 \oplus U^1 &= U, \\ F^0 \oplus F^1 &= F, \end{aligned} \tag{A1}$$

and there exists the operator

$$L_1^{-1} \in \mathscr{L}\left(F^1; U^1\right). \tag{A2}$$

Remark 8. Condition (A1) holds, for example, in the case where M is strongly (L, p)-sectorial on the right (left) or when the space U (F) is reflexive [2, page 69]. Condition (A2) holds in the case when the operator M is strongly (L, p)-sectorial or when it is (L, p)-sectorial, condition (A1) is fulfilled and $U^1 = \operatorname{im} L^1$.

Condition (A1) is equivalent to the existence of the projector P (Q) along U^0 (F^0) on U^1 (F^1).

Theorem 9 (see [2, pp. 69, 71, 73]). *Let the operator M be (L, p)-sectorial and let conditions* (A1), (A2) *be fulfilled. Then*

(i) *the projector P (Q) can be represented as*

$$\begin{aligned} P &= \mathscr{U}^0 = s - \lim_{t \to 0+} \mathscr{U}^t, \\ Q &= \mathscr{F}^0 = s - \lim_{t \to 0+} \mathscr{F}^t; \end{aligned} \tag{17}$$

(ii) *the operator $M_1 \in \operatorname{cl}(U^1; F^1)$ and the operator $M_0 \in \operatorname{cl}(U^0; F^0)$;*

(iii) *the operator $S = L_1^{-1}M_1 \in \operatorname{cl}(U^1)$ is sectorial.*

The solution to (12) is called *a solution* to a Cauchy problem if it also satisfies the condition

$$\lim_{t \to 0+} v(t) = v_0. \tag{18}$$

Definition 10. The set $\mathscr{P} \subset U$ is called *a phase space* of (12), if

(i) any solution $v = v(t)$ of (12) lies in $\mathscr{P}$; that is, $v(t) \in \mathscr{P}$ for all $t \in \mathbf{R}_+$;

(ii) for any $v_0 \in \mathscr{P}$ there exists a unique solution of problem (12), (18).

Theorem 11 (see [2, p. 67]). *Let the operator M be (L, p)-sectorial. Then phase space of (10) and (11) coincides with the image of semigroup* $\operatorname{im} \mathscr{U}^{\bullet}$ $(\operatorname{im} \mathscr{F}^{\bullet})$.

3. The Spaces of "Noises"

Let $\Omega \equiv (\Omega, A, P)$ be a complete probability space and let $\mathbf{R}$ be the set of real numbers endowed with Boreal σ-algebra. The measurable mapping $\xi : \Omega \to \mathbf{R}$ is called *a random variable*. The set of random variables with zero mean and finite variances forms a Hilbert space with the scalar product $(\xi_1, \xi_2) = \mathbf{E}\xi_1\xi_2$, where $\mathbf{E}$ denotes the mathematical expectation. This Hilbert space will be denoted by $\mathbf{L}_2$. The random variables $\xi \in \mathbf{L}_2$, with normal (Gaussian) distribution, will be very important later on; they are called Gaussian random variables. Let A_0 be a σ-subalgebra of σ-algebra A. Construct the space $\mathbf{L}_2^0$ of random variables, measurable with respect to A_0. Obviously, $\mathbf{L}_2^0$ is a subset of $\mathbf{L}_2$; denote by $\Pi : \mathbf{L}_2 \to \mathbf{L}_2^0$ the orthoprojector. Let $\xi \in \mathbf{L}_2$, then $\Pi\xi$ is called *conditional expectation* of the random variable ξ and is denoted by $\mathbf{E}(\xi \mid A_0)$. It is easy to see that $\mathbf{E}(\xi \mid A_0) = \mathbf{E}\xi$, if $A_0 = \{\varnothing, \Omega\}$; and $\mathbf{E}(\xi \mid A_0) = \xi$, if $A_0 = A$. Finally, the minimal σ-subalgebra $A_0 \subset A$, regarding which random variable ξ is measurable, is called *the σ-algebra generated by ξ*.

