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Research Article

A Decomposition of the Dual Space of Some Banach Function Spaces

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We give a decomposition for the dual space of some Banach Function Spaces as the Zygmund space EXP_α of the exponential integrable functions, the Marcinkiewicz space $L^{p,\infty}$, and the Grand Lebesgue Space $L^{p),\theta}$.

1. Introduction

Let Ω be a set of Lebesgue measure $|\Omega| < +\infty$.

In this paper, we deal with the following issue. What is the difference between the dual space X^* and the associate space X' of a Banach Function Space X?

By associate space X' of X we mean the space determined by the associate norm ρ':

$$\rho'(g) = \sup\left\{ \int_\Omega f g \, dx : f \in \mathcal{M}^+, \rho(f) \le 1 \right\} \tag{1.1}$$

as in Definitions 2.3 and 2.4.

If X is a reflexive Banach Function Space, then the dual space X^* is canonically isometrically isomorphic to the associate space X' [1, page 23]. On the other hand, for example, if we consider the Orlicz space $\mathrm{EXP}(\Omega)$ of exponentially integrable functions, which is not reflexive, the associate space $(\mathrm{EXP}(\Omega))'$ coincides with the Zygmund space $L\log L(\Omega)$, while the dual can be represented by

$$(\mathrm{EXP}(\Omega))^* = L\log L(\Omega) \oplus \left(\exp(\Omega)\right)^{\perp}, \tag{1.2}$$

where $\exp(\Omega)$ is the closure of $L^\infty(\Omega)$ with respect to the EXP norm (see [2, Chapter IV], [3] and also Corollary 3.4).

Our aim is to show that the decomposition for the dual space as in (1.2) holds in a more general setting: namely, if X is a rearrangement invariant Banach Function Space on Ω such that its fundamental function φ_X verifies

$$\varphi_X(0+) = 0, \tag{1.3}$$

then,

$$X^* = X' \oplus (X_b)^\perp, \tag{1.4}$$

where X_b denotes the closure of $L^\infty(\Omega)$ in X. We stress that, due to assumption (1.3), our argument is much shorter than the corresponding one, treated in Zaanen ([4, Section 70, Theorem 2, page 467]) in the more abstract setting of normed Köethe spaces. (See also [2, Chapter IV, Proposition 2.8 and Theorem 2.11]).

In Section 3, we consider our decomposition in the particular case of EXP_α spaces, Marcinkiewicz spaces, and the Grand Lebesgue Spaces, specifying case by case the expression of the associate space.

Let us note that in general a Banach Function Space X can be identified with a closed subspace of $(X')^*$ [1], while the spaces mentioned in our particular cases verify

$$X = (X')^* \tag{1.5}$$

as shown in Theorem 3.7.

2. Preliminaries

Let Ω be a set of Lebesgue measure $|\Omega| < +\infty$ and let $\mathcal{M}_o^+$ be the set of all measurable functions, whose values lie in $[0, +\infty]$, finite a.e. in Ω.

Definition 2.1. A mapping $\rho : \mathcal{M}_o^+ \to [0, +\infty]$ is called a *Banach function norm* if, for all $f,\ g,\ f_n\ (n = 1, 2, 3, \ldots)$ in $\mathcal{M}_o^+$, for all constants $a \geq 0$, and for all measurable subsets $E \subset \Omega$, the following properties hold.

$$\begin{gathered}
\rho(f) = 0 \Longleftrightarrow f = 0 \text{ a.e. in } \Omega, \\
\rho(af) = a\rho(f), \\
\rho(f + g) \leq \rho(f) + \rho(g), \\
0 \leq g \leq f \text{ a.e. in } \Omega \Longrightarrow \rho(g) \leq \rho(f)\ \ , \\
0 \leq f_n \uparrow f \text{ a.e. in } \Omega \Longrightarrow \rho(f_n) \uparrow \rho(f), \\
|E| < +\infty \Longrightarrow \rho(\chi_E) < +\infty, \\
|E| < +\infty \Longrightarrow \int_E f\,dx \leq C_E \rho(f)
\end{gathered} \tag{2.1}$$

for some constant C_E, $0 < C_E < \infty$, depending on E and ρ, but independent of f.

