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Research article

Vertex-edge perfect Roman domination number

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Abstract: A vertex-edge perfect Roman dominating function on a graph $G = (V, E)$ (denoted by ve-PRDF) is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ such that for every edge $uv \in E$, $\max\{f(u), f(v)\} \neq 0$, or u is adjacent to exactly one neighbor w such that $f(w) = 2$, or v is adjacent to exactly one neighbor w such that $f(w) = 2$. The weight of a ve-PRDF on G is the sum $w(f) = \sum_{v\in V} f(v)$. The vertex-edge perfect Roman domination number of G (denoted by $\gamma^p_{veR}(G)$) is the minimum weight of a ve-PRDF on G. In this paper, we first show that vertex-edge perfect Roman dominating is NP-complete for bipartite graphs. Also, for a tree T, we give upper and lower bounds for $\gamma^p_{veR}(T)$ in terms of the order n, l leaves and s support vertices. Lastly, we determine $\gamma^p_{veR}(G)$ for Petersen, cycle and Flower snark graphs.

Keywords: vertex-edge perfect domination number; trees; cycles; Petersen graph; bipartite graph
Mathematics Subject Classification: 05C05, 05C69

1. Introduction

Let $G = (V, E)$ be a graph where V and E denote the set of vertices and the set of edges respectively. The order of G is $|V|$. Two vertices, x and y, in V are adjacent when they are linked by an edge, i.e., $xy \in E$. For $v \in V$, the set $N(v) = \{u : uv \in E\}$ is known as the open neighborhood of a vertex v while the set $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v. The cardinality of the open neighborhood of v is called the degree of v and denoted by $\mathrm{d}(v)$. Two edges are adjacent when they share a common vertex. The length of a path is the number of edges in it. The path of length n is denoted by P_{n+1}. In a tree graph, a leaf is a vertex with degree one and a support vertex is a vertex in the open neighborhood of a leaf. The cycle graph is usually denoted by C_n where n is the order of C_n.

If G is a connected graph and $x, y \in V(G)$, the distance between x and y denoted by $\mathrm{dist}_G(x, y)$, is the length of a shortest path between x and y. We shall omit G and write $\mathrm{dist}(x, y)$ instead of

$\text{dist}_G(x, y)$ if G is known from the context. The diameter of G, denoted by diam(G) is defined by $\text{diam}(G) = \max\{\text{dist}(x, y) : (x, y) \in V \times V\}$. A diametral path of G is a path witnessing diam(G).

A rooted tree is a tree in which a special vertex called the root is distinguished from the other vertices of the tree. Let T be a tree rooted at a vertex r. If $uv \in E(T)$ and $\text{dist}(r, v) < \text{dist}(r, u)$, we say that v is the parent of u and u is a child of v. A double star graph is a tree containing exactly two non-leaf vertices.

A dominating set of G is a subset D of V such that each vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D. The domination number of G denoted by $\gamma(G)$ is the minimum size of a dominating set. The study of domination number has received much attention in the literature and for basic definitions and concepts relating to this subject we refer the reader to [5]. Some variations on domination number are introduced in the literature such as perfect, edge, vertex-edge, Roman, and perfect Roman [4, 6–8, 11–13].

A perfect dominating set of G is a subset S of V such that each vertex $v \in V(G) \setminus S$ satisfies that $| N(v) \cap S | = 1$. The perfect domination number denoted by $\gamma^p(G)$ is the minimum size of a perfect dominating set. An edge dominating set of G is a subset H of E such that each edge in $E \setminus H$ is adjacent to at least one edge in H. The edge domination number of G denoted by $\gamma_e(G)$ is the minimum size of an edge dominating set. A vertex-edge dominating set of G, briefly ve-dominating set, is a subset S of V such that every edge $e \in E$ has an end point in S. The ve-domination number of G denoted by $\gamma_{ve}(G)$ is the minimum size of a ve-dominating set.

A function $f : V(G) \to \{0, 1, 2\}$ on a graph G is called a Roman dominating function denoted by RDF when every vertex v with $f(v) = 0$ is adjacent to at least one vertex u with $f(u) = 2$. The weight of f denoted by $w(f)$ is the sum $\sum_{v \in V(G)} f(v)$. The Roman domination number of G denoted by $\gamma_R(G)$ is the minimum weight of a RDF. The concept of Roman domination is one of the most important variation of domination. There is a large literature that covers this subject, see for example [3]. There are some variations of Roman domination appeared in the literature such as perfect, edge, vertex-edge and perfect Roman {3}-domination [1, 2, 6, 10].

The study of vertex-edge Roman domination was considered by Naresh Kumar and Venkatakrishnan [9, 10]. A vertex-edge Roman dominating function on a graph G denoted by ve-RDF is a function $f : V(G) \to \{0, 1, 2\}$ having the property that for every edge $uv \in E$, either $\max\{f(u), f(v)\} \neq 0$, or there exists $w \in N(u) \cup N(v)$ such that $f(w) = 2$. The vertex-edge Roman domination number of a graph G denoted by $\gamma_{veR}(G)$ is the minimum weight of a ve-RDF, i.e.,

$$\gamma_{veR}(G) = \min\{w(f) : f \text{ is a ve-RDF on } G\}.$$

Our aim in this work is to apply the analogue of perfect domination on ve-RDF and establish the variation vertex-edge perfect Roman dominating as follows.

