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# Cartesian Stiffness Matrix Mapping of a Translational Parallel Mechanism with Elastic Joints

Regular Paper

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Abstract This paper is devoted to calculating the Cartesian stiffness matrix of a translational parallel manipulator with elastic joints. The calculation takes into account the contribution of the Jacobian variation because of the change of manipulator configuration due to the elasticity and it covers the entire theoretical workspace of the manipulator. Three kineto-static adimensional indices are proposed to measure the response of the manipulator in terms of stiffness.

Keywords Stiffness Matrix, elastic joints, translational parallel manipulators

## 1. Introduction

Stiffness is one of the most important properties of a mechanism. Broadly speaking, the stiffness matrix *maps the applied loads with the displacements of the rigid bodies in static conditions*. As a result, stiffness clearly affects the accuracy and repeatability of the location of the end-effector. Stiffness depends on the manipulator configuration and on the direction of the applied loads. In the literature, the methods of calculating stiffness can be classified as follows: *a*) finite element analysis (FEA) *b*) matrix structural analysis (MSA) and *c*) the virtual joint method (VJM).

a) The FEA method, extensively used in structural mechanics, is reliable and accurate as the numerical model can duplicate the entire mechanism faithfully [1, 2]. Its accuracy is limited by the intrinsic parameters of the discretization mesh. On account of its reliability and accuracy, this method is used for validating other stiffness analysis techniques [3, 4, 5] and for comparative studies [6]. However, because of the repeated re-meshing routines required to cover the entire mechanism workspace, it has high computational costs. Moreover, it does not establish the analytical relationship between stiffness, dimensions and the free shape of the mechanism.

b) The MSA method incorporates the main ideas of FEA. The structural model of a mechanism is obtained as a combination of beam elements and nodes. Therefore, the MSA can be thought of as a simplification of the FEA, as it brings about a reduction of the computational expenses and, in some cases, allows the analytical stiffness matrix to

be obtained formally. A single element is represented by the Euler-Bernoulli beam with a 12 × 12 stiffness matrix. Much like FEA, the assembly of the stiffness matrices produces the desired 6 × 6 matrix for the whole mechanism. In [7], under the assumption that the links are not subject to bending, this approach was used for the calculation of the stiffness of a Stewart platform. This approach was also used in [8, 9] and, recently, for the Delta-type mechanism [10].

c) The VJ method (lumped model method) is based on the development of the standard rigid model to which virtual joints (localized springs) are added, which describe the elastic deformations of the mechanism components (links, joints and actuators). This approach was originally followed by Gosselin [11], who calculated the mechanism stiffness by considering the actuators one-dimensional linear springs, the links rigid and the passive joints perfect (*standard calculation*). The same author developed the method by modelling the links' flexibility as lumped linear/torsional springs connecting rigid bodies [12]. In general, there are numerous works based on modifications or simplifications of this method [13, 14, 15, 16, 17, 18]. In [19, 20, 21] a 6 degrees of freedom (DOFs) lumped virtual spring is proposed to model the link flexibility in order to consider the coupling between the linear and rotational deflections.

In the literature pertaining to the stiffness matrix of mechanisms there are also several papers whose main goal was to inspect the mathematical nature, *symmetry, positive-definiteness*, of the Cartesian stiffness. Griffis and Duffy [22], Ciblak and Lipkin [23], among others, discussed the asymmetric nature of the Cartesian stiffness matrix. Howard et al. [24], Zefran and Kumar [25, 26] investigated the symmetry of the Cartesian matrix and derived such a matrix by a formulation based on Lie groups. These researchers concluded that the Cartesian stiffness matrix of the elastic structure coupling two rigid bodies is asymmetric in general and becomes symmetric if the connection is not subjected to any pre-loading. Chakarov [27] studied the impedance control problems of manipulators touching the environment and developed a formulation of the Cartesian stiffness matrix without fully explicating all its terms. More recently, the same author developed the earlier work by Freeman et al. [28, 29, 30] and focused his attention on the antagonistic stiffness of redundantly actuated mechanisms [31]. Kövecses-Angeles [32] and Quennouelle-Gosselin [33] discussed the Cartesian stiffness matrix of the mechanisms in a detailed manner and ascertained that the Cartesian stiffness matrix still remains symmetric even when loading the end-effector.

