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Research article

Regularity results for solutions of micropolar fluid equations in terms of the pressure

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Abstract: This paper is devoted to investigating regularity criteria for the 3D micropolar fluid equations in terms of pressure in weak Lebesgue space. More precisely, we prove that the weak solution is regular on $(0, T]$ provided that either the norm $\|\pi\|_{L^{\alpha,\infty}(0,T;L^{\beta,\infty}(\mathbb{R}^3))}$ with $\frac{2}{\alpha} + \frac{3}{\beta} = 2$ and $\frac{3}{2} < \beta < \infty$ or $\|\nabla\pi\|_{L^{\alpha,\infty}(0,T;L^{\beta,\infty}(\mathbb{R}^3))}$ with $\frac{2}{\alpha} + \frac{3}{\beta} = 3$ and $1 < \beta < \infty$ is sufficiently small.

Keywords: micropolar fluid flow; weak solution; regularity criterion; weak Lebesgue spaces
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1. Introduction

In this paper, we are concerned with the 3D incompressible micropolar fluid equations in the whole space:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla\pi - \nabla \times \omega = 0, \\ \partial_t \omega - \Delta\omega - \nabla(\nabla \cdot \omega) + 2\omega + (u \cdot \nabla)\omega - \nabla \times u = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \ \omega(x, 0) = \omega_0(x), \end{cases} \tag{1.1}$$

where $u = u(x, t) \in \mathbb{R}^3$, $\omega = \omega(x, t) \in \mathbb{R}^3$ and $\pi = \pi(x, t)$ denote the unknown velocity vector field, the micro-rotational velocity and the unknown scalar pressure of the fluid at the point $(x, t) \in \mathbb{R}^3 \times (0, T)$, respectively, while u_0, ω_0 are given initial data with $\nabla \cdot u_0 = 0$ in the sense of distributions.

The system (1.1) was first studied by Eringen in [12]. It is a special model of microfluids which exhibits the microrotational effects and microrotational inertia and can be viewed as a non-Newtonian fluid. In a physical sense, micropolar fluid may represent fluids that consists of rigid, randomly oriented (or spherical) particles suspended in a viscous medium where the deformation of fluid particles is ignored. It describes many phenomena such as animal blood and certain anisotropic fluids, e.g., liquid crystals which cannot be characterized appropriately by the Navier-Stokes equations. For more detailed background we refer the readers to see [23, 28] and the references therein. Besides their physical applications, micropolar fluid equations are also mathematically significant. The existence of weak solutions was established by Galdi and Rionero in [17]. Yamaguchi [33] obtained the existence of global strong solutions. Yuan [34] established classical Serrin-type regularity criterion which only need the velocity u or its gradient ∇u. Later, many works about regularity criterion of micropolar equations have been proven (see e.g., [7, 9, 13, 32] and the references therein).

When the micro-rotational term ($\omega = 0$) is neglected, the micropolar flows system reduces to the well known Navier-Stokes equations. There has been a lot of progress about the question of wether a solution of the 3D Navier-Stokes equations or the micropolar fluid equations can develop a finite time singularity from smooth initial data with finite energy. For example, Beirão da Veiga [1], Berselli and Galdi [4] and Zhou [35–38] proved the following following regularity criteria

$$\pi \in L^p(0, T; L^q(\mathbb{R}^3)) \text{ with } \frac{2}{p} + \frac{3}{q} = 2, \ \frac{5}{2} \le q \le \infty$$

or

$$\nabla\pi \in L^p(0, T; L^q(\mathbb{R}^3)) \text{ with } \frac{2}{p} + \frac{3}{q} = 3, \ \frac{5}{3} \le q < 3.$$

In [29, 30], Suzuki proved the regularity criteria in the Lorentz space under the assumption for the pressure via the truncation method introduced by Beirão da Veiga [2]; namely, if

$$\|\pi\|_{L^{\alpha,\infty}(0,T;L^{\beta,\infty}(\mathbb{R}^3))} \le \delta, \quad \text{with } \frac{2}{\alpha} + \frac{3}{\beta} = 2 \text{ and } \frac{5}{2} < \beta < \infty$$

or

$$\|\nabla\pi\|_{L^{\alpha,\infty}(0,T;L^{\beta,\infty}(\mathbb{R}^3))} \le \delta, \quad \text{with } \frac{2}{\alpha} + \frac{3}{\beta} = 3 \text{ and } \frac{5}{3} \le \beta < 3,$$

is regular. A natural question proposed by Suzuki in [30] is what may happen about the case $\frac{3}{2} < \beta < \frac{5}{2}$ guarantees the regularity of the leray-Hopf weak solutions. The goal of this paper is to give an answer to the question mentioned above and we recover the result in [29, 30].