Let $I \subset \mathbf{R}$ be some interval. Consider two mappings: the first one $f : I \to \mathbf{L}_2$, which maps each $t \in I$ to a random variable $\xi \in \mathbf{L}_2$, and the second one $g : \mathbf{L}_2 \times \Omega \to \mathbf{R}$, which maps every pair (ξ, ω) to the point $\xi(\omega) \in \mathbf{R}$. The composition $\eta : I \times \Omega \to \mathbf{R}$, $\eta = \eta(t,\omega) = g(f(t),\omega)$, is called *a (one-dimensional) random process*. Thus, for every fixed $t \in I$ the random process $\eta = \eta(t,\cdot)$ is a random variable; that is, $\eta(t,\cdot) \in \mathbf{L}_2$, and for every fixed $\omega \in \Omega$ the random process $\eta = \eta(\cdot,\omega)$ is called *the (sample) trajectory*. The random process η is called *continuous* if almost surely (a.s.) all its trajectories are continuous; that is, for almost every (a.e.) $\omega \in \Omega$ the trajectories $\eta(\cdot,\omega)$ are continuous. The set of continuous random processes form a Banach space, which will be denoted by $\mathbf{CL}_2$. The continuous random process, whose random variables are Gaussian, is called *Gaussian*.

The (one-dimensional) Wiener process $\beta = \beta(t)$, modeling Brownian motion on the line in Einstein-Smolukhovsky theory, is one of the most important examples of the continuous Gaussian random processes. It has the following properties:

(W1) a.s. $\beta(0) = 0$; a.s. all its trajectories $\beta(t)$ are continuous, and for all $t \in \overline{\mathbf{R}}_+$ $(= \{0\} \cup \mathbf{R}_+)$ the random variable $\beta(t)$ is Gaussian;

(W2) the mathematical expectation $\mathbf{E}(\beta(t)) = 0$ and autocorrelation function $\mathbf{E}((\beta(t) - \beta(s))^2) = |t - s|$ for all $s, t \in \overline{\mathbf{R}}_+$;

(W3) the trajectories $\beta(t)$ are nondifferentiable at any point $t \in \overline{\mathbf{R}}_+$ and have unbounded variation on any small interval.

Example 12. There exists a random process β, satisfying properties (W1), (W2); moreover, it can be represented in the form

$$\beta(t) = \sum_{k=0}^{\infty} \xi_k \sin\frac{\pi}{2}(2k+1)t, \tag{19}$$

where ξ_k are independent Gaussian variables, $\mathbf{E}\xi_k = 0$, and $\mathbf{D}\xi_k = [(\pi/2)(2k+1)]^{-2}$, where $\mathbf{D}$ denotes the dispersion.

The random process β, satisfying properties (W1)-(W2), will be called *Brownian motion.*

Now fix $\eta \in \mathbf{CL}_2$ and $t \in I$ $(= (\varepsilon,\tau) \subset \mathbf{R})$ and by N_t^η denote the σ-algebra, generated by the random variable $\eta(t)$. For the sake of brevity, we introduce the notation $\mathbf{E}_t^\eta = \mathbf{E}(\cdot \mid N_t^\eta)$.

Definition 13. Let $\eta \in \mathbf{CL}_2$, and the random variable

$$\begin{gathered} D\eta(t,\cdot) = \lim_{\Delta t \to 0+} \mathbf{E}_t^\eta\left(\frac{\eta(t+\Delta t,\cdot) - \eta(t,\cdot)}{\Delta t}\right), \\ \left(D_*\eta(t,\cdot) = \lim_{\Delta t \to 0+} \mathbf{E}_t^\eta\left(\frac{\eta(t,\cdot) - \eta(t-\Delta t,\cdot)}{\Delta t}\right)\right) \end{gathered} \tag{20}$$

is called *a forward $D\eta(t,\cdot)$ (a backward $D_*\eta(t,\cdot)$) mean derivative of the random process η at the point $t \in (\varepsilon,\tau)$* if the limit exists in the sense of uniform metric on $\mathbf{R}$. The random process η is called *forward (backward) mean differentiable on* (ε,τ), if for every point $t \in (\varepsilon,\tau)$ there exists the forward (backward) mean derivative.