Conversely, there are not many papers providing the formulation of the mechanism stiffness matrix in the joint space (a.k.a. Lagrangian) and the relationship with the Cartesian matrix. In the cited work Kövecses-Angeles and previously Chen and Kao [34], the latter who dealt with serial manipulators and considered only the actuated joints elastic, work explicitly on this topic.

In this paper the formulation of the Cartesian stiffness matrix proposed by the same author in [41] is applied to a translational parallel mechanism (TPM) to calculate the Cartesian stiffness matrix in the entire workspace. The formulation is general, as it is based on the development of the principle of virtual work and on the definition of the Cartesian stiffness matrix. Besides, three kineto-static indices are proposed and calculated in all points of the workspace with the aim of measuring different aspects of the stiffness property of the TPM.

## 2. Kinematic equations

The manipulator under study is a not-overconstrained TPM [37, 38, 39], a variant of the 3 – *RRPRR* [1] architecture. Each leg is composed of the *PU P R* kinematic chain with the *P* joints connected to the base along with

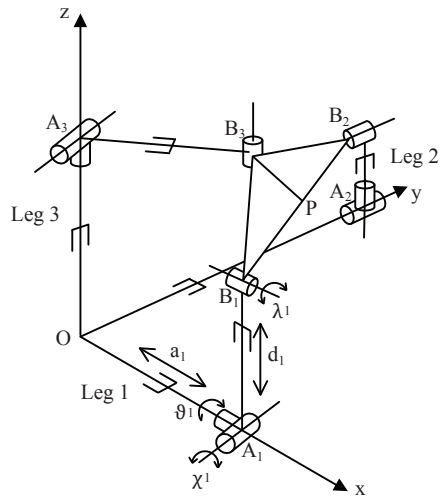



**Figure 1**. The 1-*PU P R* manipulator (all the joints are elastic).

orthogonal directions. The manipulator is shown in Figure 1. As proved in [40], to have zero angular velocity of the end-effector (E.E., a.k.a. moving platform) the joint variables of each leg has to guarantee the following conditions:

$$\chi_i = 0, \qquad \lambda_i = -\theta_i \qquad i=1,2,3 \qquad (1)$$

Thus, the vector of the remaining joint variables can be partitioned in a vector $q$ of independent (Lagrangian) coordinates and in a vector $\xi$ of dependent (constrained) coordinates. The number of independent coordinates is to be equal to the number of the E.E. degrees of freedom (DOFs). However, the choice of which joint variables to

[1] In this paper, *R* and *P* denote a revolute and prismatic joint, respectively, whereas *U* denotes a universal joint.

include in $\mathbf{q}$ and $\boldsymbol{\xi}$ is arbitrary:

$$\mathbf{q}=\left(\begin{array}{ccc} a_1 & \theta_1 & d_1 \end{array}\right)^T$$
$$\boldsymbol{\xi}=\left(\begin{array}{cccccc} a_2 & \theta_2 & d_2 & a_3 & \theta_3 & d_3 \end{array}\right)^T$$

*2.1 Position Equations*

These equations map the Lagrangian vector space into the Cartesian one:

$$\Psi(\mathbf{x}, \mathbf{q}) = 0 \tag{2}$$

$\mathbf{x} = (x \;\; y \;\; z)^T$ is the vector of the Cartesian coordinates of the reference point $P$ of the E.E. Equations (2) are scalar equations obtained from the vector loop-equation involving leg 1: $\overline{OA_1} + \overline{A_1B_1} + \overline{B_1P} - \overline{OP} = 0$.

$$\Psi_1 = x - a_1 + b_P c_b = 0$$
$$\Psi_2 = y - b_P s_b + d_1 s_{\theta_1} = 0$$
$$\Psi_3 = z - d_1 c_{\theta_1} = 0$$

