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On 2-primitive triangle decompositions of cocktail party graphs

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ON 2-PRIMITIVE TRIANGLE DECOMPOSITIONS OF COCKTAIL PARTY GRAPHS

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APPROVAL OF THESIS

We, the faculty supervising the work of Ian P. Waddell, affirm that the thesis, *On 2-primitive triangle decompositions of cocktail party graphs*, meets the high academic standards for original scholarship and creative work established by the Department of Mathematics and the College of Science. The work also conforms to the formatting guidelines of Marshall University. With our signatures, we approve the manuscript for publication.

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ABSTRACT

A decomposition of a graph Γ is a collection $\mathcal{C}$ of subgraphs, perhaps nonisomorphic, that partition the edges of Γ. Analogously, consider a group of truck drivers whose non-overlapping routes jointly cover all of the roads between a set of cities; that is, each road is traversed by precisely one driver. In this scenario, the cities are the vertices of the graph, the roads are the edges between vertices, and the drivers' routes are the subgraphs in the decomposition. Given a graph H, we call $\mathcal{C}$ an H-decomposition of Γ if each subgraph in $\mathcal{C}$ is isomorphic to the graph H. Continuing with the previous analogy, this would imply that each truck driver travels a route which is identical in connectivity between neighboring cities, but differs in locale.

A subdecomposition of $\mathcal{C}$ refers to a nonempty subset of $\mathcal{C}$ which partitions the edges of an induced subgraph of Γ, and $\mathcal{C}$ is said to be t-primitive when there exist no proper subdecompositions of $\mathcal{C}$ containing t or more subgraphs. To visualize this, we consider a scenario in which t of our truck drivers, along with their respective routes, are infected with an illness that spreads to any healthy driver that travels directly between two sick towns. Assuming that an infected driver will infect their entire route, we ask the natural question of whether this illness spreads to the entire collection of drivers and cities, or whether it ends up confined to some subset of them. If any t sick drivers result in universal infection, the highway network which their routes partition is t-primitive.

In this work we examine decompositions of cocktail party graphs into triangles. In particular, we establish the existence of 2-primitive triangle decompositions of cocktail party graphs with $6k+2$ vertices for each nonnegative integer k. Coupled with the results of a recent undergraduate capstone, this work completes the classification of when such decompositions exist for all cocktail party graphs.

CHAPTER 1

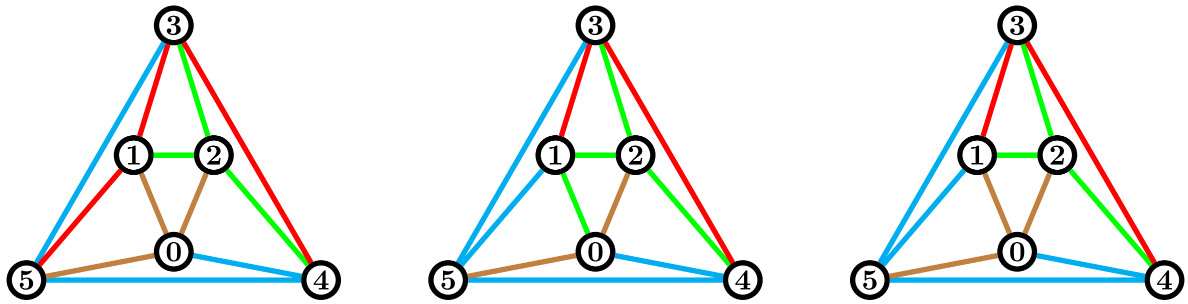
Introduction

In daily life, we frequently encounter systems which can be modeled by graph theory, and can answer questions about those systems by understanding their graph theoretic properties. Consider a metropolitan area, in which a collection of cities is joined by a highway network, and suppose that this area is supplied by a group of truck drivers. Suppose that each truck driver, needing to fulfill their supply network, drives a unique route along the highway network; in particular, assume that each highway is traversed by exactly one driver and that the group's non-intersecting routes jointly supply every highway in the metropolitan area. Now, suppose that some number, say t, of our drivers contract an illness and infect the cities that they visit and the highways that they traverse. Assuming that a highway between two infected cities will infect the driver of the highway between them, we ask the natural question of whether the entire metropolitan area (cities, highways, and drivers) will be infected as infected drivers carry the disease along their routes.

Suppose that the images in Figure 1 represent three such metropolitan areas, where the numbered circles represent cities, the colored lines represent the highways between cities, and each color corresponds to a truck route.

Under the infection rules we have outlined, the entirety of route collection $\mathfrak{A}$ would become infected if any single driver were infected. For example, infecting the driver of the brown route would spread the disease to every brown highway and cities 0, 1, 2, and 5, causing the driver of the green route to be infected by the highway between 1 and 2. Once the brown and green routes are infected, the disease spreads to every city and highway in the area.

Route collection $\mathfrak{B}$, by contrast, is resilient to the infection of a single driver. For example, if the driver of the red route is infected, then the disease is contained to the red highways and cities 1, 3, and 4 since none of the other colored routes traverse a highway directly between cities infected by the driver of the red route. However, one can check that, if any pair of drivers becomes sick, then the disease will inevitably spread to the entire area.



Route Collection $\mathfrak{A}$ Route Collection $\mathfrak{B}$ Route Collection $\mathfrak{C}$

Figure 1. A graph with decompositions of varying primitivity

Given the same metropolitan area, three different route collections $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ are shown. Route collection $\mathfrak{A}$ is 1-primitive, route collection $\mathfrak{B}$ is 2-primitive, but not 1-primitive, and route collection $\mathfrak{C}$ is not 2-primitive.

Finally, route collection $\mathfrak{C}$ can survive the infection for certain pairs of drivers. In particular, if the red and green routes are infected, then the infection is contained in the red and green highways and cities 1, 2, 3, and 4. No other driver traverses a road between cities infected by the drivers of the red and green routes.

From this example, we see that the answer to our question depends on several factors. First, the likelihood of survival may be influenced by the number of drivers that are infected. If many drivers are initially infected, one would generally expect the outcome of universal infection, while we might expect some uninfected survivors in a scenario when few are initially infected. Second, the routes of the infected drivers determine which local cities and highways, and by extension which other drivers, will contract the illness. It would be plausible to expect that infecting a driver with an extensive route covering a substantial quantity of highways between cities would spread the infection more readily than one which does not have an extensive route. Finally, the size of the metropolitan area could influence the spread of the infection, and one might suspect that a large area could be more resilient to infection than a small area.

Indeed, the answer may depend heavily on both the metropolitan area and the routes of the infected drivers. Thus, an interesting question to consider is the following: given a metropolitan area with a specified collection of routes, is there a threshold number of infected drivers which guarantees universal infection, and, if so, can we quantify its size?

In this scenario, the metropolitan area can be viewed as a graph Γ, the cities as the vertices of Γ, and the roads as the edges of Γ. The unique routes driven by the truckers constitute a decomposition of Γ; that is, a collection of subgraphs $\mathcal{C}$ such that this collection partitions the edges of Γ. Of particular interest are scenarios in which all of the drivers have similar (isomorphic) routes. Here, the subgraphs in $\mathcal{C}$ are all isomorphic to some graph H, and thus to each other, and the corresponding decomposition is referred to as an H-decomposition of Γ.

Given a subcollection $\mathcal{S} \subseteq \mathcal{C}$ of the subgraphs, $\mathcal{S}$ is referred to as a *subdecomposition* of $\mathcal{C}$ if $\mathcal{S}$ is a decomposition of an induced subgraph of Γ, which is defined later. Decompositions may be classified according to the nature of their proper subdecompositions. In particular, a decomposition is called *primitive* if it contains no proper subdecompositions, and this property has been a popular subject of focus in the literature. In 2000, Rodger and Spicer [7] classified primitive $(K_4 - e)$-decompositions of complete graphs, in 2012 Dinavahi and Rodger [3] classified primitive P_m-decompositions of complete graphs for all positive values of m, and in 2022, Asplund et al. [1] examined primitive C_m-decompositions of complete graphs and cocktail party graphs for $m \geq 4$.

However, there are some graphs for which, given a graph H, a primitive H-decomposition does not exist. For example, decomposing a graph into subgraphs which are each isomorphic to C_3 is not primitive in general, since an individual C_3 subgraph is always an induced subgraph. A 1969 paper by Doyen [4], whose work inspired the methods in this paper, examined the existence of primitive C_3-decompositions of complete graphs when the number of vertices is congruent to 1 or 3 modulo 6, with the exception that he did not consider an individual C_3 subgraph as a proper subdecomposition. For the purpose of describing these nearly primitive decompositions, Schroeder [8] introduced the phrase *t-primitivity*, a generalized concept of primitivity which describes a decomposition containing no proper subdecompositions with t or more subgraphs.

With this new definition in mind, we can now revisit Figure 1, where each image depicts a graph imbued with a decomposition whose subgraphs correspond to the colored edges. By the previous discussion, we can conclude that the decomposition $\mathfrak{A}$ is 1-primitive, the decomposition $\mathfrak{B}$ is 2-primitive, but not 1-primitive, and the decomposition $\mathfrak{C}$ is not 2-primitive. This new definition also allows our infection question to be equivalently restated as two separate questions. Given a graph with a decomposition, can we find t such that the decomposition is t-primitive? Furthermore, given a graph and t, can we find a t-primitive decomposition?

In this paper, we examine C_3-decompositions of cocktail party graphs with n vertices. In a 2021 undergraduate capstone, Stamm [9] leveraged the methods of Doyen [4] in the study of 2-primitive C_3-decompositions of complete and cocktail party graphs to prove the following theorem:

Theorem 1.1. *Let $k \geq 1$. There exists a 2-primitive C_3-decomposition of cocktail party graphs with $6k$ vertices.*

In this paper, we complete the classification of 2-primitive C_3-decompositions of cocktail party graphs by investigating cocktail party graphs with $6k+2$ vertices for nonzero integers k. We begin in Chapter 2 by presenting some relevant definitions and results from abstract algebra, defining our graph theoretic objects, and describing constructions of C_3-decompositions of complete and cocktail party graphs with $6k+3$ and $6k+2$ vertices, respectively, for $k \geq 1$. Following these preliminaries, we present and prove a modified version of Doyen's construction of a 2-primitive C_3-decomposition of complete graphs with $6k+3$ vertices in Chapter 3. Ultimately, in Chapter 4, we parallel Doyen's methods to demonstrate the following theorem as the main result:

Theorem 1.2. *Let $k \geq 1$. There exists a 2-primitive C_3-decomposition of cocktail party graphs with $6k+2$ vertices.*

Theorem 1.1 and Theorem 1.2, in conjunction with necessary numerical conditions on k, classify the existence of a 2-primitive C_3-decomposition of all cocktail party graphs, which is summarized in the following theorem:

Theorem 1.3. *For $k \geq 1$, there exists a 2-primitive C_3-decomposition of cocktail party graphs with $6k+\ell$ vertices if and only if $\ell = 0$ or $\ell = 2$.*

After the main results are presented in Chapter 4, we conclude in Chapter 5 with some future directions and open questions.

CHAPTER 2

PRELIMINARIES

We begin this chapter by recalling some group theoretic terminology and defining the notation used for group operations, which are central to the construction of our C_3-decompositions, as well as in arguments concerning 2-primitivity. Additionally, we prove some results about cosets that we later leverage to demonstrate the 2-primitivity of our C_3-decompositions in the main theorems. Subsequently, we formally define the graph theoretic terminology and notation, followed by an examination of the necessary numerical conditions for a C_3-decomposition of a cocktail party graph to exist. We conclude the chapter by providing the constructions of C_3-decompositions of complete and cocktail party graphs with $6k+3$ and $6k+2$ vertices, respectively, that are discussed throughout the paper.