Definition 1. *A vertex-edge perfect Roman dominating function, denoted by ve-PRDF on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ having the property that for every edge $uv \in E$, $\max\{f(u), f(v)\} \neq 0$, or u is adjacent to exactly one neighbor w such that $f(w) = 2$, or v is adjacent to exactly one neighbor w such that $f(w) = 2$. The weight of a ve-PRDF on G is the sum $w(f) = \sum_{v \in V} f(v)$. The vertex-edge perfect Roman domination number of G denoted by $\gamma^p_{veR}(G)$ is the minimum weight of a ve-PRDF on G.*

If f is a ve-PRDF on G and $H \subseteq G$, we denote the sum $\sum_{v \in H} f(v)$ by $f(H)$. We say that the edge uv is dominated if it satisfies the condition in Definition 1.

It is clear that every vertex-edge perfect Roman dominating function is a vertex-edge Roman dominating function. So, $\gamma_{veR}(G) \leq \gamma^{p}_{veR}(G)$ for every graph G, and every perfect Roman dominating function is a vertex-edge perfect Roman dominating function. So, $\gamma^{p}_{veR}(G) \leq \gamma^{p}_{R}(G)$ for any graph G.

All graphs considered in this work are finite, simple and undirected. This paper is organized as follows. In Section 2, we show that vertex-edge perfect Roman domination is NP-complete for bipartite graphs. In Section 3, we give an upper bound and a lower bound for vertex-edge perfect Roman domination number of trees. In the last section, we determine the vertex-edge perfect Roman domination number of Petersen, cycle and Flower snark graphs.

2. Complexity

In this section we prove that the decision problem associated with vertex-edge perfect Roman domination is NP-complete for bipartite graphs. We give a polynomial time reduction from the well known NP-complete problem, EXACT 3-COVER (X3C). Consider the following decision problems.

Vertex-edge perfect Roman domination (ve-PRD)

Instance: Graph $G = (V, E)$, positive integer $k \leq |V|$.

Question: Does G admit a ve-PRDF of weight at most k?

Exact 3-cover (X3C)

Instance: A set X with $|X| = 3q$, a collection C of 3-element subsets of X.

Question: Does (X, C) have an exact cover? That is, is there a sub-collection $C' \subseteq C$ such that every element of X is contained in exactly one element of C'?

Theorem 1. *ve-PRD is NP-complete for bipartite graphs.*

Proof. It is clear that ve-PRD is in NP class as we can check in polynomial time if a given function $f : V \longrightarrow \{0, 1, 2\}$ is a ve-PRDF of weight at most k. Now we describe a polynomial-time transformation from any instance of X3C to an instance of ve-PRD such that one of them has a solution if and only if the other instance has a solution.

Let $X = \{x_1, x_2, \cdots, x_{3q}\}$ and $C = \{C_1, C_2, \cdots C_t\}$ be an arbitrary instance of X3C. For every $i \in [3q]$, set $P_i := s_i t_i u_i v_i w_i x_i$. Let $O = \cup_{i\in[3q]} P_i$. For every $j \in [t]$, set $Q_j := a_j b_j d_j g_j c_j$. Let $Q = \cup_{j\in[t]} Q_j$. Finally, let G be the graph obtained from the disjoint union of O and Q by adding edges $x_i c_j$ if $x_i \in C_j$ see Figure 1. Set $k = 7q + 2t$.

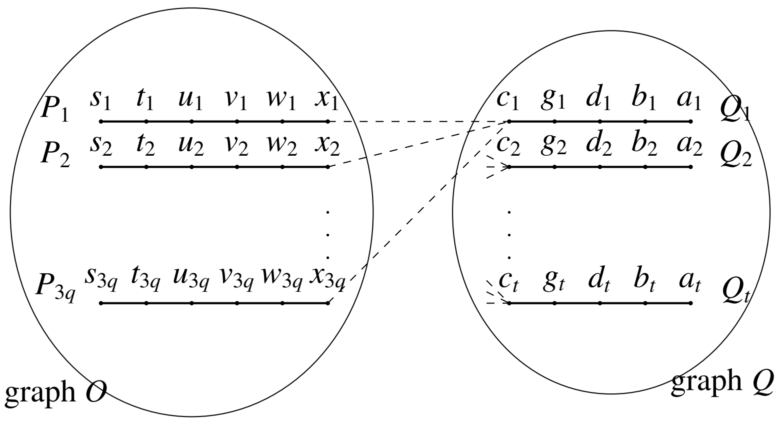

Figure 1. The bipartite graph G.