Regarding 3D micropolar fluid equations, many interesting results have been obtained (see [24] and the references therein). Dong et al. [11] (see also Yuan [34]) showed that the weak solution becomes regular if the pressure satisfies

$$\pi \in L^q(0, T; L^{p,\infty}(\mathbb{R}^3)), \quad \text{for } \frac{2}{q} + \frac{3}{p} \le 2, \quad \frac{3}{2} < p \le \infty,$$

or

$$\pi \in L^1(0, T; \dot{B}^0_{\infty,\infty}(\mathbb{R}^3)),$$

where $L^{p,\infty}$ and $\dot{B}^0_{\infty,\infty}$ denote weak Lebesgue space and homogeneous Besov space. For more regularity criteria results for the 3D micropolar fluid equations we refer the readers to [6–10, 13–16, 20, 21] and the references therein.

Inspired by the regularity results of the Navier-Stokes equations cited above (see e.g., [26, 27, 29, 30]), this paper is devoted to study the regularity criterion for weak solutions to 3D micropolar equations in weak Lebesgue space. More precisely, it is shown that if the pressure belongs to some weak Lebesgue spaces in both time and spatial directions, then the weak solutions are regular on $[0, T]$. The method presented here may be applicable to similar situations involving other partial differential equations.

Before stating the main result, let us first recall the definitions of the Lorentz spaces and the weak solutions to the (1.1). For the functional space, $L^p(\mathbb{R}^3)$ denotes the usual Lebesgue space of real-valued functions with norm $\|\cdot\|_{L^p}$:

$$\|f\|_{L^p} = \begin{cases} \left(\int_{\mathbb{R}^3} |f(x)|^p\, dx\right)^{\frac{1}{p}}, & \text{for} \quad 1 \leq p < \infty, \\ ess \sup\limits_{x\in\mathbb{R}^3} |f(x)|, & \text{for} \quad p = \infty. \end{cases}$$

To prove Theorem 1.2 we use the theory of weak Lebesgue spaces and introduce the following notations. $L^r_w(\mathbb{R}^3)$ denotes the weak L^r-space which is defined as

$$L^r_w(\mathbb{R}^3) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^3) : \|f\|_{L^r_w} = \sup_{\lambda>0} \lambda \left| \left\{ x \in \mathbb{R}^3 : |f(x)| > \lambda \right\} \right|^{\frac{1}{r}} < \infty \right\}$$

$L^{p,q}(\mathbb{R}^3)$ $(1 \leq p, q \leq \infty)$ denotes the Lorentz space, the norm of which is defined as follows :

$$\begin{aligned} \|f\|_{L^{p,q}} &= \left(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}, \quad \text{for} \quad 1 \leq q < \infty, \\ \|f\|_{L^{p,\infty}} &= \sup_t (t^{\frac{1}{p}} f^*(t)), \quad \text{for} \quad q = \infty, \end{aligned}$$

where

$$f^*(t) = \inf \{\lambda > 0 : m(\lambda, f) \leq t\}, \quad m(\lambda, f) = \left| \left\{ x \in \mathbb{R}^3 : |f(x)| > \lambda \right\} \right|.$$

For $1 < p < \infty$, it is well known that

$$L^{p,p}(\mathbb{R}^3) = L^p(\mathbb{R}^3), \quad L^{p,\infty}(\mathbb{R}^3) = L^r_w(\mathbb{R}^3), \quad L^{p,q_1}(\mathbb{R}^3) \subset L^{p,q_2}(\mathbb{R}^3), \quad \text{for} \quad q_1 \leq q_2.$$

For details, refer to [3] and [31].

Definition 1.1 (weak solutions [17,23]). *Let* $(u_0, \omega_0) \in L^2(\mathbb{R}^3)$ *with* $\nabla\cdot u_0 = 0$ *in the sense of distribution and* $T > 0$*. A measurable function* $(u(x, t), \omega(x, t))$ *on* $\mathbb{R}^3 \times (0, T)$ *is called a weak solution of (1.1) on* $[0, T)$ *if* (u, ω) *satisfies the following properties :*

(i) $(u, \omega) \in L^\infty\left((0, T) ; L^2(\mathbb{R}^3)\right) \cap L^2\left((0, T) ; H^1(\mathbb{R}^3)\right)$;

(ii) $\nabla \cdot u = 0$ *in the sense of distribution;*

(iii) (u, ω) *verifies (1.1) in the sense of distribution.*

Now, our main result reads as follows:

Theorem 1.2. *Let* $(u_0, \omega_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$ *in* $\mathbb{R}^3$. *Suppose that* (u, ω) *is a weak solution of (1.1) in* $(0, T)$. *Then, there exists a constant* $\delta > 0$ *such that* (u, ω) *is a regular solution on* $(0, T]$ *provided that the pressure satisfies one the following conditions :*

$$\|\pi\|_{L^{\alpha,\infty}(0,T;L^{\beta,\infty}(\mathbb{R}^3))} \leq \delta, \quad \text{with } \frac{2}{\alpha} + \frac{3}{\beta} = 2 \text{ and } \frac{3}{2} < \beta < \infty \tag{1.2}$$

or

$$\|\nabla\pi\|_{L^{\alpha,\infty}(0,T;L^{\beta,\infty}(\mathbb{R}^3))} \leq \delta, \quad \text{with } \frac{2}{\alpha} + \frac{3}{\beta} = 3 \text{ and } 1 < \beta < \infty. \tag{1.3}$$

This allows us to obtain the regularity criterion of weak solutions via only the pressure.

Remark 1.1. *If we ignore the influence of the micro-rotational velocity, system (1.1) reduces to the 3D Navier-Stokes equations. Then, the conclusion of Theorem 1.2 holds true for 3D Navier-Stokes equations and we notice that our criterion (1.2) becomes the result of Ji et al. [19] for the Navier-Stokes equations. Therefore, our result can be viewed as as an affirmative answer to a question proposed by Suzuki in [30], Remark 2.4, p.3850.*

We recall the following result according to Dong et al. [11] that will be used in the proof of Theorem 1.2.

Lemma 1.3. *Suppose* $(u_0, \omega_0) \in L^s(\mathbb{R}^3)$, $s > 3$ *with* $\nabla \cdot u_0 = 0$ *in* $\mathbb{R}^3$. *Then, there exists* $T > 0$ *and a unique strong solution* (u, ω) *of the 3D micropolar fluid equations (1.1) such that*

$$(u, \omega) \in (L^\infty \cap C)\left([0, T); L^s(\mathbb{R}^3)\right).$$

Moreover, let $(0, T_0)$ *be the maximal interval such that* (u, ω) *solves (1.1) in* $C\left((0, T_0); L^s(\mathbb{R}^3)\right)$, $s > 3$. *Then, for any* $t \in (0, T_0)$,

$$\|(u, \omega)(\cdot, t)\|_{L^s} \geq \frac{C}{(T_0 - t)^{\frac{s-3}{2s}}}$$

with the constant C independent of T_0 *and s.*

By a strong solution we mean a weak solution (u, ω) such that

$$(u, \omega) \in L^\infty\left((0, T); H^1(\mathbb{R}^3)\right) \cap L^2\left((0, T); H^2(\mathbb{R}^3)\right).$$

It is well-known that strong solution are regular (say, classical) and unique in the class of weak solutions.

Additionally, let us recall the following lemma which can be viewed as the generalization of the Gronwall lemma.

Lemma 1.4 ([5]). *Let* φ *be a measurable positive function defined on the interval* $[0, T]$. *Suppose that there exists* $\epsilon_0 > 0$ *and a constant* $\kappa > 0$ *such that for all* $0 < \epsilon < \epsilon_0$ *and a.e.* $t \in [0, T]$, φ *satisfies the inequality*

$$\frac{d\varphi}{dt} \leq \kappa\lambda^{1-\epsilon}\varphi^{1+2\epsilon},$$

where $0 < \lambda \in L^{1,\infty}(0, T)$ *with*

$$\kappa \|\lambda\|_{L^{1,\infty}(0,T)} < \frac{1}{2}.$$

Then φ *is bounded on* $[0, T]$.

The following lemma will be frequently used when we apply Lemma 1.4.

Lemma 1.5 ([19]). *Assume that the pair (α,β) satisfies $\frac{2}{\beta}+\frac{3}{\alpha}=a$ with $a,\alpha\geq 1$ and $\beta>0$. Then, for every $\kappa\in[0,1]$ and given $b,c_0\geq 1$, there exist $\beta_\kappa>0$ and $\min(\alpha,b)\leq\alpha_\kappa\leq\max(\alpha,b)$ such that*

$$\begin{cases} \frac{2}{\beta_\kappa}+\frac{3}{\alpha_\kappa}=a \\ \frac{\beta_\kappa}{\alpha_\kappa}=\frac{\beta(1-\kappa)}{\alpha}+\frac{\kappa c_0}{b} \end{cases}. \tag{1.4}$$

2. Proof of Theorem 1.2

We are now in a position to prove Theorem 1.2.