2.1 Abstract algebra

Given a nonempty set G with an associative binary operation $*$, we say that G is a *group* if there is an element $e \in G$ satisfying that $a * e = e * a = a$ for every $a \in G$, and for each $a \in G$ there exists an inverse element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$. A subset $H \subseteq G$ is a *subgroup* of G if H is a group under $*$, and we denote this relationship $H \leq G$. In this paper, we work with abelian groups, which satisfy the additional property that $*$ is commutative. For this reason, the operation $*$ is generally written in additive notation; for the rest of this paper, we denote the operation $*$ by $+$, e by 0, and a^{-1} by $-a$. Note that for an abelian group of odd order, the operations of doubling and halving an element are well-defined. In general, for $x \in G$, we adopt the common notational conventions that $2x = x + x$ and $x/2$ is the unique element $y \in G$ such that $2y = x$. Given a set $A \subseteq G$, we also use the notation $2A = \{2a : a \in A\}$ and $\frac{1}{2}A = \{a/2 : a \in A\}$. Selecting a fixed value $a \in G$, we define the *coset* $a + H$ of a subgroup H to be $a + H = \{a + h : h \in H\}$. The set of cosets of H forms a partition of the group G. Readers seeking more information on group theory are directed to the textbook by Fraleigh [5].

The following lemma provides sufficient conditions for when a set is a coset of a nontrivial subgroup of an abelian group of odd order.

Lemma 2.1. *Let G be a finite abelian group of odd order and $A \subseteq G$ be nontrivial. If $(x+y)/2 \in A$ and $2z - x \in A$ for all $x, y, z \in A$, then A is a coset of a nontrivial subgroup of G.*

Proof. Let $a \in A$ and define $B = -a + A$. It is sufficient to show that B is a nontrivial subgroup of G. Clearly, $0 \in B$, and B is nontrivial since A is nontrivial by the hypothesis. Furthermore, given $b \in B$, we note that $a + b \in A$ and that $2a - (a + b) = a - b \in A$ by the hypothesis. Hence, $-b \in B$ whenever $b \in B$, and thus B contains its elements' inverses.

Now we need only show that B is closed under addition. Let $b, c \in B$. Then, we have that $a+b$ and $a+c$ belong to A. By the hypothesis, we have that $(a+b+a+c)/2 = a+(b+c)/2$ belongs to A and hence that $2(a + (b + c)/2) - a = a + b + c$ belongs to A. Thus, $b + c \in B$. $\square$

Lemma 2.1 allows us to demonstrate that a set is a coset of a nontrivial subgroup simply by testing for the inclusion of carefully chosen elements. The next lemma provides a method for describing the intersection of a coset of a nontrivial subgroup of $\mathbb{Z}_n$ with general subsets of $\mathbb{Z}_n$. This will be instrumental in the proof of Lemma 2.3, which provides a powerful diagnostic for testing whether a coset of a nontrivial subgroup of $\mathbb{Z}_n$ is equal to $\mathbb{Z}_n$.

Lemma 2.2. *Let A be a coset of a nontrivial subgroup $H \leq \mathbb{Z}_n$, where $H = \langle d \rangle$ for some positive integer d such that d divides n. If $Y \subseteq \mathbb{Z}_n$ contains at least d consecutive integers, then $A \cap Y \neq \emptyset$.*

Proof. It is sufficient to assume that Y contains precisely d consecutive integers. The result follows for any set containing more than d consecutive integers by extracting a subset containing precisely d consecutive integers. Since $H = \langle d \rangle$, there exists some $x \in \mathbb{Z}_n$ such that $0 \leq x \leq d - 1$ and $A = x + H$. Observe that for each $y \in Y$, we have that $0 \leq y \pmod{d} \leq d-1$. Furthermore, given $y, y' \in Y$, note that $-(d - 1) \leq y - y' \leq d - 1$, and hence $y \equiv y' \pmod{d}$ if and only if $y = y'$. Thus, since $|Y| = d$, there exists a unique element $a \in Y$ such that $a \equiv x \pmod{d}$. Equivalently, $a - x \equiv 0 \pmod{d}$. However, this implies that $a - x$ is a multiple of d, and hence $a - x \in H$. Additionally, $a - x + x = a$ is in A since $A = x + H$. So, $a \in A \cap Y$, as desired. $\square$

Lemma 2.3. *Let $k \geq 1$ and $A \subseteq \mathbb{Z}_{2k+1}$. Let ϕ be the permutation of $\mathbb{Z}_{2k+1}$ given by the cycle $\phi = (1 \cdots k)$. If A and $\phi(A)$ are both cosets of a nontrivial subgroup of $\mathbb{Z}_{2k+1}$, then $A = \mathbb{Z}_{2k+1}$.*

Proof. Observe that the result follows immediately if $2k+1$ is prime, since $\mathbb{Z}_{2k+1}$ has no proper nontrivial subgroups. So, let $k \geq 4$. First, note that $|A| = |\phi(A)|$, and hence A and $\phi(A)$ are cosets of the same nontrivial subgroup $H \leq \mathbb{Z}_{2k+1}$ since, for each divisor of $2k+1$, there exists a unique subgroup of $\mathbb{Z}_{2k+1}$ with that order. Furthermore, H is cyclic since $\mathbb{Z}_{2k+1}$ is cyclic. Thus, there exists an integer d such that d divides $2k+1$ and d generates H, and $d \leq (2k+1)/3$ since $2k+1$ is odd and $|H| \geq 3$. Now, define the sets $Y = \{1, \ldots, k-1\}$ and $Z = \{k+1, \ldots, 2k\}$. Since $k \geq 4$,

$$k - 1 \geq \frac{2k+1}{3} \geq d,$$

and it follows that both Y and Z have at least d consecutive integers. By Lemma 2.2, $A \cap Y$ and $A \cap Z$ are nonempty. Let $y \in A \cap Y$ and $z \in A \cap Z$. Then, since $y, z \in A$ and A is a coset of H, $y - z \in H$. Furthermore, since $y \in Y$ and $z \in Z$, $\phi(y) = y + 1$ and $\phi(z) = z$, and hence $y + 1$ and z are elements of $\phi(A)$. Since $\phi(A)$ is also a coset of H, $(y+1) - z \in H$. Since H is a subgroup, H is closed under addition and contains the additive inverse of each of its elements. It follows that $(y+1) - z - (y - z) = 1$ is an element of H; so $H = \langle 1 \rangle$. Therefore, $A = H = \mathbb{Z}_{2k+1}$. $\square$

The previous lemma is the main tool used in our modified version of Doyen's [4] proof of 2-primitivity at the end of Chapter 3, as well as the proof of Theorem 1.2 at the end of Chapter 4.

2.2 Basic graph theory

A *graph* Γ is a collection of *vertices* connected by *edges*, where the vertex set is usually denoted by $V(\Gamma)$ and the edge set by $E(\Gamma)$. Given two vertices $x, y \in V(\Gamma)$, an edge in $E(\Gamma)$ between x and y may be denoted by $\{x, y\}$ or simply xy, and x and y are said to be *adjacent* if $xy \in E(\Gamma)$. Additionally, the vertices x and y are referred to as the *endpoints* of the edge xy, and we say that xy is *incident* to x and y.

Given a graph Γ, a *subgraph* Γ' is a subcollection of vertices and edges of Γ such that $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$. An *induced subgraph* of a graph Γ is a subgraph formed by deleting all vertices in some vertex set $V' \subseteq V(\Gamma)$ and all edges $e \in E(\Gamma)$ such that e is incident to a vertex in V'. Consider route collection $\mathfrak{C}$ in Figure 1; the red and green edges constitute the induced subgraph formed by deleting vertices 0 and 5 and their incident edges. However, the cyan edges do not correspond to an induced subgraph in any of the three route collections.

The *degree* of a vertex is given by the number of edges which are incident to that vertex. A *k-regular* graph is a graph in which every vertex has degree k. A *complete graph* with n vertices, denoted K_n, contains every edge between distinct vertices; equivalently, it is an $(n-1)$-regular graph with n vertices. A *perfect matching*, which sometimes referred to as a 1-factor, is a 1-regular graph. A *cocktail party graph* is a $(n-2)$-regular graph with n vertices formed by deleting the edges of a perfect matching from a complete graph with n vertices. Throughout the paper, we denote cocktail party graphs with n vertices by $K_n - I$, where I is the perfect matching removed from K_n. In particular, we generally interpret I as the set of edges removed from K_n.

A *decomposition* of Γ is a collection $\mathcal{C}$ of subgraphs of Γ whose edges partition the edges of Γ. We use the notation $E(\mathcal{C})$ to refer to the set of edges appearing in $\mathcal{C}$ and $V(\mathcal{C})$ as their incident vertices. A *subdecomposition* $\mathcal{S}$ is a nonempty subset of $\mathcal{C}$ that is a decomposition of an induced subgraph of Γ. As previously discussed, the red and green routes of route collection $\mathfrak{C}$ decompose an induced subgraph; therefore, the red and green routes form a subdecomposition of route collection $\mathfrak{C}$. A decomposition $\mathcal{C}$ is said to be *t-primitive* if the only subdecomposition of $\mathcal{C}$ containing at least t elements is $\mathcal{C}$ itself.

In this paper, we are concerned with finding *triangle decompositions* of graphs; that is, we seek decompositions of graphs into subgraphs isomorphic to cycles of length 3, denoted as C_3. As previously discussed, a 1-primitive triangle decomposition of Γ is impossible if $V(\Gamma) > 3$. In much of the literature, including Doyen's [4] original paper, triangle decompositions are referred to as *Steiner triple systems*. When describing the elements of a triangle decomposition, we frequently refer to these subgraphs as *triangles* or *triples*. We often represent a C_3 subgraph by the triple corresponding to its vertices. For example, the triple $\{x, y, z\}$ corresponds to a graph of vertex set $\{x, y, z\}$ and edge set $\{xy, xz, yz\}$.

In practice, we typically use the following observation to classify a decomposition as t-primitive by analyzing the vertex sets of its subdecompositions.

Observation 2.4. Let Γ be a graph and $\mathcal{C}$ a decomposition of Γ. Let $\mathcal{S}$ be a subdecomposition of $\mathcal{C}$. Then, $\mathcal{S} = \mathcal{C}$ if and only if $V(\mathcal{S}) = V(\mathcal{C}) = V(\Gamma)$. Therefore, if $V(\mathcal{S}) = V(\Gamma)$ for all subdecompositions $\mathcal{S}$ of $\mathcal{C}$ for which $|\mathcal{S}| \geq t$, then $\mathcal{C}$ is t-primitive.

Readers seeking more information on graph and design theory are directed to the often-cited textbook by Lindner and Rodger [6], and discussion on primitivity can be found in papers by Asplund et al. [1] and Schroeder [8].

2.3 Decompositions of graphs

Given a particular family of graphs and a decomposition, it is of primary interest to determine what numerical conditions need to be satisfied for a decomposition to exist. For a complete graph with n vertices, a triangle decomposition is possible only if $n \equiv 1$ or $3 \pmod 6$ [6]. In the following lemma, we establish a similar condition on the number of vertices in a cocktail party graph with a triangle decomposition.

Lemma 2.5. *Let $m \geq 1$. If $K_{2m} - I$ has a C_3-decomposition, then $m \equiv 0$ or $1 \pmod 3$.*

Proof. Suppose that $K_{2m} - I$ has a C_3-decomposition for some $m \geq 1$. Let $m = 3q + r$ for some integers q and r such that $r \in \{0, 1, 2\}$. Let x denote the number of edges in $K_{2m} - I$. Then,

$$x = \binom{2m}{2} - m = \frac{2m(2m-1)}{2} - m = m(2m-1) - m = m(2m-2) = 2m(m-1).$$

Furthermore, 3 must divide x since $K_{2m} - I$ has a C_3-decomposition. By our definition of m,

$$x = 2m(m-1) = 2(3q+r)(3q+r-1) = 3(6q^2 + 4qr - 2q) + 2r(r-1).$$

Since 3 divides x, we have that $r = 0$ or 1, so it follows that $m \equiv 0$ or $1 \pmod 3$. $\square$

Note that in the prior lemma, we choose to write our cocktail party graph as $K_{2m} - I$ out of convenience, since cocktail party graphs only exist when the number of vertices is even. This lemma can be equivalently stated as $K_{6k+\ell} - I$ has a C_3-decomposition only if $\ell = 0$ or $\ell = 2$, as seen in Theorem 1.3. This condition captures both the fact that the number of vertices must be even in order for the perfect matching I to exist, and that the number of edges must be divisible by 3 in order for a triangle decomposition to be possible.

2.4 The Bose construction

In Doyen's original paper, he constructed a 2-primitive triangle decomposition of K_{6k+3} by starting with the Bose construction [2]. We first make a few observations about solutions to linear equations in G, which we will make frequent use of throughout the paper. Afterward, we present the Bose construction and show it decomposes K_{6k+3} for all positive integers k.