Definition 2.2. If ρ is a Banach function norm, the Banach space

$$X = \{f \in \mathcal{M} : \rho(|f|) < +\infty\} \tag{2.2}$$

is called a *Banach Function Space*.

For each $f \in X$, define

$$\|f\|_X = \rho(|f|). \tag{2.3}$$

Recall that the simple functions are contained in every Banach Function Spaces X [1].

Definition 2.3. If ρ is a function norm, its *associate norm* ρ' is defined on $\mathcal{M}_o^+$ by

$$\rho'(g) = \sup\left\{\int_\Omega fg\,dx : f \in \mathcal{M}^+, \rho(f) \le 1\right\}. \tag{2.4}$$

Definition 2.4. Let ρ be a function norm and let $X = X(\rho)$ be the Banach Function Space determined by ρ. Let ρ' be the associate norm of ρ. The Banach Function Space $X' = X(\rho')$ determined by ρ' is called the *associate space* of X.

In particular, from the definition of $\|f\|_X$, it follows that the norm of a function g in the associate space X' is given by

$$\|g\|_{X'} = \sup\left\{\int_\Omega fg\,dx \,:\, f \in X, \|f\|_X \le 1\right\}. \tag{2.5}$$

Definition 2.5. A function f in a Banach Function Space X is said to have *absolutely continuous norm* in X if $\|f\chi_{E_n}\|_X \to 0$ for every sequence $\{E_n\}_{n=1}^\infty$ satisfying $E_n \to \emptyset$ a.e. The set of all functions in X of absolutely continuous norm is denoted by X_a. If $X = X_a$, then the space X itself is said to have *absolutely continuous norm.*

Definition 2.6. Let $f \in \mathcal{M}_o$. The function

$$\mu_f(\lambda) = |\{x \in \Omega : |f(x)| > \lambda\}| \quad \forall\lambda \ge 0 \tag{2.6}$$

is called the *distribution function* of f. The *decreasing rearrangement* of f, f^*, is defined on $[0, |\Omega|]$ by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \le t\}, \tag{2.7}$$

where here we use the convention $\inf\emptyset = +\infty$.

Two functions having the same distribution function are called equimeasurable.

Let us recall that a function norm ρ is said to be *rearrangement invariant* (briefly, "r.i.") if $\rho(f) = \rho(g)$ for every couple of equimeasurable functions. The Banach Function Space arising from a r.i. function norm is called a *rearrangement-invariant space*.

By $f^{**} : (0,\infty) \to [0,\infty]$, we denote the function given by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds, \qquad (t > 0). \tag{2.8}$$

The function f^{**} is nonincreasing and verifies $f^*(t) \le f^{**}(t) \quad (t > 0)$.

Definition 2.7. Let X be a r.i. Banach Function Space determined by a function norm ρ. For each $t \in [0,|\Omega|]$, let $E_t \subseteq \Omega$ be a set of measure t. The *fundamental function* of X, $\varphi_X(t)$, is defined by

$$\varphi_X(t) = \rho(\chi_{E_t}) = \|\chi_{E_t}\|_X. \tag{2.9}$$

Definition 2.8. Let $1 \le p \le \infty$ and $\alpha \in \mathbf{R}$, then the *Zygmund space* $L^p(\log L)^\alpha(\Omega)$ is the set of all measurable functions f in Ω for which the quantity

$$\|f\|_{L^p(\log L)^\alpha(\Omega)} = \left\| \left(1 + \log\left(\frac{|\Omega|}{t}\right)\right)^\alpha f^*(t) \right\|_{L^p(0,|\Omega|)} \tag{2.10}$$

is finite.

For $p = 1$ and $\alpha = 1$ we will replace $L^1(\log L)^1(\Omega)$ by $L(\log L)(\Omega)$.