With $b_p = \overline{\|B_1 P\|}$, $c_b = \cos(\pi/4)$, $s_b = \sin(\pi/4)$. $c\angle$ and $s\angle$ indicate $\cos(\angle)$ and $\sin(\angle)$, respectively. Time derivative of the location equations leads easily to:

$$\dot{\mathrm{x}} = -\left(\frac{\partial \Psi}{\partial \mathbf{x}}\right)^{-1} \frac{\partial \Psi}{\partial \mathbf{q}} \dot{\mathbf{q}} = \mathbf{J}_{\mathrm{x}}^{-1} \mathbf{J}_{\mathrm{q}} \dot{\mathbf{q}} = \mathbf{J}\dot{\mathbf{q}} \tag{3}$$

with

$$\mathbf{J} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mathrm{d}_1 \mathrm{c}_{\theta 1} & -\mathrm{s}_{\theta 1} \\ 0 & -\mathrm{d}_1 \mathrm{s}_{\theta 1} & \mathrm{c}_{\theta 1} \end{pmatrix}$$

$\mathbf{J}$ is the Jacobian of the TPM.

*2.2 Constraint Equations*

These equations generate the subspace of the configuration space of the TPM and can be expressed as:

$$\Phi(\mathbf{q}, \boldsymbol{\xi}) = 0 \tag{4}$$

More explicitly they are the geometric loop conditions:

$$\overline{OA_1} + \overline{A_1B_1} + \overline{B_1B_2} - \overline{OA_2} - \overline{A_2B_2} = 0$$
$$\overline{OA_1} + \overline{A_1B_1} + \overline{B_1B_3} - \overline{OA_3} - \overline{A_3B_3} = 0$$

which lead to:

$$\Phi_1 = a_1 - b c_b - d_2 s_{\theta 2} = 0$$
$$\Phi_2 = -d_1 s_{\theta 1} + b s_b - a_2 = 0$$
$$\Phi_3 = d_1 c_{\theta 1} - d_2 c_{\theta 2} = 0$$
$$\Phi_4 = d_1 s_{\theta 1} + d_3 s_{(\theta_3 + \pi/4)} = 0$$
$$\Phi_5 = d_1 c_{\theta 1} - a_3 = 0$$
$$\Phi_6 = a_1 - b - d_3 c_{(\theta_3 + \pi/4)} = 0$$

Time derivative of the constraint equations leads to:

$$\dot{\boldsymbol{\xi}} = -\left(\frac{\partial \Phi}{\partial \boldsymbol{\xi}}\right)^{-1} \frac{\partial \Phi}{\partial \mathbf{q}} \dot{\mathbf{q}} = -\mathbf{D}_{\xi}^{-1} \mathbf{D}_q \dot{\mathbf{q}} = \mathbf{D}\dot{\mathbf{q}} \tag{5}$$

Because of the intrinsic nature (*multiple closed chains*) of parallel mechanisms, $\mathbf{D}q$ and $\mathbf{D}\xi$ can be partitioned as:

$$\mathbf{D}_q = \begin{pmatrix} \mathbf{D}_{q12} \\ \mathbf{D}_{q13} \end{pmatrix}$$
$$\mathbf{D}_{\xi} = \begin{pmatrix} \mathbf{D}_{\xi 12} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{D}_{\xi 13} \end{pmatrix}$$

such that:

$$\mathbf{D} = \begin{pmatrix} -\mathbf{D}_{\xi 12}^{-1} & -\mathbf{D}_{q12} \\ -\mathbf{D}_{\xi 13}^{-1} & -\mathbf{D}_{q12} \end{pmatrix}$$

where:

$$\mathbf{D}_{q12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -d_1 c_{\theta 1} & -s_{\theta 1} \\ 0 & -d_1 s_{\theta 1} & c_{\theta 1} \end{pmatrix}, \mathbf{D}_{q13} = \begin{pmatrix} 0 & d_1 c_{\theta 1} & s_{\theta 1} \\ 0 & -d_1 s_{\theta 1} & c_{\theta 1} \\ 1 & 0 & 0 \end{pmatrix}$$
$$\mathbf{D}_{\xi 13} \begin{pmatrix} 0 & -d_3 c_{(\theta_3 + \pi/4)} & -s_{(\theta_3 + \pi/4)} \\ 1 & 0 & 0 \\ 0 & -d_3 s_{(\theta_3 + \pi/4)} & c_{(\theta_3 + \pi/4)} \end{pmatrix}, \mathbf{D}_{\xi 12} = \begin{pmatrix} 0 & d_2 c_{\theta 2} & s_{\theta 2} \\ 1 & 0 & 0 \\ 0 & -d_2 s_{\theta 2} & c_{\theta 2} \end{pmatrix}$$