Observation 2.6. Let $k \geq 1$, G be an abelian group of odd order $2k+1$, and ϕ be a permutation on G. Then, the following statements hold:

1. For distinct $u, v \in G$, there is a unique element $w \in G$ such that $\phi(u) + \phi(v) = 2\phi(w)$.

2. For distinct $u, w \in G$, there is a unique element $v \in G$ such that $\phi(v) = 2\phi(w) - \phi(u)$.

Definition 2.7 (The Bose construction)**.** Let G be a finite abelian group of odd order $2k+1$ for some integer $k \geq 1$, and let $G_i = \{(x, i) : x \in G\}$ for each $i \in \mathbb{Z}_3$. Label the vertices of K_{6k+3} so that $V(K_{6k+3}) = G_0 \cup G_1 \cup G_2$. Define the following triples:

1. For each $x \in G$, let $W_x = \{(x, 0), (x, 1), (x, 2)\}$.

2. For each $i \in \mathbb{Z}_3$ and distinct $x, y \in G$, let $T_{i,\{x,y\}} = \{(x, i), (y, i), (z, i + 1)\}$, where $z \in G$ is the unique element satisfying $x + y = 2z$ guaranteed by Observation 2.6.

As mentioned earlier, a triple is a C_3 subgraph; the definition above specifies the vertices used by a subgraph, and every edge is assumed present. We denote the union of these triples by $\mathcal{B}(G)$.

Visually, the Bose construction can be interpreted as a division of the vertices into three rows, where each row is a "copy" of the group G. In Doyen's [4] original paper, the set W_x was referred to as the *vertical* passing through x for each $x \in G$, and the set $T_{i,\{x,y\}}$ can be viewed as forming a triangle between x and y in row i and z in row $i+1$ for $i \in \mathbb{Z}_3$ and distinct $x, y \in G$. See Figure 2.

We now demonstrate that the Bose construction provides a legitimate decomposition of K_{6k+3} for all integers $k \geq 1$.

$W_0 = \{(0,0),(0,1),(0,2)\}$	$T_{0,\{0,1\}} = \{(0,0),(1,0),(3,1)\}$	$T_{1,\{0,1\}} = \{(0,1),(1,1),(3,2)\}$	$T_{2,\{0,1\}} = \{(0,2),(1,2),(3,0)\}$
$W_1 = \{(1,0),(1,1),(1,2)\}$	$T_{0,\{0,2\}} = \{(0,0),(2,0),(1,1)\}$	$T_{1,\{0,2\}} = \{(0,1),(2,1),(1,2)\}$	$T_{2,\{0,2\}} = \{(0,2),(2,2),(1,0)\}$
$W_2 = \{(2,0),(2,1),(2,2)\}$	$T_{0,\{0,3\}} = \{(0,0),(3,0),(4,1)\}$	$T_{1,\{0,3\}} = \{(0,1),(3,1),(4,2)\}$	$T_{2,\{0,3\}} = \{(0,2),(3,2),(4,0)\}$
$W_3 = \{(3,0),(3,1),(3,2)\}$	$T_{0,\{0,4\}} = \{(0,0),(4,0),(2,1)\}$	$T_{1,\{0,4\}} = \{(0,1),(4,1),(2,2)\}$	$T_{2,\{0,4\}} = \{(0,2),(4,2),(2,0)\}$
$W_4 = \{(4,0),(4,1),(4,2)\}$	$T_{0,\{1,2\}} = \{(1,0),(2,0),(4,1)\}$	$T_{1,\{1,2\}} = \{(1,1),(2,1),(4,2)\}$	$T_{2,\{1,2\}} = \{(1,2),(2,2),(4,0)\}$
	$T_{0,\{1,3\}} = \{(1,0),(3,0),(2,1)\}$	$T_{1,\{1,3\}} = \{(1,1),(3,1),(2,2)\}$	$T_{2,\{1,3\}} = \{(1,2),(3,2),(2,0)\}$
	$T_{0,\{1,4\}} = \{(1,0),(4,0),(0,1)\}$	$T_{1,\{1,4\}} = \{(1,1),(4,1),(0,2)\}$	$T_{2,\{1,4\}} = \{(1,2),(4,2),(0,0)\}$
	$T_{0,\{2,3\}} = \{(2,0),(3,0),(0,1)\}$	$T_{1,\{2,3\}} = \{(2,1),(3,1),(0,2)\}$	$T_{2,\{2,3\}} = \{(2,2),(3,2),(0,0)\}$
	$T_{0,\{2,4\}} = \{(2,0),(4,0),(3,1)\}$	$T_{1,\{2,4\}} = \{(2,1),(4,1),(3,2)\}$	$T_{2,\{2,4\}} = \{(2,2),(4,2),(3,0)\}$
	$T_{0,\{3,4\}} = \{(3,0),(4,0),(1,1)\}$	$T_{1,\{3,4\}} = \{(3,1),(4,1),(1,2)\}$	$T_{2,\{3,4\}} = \{(3,2),(4,2),(1,0)\}$

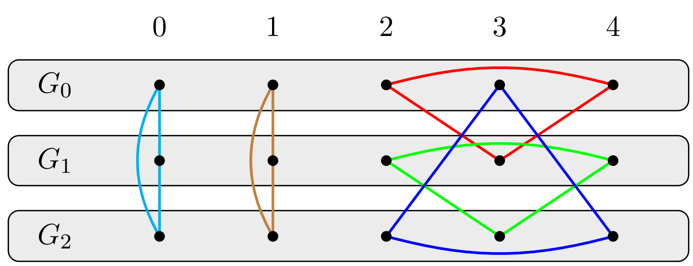


Figure 2. The Bose decomposition $\mathcal{B}(\mathbb{Z}_5)$

The set of 35 triples displayed above constitute the Bose decomposition $\mathcal{B}(\mathbb{Z}_5)$, where the far left column contains the vertical triples W_x for each integer $x \in \mathbb{Z}_5$. The three subsequent columns contain the triples of the form $T_{0,\{x,y\}}$, $T_{1,\{x,y\}}$, and $T_{2,\{x,y\}}$, respectively, for each pair of distinct integers $x, y \in \mathbb{Z}_5$. Below, five triples from the decomposition $\mathcal{B}(\mathbb{Z}_5)$ are illustrated. The verticals W_0 and W_1 are shown in cyan and brown, respectively, and the triples $T_{0,\{2,4\}}$, $T_{1,\{2,4\}}$, and $T_{2,\{2,4\}}$ are illustrated in red, green, and blue, respectively.

Lemma 2.8. *Let $k \geq 1$ and G be a finite abelian group of order $2k+1$. Then, $\mathcal{B}(G)$ yields a decomposition of K_{6k+3}.*

Proof. Each edge in K_{6k+3} is of the form $(x,i)(y,j)$, where $x, y \in G$ and $i, j \in \mathbb{Z}_3$. We first demonstrate that each edge of K_{6k+3} is associated to at least one triple in $\mathcal{B}(G)$. Suppose that $x = y$. Further suppose that $i \neq j$. Then, $(x,i)(y,j) = (x,i)(x,j) \in E(W_x)$.

Now, suppose $x \neq y$. Further suppose that $i = j$. By Observation 2.6, there is a unique $z \in G$ satisfying that $x + y = 2z$. Thus, $(x,i)(y,j) = (x,i)(y,i) \in E(T_{i,\{x,y\}})$. Next, suppose that $i \neq j$. Without loss of generality, suppose that $j = i+1$. Then, there is a unique $z \in G$ satisfying that $z = 2y - x$, and it follows that $(x,i)(y,j) = (x,i)(y,i+1) \in E(T_{i,\{x,z\}})$. Thus, $E(K_{6k+3}) \subseteq E(\mathcal{B}(G))$.

It is now sufficient to show that $|E(\mathcal{B}(G))| \leq |E(K_{6k+3})|$. First, observe that K_{6k+3} has $\binom{6k+3}{2} = 3(2k+1)(3k+1)$ edges. Since $|G| = 2k+1$ and $i \in \mathbb{Z}_3$, there $2k+1$ triples of the form V_x for some $x \in G$, and there are $3 \cdot \binom{2k+1}{2}$ triples of the form $T_{i,\{x,y\}}$ for some $x, y \in G$. Hence,

$$|E(\mathcal{B}(G))| \leq \sum_{\alpha} |E(W_\alpha)| + \sum_{\alpha \neq \beta} |E(T_{i,\{\alpha,\beta\}})| = 3(2k+1) + 9\binom{2k+1}{2} = 3(2k+1)(3k+1),$$

and it follows that $|E(\mathcal{B}(G))| \leq |E(K_{6k+3})|$, which finishes the proof. □

Note that the Bose decomposition is not 2-primitive, in general. However, using the following definition, we present a tool that we use to modernize Doyen's [4] generalization of the Bose decomposition, and as we demonstrate later, this generalized decomposition can be used to find a 2-primitive C_3-decomposition of K_{6k+3} for $k \geq 1$.

Definition 2.9. Let G be a finite abelian group. Then, we call Φ a *permutation triple of G* if Φ is an ordered triple (ϕ_0, ϕ_1, ϕ_2) of permutations of G. If $\phi_0(0) = \phi_1(0) = \phi_2(0) = 0$, we say that Φ is a *proper permutation triple of G*.

Definition 2.10 (The augmented Bose decomposition)**.** Let G be a finite abelian group of odd order $2k+1$ for some integer $k \geq 1$, and let $G_i = \{(x,i) : x \in G\}$ for each $i \in \mathbb{Z}_3$. Label the vertices of K_{6k+3} so that $V(K_{6k+3}) = G_0 \cup G_1 \cup G_2$. Let Φ be a proper permutation triple of G, and define the following triples:

1. For each $x \in G$, let $W_x = \{(x,0),(x,1),(x,2)\}$.

2. For each $i \in \mathbb{Z}_3$, $T_{i,\{x,y\}} = \{(x,i),(y,i),(z,i{+}1)\}$, where $z \in G$ is the unique element satisfying $\phi_i(x) + \phi_i(y) = 2\phi_i(z)$ guaranteed by Observation 2.6.

We denote the union of these triples by $\mathcal{C}(G,\Phi)$ and refer to it as an *augmented Bose decomposition derived from G acted on by Φ*. Figure 3 gives an example of the augmented Bose decomposition $\mathcal{C}(\mathbb{Z}_5,\Phi)$, where $\Phi = ((234),(12),(1423))$.

Observe that $\mathcal{C}(G,\Phi)$ is also a legitimate triangle decomposition of K_{6k+3}; given $x, y \in G$, the equation $\phi_i(x) + \phi_i(y) = 2\phi_i(z)$ is solved by a unique element $z \in G$. This condition, as in the Bose decomposition, ensures that no two triangles share an edge. We also note that, in Doyen's original argument, there are no restrictions on ϕ_0, ϕ_1, and ϕ_2 to fix zero, but we require this for our constructions later. In particular, this constraint is relevant for the following construction for a triangle decomposition of $K_{6k+2} - I$, which is used as the primary subject of focus in generating a 2-primitive triangle decomposition of $K_{6k+2} - I$ in Chapter 4.

Definition 2.11 (The deleted Bose decomposition)**.** Let $k \geq 1$, G an abelian group of order $2k+1$, and Φ a proper permutation triple of G. Define $\mathcal{D}(G,\Phi)$ as the set obtained by deleting all triples from $\mathcal{C}(G,\Phi)$ which use $(0,0)$ as one of its vertices; that is,

$$\mathcal{D}(G,\Phi) = \mathcal{C}(G,\Phi) \setminus \{T \in \mathcal{C}(G,\Phi) : (0,0) \in V(T)\}.$$

Furthermore, note that $\mathcal{D}(G,\Phi)$ is a decomposition of $K_{6k+2} - I$, where

$$\begin{aligned} V(K_{6k+2} - I) = [G_0 \backslash \{(0,0)\}] \cup G_1 \cup G_2, \text{ and} \\ I = \{(0,1)(0,2)\} \cup \{(x,0)(z,1) : \phi_0(x) = 2\phi_0(z)\} \\ \cup \{(x,2)(y,2) : \phi_2(x) + \phi_2(y) = 0\}. \end{aligned}$$

Recall that Figure 3 gave an example of the augmented Bose decomposition $\mathcal{C}(\mathbb{Z}_5,\Phi)$, where $\Phi = ((234),(12),(1423))$. Removing the triples highlighted in red produces the deleted Bose decomposition $\mathcal{D}(\mathbb{Z}_5,\Phi)$.