Now let the random process $\eta \in \mathbf{CL}_2$ be forward (backward) mean differentiable on (ε,τ). Its forward (backward) mean derivative is also a random process; we denote it by $D\eta$ $(D_*\eta)$. If the random process $\eta \in \mathbf{CL}_2$ is forward (backward) mean differentiable on (ε,τ), then *the symmetric mean derivative*

$$D_S\eta = \frac{1}{2}(D + D_*)\eta \tag{21}$$

can be defined. Since the mean derivatives were introduced by Nelson [19], and the theory of these derivatives was developed by Gliklikh [7], the symmetric mean derivative D_S or the random process η will henceforth be called *the Nelson-Gliklikh derivative* for brevity and will be denoted by $\overset{\circ}{\eta}$; that is, $D_S\eta \equiv \overset{\circ}{\eta}$. By $\overset{\circ}{\eta}{}^{(l)}$, $l \in \mathbf{N}$ denote the lth Nelson-Gliklikh derivative of the random process η. Note that if the trajectories of the random process η are a.s. continuously differentiable in a "common sense" on (ε,τ), then the Nelson-Gliklikh derivative of η coincides with the "regular" derivative.

Theorem 14 (see [14]). *Let $\overset{\circ}{\beta}{}^{(l)}(t) = (-1)^{l+1}\prod_{i=1}^{l-1}(2i-1)(2t)^{-l}\beta(t)$ for all $t \in \mathbf{R}_+$ and $l \in \mathbf{N}$.*

Now let $V \equiv (V, \langle\cdot,\cdot\rangle)$ be a real separable Hilbert space; consider the operator $K \in \mathscr{L}(V)$ with spectrum $\sigma(K)$ whose elements are nonnegative, discrete, with finite multiplicity tending only to zero. By $\{\lambda_j\}$ denote the sequence of eigenvalues of operator K, numbered in decreasing order according to their multiplicity. Note that the linear span of related orthonormal eigenfunctions $\{\varphi_j\}$ of operator K is dense in V. Suppose that the operator K is nuclear (i.e., its trace $\operatorname{Tr} K = \sum_{j=1}^{\infty}\lambda_j < +\infty$).

Take the sequence of independent random processes $\{\eta_j\}$ and define the K-random process

$$\Theta_K(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \eta_j(t) \varphi_j \tag{22}$$

provided that the series (22) converges uniformly on any compact subset of I. Note that if $\{\eta_j\} \subset \mathbf{CL}_2$ and the K-random process Θ_K exists, then a.s. its trajectories are continuous. Denote the space of such processes by the symbol $\mathbf{C}_K \equiv \mathbf{C}_K(I \times \Omega; V)$. Consider in $\mathbf{C}_K$ the subspace $\mathbf{C}_K\mathbf{L}_2$ of random processes, whose random variables belong to $\mathbf{L}_2(\Omega; V) = \{\xi : \int_\Omega \|\xi(\omega)\|^2 d\mathbf{P}(\omega) < +\infty\}$; that is, $\eta \in \mathbf{C}_K\mathbf{L}_2$, if $\eta(t,\cdot) \in \mathbf{L}_2(\Omega; V)$ for each $t \in I$. Note that the space $\mathbf{C}_K\mathbf{L}_2$ contains, in particular, those K-random processes for which almost surely all trajectories are continuous, and all (independent) random variables are Gaussian.

We now introduce the Nelson-Gliklikh derivatives of a K-random process

$$\overset{\circ}{\Theta}{}_K^{(l)}(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \overset{\circ}{\eta}{}_j^{(l)}(t) \varphi_j \tag{23}$$

provided that the derivatives up to degree of l in the right hand side of (23) exist and the series uniformly converges on any compact subset of I.

Similarly, introduce the space $\mathbf{C}_K^l\mathbf{L}_2$ of K-random processes with a.s. continuous Nelson-Gliklikh derivatives up to order $l \in \mathbf{N}$, whose random variables belong to $\mathbf{L}_2(\Omega; V)$.

As an example consider the K-Wiener process

$$W_K(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \beta_j(t) \varphi_j, \tag{24}$$

which is defined on $\overline{\mathbf{R}}_+$.

Corollary 15. *Let* $\overset{\circ}{W}{}_K^{(l)}(t) = (-1)^{l+1} \prod_{i=1}^{l-1} (2i-1)(2t)^{-l} W_K(t)$ *for all* $t \in \mathbf{R}_+$, $l \in \mathbf{N}$, *and nuclear operator* $K \in \mathscr{L}(V)$.