Proof. First, we multiply both sides of the Eq $(1.1)_1$ by $u\,|u|^2$, and integrate over $\mathbb{R}^3$. After suitable integration by parts, we obtain

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\|u(\cdot,t)\|_{L^4}^4+\int_{\mathbb{R}^3}|\nabla u|^2|u|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla|u|^2\right|^2dx \\ \leq\;& \left|\int_{\mathbb{R}^3}\nabla\pi\cdot(|u|^2u)dx\right|+\int_{\mathbb{R}^3}|\omega|\,|u|^2\,|\nabla u|\,dx, \end{aligned} \tag{2.1}$$

where we used the following identities due to divergence free condition:

$$\begin{aligned} \int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot|u|^2\,udx &= \frac{1}{4}\int_{\mathbb{R}^3}u\cdot\nabla|u|^4\,dx=0, \\ \int_{\mathbb{R}^3}(\Delta u)\cdot|u|^2\,udx &= -\int_{\mathbb{R}^3}|\nabla u|^2|u|^2\,dx-2\int_{\mathbb{R}^3}|\nabla|u||^2|u|^2\,dx \\ &= -\int_{\mathbb{R}^3}|\nabla u|^2|u|^2\,dx-\frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla|u|^2\right|^2dx, \end{aligned}$$

$$\int_{\mathbb{R}^3}\nabla\times\omega\cdot|u|^2\,udx=-\int_{\mathbb{R}^3}|u|^2\,\omega\cdot\nabla\times udx-\int_{\mathbb{R}^3}\omega\cdot\nabla|u|^2\times udx.$$

Note that

$$|\nabla\times u|\leq|\nabla u|,\quad |\nabla|u||\leq|\nabla u|.$$

Multiplying the second equation of (1.1) by $\omega\,|\omega|^2$, then integrating the resulting equation with respect to x over $\mathbb{R}^3$ and using integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\|\omega(\cdot,t)\|_{L^4}^4+\int_{\mathbb{R}^3}|\nabla\omega|^2|\omega|^2\,dx+\int_{\mathbb{R}^3}\left|\nabla|\omega|^2\right|^2dx \\ &\quad+\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\times\omega|^2|\omega|^2\,dx+2\int_{\mathbb{R}^3}|\omega|^4\,dx \\ =\;& \int_{\mathbb{R}^3}\nabla\times u\cdot|\omega|^2\,\omega dx, \end{aligned} \tag{2.2}$$

where we have used the fact that $\nabla\mathrm{div}\omega=\nabla\times(\nabla\times\omega)+\Delta\omega$ yields

$$-\int_{\mathbb{R}^3}\nabla\mathrm{div}\omega\cdot|\omega|^2\,\omega dx$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^3} (\nabla \times (\nabla \times \omega) + \Delta\omega) \cdot |\omega|^2 \, \omega dx \\
&= \int_{\mathbb{R}^3} |\nabla \times \omega|^2 |\omega|^2 \, dx + \int_{\mathbb{R}^3} \nabla \times \omega \cdot \nabla |\omega|^2 \times \omega dx + \int_{\mathbb{R}^3} |\nabla\omega|^2 |\omega|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^3} \left|\nabla |\omega|^2\right|^2 dx \\
&\geq \int_{\mathbb{R}^3} |\nabla \times \omega|^2 |\omega|^2 \, dx - \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla \times \omega|^2 |\omega|^2 + \left|\nabla |\omega|^2\right|^2) dx + \int_{\mathbb{R}^3} |\nabla\omega|^2 |\omega|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^3} \left|\nabla |\omega|^2\right|^2 dx \\
&= -\frac{1}{2}\int_{\mathbb{R}^3} |\nabla \times \omega|^2 |\omega|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^3} \left|\nabla |\omega|^2\right|^2 dx.
\end{aligned}
$$

Combining (2.1) and (2.2) together, it follows that

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\left(\|u(\cdot,t)\|_{L^4}^4 + \|\omega(\cdot,t)\|_{L^4}^4\right) + \int_{\mathbb{R}^3} |\nabla u|^2 |u|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^3} \left|\nabla |u|^2\right|^2 dx \\
&\quad + \int_{\mathbb{R}^3} |\nabla\omega|^2 |\omega|^2 \, dx + \int_{\mathbb{R}^3} \left|\nabla |\omega|^2\right|^2 dx + 2\int_{\mathbb{R}^3} |\omega|^4 \, dx \\
\leq\ & \left|\int_{\mathbb{R}^3} \nabla\pi \cdot (|u|^2 u) dx\right| + \int_{\mathbb{R}^3} |\omega| \, |u|^2 \, |\nabla u| \, dx + \int_{\mathbb{R}^3} |u| \, |\omega|^2 \, |\nabla\omega| \, dx \\
=\ & A_1 + A_2 + A_3.
\end{aligned} \tag{2.3}
$$

By the Hölder's and Young's inequalities, the first two terms on the right-hand side of (2.3) are bounded by