$W_0 = \{(0,0),(0,1),(0,2)\}$	$T_{0,\{0,1\}} = \{(0,0),(1,0),(2,1)\}$	$T_{1,\{0,1\}} = \{(0,1),(1,1),(2,2)\}$	$T_{2,\{0,1\}} = \{(0,2),(1,2),(2,0)\}$
$W_1 = \{(1,0),(1,1),(1,2)\}$	$T_{0,\{0,2\}} = \{(0,0),(2,0),(3,1)\}$	$T_{1,\{0,2\}} = \{(0,1),(2,1),(3,2)\}$	$T_{2,\{0,2\}} = \{(0,2),(2,2),(3,0)\}$
$W_2 = \{(2,0),(2,1),(2,2)\}$	$T_{0,\{0,3\}} = \{(0,0),(3,0),(4,1)\}$	$T_{1,\{0,3\}} = \{(0,1),(3,1),(4,2)\}$	$T_{2,\{0,3\}} = \{(0,2),(3,2),(4,0)\}$
$W_3 = \{(3,0),(3,1),(3,2)\}$	$T_{0,\{0,4\}} = \{(0,0),(4,0),(1,1)\}$	$T_{1,\{0,4\}} = \{(0,1),(4,1),(1,2)\}$	$T_{2,\{0,4\}} = \{(0,2),(4,2),(1,0)\}$
$W_4 = \{(4,0),(4,1),(4,2)\}$	$T_{0,\{1,2\}} = \{(1,0),(2,0),(4,1)\}$	$T_{1,\{1,2\}} = \{(1,1),(2,1),(4,2)\}$	$T_{2,\{1,2\}} = \{(1,2),(2,2),(4,0)\}$
	$T_{0,\{1,3\}} = \{(1,0),(3,0),(0,1)\}$	$T_{1,\{1,3\}} = \{(1,1),(3,1),(0,2)\}$	$T_{2,\{1,3\}} = \{(1,2),(3,2),(0,0)\}$
	$T_{0,\{1,4\}} = \{(1,0),(4,0),(3,1)\}$	$T_{1,\{1,4\}} = \{(1,1),(4,1),(3,2)\}$	$T_{2,\{1,4\}} = \{(1,2),(4,2),(3,0)\}$
	$T_{0,\{2,3\}} = \{(2,0),(3,0),(1,1)\}$	$T_{1,\{2,3\}} = \{(2,1),(3,1),(1,2)\}$	$T_{2,\{2,3\}} = \{(2,2),(3,2),(1,0)\}$
	$T_{0,\{2,4\}} = \{(2,0),(4,0),(0,1)\}$	$T_{1,\{2,4\}} = \{(2,1),(4,1),(0,2)\}$	$T_{2,\{2,4\}} = \{(2,2),(4,2),(0,0)\}$
	$T_{0,\{3,4\}} = \{(3,0),(4,0),(2,1)\}$	$T_{1,\{3,4\}} = \{(3,1),(4,1),(2,2)\}$	$T_{2,\{3,4\}} = \{(3,2),(4,2),(2,0)\}$

Figure 3. The decompositions $\mathcal{C}(\mathbb{Z}_5, \Phi)$ and $\mathcal{D}(\mathbb{Z}_5, \Phi)$, where $\Phi = ((234),(12),(1423))$

The entire set of triples displayed represents the augmented Bose decomposition $\mathcal{C}(\mathbb{Z}_5, \Phi)$, where $\Phi = ((234),(12),(1423))$. Each of the triples highlighted in red contains the vertex $(0,0)$; removing these triples produces the deleted Bose decomposition $\mathcal{D}(\mathbb{Z}_5, \Phi)$.

CHAPTER 3

DOYEN CONSTRUCTION FOR 2-PRIMITIVE STS($6k + 3$)

We begin this chapter by defining some new, descriptive terminology that makes the hypotheses and proofs of results in our updated version of Doyen's [4] argument more concise. Namely, given a subdecomposition of the augmented Bose decomposition, the following definition is convenient for describing the vertex set of the induced graph to which the subdecomposition associates. This structure is later mirrored in Chapter 4. Following this definition is a lemma using this new notation to concisely describe the vertex set corresponding to a nontrivial subdecomposition.

Definition 3.1. Let $k \geq 1$, Φ be a proper permutation triple of an abelian group G of order $2k+1$, and $\mathcal{S}$ be a subdecomposition of $\mathcal{C}(G, \Phi)$. We say $\mathcal{S}$ has *vertex type* (V_0, V_1, V_2) if for each $i \in \mathbb{Z}_3$, $V_i = \{x \in G : (x, i) \in V(\mathcal{S})\}$.

Lemma 3.2. *Let $k \geq 1$, Φ be a proper permutation triple of an abelian group G of order $2k+1$, and $\mathcal{S}$ be a subdecomposition of $\mathcal{C}(G, \Phi)$ of vertex type (V_0, V_1, V_2) for some $V_0, V_1, V_2 \subseteq G$ and suppose that $|\mathcal{S}| \geq 2$. Then, for each $i \in \mathbb{Z}_3$, $|V_i| > 0$.*

Proof. Suppose that $\mathcal{S}$ associates to an induced subgraph Γ' on t vertices. Since $|\mathcal{S}| \geq 2$, we have that $t \geq 5$. Note that since Γ' is an induced subgraph of a complete graph, Γ' must be isomorphic to K_t. Therefore, $t \equiv 1$ or $3 \pmod 6$ and, since $|\mathcal{S}| \geq 2$, hence $t \geq 7$, which implies that $|V_i| \geq 3$ for some $i \in \mathbb{Z}_3$. Let $u, v, w \in V_i$ be distinct and $x, y, z \in G$ such that

$$\begin{aligned} \phi_i(u) + \phi_i(v) &= 2\phi_i(x), \\ \phi_i(u) + \phi_i(w) &= 2\phi_i(y), \text{ and} \\ \phi_{i+1}(x) + \phi_{i+1}(y) &= 2\phi_{i+1}(z). \end{aligned}$$

Since $u, v \in V_i$ and $\{(u, i), (v, i), (x, i + 1)\} \in \mathcal{C}(G, \Phi)$, it follows that $\{(u, i), (v, i), (x, i + 1)\} \in \mathcal{S}$ and hence $x \in V_{i+1}$. Likewise, since $u, w \in V_i$ and $\{(u, i), (w, i), (y, i + 1)\} \in \mathcal{C}(G, \Phi)$, we have that $\{(u, i), (w, i), (y, i + 1)\} \in \mathcal{S}$ and $y \in V_{i+1}$. A similar line of reasoning demonstrates that $\{(x, i + 1), (y, i + 1), (z, i + 2)\} \in \mathcal{S}$ and $z \in V_{i+2}$. Therefore, $|V_{i+1}| \geq 2$ and $|V_{i+2}| \geq 1$. $\square$

Lemma 3.2 demonstrates that for a subdecomposition of the augmented Bose decomposition containing at least two triples, each V_i is nonempty. This means that each of the three rows of vertices, or more precisely, each of the three copies of G used to label the vertices of K_{6k+3}, contributes at least one vertex.

Definition 3.3. Let $k \geq 1$, Φ be a proper permutation triple of an abelian group G of order $2k+1$, and $\mathcal{S}$ be a subdecomposition of $\mathcal{C}(G, \Phi)$. Let $A \subseteq G$. Then, $\mathcal{S}$ is *columned with respect to* A if $\mathcal{S}$ has vertex type (A, A, A).

The next lemma strengthens the conditions on the vertex type of a subdecomposition of the augmented Bose decomposition even further and allows us to precisely describe the vertex type of any subdecomposition of the augmented Bose decomposition containing at least two triples using this new definition.

Lemma 3.4. *Let $k \geq 1$, Φ be a proper permutation triple of an abelian group G of order $2k+1$, and $\mathcal{S}$ be a subdecomposition of $\mathcal{C}(G, \Phi)$ such that $|\mathcal{S}| \geq 2$. Then, there exists $A \subseteq G$ such that $\mathcal{S}$ is columned with respect to A.*

Proof. Suppose that $\mathcal{S}$ is a subdecomposition of vertex type (V_0, V_1, V_2) for some $V_0, V_1, V_2 \subseteq G$ that associates to an induced subgraph Γ' with t vertices. Since Γ' is an induced subgraph of a complete graph, Γ' is isomorphic to K_t and hence K_t has a triangle decomposition. By Lemma 3.2, $|V_i| > 0$ for each $i \in \mathbb{Z}_3$. Let $i \in \mathbb{Z}_3$ such that $|V_i|$ is maximal, and define $A = V_i$. We first demonstrate by way of contradiction that $V_{i-1} \subseteq V_i$. Assume to the contrary that there exists some $x \in V_{i-1} \backslash V_i$. Define the function $\alpha : V_i \to V_{i-1} \backslash \{x\}$ such that $\{(x, i-1), (\alpha(y), i-1), (y, i)\} \in \mathcal{S}$ for each $y \in V_i$; these triples are illustrated in Figure 4. Note that since Γ' is a complete graph and every edge is contained in a unique triple, α is well-defined. Additionally, α is an injection; otherwise, $\alpha(y) = \alpha(y')$ for some distinct y and y' in V_i would imply the existence of the triples $\{(x, i-1), (\alpha(y), i-1), (y, i)\} \in \mathcal{S}$ and $\{(x, i-1), (\alpha(y), i-1), (y', i)\} \in \mathcal{S}$, which share an edge. It follows that

$$|V_i| = |\alpha(V_i)| \leq |V_{i-1} \backslash \{x\}| = |V_{i-1}| - 1,$$

which is bounded above by $|V_i| - 1$, which is a contradiction. Thus, $V_{i-1} \subseteq V_i$.

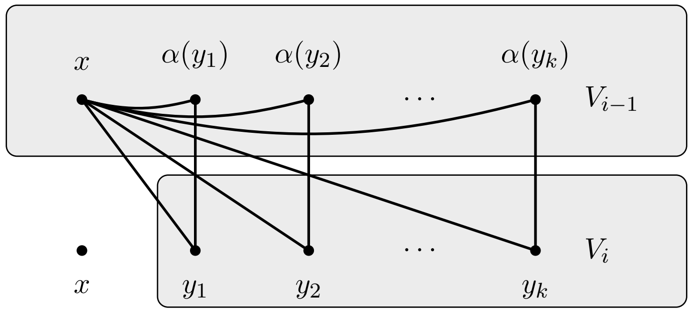


Figure 4. Illustration of the function $\boldsymbol{\alpha}$ between $\boldsymbol{V_i}$ and $\boldsymbol{V_{i-1}}$

Assuming the existence of $x \in V_{i-1} \backslash V_i$, the function α is used to map V_i into $V_{i-1} \backslash \{x\}$ injectively. This injective map, in conjunction with numerical conditions on V_i and V_{i-1}, is used to prove that $V_{i-1} \subseteq V_i$ by way of contradiction. The triples $\{(x, i-1), (\alpha(y), i-1), (y, i)\}$ are illustrated for each $y \in V_i$.

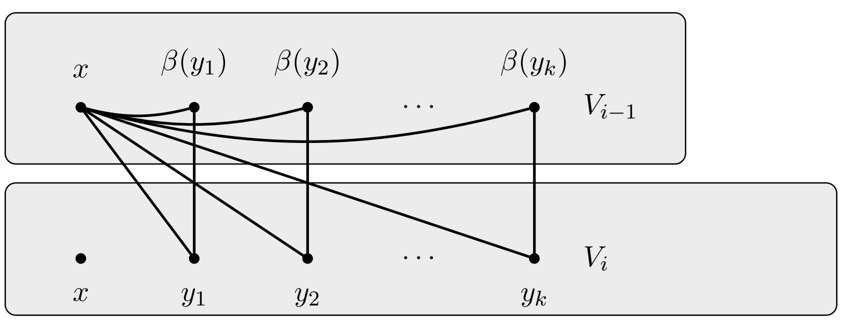


Figure 5. Illustration of the function $\boldsymbol{\beta}$ between $\boldsymbol{V_i}$ and $\boldsymbol{V_{i-1}}$

When $V_{i-1} \subseteq V_i$ and $x \in V_{i-1}$ is given, the function β is used to map $V_i \backslash \{x\}$ into $V_{i-1} \backslash \{x\}$ injectively, which is used in conjunction with numerical conditions on V_i and V_{i-1} to prove that $V_{i-1} = V_i$. The triples $\{(x, i-1), (\beta(y), i-1), (y, i)\}$ are illustrated for each $y \in V_i \backslash \{x\}$.