Assume that (X, C) has a solution C'. Define a function $f : V \longrightarrow \{0, 1, 2\}$ as follows. For every $i \in [3q]$, assign value 2 to u_i and assign value 0 to remaining vertices in P_i. For every $j \in [t]$, if $C_j \in C'$ then assign value 2 to c_j, assign value 1 to b_j and assign value 0 to the remaining vertices of Q_j. If $C_j \notin C'$, assign value 2 to d_j and assign value 0 to the remaining vertices of Q_j. As C' is an exact cover, for every $i \in [3q]$, x_i has exactly one neighbor c_j such that $f(c_j) = 2$. So for every $i \in [3q]$, the edge $w_i x_i$ is dominated and the edges $x_i c_j$ when $C_j \notin C'$ are dominated. It is clear that the remaining edges in G are dominated. Thus, f is a ve-PRDF on G of weight equals to $7q + 2t = k$.

Conversely, assume that G admits a ve-PRDF of weight at most k. Let f be a ve-PRDF on G of a minimum weight. Observe that $f(P_i) \geq 2$, and if $f(P_i) = 2$ then $f(x_i) = f(w_i) = 0$ and $f(v_i) \leq 1$. So, if $f(P_i) = 2$ then x_i has exactly one neighbor c_j such that $f(c_j) = 2$. Observe also that for every $j \in [t]$, $f(Q_j) \geq 2$, and if $f(Q_i) = 2$ then $f(c_j) = 0$. Let $p =\mid \{i \in [3q] : f(P_i) > 2\} \mid$ and $y =\mid \{j \in [t] : f(Q_j) > 2\} \mid$. Then,

$$\begin{aligned} f(G) &\geq 2(3q - p) + 3p + 2(t - y) + 3y \\ &= 6q + p + 2t + y. \end{aligned}$$

As $f(G) \leq k = 7q + 2t$, $q \geq p + y$. On the other hand, $y \geq \frac{3q-p}{3}$ as each c_j has exactly three neighbors in X. Combining those two inequalities we get $p = 0$ and $y = q$. Thus for all $i \in [3q]$, $f(P_i) = 2$ and x_i has exactly one neighbor c_j such that $f(c_j) = 2$. Hence, $C' := \{C_j : f(c_j) = 2\}$ is a solution for (X, C).

□

3. Vertex-edge perfect Roman domination of trees

In this section we prove that if T is a tree of order $n \geq 3$ with l leaves and s support vertices then $\gamma^p_{veR}(T) \leq \frac{n-l+s}{2}$. This bound is tight when $T = P_n$ and n is even. We also prove that if diam$(T) \geq 3$ then $\gamma^p_{veR}(T) \geq \frac{n-l-s+3}{2}$.

Proposition 1. *Let $n \geq 2$. Then $\gamma^p_{veR}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. We proceed by induction on n. It is easy to see that $\gamma^p_{veR}(P_2) = \gamma^p_{veR}(P_3) = 1$. This establishes the base step. Assume that $n \geq 4$. Assume that the statement holds for paths P with $2 \leq |P| < n$. Let w be one of the endpoints of P_n, let x be the unique neighbor of w, let y be the other neighbor of x, let z be the other neighbor of y. Let P_{n-2} be the graph obtained from P_n by deleting w and x. From induction hypothesis, P_{n-2} admits a ve-PRDF f' with $w(f') = \left\lfloor \frac{n-2}{2} \right\rfloor$. Define a function f on P_n as follows. Set $f(w) = 0, f(x) = 1$ and $f = f'$ otherwise. Then, f is a ve-PRDF on P_n with

$$w(f) = w(f') + 1 = \left\lfloor \frac{n-2}{2} \right\rfloor + 1 = \left\lfloor \frac{n}{2} \right\rfloor.$$

Thus $\gamma^p_{veR}(P_n) \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Assume that P_n admits a ve-PRDF g with $w(g) < \left\lfloor \frac{n}{2} \right\rfloor$. Assume that g is of a minimum weight. If $\{g(w), g(x)\} \cap \{1\} \neq \phi$, then the restriction of g on $P_n - \{w, x\}$ is a ve-PRDF on $P_n - \{w, x\}$ of weight less than $\left\lfloor \frac{n-2}{2} \right\rfloor$, a contradiction. If $\{g(w), g(x)\} \cap \{2\} \neq \phi$ then $g(y) = 0$. Define a function g' on $P_n - \{w, x\}$ as follows. Set $g'(y) = 1$ and $g' = g$ otherwise. Then, g' is a ve-PRDF on $P_n - \{w, x\}$ of weight less than $\left\lfloor \frac{n-2}{2} \right\rfloor$, a contradiction. Thus, $g(w) = g(x) = 0$ and $g(y) = 2$. As $w(g)$ is minimum, $g(z) < 2$. Set