Moreover, the K-Wiener process (24) satisfies conditions (W1) a.s. $W_K(0) = 0$, a.s. all its trajectories $\beta(t)$ are continuous, and for all $t \in \overline{\mathbf{R}}_+$ the random variable $W_K(t,\cdot)$ is Gaussian; (W2) the mathematical expectation $\mathbf{E}(W_K(t)) = 0$ and autocorrelation function

$$\mathbf{E}\left((\beta(t) - \beta(s))^2\right) = |t-s| K \quad \forall s, t \in \overline{\mathbf{R}}_+. \tag{25}$$

4. The Stochastic Sobolev Type Equation with Relatively p-Sectorial Operator

Let U, F, V be real separable Hilbert spaces. Let the operator M be (L,p)-sectorial, let $p \in \{0\} \cup \mathbf{N}$ and conditions (A1), (A2) be fulfilled, and let the operator $N \in \mathscr{L}(V; F)$. Let $I = [0, \tau)$. Let the operator $K \in \mathscr{L}(V)$ be nuclear with eigenvalues $\{\lambda_j\} \subset \mathbf{R}_+$. Consider the linear stochastic Sobolev type equation (3) with condition (4).

Remark 16. Due to Theorem 6 condition (4) is equivalent to the following condition:

$$\lim_{t \to 0+} P(\eta(t) - \xi_0) = 0. \tag{26}$$

Definition 17. The K-random process $\eta \in \mathbf{C}_K^1\mathbf{L}_2$ is called *a (classical) solution of* (3), if a.s. all its trajectories satisfy (3) with some K-random process $w \in \mathbf{C}_K\mathbf{L}_2$ for all $t \in (0,\tau)$. The solution of (3) is called *a solution of weakened Showalter-Sidorov problem* (3), (4), if it also satisfies condition (4).

Suppose that the K-random process $w = w(t)$, $t \in [0,\tau)$, satisfies condition

$$\begin{gathered} (I - Q) N w \in \mathbf{C}_K^{p+1}\mathbf{L}_2, \\ Q N w \in \mathbf{C}_K\mathbf{L}_2. \end{gathered} \tag{27}$$

Theorem 18. *Let the operator M be (L,p)-sectorial, let $p \in \{0\} \cup \mathbf{N}$ and conditions* (A1), (A2) *be fulfilled, and let operator $N \in \mathscr{L}(V; F)$. For any K-random process $w = w(t)$ satisfying (27) and for any U-valued random variable $\xi_0 \in \mathbf{L}_2$, independent of w, there exists a unique solution $\eta \in \mathbf{C}_K^1\mathbf{L}_2$ to problem (3), (4), given by*

$$\begin{aligned} \eta(t) = {} & \mathscr{U}^t \xi_0 + \int_0^t \mathscr{U}^{t-s} L_1^{-1} Q N w(s)\, ds \\ & - \sum_{q=0}^{p} H^q M_0^{-1} (\mathbf{I} - Q) N \overset{\circ}{w}{}^{(q)}(t). \end{aligned} \tag{28}$$

Proof. Proof of the theorem is analogous to the deterministic case [2]. Acting on (3) and condition (4) by projectors Q and $\mathbf{I} - Q$ and using Theorems 7 and 9, reduce it to the equivalent system of two independent problems

$$H \overset{\circ}{\eta}{}^0 = \eta^0 + M_0^{-1} (\mathbf{I} - Q) N w, \tag{29}$$

$$\begin{gathered} \overset{\circ}{\eta}{}^1 = S \eta^1 + L_1^{-1} Q N w, \\ \lim_{t \to 0+} \eta^1(t) = \xi_0^1, \end{gathered} \tag{30}$$

where $\eta^0 = (I - P)\eta$, $\eta^1 = P\eta$, $\xi_0^1 = P\xi_0$. Since the operator H is nilpotent, it follows from (29) that necessarily

$$\eta^0(t) = -\sum_{q=0}^{p} H^q M_0^{-1} (\mathbf{I} - Q) N \overset{\circ}{w}{}^{(q)}(t). \tag{31}$$