Now, let $x \in V_{i-1}$ and define the function $\beta : V_i \backslash \{x\} \to V_{i-1} \backslash \{x\}$ such that for each $y \in V_i \backslash \{x\}$, $\{(x, i-1), (\beta(y), i-1), (y, i)\} \in \mathcal{S}$; these triples are illustrated in Figure 5. Note that β is well-defined and injective for precisely the same reasons as α. So, we have that

$$|V_i| - 1 = |V_i \backslash \{x\}| = |\beta(V_i \backslash \{x\})| \leq |V_{i-1} \backslash \{x\}| = |V_{i-1}| - 1.$$

Equivalently, $|V_i| \leq |V_{i-1}|$, and it follows that $|V_i| = |V_{i-1}|$ since $|V_i|$ was assumed maximal. Since $V_{i-1} \subseteq V_i$ and $|V_i| = |V_{i-1}|$, it follows that $V_i = V_{i-1}$. Since $|V_{i-1}|$ is now maximal, an identical line of argument demonstrates that $V_{i-1} = V_{i-2}$, and we have that $V_{i-2} = V_{i-1} = V_i = A$. □

Lemma 3.4 indicates that all subdecompositions of the augmented Bose decomposition containing at least two triples possess a large amount of inherent structure in their vertex set. If $x \in V_i$ for any $i \in \mathbb{Z}_3$, then the vertical $W_x \in \mathcal{S}$. Of particular importance is the observation that $V(\mathcal{S}) = A \times \mathbb{Z}_3$. The next lemma provides the final piece of information to construct a 2-primitive decomposition of K_{6k+3}.

Lemma 3.5. *Let $k \geq 1$, Φ be a proper permutation triple of an abelian group G of order $2k+1$, and $\mathcal{S}$ be a subdecomposition of $\mathcal{C}(G, \Phi)$, which is columned with respect to some $A \subseteq G$, and suppose that $|\mathcal{S}| \geq 2$. Then, the set $\phi_i(A)$ is a coset of a nontrivial subgroup of G for each $i \in \mathbb{Z}_3$.*

Proof. Let $i \in \mathbb{Z}_3$. We show that the conditions of Lemma 2.1 hold for $\phi_i(A)$. Let $x, y \in \phi_i(A)$. First we show that $(x+y)/2 \in \phi_i(A)$. Since $x, y \in \phi_i(A)$, there exist $u, v \in A$ such that $\phi_i(u) = x$ and $\phi_i(v) = y$. Furthermore, since $u, v \in A$, we have that $u, v \in V_i$. Define $z = (x+y)/2$ and let $w \in G$ be the element satisfying $\phi_i(w) = z$. By construction, $\phi_i(u) + \phi_i(v) = 2\phi_i(w)$, and hence $\{(u, i), (v, i), (w, i+1)\} \in \mathcal{C}(G, \Phi)$. Since $u, v \in V_i$, it follows that $\{(u, i), (v, i), (w, i+1)\} \in \mathcal{S}$ and hence $w \in V_{i+1}$. So, $w \in A$ and hence $z \in \phi_i(A)$.

Now, let $x, z \in \phi_i(A)$. We demonstrate that $2z - x \in \phi_i(A)$. Since $x, z \in \phi_i(A)$, there exist $u, w \in A$ such that $\phi_i(u) = x$ and $\phi_i(w) = z$. Furthermore, since $u, w \in A$, we have that $u, w \in V_i$. Define $y = 2z - x$ and let $v \in G$ be the element satisfying $\phi_i(v) = y$. By construction, $\phi_i(u) + \phi_i(v) = 2\phi_i(w)$ and hence $\{(u, i), (v, i), (w, i+1)\} \in \mathcal{S}$. Therefore, $v \in V_i$ and hence $v \in A$. Thus, $y \in \phi_i(A)$. □

By cleverly selecting our permutation triple Φ, we can now construct a 2-primitive triangle decomposition of K_{6k+3} by leveraging the algebraic results in Chapter 2.

Theorem 3.6. *There exists a 2-primitive triangle decomposition of K_{6k+3} for all $k \geq 1$.*

Proof. Let $k \geq 1$ and Φ be the permutation triple (id, ϕ, id) of $\mathbb{Z}_{2k+1}$, where id represents the identity permutation and ϕ is the permutation given in Lemma 2.1; note that Φ is proper. We claim that $\mathcal{C}(\mathbb{Z}_{2k+1}, \Phi)$ is a 2-primitive triangle decomposition of K_{6k+3}. Suppose that $\mathcal{S}$ is a subdecomposition of $\mathcal{C}(\mathbb{Z}_{2k+1}, \Phi)$ such that $|\mathcal{S}| \geq 2$. By Lemma 3.4, there exists $A \subseteq G$ such that $\mathcal{S}$ is columned with respect to A. By Lemma 3.5, $A = id(A)$ and $\phi(A)$ are cosets of a nontrivial subgroup of $\mathbb{Z}_{2k+1}$, and hence $A = \mathbb{Z}_{2k+1}$ by Lemma 2.3.

So, $V(\mathcal{S}) = A \times \mathbb{Z}_3 = V(\mathcal{C}(\mathbb{Z}_{2k+1}, \Phi))$, and the result follows from Observation 2.4. $\square$

CHAPTER 4

CONSTRUCTION OF 2-PRIMITIVE DECOMPOSITION OF $K_{6k+2} - I$

Throughout this chapter, we parallel the results demonstrated for the augmented Bose decomposition in Chapter 3 to build up to a proof Theorem 1.2. We start with a lemma that allows us to classify induced subgraphs of cocktail party graphs into two distinct categories, which allows us to sidestep large amounts of case-by-case analysis. We follow with a definition analagous to Definition 3.1 that allows us to concisely discuss the vertex set of any subdecomposition of the deleted Bose decomposition containing at least two triples.

Lemma 4.1. *Let $m \geq 1$ and suppose $K_{2m} - I$ has a decomposition $\mathcal{C}$ such that each vertex in each subgraph of $\mathcal{C}$ has even degree. If $\mathcal{S}$ is a subdecomposition of $\mathcal{C}$, then the induced subgraph to which $\mathcal{S}$ associates is either a complete graph or a cocktail party graph.*

Proof. Assume that $\mathcal{S}$ is a subdecomposition with associated induced subgraph Γ' with t vertices, and let $x \in V(\Gamma')$. Clearly, $\deg_{\Gamma'}(x) \leq t-1$. Also, since Γ' is an induced subgraph of $K_{2m} - I$, it follows that for each vertex $x \in V(\Gamma')$, there is at most one vertex $y \in V(\Gamma')$ such that $xy \notin E(\Gamma')$, since exactly one edge of I is incident to x. Thus, $\deg_{\Gamma'}(x) \geq t-2$. Now, observe that $\deg_{\Gamma'}(x)$ must be even because $\deg_{\Gamma'}(x) = \sum_{C \in \mathcal{C}} \deg_C(x)$, and each term in this sum is even by the hypothesis. We have already established that the degree of each vertex is either $t-1$ or $t-2$; however, precisely one of these values is even. If t is odd, then Γ' must be $(t-1)$-regular and is therefore isomorphic to a complete graph. Otherwise, Γ' is $(t-2)$-regular and is isomorphic to a cocktail party graph. $\square$

Corollary 4.2. *Let $m \geq 1$. If $K_{2m} - I$ has a C_3-decomposition with subdecomposition $\mathcal{S}$, then the induced subgraph to which $\mathcal{S}$ associates is either complete or a cocktail party graph.*

In light of the previous corollary, we make the following observation about the correspondence between the type of induced subgraph to which a subdecomposition of the deleted Bose decomposition associates and the inclusion of zero in the vertex set of the subdecomposition.

Observation 4.3. Let $k \geq 1$, G be an abelian group of order $2k+1$, and Φ be a proper permutation triple of G. Let $\mathcal{S}$ be a subdecomposition of $\mathcal{D}(G, \Phi)$ that associates to Γ'. If Γ' is a complete graph, then $0 \notin V_1 \cap V_2$; otherwise, Γ' is a cocktail party graph and either $0 \in V_1 \cap V_2$ or $0 \notin V_1 \cup V_2$.

Definition 4.4. Let $k \geq 1$, G be an abelian group of order $2k+1$, and Φ be a proper permutation triple of G. Let $\mathcal{S}$ be a subdecomposition of $\mathcal{D}(G, \Phi)$. We say $\mathcal{S}$ has *vertex type* $(V_0, V_1, V_2; v_0, v_1, v_2)$ if for each $i \in \mathbb{Z}_3$, $V_i = \{x \in G : (x, i) \in V(\mathcal{S})$ and $v_i = |V_i \backslash \{0\}|$.

Lemma 4.5. *Let $k \geq 1$, G be an abelian group of order $2k+1$, and Φ be a proper permutation triple of G. Let $\mathcal{S}$ be a subdecomposition of $\mathcal{D}(G, \Phi)$ of vertex type $(V_0, V_1, V_2; v_0, v_1, v_2)$ and suppose $|\mathcal{S}| \geq 2$. Then, $v_i > 0$ for each $i \in \mathbb{Z}_3$.*

Proof. Suppose that $\mathcal{S}$ associates to a subgraph Γ' with t vertices. By Corollary 4.2, Γ' is either a complete graph or a cocktail party graph. Note that $t = |V_0| + |V_1| + |V_2|$, and observe that $t - 2 \leq v_0 + v_1 + v_2 \leq t$ since $v_0 = |V_0|$, $|V_1| - 1 \leq v_1 \leq |V_1|$, and $|V_2| - 1 \leq v_2 \leq |V_2|$.

Suppose that Γ' is a complete graph. Then, since $|\mathcal{S}| \geq 2$ and $t \equiv 1$ or $3 \pmod 6$, it follows that $t \geq 7$. If $v_i > 0$ for each $i \in \mathbb{Z}_3$, then the result holds. Assume $v_j = 0$ for some $j \in \mathbb{Z}_3$. It follows that $v_i \geq 3$ for some $i \in \mathbb{Z}_3$; otherwise, $v_0+v_1+v_2 \leq 4$ and hence $t \leq 6$. Let $x, y, z \in V_i$ be distinct and nonzero. Since Γ' is complete, $(x, i)(y, i)$, $(x, i)(z, i)$, and $(y, i)(z, i)$ are each contained in $E(\Gamma')$. Let $u, v, w \in G$ satisfy the following:

$$\begin{aligned}
&\phi_i(x) + \phi_i(y) = 2\phi_i(u)\\
&\phi_i(x) + \phi_i(z) = 2\phi_i(v), \text{ and}\\
&\phi_i(y) + \phi_i(z) = 2\phi_i(w).
\end{aligned}$$

Then, $\{(x, i), (y, i), (u, i+1)\}$, $\{(x, i), (z, i), (v, i+1)\}$, and $\{(y, i), (z, i), (w, i+1)\}$ are each triples in $\mathcal{S}$, and at most one of u, v, and w is zero. So, v_{i+1} is nonzero. A similar line of reasoning using u, v, and w produces a nonzero element of V_2, completing the proof of the result when Γ' is complete.

Now, suppose that Γ' is a cocktail party graph. Then, since $t \equiv 0$ or $2 \pmod 6$ and $|\mathcal{S}| \geq 2$, we have that $t \geq 6$. As before, assume that $v_i = 0$ for some $i \in \mathbb{Z}_3$. Then, $v_i \geq 2$ for some $i \in \mathbb{Z}_3$, else $v_0 + v_1 + v_2 \leq 2$ and hence $t \leq 4$. Observe that $v_0 = v_1$ since the missing 1-factor I creates a natural pairing between V_0 and the nonzero vertices of V_1. Thus, we establish the result in two different cases: when $v_0 = v_1 \neq 0$ and $v_2 = 0$ and when $v_2 \neq 0$ and $v_0 = v_1 = 0$.