With these notations, the usual space EXP_α of the *exponentially integrable functions* corresponds to the Zygmund space $(L^\infty/(\log L)^{1/\alpha})(\Omega)$ and consists of all measurable functions f in Ω for which the quantity

$$\|f\|_{L^\infty/(\log L)^{1/\alpha}(\Omega)} = \|f\|_{\mathrm{EXP}_\alpha(\Omega)} = \sup_{0<t<|\Omega|} \left(1 + \log\left(\frac{|\Omega|}{t}\right)\right)^{-1/\alpha} f^*(t) \tag{2.11}$$

is finite.

All these spaces are particular cases of the Orlicz spaces.

Let $\phi : [0,\infty) \to [0,\infty)$ be a right-continuous, increasing function, such that $\phi(0) = 0$ and $\lim_{t\to\infty}\phi(t) = \infty$, then the function defined by

$$\Phi(t) = \int_0^t \phi(s)\,ds \tag{2.12}$$

is called *N function*; it is a continuous, convex, increasing function such that $\lim_{t\to\infty}(\Phi(t)/t) = +\infty$ and $\lim_{t\to 0}(\Phi(t)/t) = 0$.

Definition 2.9. The *Orlicz space* $L^\Phi(\Omega)$ consists of all measurable functions f on Ω for which there exists some $\lambda > 0$ such that

$$⨍_\Omega \Phi\left(\frac{|f|}{\lambda}\right) < \infty, \tag{2.13}$$

where $⨍_\Omega$ stands for $(1/|\Omega|)\int_\Omega$.

This is a Banach space with respect to the Luxemburg norm:

$$\|f\|_{L^\Phi}(\Omega) = \inf\left\{\lambda > 0 : \fint_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\}. \tag{2.14}$$

The Orlicz spaces are a standard example of rearrangement-invariant Banach Function Space: the associate space of $L^\Phi(\Omega)$ is given by $L^{\tilde{\Phi}}(\Omega)$, where $\tilde{\Phi}$ denotes the complementary function of Φ, defined by

$$\tilde{\Phi}(t) = \max\{st - \Phi(s) : s \ge 0\}. \tag{2.15}$$

Moreover, we notice that, for $\Phi(t) = t^p$, $\Phi(t) = t^p(\log t)^\alpha$, and $\Phi(t) = e^{t^\alpha} - 1$, the Orlicz space associated reduces, respectively, to the spaces $L^p(\Omega)$, $L^p(\log L)^\alpha(\Omega)$ and to $\mathrm{EXP}_\alpha(\Omega)$.

Definition 2.10. Given $1 \le p, q \le \infty$, the *Lorentz space* $L^{p,q}(\Omega)$ consists of all measurable functions f in Ω for which

$$\|f\|_{p,q} = \begin{cases} \displaystyle\int_0^\infty \left[t^{1/p} f^*(t)\right]^q \frac{dt}{t}, & 0 < q < \infty, \\ \displaystyle\sup_{t>0} t^{1/p} f^*(t), & q = \infty \end{cases} \tag{2.16}$$

is finite.

The space $L^{p,\infty}(\Omega) = Weak\text{-}L^p(\Omega)$ is known as the *Marcinkiewicz space*, and it is another example of r.i. Banach Function Space.

The quantity (2.16) is not a norm since the triangle inequality may fail; however, for $p > 1$, replacing $f^*(t)$ with $f^{**}(t)$, we obtain a norm equivalent to (2.16).

In particular, for $q = \infty$, in the case of a nonatomic measure space, (2.16) is equivalent to

$$\sup\left\{|E|^{1/p-1}\int_E |f|\,dx,\ E \subset \Omega \text{ measurable}\right\}. \tag{2.17}$$

Now, we recall the definitions of Grand and Small Lebesgue Spaces, introduced, respectively, in [5] and in [6].