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |\omega| \, |u|^2 \, |\nabla u| \, dx + \int_{\mathbb{R}^3} |u| \, |\omega|^2 \, |\nabla\omega| \, dx \\
\leq\ & \||\omega||u|\|_{L^2} \, \||u||\nabla u|\|_{L^2} + \||\omega||u|\|_{L^2} \, \||\omega||\nabla\omega|\|_{L^2} \\
\leq\ & \frac{1}{2}\||u||\nabla u|\|_{L^2}^2 + \frac{1}{2}\||\omega||u|\|_{L^2}^2 + \frac{1}{2}\||\omega||\nabla\omega|\|_{L^2}^2 + \frac{1}{2}\||\omega||u|\|_{L^2}^2 \\
\leq\ & \frac{1}{2}\||u||\nabla u|\|_{L^2}^2 + \frac{1}{2}\||\omega||\nabla\omega|\|_{L^2}^2 + \|u\|_{L^4}^2 \, \|\omega\|_{L^4}^2 \\
\leq\ & \frac{1}{2}\||u||\nabla u|\|_{L^2}^2 + \frac{1}{2}\||\omega||\nabla\omega|\|_{L^2}^2 + \frac{1}{2}\left(\|u\|_{L^4}^4 + \|\omega\|_{L^4}^4\right).
\end{aligned} \tag{2.4}
$$

Now, we estimate the terms A_1 under the assumption (1.2) and (1.3).

Case 1. If (1.2) holds, estimating A_1 under the assumption (1.2), using the divergence free condition $(1.1)_3$ after integration by parts and Cauchy-Schwarz inequality results in

$$
\begin{aligned}
A_1 &= \left|\int_{\mathbb{R}^3} \nabla\pi \cdot (|u|^2 u) dx\right| = \left|\int_{\mathbb{R}^3} \pi \cdot \mathrm{div}(|u|^2 u) dx\right| \\
&\leq 2\int_{\mathbb{R}^3} |\pi| \, |u|^2 \, |\nabla u| \, dx \leq 2 \, \|\pi u\|_{L^2} \, \||u||\nabla u|\|_{L^2} \\
&\leq C\int_{\mathbb{R}^3} |\pi|^2 \, |u|^2 \, dx + \frac{1}{2}\||u||\nabla u|\|_{L^2}^2 .
\end{aligned} \tag{2.5}
$$

Let us estimate the integral

$$
I = \int_{\mathbb{R}^3} |\pi|^2 \, |u|^2 \, dx
$$

on the right-hand side of (2.5). Before turning to estimate I, it is well-known that for the micropolar fluid equations in $\mathbb{R}^3$, we have the following relationship between π and u and Calderòn-Zygmund inequality

$$-\Delta\pi = \operatorname{div}(u\cdot\nabla u) = \sum_{i,j=1}^{3}\partial_i\partial_j(u_iu_j),$$
$$\|\pi\|_{L^q} \le C\,\|u\|_{L^{2q}}^2, \quad 1<q<\infty. \tag{2.6}$$

By a simple interpolation argument, we get from (2.6) that

$$\|\pi\|_{L^{q,\sigma}} \le C\,\|u\|_{L^{2q,2\sigma}}^2, \quad \text{for } 1<q<\infty \text{ and } 1<\sigma<\infty. \tag{2.7}$$

Then, we can estimate I as follows

$$\begin{aligned}
I &\le \|\pi\|_{L^{\beta,\infty}}\|\pi\|_{L^{\frac{2\beta}{\beta-1},2}}\left\||u|^2\right\|_{L^{\frac{2\beta}{\beta-1},2}} \le C\,\|\pi\|_{L^{\beta,\infty}}\left\||u|^2\right\|_{L^{\frac{2\beta}{\beta-1},2}}^2\\
&\le C\,\|\pi\|_{L^{\beta,\infty}}\left\||u|^2\right\|_{L^{2,2}}^{2-\frac{3}{\beta}}\left\||u|^2\right\|_{L^{6,2}}^{\frac{3}{\beta}}\\
&\le C\,\|\pi\|_{L^{\beta,\infty}}\left\||u|^2\right\|_{L^{2}}^{2-\frac{3}{\beta}}\left\|\nabla|u|^2\right\|_{L^{2}}^{\frac{3}{\beta}}\\
&\le C\,\|\pi\|_{L^{\beta,\infty}}^{\frac{2\beta}{2\beta-3}}\|u\|_{L^4}^4+\frac{1}{2}\left\|\nabla|u|^2\right\|_{L^2}^2,
\end{aligned}$$

where we have used the following interpolation inequality in the Lorentz spaces (see [3]):

$$\|f\|_{L^{\frac{2\beta}{\beta-1},2}} \le C\,\|f\|_{L^{2,2}}^{1-\frac{3}{2\beta}}\|f\|_{L^{6,2}}^{\frac{3}{2\beta}}.$$