First, suppose that $v_0 \neq 0$. It follows that $v_0 \geq 2$, and hence $v_1 \geq 2$, by the previous discussion. Let $x \in V_1$ be nonzero. If $0 \in V_1 \cap V_2$, let $z \in G$ such that $\phi_1(0)+\phi_1(x) = 2\phi_1(z)$. Then, $\{(0,1),(x,1),(z,2)\} \in \mathcal{S}$. Since $\phi_1(0) = 0$ and $x \neq 0$, it follows that $2\phi_1(z) \neq 0$ and hence $z \neq 0$. Thus, $v_2 > 0$ if $0 \in V_1 \cup V_2$. So, $0 \notin V_1 \cup V_2$, and $v_1 \geq 3$. Note that there exists a unique $x' \in G$ such that $\phi_1(x) + \phi_1(x') = 0$. Choose $y \in V_1 \backslash \{x'\}$; so $\phi_1(x) + \phi_1(y) \neq 0$. Let $z \in G$ such that $\phi_1(x) + \phi_1(y) = 2\phi_1(z)$. Then, $\{(x,1),(y,1),(z,2)\} \in \mathcal{S}$. Since $\phi_1(0) = 0$ and $\phi_1(x) + \phi_1(y) \neq 0$, we have that $z \neq 0$. Therefore, $v_2 > 0$, which is a contradiction.

So, $v_0 = v_1 = 0$ and $v_2 \neq 0$. It follows that $v_2 \geq 4$. Let $x \in V_2$ be nonzero. Note that there exists a unique nonzero $x' \in G$ such that $\phi_2(x) + \phi_2(x') = 0$. Select $y \in V_2 \backslash \{x'\}$; so $\phi_2(x) + \phi_2(y) \neq 0$. Let $z \in G$ such that $\phi_2(x) + \phi_2(y) = 2\phi_2(z)$. Then, $\{(x,2),(y,2),(z,0)\} \in \mathcal{S}$. Furthermore, $z \neq 0$ since $\phi_2(0) = 0$ and $\phi_2(x) + \phi_2(y) \neq 0$. It follows that $v_0 > 0$, which is a contradiction. So, v_0, v_1, and v_2 are all positive when Γ' is a cocktail party graph. $\square$

While Lemma 3.2 guarantees a nonempty contribution from each V_i in a subdecomposition of the augmented Bose decomposition containing at least two triples, the result of Lemma 4.5 guarantees that each V_i in a subdecomposition of the deleted Bose decomposition containing at least two triples contributes a nonzero vertex; in particular, $V_i \backslash \{0\}$ is nonempty for each $i \in \mathbb{Z}_3$. Since an analysis of $V(\mathcal{S})$ requires particular attention to the vertices with first ordinate zero, Lemma 4.5 is useful when we need to separately describe the behavior of the nonzero vertices of V_i.

Continuing with the parallels to Chapter 3, we modify Definition 3.3 to introduce columned terminology pertaining to subdecompositions of the deleted Bose decomposition containing at least two triples, and subsequently mirror the result of Lemma 3.4 using this new definition.

Definition 4.6. Let $k \geq 1$, G be an abelian group of order $2k+1$, and Φ be a proper permutation triple of G. Let $\mathcal{S}$ be a subdecomposition of $\mathcal{D}(G, \Phi)$ and $A \subseteq G$. Then, $\mathcal{S}$ is *columned with respect to A* if $\mathcal{S}$ has vertex type $(A \backslash \{0\}, A, A)$.

Lemma 4.7. *Let $k \geq 1$, G be an abelian group of order $2k + 1$, and Φ be a proper permutation triple of G. Let $\mathcal{S}$ be a subdecomposition of $\mathcal{D}(G, \Phi)$ and suppose $|\mathcal{S}| \geq 2$. If $\mathcal{S}$ associates to a complete graph, then there exists $A \subseteq G$ such that $\mathcal{S}$ is columned with respect to A and $0 \notin A$.*

Proof. Suppose that $\mathcal{S}$ has vertex type $(V_0, V_1, V_2; v_0, v_1, v_2)$. Let $A = V_0$. Then, $A \subseteq G$ and $0 \notin A$. The result follows if we show that $V_0 = V_1 = V_2$.

Suppose that $\mathcal{S}$ associates to a complete graph Γ'. By Lemma 4.5, we know that $v_i > 0$ for each $i \in \mathbb{Z}_3$. It is sufficient to show that if $i \in \mathbb{Z}_3$ and $v_i = \max\{v_0, v_1, v_2\}$, then $V_i = V_{i-1}$. We demonstrate this fact on a case-wise basis, where each case presents an argument based on set containment.

<u>Case 0</u>: Suppose that $v_0 = \max\{v_0, v_1, v_2\}$. By way of contradiction, we demonstrate that $V_2 \subseteq V_0$. Assume to the contrary that there exists some $x \in V_2 \backslash V_0$. Define the function $\alpha_0 : V_0 \to V_2 \backslash \{x\}$ such that $\{(x, 2), (\alpha_0(y), 2), (y, 0)\} \in \mathcal{S}$ for each $y \in V_0$. Since Γ' is a complete graph and the triples in $\mathcal{S}$ partition $E(\Gamma')$, α_0 is well-defined and injective. It follows that

$$v_0 = |V_0| = |\alpha_0(V_0)| \leq |V_2 \backslash \{x\}| = |V_2| - 1.$$

Equivalently, $|V_2| \geq v_0 + 1$. Observe that if $0 \notin V_2$, then $|V_2| = v_2$, and so $v_2 \geq v_0 + 1$, which is a contradiction. Thus, $0 \in V_2$, and $|V_2| = v_2 + 1$. Hence, $v_2 \geq v_0$, and it follows from the maximality of v_0 that $v_2 = v_0$.

In order to generate a contradiction to the assumption that $x \in V_2 \backslash V_0$, we further demonstrate by way of contradiction that $V_1 \subseteq V_2$. To that end, suppose that there exists some $z \in V_1 \backslash V_2$ and define the function $\alpha_1 : V_2 \to V_1 \backslash \{z\}$ such that $\{(z, 1), (\alpha_1(y), 1), (y, 2)\} \in \mathcal{S}$ for each $y \in V_2$. Note that α_1 is a well-defined injection for the same reasons as α_0, so we have that

$$|V_2| = |\alpha_1(V_2)| \leq |V_1 \backslash \{z\}| = |V_1| - 1.$$

That is, $|V_2| \leq |V_1| - 1$. It was previously demonstrated that $0 \in V_2$ necessarily. As a consequence of Γ' being complete, it follows that $0 \notin V_1$, else there would be a missing edge between $(0, 1)$ and $(0, 2)$ in Γ', which is fixed under ϕ_1. So, $|V_1| = v_1$, and therefore

$$v_0 + 1 = v_2 + 1 = |V_2| \leq |V_1| - 1 = v_1 - 1.$$

This is equivalent to $v_0 \leq v_1 - 2$, which is a contradiction. Therefore, $V_1 \subseteq V_2$.

With this, we now finally contradict the assumption that $x \in V_2 \backslash V_0$. Let $z \in V_1$, and therefore $z \in V_2$. Define the function $\beta_0 : V_2\backslash\{z\} \to V_1\backslash\{z\}$ such that $\{(z,1),(\beta_0(y),1),(y,2)\} \in \mathcal{S}$ for each $y \in V_2\backslash\{z\}$. As with α_0 and α_1, β_0 is a well-defined injection. So,

$$v_2 = |V_2| - 1 = |V_2\backslash\{z\}| = |\beta_0(V_2\backslash\{z\})| \leq |V_1\backslash\{z\}| = |V_1| - 1 = v_1 - 1.$$

However, $v_2 = v_0$, so the above implies $v_0 \leq v_1 - 1$, contradicting the maximality of v_0. So, $V_2 \subseteq V_0$.

Now, let $x \in V_2$. Then, $x \in V_0$. Define the function $\beta_1 : V_0\backslash\{x\} \to V_2\backslash\{x\}$ such that $\{(x,2),(\beta_1(y),2),(y,0)\} \in \mathcal{S}$ for each $y \in V_0\backslash\{x\}$. As before, β_1 is a well-defined injection. Thus,

$$|V_0| - 1 = |V_0\backslash\{x\}| = |\beta_1(V_0\backslash\{x\})| \leq |V_2\backslash\{x\}| = |V_2| - 1.$$

Equivalently, $|V_0| \leq |V_2|$. Since $V_2 \subseteq V_0$, this implies that $V_0 = V_2$, which completes the demonstration of the first case.

Case 1: Suppose $v_1 = \max\{v_0, v_1, v_2\}$. As in the first case, we first demonstrate that $V_0 \subseteq V_1$ by way of contradiction.

Suppose, contrarily, that there exists some $x \in V_0\backslash V_1$. Define $\alpha : V_1 \to V_0\backslash\{x\}$ as the function such that $\{(x,0),(\alpha(y),0),(y,1)\} \in \mathcal{S}$ for each $y \in V_1$. Again, α is a well-defined injection, and it follows that

$$|V_1| = |\alpha(V_1)| \leq |V_0\backslash\{x\}| = |V_0| - 1 = v_0 - 1.$$

This is equivalent to $v_0 \geq |V_1| + 1$, but $|V_1| \geq v_1$. Thus, $v_0 \geq v_1 + 1$, which is a contradiction. Hence, $V_0 \subseteq V_1$.

Now, let $x \in V_0$. Then, $x \in V_1$ also. Define the function $\beta : V_1\backslash\{x\} \to V_0\backslash\{x\}$ by $\{(x,0),(\beta(y),0),(y,1)\} \in \mathcal{S}$ for each $y \in V_1\backslash\{x\}$. Again, β is a well-defined injection, so

$$|V_1| - 1 = |V_1\backslash\{x\}| = |\beta(V_1\backslash\{x\})| \leq |V_0\backslash\{x\}| = |V_0| - 1.$$

Hence, $|V_1| \leq |V_0|$, and we have that $V_1 = V_0$, completing the demonstration of the second case.

Case 2: Suppose $v_2 = \max\{v_0, v_1, v_2\}$. We again proceed by showing that $V_1 \subseteq V_2$ by way of contradiction. Assume to the contrary that there exists some $x \in V_1 \backslash V_2$. Define the function $\alpha_0 : V_2 \to V_1 \backslash \{x\}$ such that $\{(x, 1), (\alpha_0(y), 1), (y, 2)\} \in \mathcal{S}$ for each $y \in V_2$. Again, α_0 is well-defined and injective, so

$$|V_2| = |\alpha_0(V_2)| \leq |V_1 \backslash \{x\}| = |V_1| - 1.$$

Thus, $|V_1| \geq |V_2| + 1$. Note that if $0 \notin V_1$, then $|V_1| = v_1$, and so $v_1 \geq |V_2| + 1 \geq v_2 + 1$. However, this would contradict the maximality of v_2. It follows that $0 \in V_1$, and thus $|V_1| = v_1 + 1$. Hence, $v_1 \geq v_2$ and we have that $v_1 = v_2$.

A similar line of reasoning to that used in Case 0 is employed to contradict the existence of $x \in V_1 \backslash V_2$, in which we next demonstrate by way of contradiction that $V_0 \subseteq V_1$. Assume, contrarily, that there exists some $z \in V_0 \backslash V_1$ and define the function $\alpha_1 : V_1 \to V_0 \backslash \{z\}$ such that $\{(z, 0), (\alpha_1(y), 0), (y, 1)\} \in \mathcal{S}$ for each $y \in V_1$. As usual, α_1 is a well-defined injection, so

$$v_1 + 1 = |V_1| = |\alpha_1(V_1)| \leq |V_0 \backslash \{z\}| = |V_0| - 1 = v_0 - 1.$$

Thus, $v_0 \geq v_1 + 2$ and hence $v_0 \geq v_2 + 2$, which contradicts the maximality of v_2. Hence, $V_0 \subseteq V_1$.

As before, we are now equipped to demonstrate that $V_1 \subseteq V_2$. Let $z \in V_0$. Then, $z \in V_1$. Define the function $\beta_0 : V_1 \backslash \{z\} \to V_0 \backslash \{z\}$ such that $\{(z, 0), (\beta_0(y), 0), (y, 1)\} \in \mathcal{S}$ for each $y \in V_2 \backslash \{z\}$. Since β_0 is another well-defined injection,

$$v_2 = v_1 = |V_1| - 1 = |V_1 \backslash \{z\}| = |\beta_0(V_1 \backslash \{z\})| \leq |V_0 \backslash \{z\}| = |V_0| - 1 = v_0 - 1.$$

Equivalently, $v_0 \geq v_2 + 1$, which is a contradiction. Hence, there does not exist an element $x \in V_1 \backslash V_2$, and we have that $V_1 \subseteq V_2$.