Definition 2.11. Let $1 < p < +\infty$ and $\theta \ge 0$; the *Grand Lebesgue Space* $L^{p),\theta}$ is the Banach Function Space of all measurable functions f on Ω such that

$$\|f\|_{p),\theta} = \sup_{0<\varepsilon<p-1}\left(\varepsilon^\theta \fint_\Omega |f|^{p-\varepsilon}\,dx\right)^{1/(p-\varepsilon)} \tag{2.18}$$

is finite.

Notice that

$$L^{p),0}(\Omega) = L^p(\Omega), \qquad L^{p),1}(\Omega) = L^{p)}(\Omega). \tag{2.19}$$

If $1 < p < +\infty$ and p' is its Hölder conjugate exponent, according to [7], the *Small Lebesgue Space* $L^{(p',\theta}$ can be identified as the set of all measurable functions f on Ω such that

$$\|f\|_{(p',\theta} = \sup\left\{\left|\int_\Omega fg\, dx\right| : \|f\|_{L^{p),\theta}(\Omega)} \le 1\right\} \tag{2.20}$$

is finite.

The Grand and Small Lebesgue Spaces are r.i. Banach Function Spaces [7].

Definition 2.12. A vector space V is the *direct sum* of its subspaces U and W, denoted by $V = U \oplus W$, if and only if

$$\begin{gathered} V = U + W = \{u + w : u \in U, w \in W\}, \\ V \cap W = \{0\}. \end{gathered} \tag{2.21}$$

Elements v of the direct sum $U \oplus W$ are representable uniquely in the form

$$u + w : u \in U, \quad w \in W. \tag{2.22}$$

Definition 2.13. Let X be a Banach space and $M \subset X$ a vectorial subspace of X. The *orthogonal space* $M^\perp$ of M is

$$M^\perp = \{f \in X^* : \langle f, x\rangle = 0, \quad \forall x \in M\}, \tag{2.23}$$

where $\langle .,. \rangle$ is the duality inner product.

It is known that $M^\perp$ is a closed subspace of X^*.

We conclude this section by recalling some classical results, which will be useful in the sequel.

Theorem 2.14 (Hölder's inequality [1]). *Let X be a Banach Function Space with associate space X'. If $f \in X$ and $g \in X'$, then fg is integrable and*

$$\int_\Omega fg\, dx \le \|f\|_X \|g\|_{X'}. \tag{2.24}$$

Lemma 2.15 (see [1, Lemma 2.6, page 10]). *In order that a measurable function g belongs to the associate space X', it is necessary and sufficient that fg is integrable for every f in X.*

Theorem 2.16 (see [1, Theorem 2.7, page 10]). *Every Banach Function Space X coincides with its second associate space $X'' = (X')'$.*

Theorem 2.17 (see [1, Theorem 2.9, page 13]). *The associate space X' of a Banach Function Space X is canonically isometrically isomorphic to a closed norm-fundamental subspace of the Banach space dual X^* of X.*

Proposition 2.18 (see [1, Proposition 2.10, page 13]). *If X and Y are Banach Function Spaces and $X \subset Y$ (continuous embedding), then $Y' \subset X'$ (continuous embedding).*

Theorem 2.19 (see [1, Theorem 3.11, page 18]). *Let X be a Banach Function Space. Then, $X_a \subseteq X_b \subseteq X$.*

Corollary 2.20. *If $X_a = X$, then $X_b = X$.*

Theorem 2.21 (see [1, Theorem 3.13, page 19]). *The subspaces X_a and X_b coincide if and only if the characteristic function χ_E has absolutely continuous norm for every set E of finite measure.*

Theorem 2.22 (see [1, Corollary 4.2, page 23]). *Let X be a Banach Function Space. If X_a contains the simple functions, then $(X_a)^* = X'$.*

Theorem 2.23 (see [1, Corollary 4.3, page 23]). *The Banach space dual X^* of a Banach Function Space X is canonically isometrically isomorphic to the associate space X' if and only if X has absolutely continuous norm.*

Theorem 2.24 (see [1, Theorem 5.5, page 67]). *Let (Ω, μ) be a totally σ-finite nonatomic measure space and let X be an arbitrary rearrangement-invariant space over (Ω, μ). The following conditions on X are equivalent:*

(i) $\lim_{t\to 0^+} \varphi_X(t) = 0$;

(ii) $X_a = X_b$;

(iii) $(X_b)^* = X'$,

where $\varphi_X(t)$ is the fundamental function of X.