$g'(y) = 0, g'(z) = 1$ and $g'(v) = g(v)$ for all $v \in P_n - \{w, x, y, z\}$. Then, g' is a ve-PRDF on $P_n - \{w, x\}$ of weight less than $\left\lfloor \frac{n-2}{2} \right\rfloor$, a contradiction. Hence, $\gamma^p_{veR}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$ as desired. □

Theorem 2. *If T is a tree of order $n \geq 3$ with l leaves and s support vertices then $\gamma^p_{veR}(T) \leq \frac{n-l+s}{2}$.*

Proof. We proceed by induction on n. If $n = 3$ then $T = P_3$ and $\gamma^p_{veR}(P_3) = 1$. So, the statement holds. If $n = 4$ then T is a star or $T = P_4$. If T is a star $\gamma^p_{veR}(T) = 1$. If $T = P_4$, $\gamma^p_{veR}(T) = 2$. So, the statement holds. This establishes the base step. Assume that $|T| \geq 5$ and the statement holds for any tree T' with $3 \leq |T'| < |T|$.

The statement is obvious if $\mathrm{diam}(T) = 2$. Assume that $\mathrm{diam}(T) = 3$, then T is a double star and it is easy to see that the statement holds. Assume that $\mathrm{diam}(T) \geq 4$. Let $v_0 \cdots v_d$ be a diametral path, and if there are multiple diametral paths choose $v_0 \cdots v_d$ so that $\mathrm{d}(v_{d-1})$ is maximum. Then, v_0 and v_d must be leaves and v_{d-1} is a support vertex.

Case 1. $\mathrm{d}(v_{d-1}) \geq 3$. Then, v_{d-1} is adjacent to at least two leaves. Let T' be the the tree obtained from T by deleting v_d. Then, T' has order $n' = n - 1$, with $l' = l - 1$ and $s' = s$. From induction hypothesis, T' admits a ve-PRDF f' such that $w(f') \leq \frac{n'-l'+s'}{2}$. Define a function f on T as follows. If $f'(v_{d-2}) = 2$ or $f'(v_{d-1}) \geq 1$, set $f(v_d) = 0$ and $f(a) = f'(a)$ for all $a \in T - v_d$, if $f'(v_{d-2}) < 2$ and $f'(v_{d-1}) = 0$ then v_{d-1} is adjacent to a leaf x in T' with $f'(x) \geq 1$, set $f(v_{d-1}) = 1$, $f(x) = f(v_d) = 0$ and $f(a) = f'(a)$ for all $a \in T - \{v_{d-1}, v_d, x\}$. Then, f is a ve-PRDF on T of weight

$$w(f) \leq w(f') \leq \frac{n' - l' + s'}{2} = \frac{n - 1 - l + 1 + s}{2} = \frac{n - l + s}{2}.$$

Thus, the statement holds.

Case 2. $\mathrm{d}(v_{d-1}) = 2$.

Case I. $\mathrm{d}(v_{d-2}) = 2$. Let T' be the tree obtained from T by deleting v_{d-1} and v_d. Then, $n' = n - 2$, $l' = l$ and $s' \leq s$. From induction hypothesis, T' admits a ve-PRDF f' such that $w(f') \leq \frac{n'-l'+s'}{2}$. Set $f(v_{d-1}) = 1$, $f(v_d) = 0$ and $f(a) = f'(a)$ for all $a \in T - \{v_{d-1}, v_d\}$. Then, f is a ve-PRDF on T of weight

$$w(f) = w(f') + 1 \leq \frac{n' - l' + s'}{2} + 1 \leq \frac{n - 2 - l + s}{2} + 1 = \frac{n - l + s}{2}.$$

Thus, the statement holds.

Case II. $\mathrm{d}(v_{d-2}) \geq 3$. Let T' be the tree obtained from T by deleting v_{d-1} and v_d. Then, $n' = n - 2$, $l' = l - 1$ and $s' = s - 1$. From induction hypothesis, T' admits a ve-PRDF f' such that $w(f') \leq \frac{n'-l'+s'}{2}$. Set $f(v_{d-1}) = 1$, $f(v_d) = 0$ and $f(a) = f'(a)$ for all $a \in T - \{v_{d-1}, v_d\}$. Then, f is a ve-PRDF on T of weight

$$w(f) = w(f') + 1 \leq \frac{n' - l' + s'}{2} + 1 \leq \frac{n - 2 - l + 1 + s - 1}{2} + 1 = \frac{n - l + s}{2}.$$

Thus, the statement holds. □

The following result is clear as the number of leaves is always greater than or equal to the number of support vertices.

Corollary 1. *If T is a tree of order $n \geq 2$ then $\gamma^p_{veR}(T) \leq \frac{1}{2}n$.*

The next statement gives a lower bound of vertex-edge perfect Roman domination number of trees with diameter greater than or equal to 3.

Theorem 3. *If T is a tree with* $\mathrm{diam}(T) \geq 3$*, l leaves and s support vertices then* $\gamma^p_{veR}(T) \geq \frac{n-l-s+3}{2}$.