## 3. Cartesian stiffness matrix

The Cartesian stiffness matrix is defined as the linear transformation between a variation of the force $\mathbf{f}$ applied on the E.E. and a variation of $\mathbf{x}$ :

$$\mathbf{K}_C = \frac{d\mathbf{f}}{d\mathbf{x}} \tag{6}$$

$\mathbf{K}_C$ can be expressed as follows.

Let's consider the Principle of Virtual Work [42] which a parallel mechanism, constituted by rigid bodies connected by elastic joints and subjected to external force on the E.E., has to obey:

$$\mathbf{f}^T \delta \mathrm{x} - \mathbf{f}_q^T \delta \mathbf{q} - \mathbf{f}_{\xi}^T \delta \boldsymbol{\xi} = 0 \tag{7}$$

where $\mathbf{f_q}$ and $\mathbf{f}_{\xi}$ are the conservative forces exerted by the springs (including the pre-loads) to obtain the virtual displacements, $\delta\mathbf{q}$ and $\delta\boldsymbol{\xi}$, respectively:

$$\mathbf{f}_q = \left(k_{q1}(q_1 - q_{l0}) \;\; k_{q2}(q_2 - q_{2_{l0}}) \;\; k_{q3}(q_3 - q_{3_{l0}})\right)^T$$
$$\mathbf{f}_{\xi} = \left(k_{\xi 1}(\xi_1 - \xi_{1_{l0}}) \;\; \cdots \;\; k_{\xi 6}(\xi_6 - \xi_{6_{l0}})\right)^T \tag{8}$$

$q_{i_{|0}}$ $(i=1,2,3)$ and $\xi_{j_{|0}}$ $(j=1, \cdots, 6)$ are the coordinates at the unloaded configuration of the manipulator, $k_{q_i}, k_{\xi_j}$ are the constants associated to the extension-compression of the springs along $q_i$ and $\xi_j$. It can be noticed here that $\mathbf{f}$ is not subjected to any restriction and thus it can be non-conservative. When using Eqs. (3) and (5) in Eq. (7) then $\mathbf{f}$ can be expressed as:

$$\mathbf{f} = \mathbf{J}^{-T}\left(\mathbf{f}_q + \mathbf{D}^T\mathbf{f}_\xi\right) \tag{9}$$

and, by using the chain rule in the derivation process, Eq. (6) leads to:

$$\mathbf{K}_C = \left[\frac{d}{d\mathrm{q}}\left(\mathbf{J}^{-T}\mathbf{f}_q + \mathbf{J}^{-T}\mathbf{D}^T\mathbf{f}_\xi\right)\right]\mathbf{J}^{-1} \tag{10}$$

Finally, Eq. (10) can be written as follows:

$$\mathbf{K}_C = \mathbf{J}^{-T}\left(\mathbf{K}_q + \frac{d\mathbf{D}^T}{d\mathbf{q}}\mathrm{f}_\xi + \mathbf{D}^T\mathbf{K}_\xi\mathbf{D} - \frac{d\mathbf{J}^T}{d\mathbf{q}}\mathrm{f}\right)\mathbf{J}^{-1} \tag{11}$$

where

$$\mathbf{K}_q = \frac{d\mathbf{f}_q}{dq} = diag\,(\,k_{q_i}),\ (i=1,2,3)$$

and

$$\mathbf{K}_\xi = \frac{d\mathbf{f}_\xi}{d\boldsymbol{\xi}} = diag\,(k_{\xi_j}),\,(j=1,\ \cdots,6)$$

according to the definitions of $\mathbf{f}_q$ and $\mathbf{f}_\xi$ and $\dfrac{d\mathbf{J}^{-T}}{d\mathbf{q}} = -\mathbf{J}^{-T}\dfrac{d\mathbf{J}^T}{d\mathbf{q}}\mathbf{J}^{-T}$ [33].