Since the operator $S \in \mathrm{cl}(U^1)$ the solution of problem (30) exists and can be represented in the form

$$\eta^1(t) = \mathscr{U}^t \xi_0^1 + \int_0^t \mathscr{U}^{t-s} L_1^{-1} Q N w(s)\, ds. \tag{32}$$

□

Consider the *weakened Showalter-Sidorov problem* (4) for equation

$$L \overset{\circ}{\eta} = M\eta + N \overset{\circ}{W}_K, \tag{33}$$

here the right hand side includes the Nelson-Gliklikh derivative of the K-Wiener process $W_K(t)$. The white noise $w(t) = (2t)^{-1}W_K(t)$ does not satisfy condition (27). One of feasible approaches to overcome this difficulty was proposed in [10, 11]. The advantage of this approach comes from transformation of the second summand in the right hand side of (28) as follows:

$$\begin{aligned}\int_\varepsilon^t \mathcal{U}^{t-s} L_1^{-1} QN \overset{\circ}{W}_K(s)\, ds &= L_1^{-1} QNW_K(t) - \mathcal{U}^{t-\varepsilon} L_1^{-1} QNW_K(\varepsilon) \\ &\quad - SP \int_\varepsilon^t \mathcal{U}^{t-s} L_1^{-1} QNW_K(s)\, ds.\end{aligned} \tag{34}$$

By virtue of the definition of Nelson-Gliklikh derivative for all $\varepsilon \in (0,t)$, $t \in \mathbf{R}_+$, we can make integration by parts. Letting $\varepsilon \to 0$ in (34) we get

$$\begin{aligned}\int_0^t \mathcal{U}^{t-s} L_1^{-1} QN \overset{\circ}{W}_K(s)\, ds &= L_1^{-1} QNW_K(t) - SP \int_0^t \mathcal{U}^{t-s} L_1^{-1} QNW_K(s)\, ds.\end{aligned} \tag{35}$$

Theorem 19. *Let the operator M be (L,p)-sectorial and let $p \in \{0\} \cup \mathbf{N}$ and conditions* (A1), (A2) *be fulfilled. For any $N \in \mathscr{L}(V;F)$ and for any U-valued random variable $\xi_0 \in \mathbf{L}_2$, independent of W_K, there exists a unique solution $\eta = \eta(t)$ of problem* (4), (33)*, given by*

$$\begin{aligned}\eta(t) &= \mathcal{U}^t \xi_0 \\ &\quad + L_1^{-1}\left[QNW_K(t) - M_1 \int_0^t \mathcal{U}^{t-s} L_1^{-1} QNW_K(s)\, ds\right] \\ &\quad - \sum_{q=0}^{p} H^q M_0^{-1} (\mathbf{I} - Q) N \overset{\circ}{W}_K^{(q+1)}(t).\end{aligned} \tag{36}$$

5. Dzektser Stochastic Model

Let $D \subset \mathbf{R}^n$ be a bounded domain with a boundary ∂D of class C^∞. Consider a boundary value

$$\Delta\eta(x,t) = \eta(x,t) = 0, \quad (x,t) \in \partial D \times [0,T] \tag{37}$$

and initial value

$$\lim_{t \to 0+} (\lambda - \Delta)(\eta(t) - \xi_0) = 0 \tag{38}$$

problems for the stochastic equation

$$(\lambda - \Delta)\overset{\circ}{\eta} = (\beta\Delta - \alpha\Delta^2)\eta + w. \tag{39}$$

Here the parameters $\lambda, \beta \in \mathbf{R}$, $\alpha \in \mathbf{R}_+$. This model describes evolution of free surface of filtered liquid.

Define the space $U = W_2^2(D) \cap \overset{\circ}{W}{}_2^1(D)$ and the space $F = L_2(D)$ with the scalar product

$$\langle u, v\rangle = \int_D uv\, dx. \tag{40}$$

Denote by $\{\lambda_j\}$ the sequence of eigenvalues of the homogeneous Direchlet problem for the operator Δ, numbered in nonincreasing order with regard to multiplicities and tending to $-\infty$. By $\{\varphi_j\}$ denote the orthonormal (in the sense of $L_2(D)$) family of corresponding eigenfunctions $\varphi_j \in C^\infty(D)$, $j \in \mathbf{N}$. Introduce the F-valued K-random process. Define the operator $\Lambda = (-1)^{m-1}\Delta^m$ with the domain

$$\begin{aligned}\operatorname{dom}\Lambda = \Big\{W_2^{2m}(D) : \Delta^k u(x) = 0,\ x \in \partial D,\ k \\ \in 0, 1, \ldots, m-1\Big\}, \quad m \in \mathbf{N}.\end{aligned} \tag{41}$$