Proof. We proceed by induction on n. Since $\mathrm{diam}(T) \geq 3$, $|T| \geq 4$. If $|T| = 4$, $T = P_4$. Then $\gamma^p_{veR}(P_4) = 2 > \frac{3}{2}$. This establishes the base step. Assume that $|T| \geq 5$. Assume that the statement holds for any tree T' with $\mathrm{diam}(T') \geq 3$ and $|T'| < |T|$. Throughout the proof we denote the order, the number of leaves and the number of support vertices of T' by n', l' and s', respectively.

If $\mathrm{diam}(T) = 3$ then T is a double star, so $\gamma^p_{veR}(T) = 2 > \frac{3}{2}$. Thus, the statement holds. So, we can assume that $\mathrm{diam}(T) \geq 4$. Let $v_0 \cdots v_d$ be a diametral path, and if there are more than one candidate then choose $v_0 \cdots v_d$ such that $\mathrm{d}(v_1)$ is maximum. Let f be a ve-PRDF on T of a minimum weight, i.e., $w(f) = \gamma^p_{veR}(T)$.

Claim 1. *If* $\mathrm{d}(v_1) > 2$ *then the statement holds.*

Proof. Since the path $v_0 \cdots v_d$ is a diametral path, v_1 is adjacent to at least two leaves, v_0 and say z. Let T' be the tree obtained from T by deleting z. Then, $n' = n - 1$, $l' = l - 1$ and $s' = s$. If $f(v_1) \geq 1$ then the restriction of f on T' is a ve-PRDF on T', so $w(f) \geq \gamma^p_{veR}(T')$. Assume that $f(v_1) = 0$. Then, $f(z) = 0$. So there exists $y \in N(v_1) \setminus \{z, v_0\}$ such that $f(y) = 2$. Then, the restriction of f on T' is a ve-PRDF on T', so $w(f) \geq \gamma^p_{veR}(T')$. Thus, in all cases we have

$$w(f) \geq \gamma^p_{veR}(T') \geq \frac{n' - l' - s' + 3}{2} = \frac{n - 1 - l + 1 - s + 3}{2} = \frac{n - l - s + 3}{2}.$$

Thus, the statement holds. □

So we may assume that $\mathrm{d}(v_1) = 2$.

Claim 2. *If there exists $i \in \{2, \cdots, d-2\}$ such that v_i is a support vertex in T then the statement holds.*

Proof. Denote the leaf adjacent to v_i in T by x. Let T' be the tree obtained from T by deleting x. Then, $n' = n - 1, l' = l - 1$ and $s' \leq s$. From induction hypothesis, $\gamma^p_{veR}(T') \geq \frac{n'-l'-s'+3}{2}$. If $f(v_i) \geq 1$ then the restriction of f on T' is a ve-PRDF on T', so $w(f) \geq \gamma^p_{veR}(T')$. Assume that $f(v_i) = 0$. Then either $f(x) = a \geq 1$ or there exists $y \in N(v_i) \backslash \{x\}$ such that $f(y) = 2$. If $f(x) = a \geq 1$, define a ve-PRDF f' on T' as follows. Let $f'(v_i) = 1$ and $f' = f$ otherwise, so $w(f) \geq w(f') \geq \gamma^p_{veR}(T')$. If there exists $y \in N(v_i) \backslash \{x\}$ such that $f(y) = 2$ then the restriction of f on T' is a ve-PRDF on T', so $w(f) \geq \gamma^p_{veR}(T')$. Thus, in all cases we have

$$w(f) \geq \gamma^p_{veR}(T') \geq \frac{n' - l' - s' + 3}{2} \geq \frac{n - 1 - l + 1 - s + 3}{2} = \frac{n - l - s + 3}{2}.$$

Thus, the statement holds. □

So, we may assume that the set $\{v_2, \cdots, v_{d-2}\}$ does not contain a support vertex in T. We have two cases.

Case 1. $\mathrm{d}(v_2) > 2$. Since v_2 is not adjacent to any leaf and the path $v_0 \cdots v_d$ is a diametral path, v_2 is adjacent to a support vertex y where $y \notin \{v_1, v_3\}$ and y is adjacent to a leaf x. From the way of choosing the path $v_0 \cdots v_d$, let T' be the tree obtained from T by deleting x and y. Then, $\mathrm{diam}(T') = \mathrm{diam}(T)$,

$n' = n-2, l' = l-1$ and $s' = s-1$. If $f(v_2) \geq 1$ then the restriction of f on T' is a ve-PRDF on T', so $w(f) \geq \gamma^p_{veR}(T')$. Assume that $f(v_2) = 0$, then $f(x) + f(y) \geq 1$. Define a ve-PRDF f' on T' as follows. Let $f'(v_2) = 1$ and $f' = f$ otherwise. So, $w(f) \geq \gamma^p_{veR}(T')$. Therefore, in all cases we have

$$w(f) \geq \gamma^p_{veR}(T') \geq \frac{n' - l' - s' + 3}{2} = \frac{n-2-l+1-s+1+3}{2} = \frac{n-l-s+3}{2}.$$

Thus the statement holds.