## 4. Measures of the stiffness property

The Cartesian stiffness matrix of a mechanism is a symmetric (positive semi-definite) matrix. To obtain $\mathbf{K}_C$ allows one to know the 6 coefficients of the linear transformation between $d\mathbf{f}$ and $d\mathbf{x}$ in a such configuration (*i.e.*, a point of the workspace). However, this does not mean we are able to express an unambiguous measure of the stiffness property of the manipulator in that configuration.

Similarly to the kineto-static performance indices [43] which involve the Jacobian $\mathbf{J}$ to measure the kinematic and static (with no elastic joints) properties of the manipulator, here, some indices are proposed to estimate the stiffness properties of the manipulator by using $\mathbf{K}_C$. As much as $\mathbf{J}$, also $\mathbf{K}_C$ can have elements with non-homogenous dimensions and, thus, one should be careful in manipulating the matrix and proposing performance indices. This issue is completely skipped in this case where all the elements of $\mathbf{K}_C$ have a dimension of Newton per unit length. Therefore, three measures of the static properties of the manipulator in a point of the workspace are proposed. In general, these measures are valid for manipulators with only one task.[2]

*4.1 Averaged stiffness:*

$$S_{avg} = \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right) \tag{12}$$

$\sigma_{\mathrm{i}}, (\mathrm{i}=1,2,3)$ are the eigenvalues of $\mathbf{K}_C$. $\sigma_i$, in a such point of the Cartesian workspace, is the factor mapping the elongation $d\mathbf{s}_i$ along the principal direction $i$ (*eigenvector*) into the variation of the force in that direction $d\mathbf{f}_i$, such that:

$$\sigma_i d\mathbf{s}_i = d\mathbf{f}_i$$

As the principal direction is defined as the direction along with the matrix becomes diagonal according to the standard eigenvalue problem. It is worth noting that numerically $S_{avg}$ is equivalent to $1/3\mathrm{tr}\,(\mathbf{K}_C)$ However, they have different physical meanings. Indeed, $S_{avg}$ is the full stiffness property of the manipulator along the principal directions, whereas the measure $1/3\mathrm{tr}\,(\mathbf{K}_C)$ exclude the out-of-diagonal elements thus neglecting the stiffness coupling effects.

*4.2 Stiffness uniformity:*

$$\zeta = \sqrt{\frac{\sigma_{\mathbf{mn}}}{\sigma_{\mathbf{mx}}}} \tag{13}$$

With $\sigma_{mn} = \min\,(\sigma_i)$ and $\sigma_{mx} = \max\,(\sigma_i)$. $\zeta$ is identical to the measure proposed by Gosselin [11] with only the independent joints considered elastic. It provides an adimensional measure of the uniformity of the stiffness of the manipulator in a such configuration.

*4.3 Energy of deformation:*

$$\delta D = \left(\frac{d\varepsilon_{\mathbf{p}}}{d\mathbf{q}}\right)^T \delta\mathbf{q}$$

where $\delta$D is the energy of deformation stored in the manipulator, $\varepsilon_p$ is the potential function of the manipulator, (*i.e.*, associated with the conservative forces $\mathbf{f}_q, \mathbf{f}_\xi$). According to the Principle of Virtual Work, $\delta$D can be equivalently written as:

[2]mixed-mode manipulators require the definition of some parameters to match the elements' matrix dimension [43].

$$\delta D = \mathbf{f}^T \delta \mathbf{x} \tag{15}$$

$\mathbf{f}$ is obtained from Eq. (9) and $\delta x = \mathbf{K}_C^{-1}\mathbf{f}$ .

## 5. Numerical calculation of Kc and of the indices

It is presented here the numerical procedure followed to calculate, in the entire workspace of the manipulator, both $\mathbf{K}_C$ and three adimensional indices defined from the proposed measures. Then, the results obtained are shown.