Let $x \in V_1$. Then, $x \in V_2$. Define the function $\beta_1 : V_2 \backslash \{x\} \to V_1 \backslash \{x\}$ such that $\{(x, 1), (\beta_1(y), 1), (y, 2)\} \in \mathcal{S}$ for each $y \in V_2 \backslash \{x\}$. As before, β_1 is a well-defined injection. Thus,

$$|V_2| - 1 = |V_2 \backslash \{x\}| = |\beta_1(V_2 \backslash \{x\})| \leq |V_1 \backslash \{x\}| = |V_1| - 1.$$

Equivalently, $|V_2| \leq |V_1|$. Since $V_1 \subseteq V_2$, this implies that $V_2 = V_1$, completing the demonstration of the final case. Therefore, $V_0 = V_1 = V_2$. □

Note that although the proof of Lemma 4.7 is considerably more complex than that of Lemma 3.4, it demonstrates that when a subdecomposition of the deleted Bose decomposition containing at least two triples associates to an induced subgraph which is complete, the vertex type of the subdecomposition exactly matches the vertex type described by Lemma 3.4, as one might intuitively expect. The following lemma provides the analagous result for subdecompositions of the deleted Bose decomposition containing at least two triples that associate to cocktail party graphs, and we subsequently summarize our collective results in a corollary that describes the one-to-one correspondence between the vertex type of nontrivial subdecomposition of the deleted Bose decomposition containing at least two triples and the type of induced subgraph to which the subdecomposition associates.

Lemma 4.8. *Let $k \geq 1$, G be an abelian group of order $2k+1$, and Φ be a proper permutation triple of G. Let $\mathcal{S}$ be a subdecomposition of $\mathcal{D}(G, \Phi)$ and suppose $|\mathcal{S}| \geq 2$. If $\mathcal{S}$ associates to a cocktail party graph, then there exists $A \subseteq G$ such that $\mathcal{S}$ is columned with respect to A and $0 \in A$.*

Proof. Suppose that $\mathcal{S}$ has vertex type $(V_0, V_1, V_2; v_0, v_1, v_2)$ and that $\mathcal{S}$ associates to a cocktail party graph Γ'. We achieve the result by demonstrating the following claims in sequence: $v_2 \leq v_0 = v_1$, $0 \in V_1 \cap V_2$, $V_0 \subseteq V_1$, $V_0 \cup \{0\} = V_1$, and finally $V_1 \subseteq V_2$.

Because the edges of the missing 1-factor I create a natural pairing between the sets V_0 and $V_1 \backslash \{0\}$, we have that $v_0 = v_1$. We initially demonstrate that $v_2 \leq v_0 = v_1$ by way of contradiction. Assume to the contrary that $v_2 > v_0 = v_1$. To generate the needed contradiction, we first demonstrate by way of contradiction that $V_1 \subseteq V_2$. Suppose, contrarily, that there exists some nonzero $x \in V_1 \backslash V_2$. Since x is nonzero, we have also that $\phi_1(x)$ is nonzero, and hence $(x, 1)(y, 2) \notin I$ for all $y \in V_2$. Define the function $\alpha : V_2 \to V_1 \backslash \{x\}$ such that for each $y \in V_2$, $\{(x, 1), (y, 2), (\alpha(y), 1)\} \in \mathcal{S}$; observe that α is well-defined and injective. It follows that

$$|V_2| = |\alpha(V_2)| \leq |V_1 \backslash \{x\}| = |V_1| - 1.$$

If $0 \in V_1 \cap V_2$, then $|V_1| = v_1 + 1$ and $|V_2| = v_2 + 1$; otherwise, $0 \notin V_1 \cup V_2$ and we have that $|V_1| = v_1$ and $|V_2| = v_2$. However, both of these situations result in $v_2 \leq v_1 - 1$, which is a contradiction. Hence we have that $V_1 \subseteq V_2$, and since $v_2 > v_1$, $V_1 \subsetneq V_2$.

Now, let $x \in V_1$ be nonzero and define the function $\beta : V_2 \backslash \{x\} \to V_1 \backslash \{x\}$ such that $\{(x, 1), (y, 2), (\beta(y), 1)\} \in \mathcal{S}$ for each $y \in V_2 \backslash \{x\}$. Note that β is a well-defined injection for the same reasons as α, and observe that

$$|V_2| - 1 = |V_2 \backslash \{x\}| = |\beta(V_2 \backslash \{x\})| \leq |V_1 \backslash \{x\}| = |V_1| - 1.$$

Equivalently, $|V_2| \leq |V_1|$, but this is again a contradiction. It follows that $v_2 > v_0 = v_1$ is impossible, and hence $v_2 \leq v_0 = v_1$.

We next demonstrate by way of contradiction that $0 \in V_1 \cap V_2$. Assume to the contrary that $0 \notin V_1 \cup V_2$. As in the previous claim, it is first necessary demonstrate by way of contradiction that $V_2 \subseteq V_0$. Suppose, contrarily, that there exists some nonzero $x \in V_2 \backslash V_0$. Define the function $\alpha : V_0 \to V_2 \backslash \{x\}$ such that $\{(x, 2), (y, 0), (\alpha(y), 2)\} \in \mathcal{S}$ for each $y \in V_0$. Note that α is a well-defined injection because x is nonzero and every edge is present between vertices of V_0 and V_2 in Γ'. Therefore,

$$v_0 = |V_0| = |\alpha(V_0)| \leq |V_2 \backslash \{x\}| = |V_2| - 1.$$

Equivalently, $|V_2| \geq v_0 + 1$. Since $0 \notin V_1 \cup V_2$, we have $|V_2| = v_2$, and our inequality becomes $v_2 \geq v_0 + 1$, contradicting the maximality of v_0. Thus, there does not exist an element $x \in V_2 \backslash V_0$; that is, $V_2 \subseteq V_0$.

Now, let $x \in V_2$ be nonzero. Define the function $\beta : V_0 \backslash \{x\} \to V_2 \backslash \{x\}$ such that $\{(x, 2), (y, 0), (\beta(y), 2)\} \in \mathcal{S}$ for each $y \in V_0 \backslash \{x\}$. Since $\phi_2(0) = 0$ and every edge is present between vertices of V_0 and V_2 in Γ', β is a well-defined injection. Thus,

$$v_0 - 1 = |V_0| - 1 = |V_0 \backslash \{x\}| = |\beta(V_0 \backslash \{x\})| \leq |V_2 \backslash \{x\}| = |V_2| - 1.$$

Since $0 \notin V_1 \cup V_2$, $|V_2| = v_2$, and our inequality becomes $v_2 \geq v_0$. Therefore, it follows that $V_0 = V_2$ since $|V_0| = v_0 = v_2 = |V_2|$.

Suppose that $x \in V_2$. Then, since Γ' is a cocktail party graph, there exists some $y \in V_2$ such that $\phi_2(x) + \phi_2(y) = 2\phi_2(0) = 0$ by Observation 2.6. That is, $(x,2)(y,2) \in I$. Since $V_0 = V_2$, it follows that $x, y \in V_0$. Furthermore, since $\phi_0 = \phi_2$, we also have that $\phi_0(x) + \phi_0(y) = 2\phi_0(z) = 0$ for some $z \in V_1$. This implies that $z = 0$ and that $\{(x,0),(y,0),(z,1)\} \in \mathcal{S}$, which gives that $0 \in V_1$, which is a contradiction. Therefore, $0 \in V_1 \cap V_2$.

Now, we demonstrate that $V_0 \subseteq V_1$ by way of contradiction. Assume to the contrary that there exists some $x \in V_0 \backslash V_1$. Let z be the unique vertex in V_1 such that $(x,0)(z,1) \in I$, and define the function $\alpha : V_1 \backslash \{z\} \to V_0 \backslash \{x\}$ such that $\{(x,0),(y,1),(\alpha(y),0)\} \in \mathcal{S}$ for each $y \in V_1 \backslash \{z\}$. Since $(x,0)(y,1) \in E(\Gamma')$ for each $y \in V_1 \backslash \{z\}$, α is well-defined and injective. It follows that

$$v_1 = |V_1| - 1 = |V_1 \backslash \{z\}| = |\alpha(V_1 \backslash \{z\})| \leq |V_0 \backslash \{x\}| = |V_0| - 1 = v_0 - 1.$$

Equivalently, $v_0 \geq v_1 + 1$, but this contradicts the fact that $v_0 = v_1$. Hence, there does not exist an element $x \in V_0 \backslash V_1$, and we have that $V_0 \subseteq V_1$. Since $v_0 = v_1$, it also follows that $V_0 \cup \{0\} = V_1$.

Next, we demonstrate that $V_1 \subseteq V_2$. Let $x \in \phi_0(V_1)$ be nonzero. Then, there exists a nonzero element $u \in V_1$ such that $\phi_0(u) = x$. By Observation 2.6, there exists $v \in V_0$ such that $\phi_0(0) + \phi_0(v) = 2\phi_0(u)$. Let $y = \phi_0(v)$. Then, since $\phi_0(0) = 0$, we have that $y = 2x$, and hence $\phi_0(V_1) = 2\phi_0(V_1)$. Furthermore, since $V_0 \cup \{0\} = V_1$, we have that $\phi_0(V_0) = 2\phi_0(V_0)$, and hence $\phi_2(V_0) = 2\phi_2(V_0)$ since $\phi_0 = \phi_2$.

Now, let $u \in V_1$ be nonzero and let $x = \phi_0(u) = \phi_2(u)$; note that $x \in \phi_0(V_1)$. Since $\phi_0(V_1) = 2\phi_0(V_1)$, we also have that $\frac{1}{2}x \in \phi_0(V_1)$. Let $u' \in V_1$ such that $\phi_0(u') = \frac{1}{2}x$. Note that $u' \neq 0$ and hence $u' \in V_0$ since $V_0 \cup \{0\} = V_1$. Now, since $u' \in V_0$ and $0 \in V_2$, we have that $(0,2)(u',0) \in E(\Gamma')$ and hence there exists some $v \in V_2$ such that $\{(0,2),(v,2),(u',0)\} \in \mathcal{S}$ by Observation 2.6. Therefore, $\phi_2(0) + \phi_2(v) = 2\phi_2(u')$. Furthermore, $\phi_2(0) = 0$ and $\phi_2(u') = \frac{1}{2}x$, so we have that $\phi_2(v) = x$, and hence $u = v$. Thus, $u \in V_2$, and we have that $V_1 \subseteq V_2$. Furthermore, since $v_2 \leq v_1$, we have that $V_1 = V_2$, and hence $V_0 \cup \{0\} = V_1 = V_2$, as desired. □

Corollary 4.9. *Let $k \geq 1$, G be an abelian group of order $2k+1$, and Φ be a proper permutation triple of G. Let $\mathcal{S}$ be a subdecomposition of $\mathcal{D}(G,\Phi)$ that associates to Γ' and suppose $|\mathcal{S}| \geq 2$. Then, there exists $A \subseteq G$ such that $\mathcal{S}$ is columned with respect to A, and Γ' is either a complete graph or a cocktail party graph. Furthermore, $0 \in A$ if and only if Γ' is a cocktail party graph.*

We conclude the chapter by demonstrating an important result detailing the relationship between the type of induced subgraph to which a subdecomposition of the deleted Bose decomposition containing at least two triples associates and the nature of the elements included in the set with respect to which it is columned. Subsequently, we prove two results which are jointly analagous to Lemma 3.5 and leverage these results to prove Theorem 1.2.