Case 2. $d(v_2) = 2$. If $\mathrm{diam}(T) = 4$ then $T = P_5$, and $\gamma^p_{veR}(P_5) = 2 = \frac{n-l-s+3}{2}$. So, we may assume that $\mathrm{diam}(T) \geq 5$. Let T' be the tree obtained from T by deleting v_0 and v_1, so $\mathrm{diam}(T') \geq 3$. Then, $n' = n-2, l' = l$ and $s' = s$ (recall that v_3 is not a support vertex in T). Assume that $f(v_0) = f(v_1) = 0$. So $f(v_2) = 2$; define a ve-PRDF f' on T' as follows. Let $f'(v_2) = 0$, $f'(v_3) = \max\{1, f(v_3)\}$ and $f' = f$ otherwise. Then, $w(f) \geq w(f') + 1 \geq \gamma^p_{veR}(T') + 1$. Assume that $f(v_0) + f(v_1) \geq 1$. Then, either $f(v_2) + f(v_3) \geq 1$ or v_3 is adjacent to a vertex w such that $f(w) = 2$. So the restriction of f on T' is a ve-PRDF on T'. Thus $f(T) \geq f(T') + 1 \geq \gamma^p_{veR}(T') + 1$. Therefore, in all cases we have

$$w(f) \geq \gamma^p_{veR}(T') + 1 \geq \frac{n' - l' - s' + 3}{2} + 1 = \frac{n-2-l-s+3}{2} + 1 = \frac{n-l-s+3}{2}.$$

Thus, the statement holds. □

4. Vertex-edge perfect Roman domination of some well-known graphs

In this section we determine the vertex-edge perfect Roman domination number of Petersen, cycle and Flower snark graphs. The Petersen graph is a well-known graph and it is given in Figure 2. An independent set is a set of vertices in G where no two vertices are adjacent. The independent number of a graph G denoted by $\alpha(G)$ is the cardinality of the largest independent set. It is known that the independent number of Petersen graph is 4. Flower snark graph, which is denoted by J_n can be constructed as following:

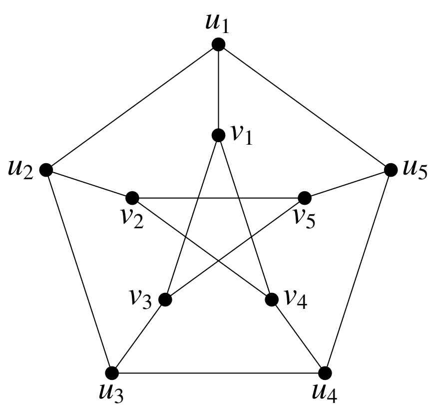

Figure 2. Petersen graph.

(1) For $n \geq 3$, take the union of n copies of $K_{1,3}$.

(2) In the i-th copy of $K_{1,3}$, denote the vertex with degree 3 by x^i and the other three vertices by $w^i y^i, z^i$.

(3) Construct cycle C_n through vertices $w^1, w^2, ..., w^n$ and cycle C_{2n} through vertices $y^1, y^2, ..., y^n, z^1, z^2, ..., z^n$.

(4) Denote the $i-th$ copy of $K_{1,3}$ by J^i and its vertices by x^i, w^i, y^i, z^i.

It is clear that there are n copies of $K_{1,3}$ in J_n, see Figure 3.

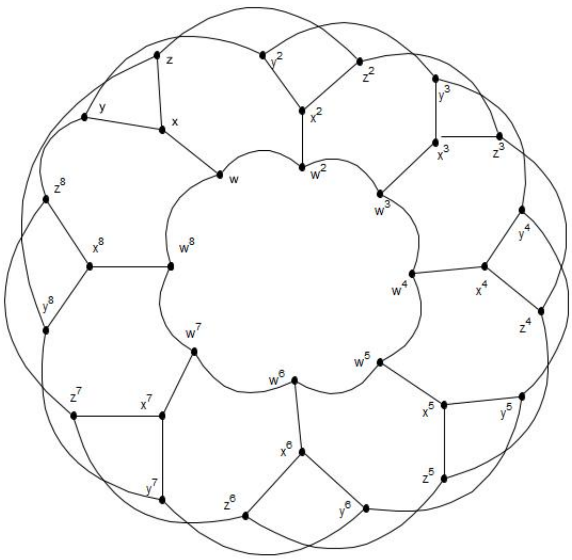

Figure 3. Flower snark graph.

Theorem 4. *The vertex-edge perfect Roman domination number of Petersen graph is* 5.

Proof. Let G be the Petersen graph with vertices labeled as in Figure 2. Set $f(v_1) = 2$, $f(v_2) = f(u_3) = f(u_5) = 1$ and $f = 0$ otherwise, then f is a ve-PRDF on G. Thus, $\gamma^p_{veR}(G) \leq 5$.