### *5.1 Procedure*

a. A symbolic calculation of $\mathbf{K}_C$ is performed using Eq. (11). All the matrices contributing to $\mathbf{K}_C$ are also available formally;
b. For each step, *i.e.*, a point *x, y, z* in the workspace, Eq. (2) and Eq. (4) are used to obtain $\mathbf{q}$ and $\boldsymbol{\xi}$, respectively. Thereby, $\mathbf{f}_q, \mathbf{f}_\xi$ and $\mathbf{f}$ can be calculated.
c. For each step, $\mathbf{K}_C$ is calculated the results from steps a) and b). $S_{avg}$, $\zeta$, $D$ are calculated according to their definitions.
d. Steps a), b) and c) are repeated to span the entire theoretical Cartesian workspace of the manipulator.
e. Once all points in the workspace were processed, $S_{avg}$ and $D$ were normalized to the maximum value reached in the workspace $W$ to define the following adimensional indices.

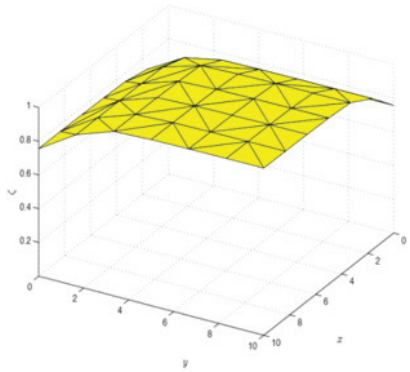
$$s_{avg} = \frac{S_{avg}}{\max\left(S_{avg}\right)_{|W}}$$

$$\chi = \frac{D}{\max\left(D\right)_{|W}}$$

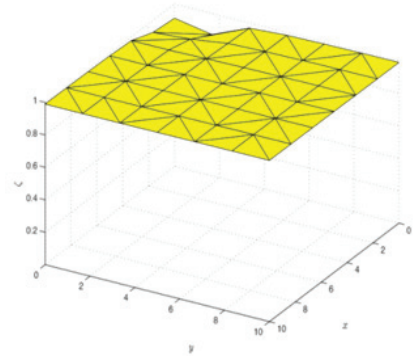
### *5.2 Results*

A numerical example was carried out to apply the procedure proposed. As $\zeta$, $s_{avg}$ and $\chi$ are adimensional, we do not need to provide geometrical and spring numerical data in this section. In the example all the prismatic joints were modelled as linear springs with the same stiffness constant. Similarly, all the rotation joints were modelled as torsional springs with the same stiffness constant. The TPM unloaded configuration was chosen to be at $x = y = z = 0$. In this position, according to Eqs. (2), (4), the values of the joints' variables $q_{i_{10}}, \xi_{j_{10}}$ were calculated. In Figures 2, 3, 4, as examples, the values assumed by the indices on the planes $z = 0$ *and* $z=10$ are shown.

According to the symmetry of the TPM architecture, the indices trends are symmetric in the workspace. Let's first consider the $\zeta$ trends (Figure 2). At x = y = 0 and z = 0 or z=10 (in general $\forall(z)$) $\zeta$ is smaller than at the rest of workspace points.

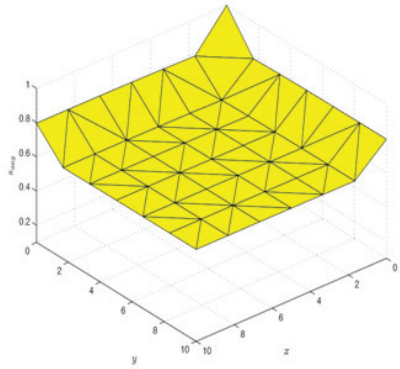


(a)

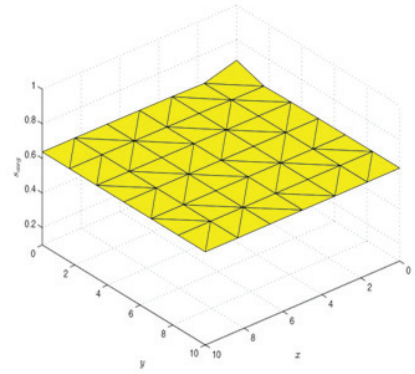


(b)