It is rather easy to find such a number m according to fixed number n (which is the dimension of the domain D) that the mentioned series converges. For example, m can be equal to n. Note that the operator Λ has the same eigenfunctions $\{\varphi_j\}$, as the Laplace operator, but its spectrum consists of eigenvalues $|\lambda_j|^m$. Since their asymptotic $|\lambda_j|^m \sim j^{2m/n} \to \infty$, $j \to \infty$, we take such number $m \in \mathbf{N}$ that the series $\sum_{j=1}^\infty |\lambda_j|^{-m}$ converges. Then the operator Λ is continuously invertable on $\operatorname{dom}\Lambda$, whereas the inverse operator (i.e., the Green operator) has the spectrum consisting of eigenvalues $\nu_j = |\lambda_j|^{-m}$. We take this operator as the nuclear operator K for F-valued K-random process.

Fix $\lambda, \beta \in \mathbf{R}$, $\alpha \in \mathbf{R}_+$, and define the operators $L = \lambda I - \Delta$, $M = \beta\Delta - \alpha\Delta^2$. The operator $L \in \mathscr{L}(U;F)$, and the operator $M \in \operatorname{cl}(U;F)$ with

$$\begin{aligned}&\operatorname{dom} M \\ &\quad = \left\{u \in W_2^4(D) : \Delta u(x) = u(x) = 0,\ x \in \partial D\right\}.\end{aligned} \tag{42}$$

Lemma 20 (see [2, p. 198]). *For any $\lambda, \beta \in \mathbf{R}$, $\alpha \in \mathbf{R}_+$ the operator M is $(L,0)$-sectorial.*

The L-spectrum of the operator M consists of all points of the form

$$\mu_k \in \mathbf{C} : \mu_k = \frac{\beta\lambda_k - \alpha\lambda_k^2}{\lambda - \lambda_k}, \quad \text{if } k : \lambda_k \neq \lambda. \tag{43}$$

By Theorem 5 there exists a holomorphic resolving semigroup for (39) in the form

$$\mathcal{U}^t = \begin{cases} \displaystyle\sum_{k=1}^\infty e^{\mu_k t} \langle \cdot, \varphi_k\rangle \varphi_k, & \text{if } \lambda_k \neq \lambda\ \forall k \in \mathbf{N}; \\ \displaystyle\sum_{k \in \mathbf{N}: k \neq l} e^{\mu_k t} \langle \cdot, \varphi_k\rangle \varphi_k, & \text{if } \exists l \in \mathbf{N} : \lambda_l = \lambda. \end{cases} \tag{44}$$

Lemma 21. *For any $\lambda, \beta \in \mathbf{R}$, $\alpha \in \mathbf{R}_+$ conditions* (A1), (A2) *are fulfilled.*

Construct subsets

$$U^0 = \begin{cases} \{0\}, & \text{if } \lambda_k \neq \lambda\ \forall k \in \mathbf{N}; \\ \{u \in U : \langle u, \varphi_k \rangle = 0,\ k \in \mathbf{N} \setminus \{l : \lambda_l = \lambda\}\}; \end{cases}$$
$$U^1 = \begin{cases} U, & \text{if } \lambda_k \neq \lambda\ \forall k \in \mathbf{N}; \\ \{u \in U : \langle u, \varphi_k \rangle = 0, \lambda_k = \lambda\}. \end{cases} \quad (45)$$

Obviously $U^0 \oplus U^1 = U$. Thus the projector P has the form

$$P = s - \lim_{t\to 0+} \mathcal{U}^t = \begin{cases} \mathbf{I}, & \text{if } \lambda_k \neq \lambda\ \forall k \in \mathbf{N}; \\ \sum_{k\in\mathbf{N}:k\neq l} \langle \cdot, \varphi_k \rangle \varphi_k, & \text{if } \exists l \in \mathbf{N} : \lambda_l = \lambda. \end{cases} \quad (46)$$

The projector Q is constructed analogously.