Combining all the estimates from above and considering the facts that $\||u||\nabla u|\|_{L^2} \le \left\|\nabla|u|^2\right\|_{L^2}$ and $\||\omega|\,|\nabla\omega|\|_{L^2} \le \left\|\nabla|\omega|^2\right\|_{L^2}$, it follows that

$$\begin{aligned}
&\frac{d}{dt}\left(\|u(\cdot,t)\|_{L^4}^4+\|\omega(\cdot,t)\|_{L^4}^4\right)+\int_{\mathbb{R}^3}|\nabla u|^2|u|^2\,dx+\int_{\mathbb{R}^3}|\nabla\omega|^2|\omega|^2\,dx+\int_{\mathbb{R}^3}|\omega|^4\,dx\\
\le\;& C\,\|\pi\|_{L^{\beta,\infty}}^{\frac{2\beta}{2\beta-3}}\left(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4\right).
\end{aligned}\tag{2.8}$$

Defining

$$H(t) = \|u(\cdot,t)\|_{L^4}^4+\|\omega(\cdot,t)\|_{L^4}^4,$$

inequality (2.8) implies that

$$\frac{d}{dt}H(t) \le C\,\|\pi\|_{L^{\beta,\infty}}^{\frac{2\beta}{2\beta-3}}H(t). \tag{2.9}$$

Applying Lemma 1.5 (with $a=b=2$, $c_0=4$), we have

$$\begin{aligned}
\|\pi\|_{L^{\beta_\kappa,\infty}}^{\alpha_\kappa} &\le \|\pi\|_{L^{\beta,\infty}}^{\alpha(1-\kappa)}\|\pi\|_{L^{2,\infty}}^{4\kappa} \le C\,\|\pi\|_{L^{\beta,\infty}}^{\alpha(1-\kappa)}\|\pi\|_{L^{2,2}}^{4\kappa}\\
&\le C\,\|\pi\|_{L^{\beta,\infty}}^{\alpha(1-\kappa)}\|u\|_{L^4}^{8\kappa}\\
&\le C\,\|\pi\|_{L^{\beta,\infty}}^{\alpha(1-\kappa)}(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4)^{2\kappa}\\
&\le C\,\|\pi\|_{L^{\beta,\infty}}^{\alpha(1-\kappa)}(H(t))^{2\kappa},
\end{aligned}\tag{2.10}$$

where we have used the following estimate (see [18, 25])

$$\|f\|_{L^{p,q_2}} \le \left(\frac{q_1}{p}\right)^{\frac{1}{q_1}-\frac{1}{q_2}} \|f\|_{L^{p,q_1}}, \quad 1 \le p \le \infty, \quad 1 \le q_1 \le q_2 \le \infty. \tag{2.11}$$

Since the pair $(\alpha_\kappa, \beta_\kappa)$ also meets $\frac{2}{\alpha_\kappa} + \frac{3}{\beta_\kappa} = 2$, using estimate (2.9) and (2.10) yields

$$\begin{aligned}\frac{d}{dt}H(t) &\le C\|\pi\|_{L^{\beta_\kappa,\infty}}^{\frac{2\beta_\kappa}{2\beta_\kappa-3}} H(t) = C\|\pi\|_{L^{\beta_\kappa,\infty}}^{\alpha_\kappa} H(t)\\ &\le C\|\pi\|_{L^{\beta,\infty}}^{\alpha(1-\kappa)} (H(t))^{1+2\kappa}.\end{aligned}$$

Integrating with respect to time, we obtain

$$H(t) \le H(0) + C\int_0^t \|\pi(\cdot,\tau)\|_{L^{\beta,\infty}}^{\alpha(1-\kappa)} (H(\tau)))^{1+2\kappa} d\tau,$$

equivalently

$$\begin{aligned}&\|u(\cdot,t)\|_{L^4}^4 + \|\omega(\cdot,t)\|_{L^4}^4\\ \le\ &\|u_0\|_{L^4}^4 + \|\omega_0\|_{L^4}^4\\ &+C\int_0^t \|\pi(\cdot,\tau)\|_{L^{\beta,\infty}}^{\alpha(1-\kappa)} (\|u(\cdot,\tau)\|_{L^4}^4 + \|\omega(\cdot,\tau)\|_{L^4}^4)^{1+2\kappa} d\tau.\end{aligned} \tag{2.12}$$

Case 2. If (1.3) holds. Let us return to estimate A_1 under the assumption (1.3). From the pressure equations $-\Delta\pi = \mathrm{divdiv}(u \otimes u)$ and the Calderòn-Zygmund Theorem, we know that

$$\|\nabla\pi\|_{L^2(\mathbb{R}^3)} \le C\,\||u|\,|\nabla u|\|_{L^2(\mathbb{R}^3)}.$$