**Figure 2.** a): $\zeta$ at $z=0$, b): $\zeta$ at $z=10$



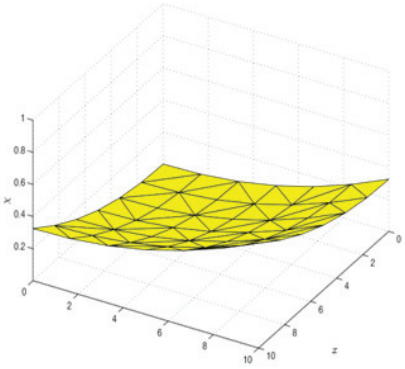
(a)



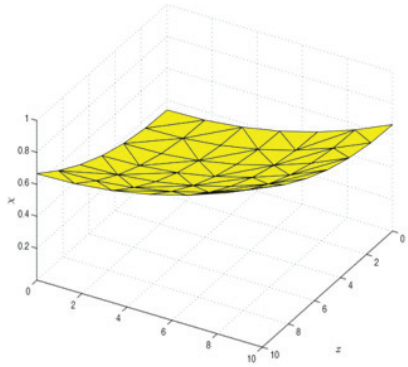
(b)

**Figure 3.** a): $s_{avg}$ at $z=0$,b): $s_{avg}$ at $z=10$

Indeed, $\mathbf{K}_C$ calculated in these points shows the value of the diagonal element $\mathbf{K}_{C33}$ much larger than the others enlarging the gap between $\sigma_{mx}$ and $\sigma_{mn}$. That is especially marked at the unloaded position where $\zeta$ reaches its minimum in the entire workspace. Physically, this agrees with the fact that at the unloaded configuration the stiffness of the TPM reaches its maximum. Investigation on $\mathbf{K}_C$ can also explain the $\zeta$ values at $x = z = 0$ and $y = z = 0$. However, $\zeta$ is larger than 0.6 when $z=0$ and increases with $z$, when $z=10$ it is very close to 1. In light of the previous discussion, the understanding of the $s_{avg}$ trends is straightforward (Figure 3).



(a)



(b)

**Figure 4.** a): $\chi$ at $z=0$ ,b): $\chi$ at $z=10$

At $x = y = 0$ and $z=0$ or $z=10$ (in general $\forall(z)$) $s_{avg}$ is larger than at the rest of workspace points. Indeed, the large value of $\mathbf{K}_{C33}$ increases $S_{avg}$ and this fact is especially marked at the unload position and also at $x = z = 0$ and $y = z = 0$. In general, $s_{avg}$ does not change so much when varying $z$ being always larger than 0.6. Figure 4 shows the $\chi$ trends. It can be noted that $\chi$ increases symmetrically with $x$ and $y$. Indeed, the energy of deformation stored in the elastic parts of the TPM rises up in moving away from the unloaded position. Accordingly, the $\chi$ values at $z=10$, $\forall(x, y)$ are larger than those at $z=0$ and the point of maximum deformation is $x = y = z = 10$.

## 6. Conclusions

In this paper the author proposed a calculation of the Cartesian stiffness matrix $\mathbf{K}_C$ of a TPM in all points of the theoretical workspace and three adimensional indices with the aim of measuring the stiffness properties of the manipulator. The calculation of $\mathbf{K}_C$ is not standard as it takes into account the contribution of the Jacobian variation because of the change of configuration due to the elasticity of the TPM. The symbolic calculation of $\mathbf{K}_C$ was implemented in a procedure running for all points of the TPM workspace. From the procedure, each contribution to $\mathbf{K}_C$ is also available. The indices $\zeta$, $s_{avg}$ and $\chi$ provide some informations on the stiffness properties of the TPM. Indeed, as proposed in the previous section, their physical interpretation allows one to better understand the behaviour of the TPM in terms of stiffness.

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