Moreover there exists the operator

$$L_1^{-1} = \begin{cases} \sum_{k=1}^{\infty} (\lambda - \lambda_k)^{-1} \langle \cdot, \varphi_k \rangle \varphi_k, & \text{if } \lambda_k \neq \lambda\ \forall k \in \mathbf{N}; \\ \sum_{k\in\mathbf{N}:k\neq l} (\lambda - \lambda_k)^{-1} \langle \cdot, \varphi_k \rangle \varphi_k, & \text{if } \exists l \in \mathbf{N} : \lambda_l = \lambda. \end{cases} \quad (47)$$

Conditions (3) and (38) take the form

$$\begin{gathered} \lim_{t\to 0+} (\eta(t) - \xi_0) = 0, \quad \text{if } \lambda_k \neq \lambda\ \forall k \in \mathbf{N}; \\ \lim_{t\to 0+} \sum_{k\neq l} \langle (\eta(t) - \xi_0), \varphi_k \rangle \varphi_k = 0, \quad \text{if } \exists l \in \mathbf{N} : \lambda_l = \lambda. \end{gathered} \quad (48)$$

Thus, we have reduced problem (37)–(39) to problem (3), (4). From Theorem 18 we have the following assertion.

Theorem 22. *For any $\lambda, \beta \in \mathbf{R}$, $\tau, \alpha \in \mathbf{R}_+$ and for any K-random process $w = w(t)$ satisfying (27) and for any U-valued random variable $\xi_0 \in \mathbf{L}_2$, independent of w, there exists a unique solution $\eta \in \mathbf{C}_K^1\mathbf{L}_2$ for problem (37)–(39), given by*

$$\eta(t) = -M_0^{-1}\xi^0(t) + \mathcal{U}^t\xi_0 + \int_0^\tau \mathcal{U}^{t-s} L_1^{-1} Q w(s)\, ds. \quad (49)$$

Here

$$M_0^{-1} = \begin{cases} \mathbf{O}, & \textit{if } \lambda_k \neq \lambda\ \forall k \in \mathbf{N}; \\ \sum_{k\in\mathbf{N}:\lambda_k=\lambda} (\beta\lambda_k - \alpha\lambda_k^2)^{-1} \langle \cdot, \varphi_k \rangle \varphi_k. \end{cases} \quad (50)$$

Consider the initial-boundary value problem (37), (38) for equation

$$(\lambda - \Delta) \overset{\circ}{\eta} = (\beta\Delta - \alpha\Delta^2) \eta + \overset{\circ}{W}_K, \quad (51)$$

where the right part includes the Nelson-Gliklikh derivative of the K-Wiener process $W_K(t)$. From Theorem 19 we have the following assertion.

Theorem 23. *For any $\lambda, \beta \in \mathbf{R}$, $\tau, \alpha \in \mathbf{R}_+$, and U-valued random variable $\xi_0 \in \mathbf{L}_2$, independent of W_K there exists a unique solution $\eta = \eta(t)$ of problem (37), (38), (51) given by*

$$\begin{aligned} \eta(t) &= \mathcal{U}^t\xi_0 + L_1^{-1}\left[QW_K(t) - M_1 \int_0^t \mathcal{U}^{t-s} L_1^{-1} Q W_K(s)\, ds\right] \\ &\quad - \sum_{q=0}^{p} H^q M_0^{-1} (\mathbf{I} - Q) \overset{\circ}{W}_K^{(q+1)}(t). \end{aligned} \quad (52)$$

Here

$$M_1 = \begin{cases} \sum_{k=1}^{\infty} (\beta\lambda_k - \alpha\lambda_k^2) \langle \cdot, \varphi_k \rangle \varphi_k, & \textit{if } \lambda_k \neq \lambda\ \forall k \in \mathbf{N}; \\ \sum_{k\in\mathbf{N}:k\neq l} (\beta\lambda_k - \alpha\lambda_k^2) \langle \cdot, \varphi_k \rangle \varphi_k, & \textit{if } \exists l \in \mathbf{N} : \lambda_l = \lambda. \end{cases} \quad (53)$$

Conflict of Interests

The autors declare that they have no conflict of interests regarding the publication of this paper.

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