Thus, we can use the Hölder's inequality to estimate A_1 as follows:

$$\begin{aligned}A_1 &= \left|\int_{\mathbb{R}^3} \nabla\pi \cdot (|u|^2 u) dx\right| \le \int_{\mathbb{R}^3} |\nabla\pi|^{\frac12} |\nabla\pi|^{\frac12} |u|^3\, dx\\ &\le \left\||\nabla\pi|^{\frac12}\right\|_{L^4(\mathbb{R}^3)} \left\||\nabla\pi|^{\frac12}\right\|_{L^{2\beta,\infty}(\mathbb{R}^3)} \left\||u|^3\right\|_{L^{\frac{4\beta}{3\beta-2},\frac43}(\mathbb{R}^3)}\\ &\le \|\nabla\pi\|_{L^2(\mathbb{R}^3)}^{\frac12} \|\nabla\pi\|_{L^{\beta,\infty}(\mathbb{R}^3)}^{\frac12} \|u\|_{L^{\frac{12\beta}{3\beta-2},4}(\mathbb{R}^3)}^3\\ &\le C\,\||u|\,|\nabla u|\|_{L^2(\mathbb{R}^3)}^{\frac12+\frac{3}{2\beta}} \|u\|_{L^4(\mathbb{R}^3)}^{3-\frac{3}{\beta}} \|\nabla\pi\|_{L^{\beta,\infty}(\mathbb{R}^3)}^{\frac12}.\end{aligned}$$

Here, we used the following interpolation inequality (see e.g., [3]):

$$\|u\|_{L^{\frac{12q}{3\beta-2},4}(\mathbb{R}^3)} \le C\|u\|_{L^4(\mathbb{R}^3)}^{1-\frac{1}{\beta}} \|u\|_{L^{12,4}(\mathbb{R}^3)}^{\frac{1}{\beta}}. \tag{2.13}$$

and the Sobolev inequality in Lorentz spaces (see e.g., [22]):

$$\|u\|_{L^{12,4}(\mathbb{R}^3)}^2 = \left\||u|^2\right\|_{L^{6,2}(\mathbb{R}^3)} \le C\,\||u|\,|\nabla u|\|_{L^2(\mathbb{R}^3)}. \tag{2.14}$$

Hence, by the Young inequality, we have the following inequality

$$A_1 \leq \frac{1}{8} \left\| |u| \, |\nabla u| \right\|_{L^2(\mathbb{R}^3)}^2 + C \left\| \nabla \pi \right\|_{L^{\beta,\infty}(\mathbb{R}^3)}^{\frac{2\beta}{3(\beta-1)}} \left\| u \right\|_{L^4(\mathbb{R}^3)}^4 .$$

Summing up the above estimates, we easily deduce

$$\begin{aligned} & \frac{d}{dt}\left(\|u(\cdot,t)\|_{L^4}^4 + \|\omega(\cdot,t)\|_{L^4}^4\right) + \int_{\mathbb{R}^3} |\nabla u|^2 |u|^2 \, dx + \int_{\mathbb{R}^3} \left|\nabla |u|^2\right|^2 dx \\ & + \int_{\mathbb{R}^3} |\nabla \omega|^2 |\omega|^2 \, dx + \int_{\mathbb{R}^3} \left|\nabla |\omega|^2\right|^2 dx + 2\int_{\mathbb{R}^3} |\omega|^4 \, dx \\ \leq \; & C \left\| \nabla \pi \right\|_{L^{\beta,\infty}(\mathbb{R}^3)}^{\frac{2\beta}{3(\beta-1)}} \|u\|_{L^4(\mathbb{R}^3)}^4 + \frac{1}{2}\left(\|u\|_{L^4}^4 + \|\omega\|_{L^4}^4\right). \end{aligned} \tag{2.15}$$

Defining

$$H(t) = \|u(\cdot,t)\|_{L^4}^4 + \|\omega(\cdot,t)\|_{L^4}^4 ,$$

which implies that

$$\frac{d}{dt}H(t) \leq C \left\| \nabla \pi \right\|_{L^{\beta,\infty}(\mathbb{R}^3)}^{\frac{2\beta}{3(\beta-1)}} H(t). \tag{2.16}$$

In view of Lemma 1.5, we infer that

$$\|\nabla \pi\|_{L^{\beta_\kappa,\infty}}^{\alpha_\kappa} \leq \|\nabla \pi\|_{L^{\beta,\infty}}^{\alpha(1-\kappa)} \|\nabla \pi\|_{L^{2,\infty}}^{4\kappa} \leq C \|\nabla \pi\|_{L^{\beta,\infty}}^{\alpha(1-\kappa)} \|\nabla \pi\|_{L^2}^{c_1\kappa} , \tag{2.17}$$

where c_1 is determined later.