Assume that there exists a ve-PRDF f on G such that $w(f) \leq 4$. Let A be the set of vertices x for which $f(x) = 0$, and C be the set of vertices x for which $f(x) = 2$. As $w(f) \leq 4$, $|A| \geq 6$. Since $\alpha(G) = 4$, there exists $y, z \in A$ such that $yz \in E(G)$. Thus, either y or z is adjacent to a vertex w such that $f(w) = 2$. So $|C| \geq 1$ and $|A| \geq 7$. Due to the symmetry of Petersen graph, we can assume that $w = v_1$. Assume there exists a vertex $w' \neq v_1$ such that $f(w') = 2$ (i.e., $|C| = 2$). Since the diameter of G is 2, either $w' \in N(v_1)$ or $\text{dist}(w', v_1) = 2$. Assume that $w' \in N(v_1)$. We may assume that $w' = u_1$, then $f(u_3) = f(u_4) = 0$ and neither u_3 nor u_4 is adjacent to a vertex labeled 2, a contradiction. Thus, $\text{dist}(w', v_1) = 2$. Let x be the unique vertex in $N(v_1) \cap N(w')$. Let $x' \in N(x)\backslash\{v_1, w'\}$ (x' is unique), then $f(x) = f(x') = 0$ and x is adjacent to two vertices labeled 2, a contradiction. Thus, $|C| = 1$.

Let D be the set of vertices at distance 2 from v_1, i. e., $D = \{v_2, v_5, u_2, u_3, u_4, u_5\}$. As $w(f) \leq 4$ and $f(v_1) = 2$, there are at least four vertices in D labeled 0. Since $\alpha(G) = 4$ and $\text{dist}(v_1, v) = 2$ for any $v \in D$, there are two adjacent vertices $y', z' \in D$ such that $f(y') = f(z') = 0$. Then, either y' or z' is adjacent to a vertex $w' \neq v_1$ such that $f(w') = 2$, a contraction. □

Theorem 5. *Let C_n be a cycle graph where $n \geq 3$. Then* $\gamma^p_{veR}(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Proof. The statement is clear when $n = 3,4,5,6$ as $\gamma^p_{veR}(C_3) = 2 = \lceil \frac{n}{2} \rceil$, $\gamma^p_{veR}(C_4) = 2 = \lceil \frac{n}{2} \rceil$, $\gamma^p_{veR}(C_5) = 3 = \lceil \frac{n}{2} \rceil$, and $\gamma^p_{veR}(C_6) = 3 = \lceil \frac{n}{2} \rceil$. For $n \geq 7$, we first show that $\gamma^p_{veR}(C_n) \leq \lceil \frac{n}{2} \rceil$. Let $V(C_n) = \{a_1, \ldots, a_n\}$ and $E(C_n) = \{a_i a_{i+1}, a_1 a_n : 1 \leq i \leq n-1\}$.

Case 1. n is an even number. Let f be as follows.

$$f(a_i) = \begin{cases} 0, & \text{if } i \text{ is odd,} \\ 1, & \text{if } i \text{ is even,} \end{cases}$$

where $1 \leq i \leq n$. Clearly $w(f) = \frac{n}{2}1 + \frac{n}{2}0 = \frac{n}{2} = \lceil \frac{n}{2} \rceil$ as required.

Case 2. n is an odd number. Let f be as follows.

$$f(a_i) = \begin{cases} 0, & \text{if } i \text{ is even,} \\ 1, & \text{if } i \text{ is odd.} \end{cases}$$

Then, $w(f) = 1 + (\frac{n-1}{2})1 + (\frac{n-1}{2})0 = \frac{n-1}{2} + 1 = \frac{n}{2} + \frac{1}{2} = \lceil \frac{n}{2} \rceil$ as required. Therefore, $\gamma^p_{veR}(C_n) \leq \lceil \frac{n}{2} \rceil$.

To Show that $\gamma^p_{veR}(C_n) = \lceil \frac{n}{2} \rceil$, the induction method will be used. Assume $n > 6$ and suppose the statement holds for n' where $n' \leq n-1$. Assume that there exists a ve-PRDF on C_n such that $w(f) < \lceil \frac{n}{2} \rceil$. Hence, there exist two different vertices $x, y \in C_n$ satisfy $f(x), f(y) \neq 0$. Choose $x, y \in C_n$ such that $\operatorname{dist}(x, y)$ is minimum and $f(x), f(y) \neq 0$. There are four cases:

Case I. When $xy \in E(C_n)$. Contract xy and call the new vertex v. Define a function f' on C_{n-1} such that: $f'(v) = max\{f(x), f(y)\}$ and $f'(a_i) = f(a_i)$ for all $a_i \in V(C_{n-1}) \setminus \{v\}$. So,

$$w(f') \leq w(f) - 1 < \left\lceil \frac{n}{2} \right\rceil - 1 = \left\lceil \frac{n-2}{2} \right\rceil \leq \left\lceil \frac{n-1}{2} \right\rceil.$$

That is a contradiction with the hypothesis.