3. Main Results

In this Section, we establish a decomposition for the dual space of a r.i. Banach Function Space.

Theorem 3.1. *Let X be a rearrangement-invariant Banach Function Space on Ω. For each $t \in [0, |\Omega|]$, let E be a subset of Ω with $|E| = t$ and let $\varphi_X(t)$ be the fundamental function of X.*

If

$$\lim_{t\to 0^+} \varphi_X(t) = 0, \tag{3.1}$$

then the following decomposition

$$X^* = X' \oplus (X_b)^{\perp} \tag{3.2}$$

holds.

Proof. Let $l \in X^*$, for all measurable sets F in Ω, we define the set function

$$\nu(F) = l(\chi_F), \tag{3.3}$$

which is σ-additive and absolutely continuous with respect to the Lebesgue measure $|F|$. Thus, ν has a locally integrable Radon-Nikodym derivative g and

$$l(f) = \int_{\Omega} fg\,dx, \quad \text{for any } f \in L^{\infty}(\Omega). \tag{3.4}$$

Since $l \in X^*$ for all $f \in X$, it is

$$l(f) \leq K\|f\|_X, \tag{3.5}$$

where K is a constant. Hence, for all $f \in L^{\infty}$,

$$\int_{\Omega} fg\,dx \leq K\|f\|_X. \tag{3.6}$$

By Lemma 2.15, it follows that $g \in X'$.

To any $g \in X'$, we can associate the functional

$$l_g : f \in X_b \longrightarrow \int_{\Omega} fg\,dx. \tag{3.7}$$

By Hölder's inequality, l_g belongs to X_b^*, which is equivalent to X' thanks to Theorem 2.24.

Finally, let l_s be defined by $l_s = l - l_g$, then $l_s(f) = \langle l_s, f \rangle = 0$ for all $f \in X_b$. Therefore, l_s belongs to $(X_b)^{\perp}$.

Hence,

$$l = l_g + l_s \in X' + X_b^{\perp}. \tag{3.8}$$

Since it is easily seen that X' and $X_b^{\perp}$, subspaces of X^*, verify $X' \cap X_b^{\perp} = \{0\}$, then the proof is complete. □

Remark 3.2. Let us point out that, by Theorem 2.24, the decomposition (3.2) can also be written as

$$X^* = (X_b)^* \oplus (X_b)^{\perp}, \tag{3.9}$$

$$X^* = (X_a)^* \oplus (X_a)^{\perp}. \tag{3.10}$$

Corollary 3.3. *Let X be an Orlicz space, then*

$$\begin{aligned} X^* &= X' \oplus (X_b)^{\perp} \\ &= (X_b)^* \oplus (X_b)^{\perp} \\ &= (X_a)^* \oplus (X_a)^{\perp}. \end{aligned} \tag{3.11}$$

Proof. If $X = L^{\Phi}(\Omega)$ is an Orlicz space, then the fundamental function is

$$\varphi_X(t) = \frac{1}{\Phi^{-1}(1/t)}, \qquad \forall t \in \,]0, |\Omega|] \tag{3.12}$$

(see [7]). Therefore, $\lim_{t\to 0^+}\varphi_X(t) = 0$ and the claim follows from Theorem 3.1 and Remark 3.2. □

Corollary 3.4. *Let* $X = \mathrm{EXP}_\alpha(\Omega)$, $\alpha > 0$, *then*

$$\begin{aligned}(\mathrm{EXP}_\alpha(\Omega))^* &= L\log^{1/\alpha}L(\Omega) \oplus \left(\exp_\alpha(\Omega)\right)^{\perp} \\ &= \left(\exp_\alpha(\Omega)\right)^* \oplus \left(\exp_\alpha(\Omega)\right)^{\perp},\end{aligned} \tag{3.13}$$

where $\exp_\alpha(\Omega)$ *denotes the closure of* $L^\infty(\Omega)$ *in* $\mathrm{EXP}_\alpha(\Omega)$.