Notice that $\frac{2}{\alpha_\kappa} + \frac{3}{\beta_\kappa} = 3$. Hence, it follows from (2.16), (2.17) and the Young inequality that

$$\begin{aligned} \frac{d}{dt}H(t) & \leq C \|\nabla \pi\|_{L^{\beta_\kappa,\infty}}^{\alpha_\kappa} H(t) \leq C \|\nabla \pi\|_{L^{\beta,\infty}}^{\alpha(1-\kappa)} \|\nabla \pi\|_{L^2}^{c_1\kappa} H(t) \\ & \leq C \|\nabla \pi\|_{L^{\beta,\infty}}^{\alpha(1-\kappa)} \left\| |u| \, |\nabla u| \right\|_{L^2}^{c_1\kappa} H(t) \\ & \leq C \|\nabla \pi\|_{L^{\beta,\infty}}^{\frac{2\alpha(1-\kappa)}{2-c_1\kappa}} H^{\frac{2}{2-c_1\kappa}}(t) + \frac{1}{8} \left\| |u| \, |\nabla u| \right\|_{L^2}^2 . \end{aligned}$$

Before going further, we take

$$c_1 = \frac{4}{3}, \quad \delta = \frac{(2-c_1)\kappa}{2-c_1\kappa},$$

i.e.,

$$1 + 2\delta = \frac{2}{2-c_1\kappa}, \quad 1 - \delta = \frac{2(1-\kappa)}{2-c_1\kappa},$$

in the last relation. Therefore, we obtain that

$$\frac{d}{dt}H(t) \leq C \|\nabla \pi\|_{L^{\beta,\infty}}^{\alpha(1-\delta)} (H(t))^{1+2\delta} .$$

Integrating with respect to time, we obtain

$$H(t) \leq H(0) + C\int_0^t \|\nabla \pi(\cdot,\tau)\|_{L^{\beta,\infty}}^{\alpha(1-\delta)} (H(\tau)))^{1+2\delta} d\tau,$$

equivalently

$$
\begin{aligned}
& \|u(\cdot,t)\|_{L^4}^4 + \|\omega(\cdot,t)\|_{L^4}^4 \\
\leq\ & \|u_0\|_{L^4}^4 + \|\omega_0\|_{L^4}^4 \\
& + C\int_0^t \|\nabla\pi(\cdot,\tau)\|_{L^{\beta,\infty}}^{\alpha(1-\delta)} (\|u(\cdot,\tau)\|_{L^4}^4 + \|\omega(\cdot,\tau)\|_{L^4}^4)^{1+2\delta} d\tau. \qquad (2.18)
\end{aligned}
$$

Since $\kappa \in [0,1]$, we know that $\delta \in [0,1]$.

Now, we are in a position to complete the proof of Theorem 1.2. From Lemma 1.3, it follows that there exists $T_0 > 0$ and the smooth solution $(\widetilde{u},\widetilde{\omega})$ of (1.1) satisfying

$$
(\widetilde{u},\widetilde{\omega})(t) \in (L^\infty \cap C)([0,T_0); L^4(\mathbb{R}^3)), \quad (\widetilde{u},\widetilde{\omega})(0) = (u_0,\omega_0).
$$

Since the weak solution (u,ω) satisfies the energy inequality, we may apply Serrin's uniqueness criterion to conclude that

$$
(u,\omega) \equiv (\widetilde{u},\widetilde{\omega}) \ \text{ on } \ [0,T_0).
$$

Thus, it is sufficient to show that $T_0 = T$. Suppose that $T_0 < T$. Without loss of generality, we may assume that T_0 is the maximal existence time for $(\widetilde{u},\widetilde{\omega})(t)$. By Lemma 1.3 again, we find that

$$
\|u(\cdot,t)\|_{L^4} + \|\omega(\cdot,t)\|_{L^4} \geq \frac{C}{(T_0-t)^{\frac{1}{8}}} \quad \text{for any } \ t \in (0,T_0). \qquad (2.19)
$$

On the other hand, from (2.12) and (2.18), we know that

$$
\sup_{0\leq t\leq T_0}\left(\|u(\cdot,t)\|_{L^4}^4 + \|\omega(\cdot,t)\|_{L^4}^4\right) \leq C(T,u_0,\omega_0) \qquad (2.20)
$$

which contradicts with (2.19). Thus, $T_0 = T$. This completes the proof of Theorem 1.2. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Prof. Maria Alessandra Ragusa is an editorial board member for AIMS Mathematics and was not involved in the editorial review and the decision to publish this article. The authors declare that they have no competing interests.

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