Lemma 4.10. *Let $k \geq 1$, G be an abelian group of order $2k+1$, and Φ be a proper permutation triple of G. Let $\mathcal{S}$ be a subdecomposition of $\mathcal{D}(G, \Phi)$ that associates to Γ' and suppose $|\mathcal{S}| \geq 2$. Let $A \subseteq G$ such that $\mathcal{S}$ is columned with respect to A. Let $u \in A$ and $v, w \in G$ such that $\phi_2(u) + \phi_2(v) = 0$ and $\phi_0(u) = 2\phi_0(w)$. If Γ' is a cocktail party graph, then $v, w \in A$; otherwise, $v, w \notin A$.*

Proof. Suppose that Γ' is a cocktail party graph. If $u = 0$, then $v = w = 0$ and the result holds trivially. So, suppose that $u \neq 0$. Observe that for each $(x, i) \in V(K_{6k+2} - I)$, there exists a unique vertex $(y, j) \in V(K_{6k+2} - I)$ such that $(x, i)(y, j) \in I$; furthermore, $(x, i) \in V(\Gamma')$ if and only if $(y, j) \in V(\Gamma')$. Since $u \in A$, $u \in V_2$ and $(u, 2) \in V(\Gamma')$. By Definition 2.11, $(u, 2)(v, 2) \in I$ since $\phi_2(u)+\phi_2(v) = 0$. Since $(u, 2) \in V(\Gamma')$, it follows that $(v, 2) \in V(\Gamma')$. Thus, $v \in V_2$ and hence $v \in A$. Similarly, since $u \in A$ and u is nonzero, $u \in V_0$ and $(u, 0) \in V(\Gamma')$. Since $\phi_0(u) = 2\phi_0(w)$, we have by Definition 2.11 that $(u, 0)(w, 1) \in I$. Thus, $(w, 1) \in V(\Gamma')$ since $(u, 0) \in V(\Gamma')$. So, $w \in V_1$ and hence $w \in A$.

Now, suppose that Γ' is a complete graph. For any vertices $(x, i), (y, j) \in V(K_{6k+2} - I)$, $(x, i)(y, j) \in E(K_{6k+2} - I)$ if and only if $(x, i)(y, j) \notin I$. Furthermore, for any $(x, i), (y, j) \in V(\Gamma')$, $(x, i)(y, j) \in E(\Gamma')$ if and only if $(x, i)(y, j) \notin I$ since Γ' is an induced subgraph of $K_{6k+2} - I$. Then, $(u, 2)(v, 2) \notin E(\Gamma')$ since $(u, 2)(v, 2) \in I$ and $(u, 0)(w, 1) \notin E(\Gamma')$ since $(u, 0)(w, 1) \in I$. However, since Γ' is complete, this implies that $(v, 2) \notin V(\Gamma')$ and $(w, 1) \notin V(\Gamma')$. Thus, $v \notin V_2$ and $w \notin V_0$, and hence $v, w \notin A$. $\square$

Lemma 4.11. *Let $k \geq 1$, G be an abelian group of order $2k + 1$, and Φ be a proper permutation triple of G. Let $\mathcal{S}$ be a subdecomposition of $\mathcal{D}(G, \Phi)$ that associates to Γ' and suppose $|\mathcal{S}| \geq 2$. Let $A \subseteq G$ such that $\mathcal{S}$ is columned with respect to A, and suppose that $\phi_0 = \phi_2$. Then, the sets $\phi_0(A)$ and $\phi_2(A)$ are cosets of a nontrivial subgroup of G.*

Proof. It is sufficient to show that $\phi_2(A)$ is a coset of a nontrivial subgroup of G since $\phi_0 = \phi_2$. We demonstrate that $\phi_2(A)$ meets the conditions of Lemma 2.1. Let $x, y \in \phi_2(A)$. We first show that $(x+y)/2 \in \phi_2(A)$. This holds trivially if $x = y$; so, suppose that $x \neq y$. Define $z = (x+y)/2$ and let $w \in G$ satisfy that $\phi_0(w) = \phi_2(w) = z$. Since $x, y \in \phi_2(A)$, there exist $u, v \in A$ such that $\phi_2(u) = x$ and $\phi_2(v) = y$. Furthermore, since $u, v \in A$, we have that $u, v \in V_2$. By construction, $\phi_2(u) + \phi_2(v) = 2\phi_2(w)$. Since $x \neq -y$, then $(u,2)(v,2) \in E(\Gamma')$ and hence $\{(u,2),(v,2),(w,0)\} \in \mathcal{S}$. Thus, $w \in V_0$ and hence $w \in A$. It follows that $z \in \phi_2(A)$. If $x = -y$, then $\phi_2(u) + \phi_2(v) = 0$ and $w = z = 0$. By Lemma 4.10, Γ' is a cocktail party graph, so $0 \in A$ and hence $0 \in \phi_2(A)$ by Corollary 4.9.

Next, let $x, z \in \phi_2(A)$; now we demonstrate that $2z - x \in \phi_2(A)$. As before, this holds trivially if $x = z$, so suppose that $x \neq z$. Since $x, z \in \phi_2(A)$, there exist $u, w \in A$ such that $\phi_2(u) = x$ and $\phi_2(w) = z$. Define $y = 2z - x$ and let $v \in G$ be the element satisfying $\phi_2(v) = y$. By construction, $\phi_2(u) + \phi_2(v) = 2\phi_2(w)$. If $z = 0$, then $w = 0$ and $\phi_2(u) + \phi_2(v) = 2\phi_2(w) = 0$. By Corollary 4.9, Γ' is a cocktail party graph. So, by Lemma 4.10, $v \in A$ and hence $y \in \phi_2(A)$. If $x = 0$, then $u = 0$ and $\phi_2(v) = 2\phi_2(w)$. Again, we have by Corollary 4.9 that Γ' is a cocktail party graph. Note that since $\phi_0 = \phi_2$, we have that $\phi_0(v) = 2\phi_0(w)$. It follows from Lemma 4.10 that $v \in A$ and hence $y \in \phi_2(A)$. Finally, if $x, z \neq 0$, then $(u,2)(w,0) \in E(\Gamma')$ and $\{(u,2),(v,2),(w,0)\} \in \mathcal{S}$. So, $v \in V_2$ and thus $y \in \phi_2(A)$. □

Lemma 4.12. *Let $k \geq 1$, G be an abelian group of order $2k+1$, and Φ be a proper permutation triple of G. Let $\mathcal{S}$ be a subdecomposition of $\mathcal{D}(G, \Phi)$ that associates to Γ' and suppose $|\mathcal{S}| \geq 2$. Let $A \subseteq G$ such that $\mathcal{S}$ is columned with respect to A. Then, the set $\phi_1(A)$ is a coset of a nontrivial subgroup of G.*

Proof. We proceed by showing that the conditions of Lemma 2.1 are met by $\phi_1(A)$. Let $x, y \in \phi_1(A)$. We first demonstrate that $(x+y)/2 \in \phi_1(A)$. This holds trivially for $x = y$; so, suppose that $x \neq y$. Since $x, y \in \phi_1(A)$, there exist $u, v \in A$ such that $\phi_1(u) = x$ and $\phi_1(v) = y$. Furthermore, since $u, v \in A$, we have that $u, v \in V_1$. Define $z = (x+y)/2$ and let $w \in G$ be the element satisfying $\phi_1(w) = z$. By construction, $\phi_1(u) + \phi_1(v) = 2\phi_1(w)$. Note that each pair of vertices in G_1 are adjacent in Γ; since $u, v \in V_1$, it follows that $(u,1)(v,1) \in E(\Gamma')$ and hence $\{(u,1),(v,1),(w,2)\} \in \mathcal{S}$.

Therefore, $w \in V_2$ and hence $w \in A$. Thus, $z \in \phi_1(A)$.

Now, let $x, z \in \phi_1(A)$. We demonstrate that $2z - x \in \phi_1(A)$. Since $x, z \in \phi_1(A)$, there exist $u, w \in A$ such that $\phi_1(u) = x$ and $\phi_1(w) = z$. Again, if $x = z$ this holds trivially, so we suppose that $x \neq z$. Define $y = 2z - x$ and let $v \in G$ be the element satisfying $\phi_1(v) = y$. By construction, $\phi_1(u) + \phi_1(v) = 2\phi_1(w)$. It follows that $y \neq x$; otherwise, $2z - x = x$, implying that $x = z$. Note that $(0,1)(0,2)$, which is the only edge between G_1 and G_2 in I, is not the image of $(u,1)(w,2)$ under ϕ_1. We have that $(u,1)(w,2) \in E(\Gamma)$, and since $(u,1),(w,2) \in V(\Gamma')$, we have that $\{(u,1),(v,1),(w,2)\} \in \mathcal{S}$. Therefore, $v \in V_1$ and hence $v \in A$. Thus, $y \in \phi_1(A)$. $\square$

Using all of the structure that we have built up concerning the vertex structure of nontrivial subdecompositions of the deleted Bose decomposition containing at least two triples, we now conclude the chapter with the demonstration of Theorem 1.2.

Proof of Theorem 1.2. Let $k \geq 1$, G an abelian group of order $2k+1$, and $\Phi = (id, \phi, id)$ be a permutation triple of G, where id represents the identity permutation and ϕ is the permutation given in Lemma 2.1; note that Φ is proper. We claim that $\mathcal{D}(\mathbb{Z}_{2k+1}, \Phi)$ is a 2-primitive triangle decomposition of K_{6k+3}. Suppose that $\mathcal{S}$ is a subdecomposition of $\mathcal{D}(\mathbb{Z}_{2k+1}, \Phi)$ such that $|\mathcal{S}| \geq 2$. If $\mathcal{S}$ associates to a complete graph, then we have by Lemma 4.7 that there exists $A \subseteq G$ such that $\mathcal{S}$ is columned with respect to A. Otherwise, if $\mathcal{S}$ associates to a complete graph, then we have by Lemma 4.8 that there exists $A \subseteq G$ such that $\mathcal{S}$ is columned with respect to A. By Lemmas 4.11 and 4.12, $A = id(A)$ and $\phi(A)$ are cosets of a nontrivial subgroup of $\mathbb{Z}_{2k+1}$, and hence $A = \mathbb{Z}_{2k+1}$ by Lemma 2.3. Therefore, $V(\mathcal{S}) = A \times \mathbb{Z}_3 = V(\mathcal{C}(\mathbb{Z}_{2k+1}, \Phi))$, and the result follows from Observation 2.4. $\square$

Theorems 1.1 and 1.2, with the necessary conditions in Lemma 2.5, prove Theorem 1.3, completing the classification of 2-primitive triangle decompositions of cocktail party graphs.

CHAPTER 5

FUTURE DIRECTIONS

As mentioned in the introduction, there are several decompositions that have been shown to be primitive in the traditional sense [1, 3, 7]. The concept of t-primitivity allows us to generalize the traditionally studied property of primitivity by rephrasing the question of primitivity as a threshold value, rather than an absolute property of a decomposition. In particular, given an H-decomposition in which H itself can be formed as an induced subgraph, t-primitivity provides a tool for meaningfully classifying the amount of structure in the decomposition. The work of Doyen [4] characterizes the existence of 2-primitive triangle decompositions of K_{6k+1} and K_{6k+3} for all positive integers k, and this paper completes the classification of when 2-primitive triangle decompositions exist for all cocktail party graphs, as summarized in Theorem 1.3. A recent paper by Schroeder [8], utilizing methods similar to those of this paper, classified the existence of 2-primitive C_4-decompositions of cocktail party graphs.

Thus, there are several graphs with decompositions that have been demonstrated to be 1-primitive, and a modest handful that have been shown to be 2-primitive. However, there are many families of graphs for which structured decompositions are known, but it is currently undetermined for which values of t these decompositions are t-primitive. It is unknown whether any decompositions exist that are t-primitive for $t \geq 3$, but not ℓ-primitive for $\ell < t$. In particular, we do not know if a decomposition exists which is 3-primitive, but not 1-primitive or 2-primitive. Thus, determining the primitivity threshold of known decompositions and constructing decompositions with a desired primitivity threshold are open research questions.

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APPENDIX A

IRB APPROVAL LETTER

MARSHALL UNIVERSITY

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Office of Research Integrity

December 5, 2022

Ian Waddell
Department of Mathematics
523 Smith Hall
Marshall University

Dear Ian:

This letter is in response to the submitted thesis abstract entitled *"On 2-Primitive Triangle Decompositions of Cocktail Party Graphs."* After assessing the abstract, it has been deemed not to be human subject research and therefore exempt from oversight of the Marshall University Institutional Review Board (IRB). The Code of Federal Regulations (45CFR46) has set forth the criteria utilized in making t/his determination. Since the information in this study does not involve human subjects as defined in the above referenced instruction, it is not considered human subject research. If there are any changes to the abstract, you provided then you would need to resubmit that information to the Office of Research Integrity for review and a determination.

I appreciate your willingness to submit the abstract for determination. Please feel free to contact the Office of Research Integrity if you have any questions regarding future protocols that may require IRB review.

Sincerely,

Bruce F. Day, ThD, CIP
Director

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