Proof. The result follows by Corollary 3.3, and by $(\mathrm{EXP}_\alpha(\Omega))' = L\log^{1/\alpha}L(\Omega)$, $\alpha > 0$, (see [1]). □

Corollary 3.5. *Let* $p \in]1,\infty[$, p' *be its Hölder conjugate exponent and* $X = L^{p,\infty}(\Omega)$, *then*

$$(L^{p,\infty}(\Omega))^* = L^{p',1}(\Omega) \oplus \left(L_b^{p,\infty}(\Omega)\right)^{\perp}. \tag{3.14}$$

Proof. The Marcinkievicz space $L^{p,\infty}(\Omega)$ is the largest of all rearrangement-invariant spaces having the same fundamental function as $L^p(\Omega)$ (see [1]), which is

$$\varphi_{L^p}(t) = t^{1/p}. \tag{3.15}$$

Moreover, the associate space of $L^{p,\infty}(\Omega)$ (see [1]) is, up to equivalence of norms, the Lorentz space $L^{p',1}(\Omega)$.

Therefore, the statement easily follows by Theorem 3.1.

A decomposition of the dual of $L^{p,\infty}$ was also given in [8]. □

Corollary 3.6. *Let* $p \in]1,\infty[$, $\theta \geq 0$ *and* $X = L^{p),\theta}(\Omega)$, *then*

$$\left(L^{p),\theta}(\Omega)\right)^* = L^{(p',\theta}(\Omega) \oplus \left(L_b^{p),\theta}(\Omega)\right)^{\perp}. \tag{3.16}$$

Proof. Let $\varphi_X(t)$ be the fundamental function of the space $L^{p),\theta}(\Omega)$, then

$$\varphi_X(t) \approx t^{1/p}\left[\log\left(\frac{1}{t}\right)\right]^{-\theta/p} \tag{3.17}$$

as $t \to 0^+$ (see [7]).

Therefore the claim easily follows by Theorem 3.1 and by the relation $(L^{p),\theta}(\Omega))' = L^{(p',\theta}(\Omega)$ (see [7]). □

In the next theorem, we show the relation between a Banach Function Space X and the dual of its associate space $(X')^*$.

Theorem 3.7. *Let X be a Banach Function Space, then the following inclusion*

$$X \subseteq (X')^* \tag{3.18}$$

holds, with equality occurring if and only if the associate space X' of X has absolutely continuous norm.

Proof. By Theorem 2.17 applied to the Banach Function Space X', we may identify $(X')'$ with a closed subspace of $(X')^*$; hence, Theorem 2.16 implies

$$X = X'' = (X')' \subseteq (X')^*. \tag{3.19}$$

Furthermore, if X' has absolutely continuous norm, that is $X' = X'_a$, since every Banach Function Space contains the simple functions, by Theorem 2.22 applied to the space X' and by Theorem 2.16, we have $(X')^* = (X'_a)^* = (X')' = X'' = X$.

On the other hand, if $X = (X')^*$, then $(X')' = X = (X')^*$, and Theorem 2.23 yields that X' has absolutely continuous norm. □

Remark 3.8. An example of a Banach Function Space verifying the proper inclusion in (3.18) is given by the Lebesgue space L^1. In fact, if $X = L^1$, then

$$(X')^* = \left(\left(L^1\right)'\right)^* = (L^\infty)^* \supset L^1, \tag{3.20}$$

as confirmed by the fact that L^∞ has not absolutely continuous norm.

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