Case II. If $\operatorname{dist}(x, y) = 2$. Assume that the vertex z is between x and y. Contract xzy and call the new vertex v. Define a function f' on C_{n-2} such that: $f'(v) = max\{f(x), f(y)\}$ and $f'(a_i) = f(a_i)$ for all $a_i \in V(C_{n-2}) \setminus \{v\}$. So,

$$w(f') \leq w(f) - 1 < \left\lceil \frac{n}{2} \right\rceil - 1 = \left\lceil \frac{n-2}{2} \right\rceil.$$

That is a contradiction with the hypothesis.

Case III. If $\operatorname{dist}(x, y) = 3$. Thus, either $f(x) = 2$ or $f(y) = 2$. Due to the symmetry of this graph, assume that $f(x) = 2$. The result follows from the following subcases:

Case III-i. If $f(y) = 2$. Assume that the vertices z and z' are between x and y. Contract $xzz'y$ and call the new vertex v. Define a function f' on C_{n-3} such that: $f'(v) = 2$ and $f'(a_i) = f(a_i)$ for all $a_i \in V(C_{n-3}) \setminus \{v\}$. So,

$$w(f') = w(f) - 2 < \left\lceil \frac{n}{2} \right\rceil - 2 = \left\lceil \frac{n-4}{2} \right\rceil \leq \left\lceil \frac{n-3}{2} \right\rceil.$$

That is a contradiction with the hypothesis.

Case III-ii. If $f(y) = 1$. From the way of choosing x and y, there will be a path $xzz'yabc$ such that $f(x) = f(c) = 2$, $f(z) = f(z') = f(a) = f(b) = 0$. Observe that $x \neq c$ as $n > 6$, and $\text{dist}(x, c) > 3$. Contract $xzz'yabc$ and call the new vertex v. Define a function f' on C_{n-6} such that: $f'(v) = 2$ and $f'(a_i) = f(a_i)$ for all $a_i \in V(C_{n-6}) \setminus \{v\}$. So,

$$w(f') = w(f) - 3 < \left\lceil \frac{n}{2} \right\rceil - 3 = \left\lceil \frac{n-6}{2} \right\rceil.$$

That is a contradiction with the hypothesis.

Case IV. If $\text{dist}(x, y) = 4$. Then there will be a path $xzz'z''y$ such that $f(x) = f(y) = 2$, $f(z) = f(z') = f(z'') = 0$. Contract $xzz'z''y$ and call the new vertex v. Define a function f' on C_{n-4} such that: $f'(v) = 2$ and $f'(a_i) = f(a_i)$ for all $a_i \in V(C_{n-4}) \setminus \{v\}$. So,

$$w(f') = w(f) - 2 < \left\lceil \frac{n}{2} \right\rceil - 2 = \left\lceil \frac{n-4}{2} \right\rceil.$$

That is a contradiction with the hypothesis.

This completes the proof.

□

Theorem 6. *Consider a graph J_n, for $n \geq 3$. Then*

$$\gamma^P_{veR}(J_n) = \begin{cases} \frac{3n}{2}, & if\ n\ is\ even, \\ \frac{3(n+1)}{2}, & if\ n\ is\ odd. \end{cases}$$

Proof. We split the problem into the following two cases.

Case 1. When n is even.

Define a function $f : V(J_n) \to \{0, 1, 2\}$ as follows:

$$f(x^i) = \begin{cases} 2, & i\ odd, \\ 1, & i\ even. \end{cases}$$

Also, let $f(w^i) = f(y^i) = f(z^i) = 0, 1 \leq i \leq n$. With the above labeling, the edges in each J^i is ve-dominated by $f(x^i)$ as well as the edges between J^i and J^{i+1}, J^{i-1} and J^i whenever $f(x^i) = 2$. Also, $f(x^i)$ ve-dominates the edges in J^i whenever $f(x^i) = 1$.

Thus, from the above labeling, we have

$$w(f) = 2\left(\frac{n}{2}\right) + 1\left(\frac{n}{2}\right) = \frac{3n}{2}.$$

Case 2. When n is odd.

Define a function $f : V(J_n) \to 0, 1, 2$ as follows:

$$f(x^i) = \begin{cases} 2, & i\ odd,\ i < n, \\ 1, & i\ even, \\ 0, & i = n. \end{cases}$$

Let $f(w^i) = f(y^i) = f(z^i) = 0, 1 \leq i \leq n-1$ and $f(w^i) = f(y^i) = f(z^i) = 1, i = n$. It is easy to see from the above labeling that all edges in J_n are *ve*-perfect Roman dominates. Thus, we have

$$w(f) = 2\left(\frac{n-1}{2}\right) + \frac{n-1}{2} + 3 = n - 1 + \frac{n-1}{2} + 3 = \frac{3n+3}{2} = 3\left(\frac{n+1}{2}\right).$$

From above, we know $\gamma^P_{veR}(J_n) \leq \frac{3n}{2}$ for n even and $\gamma^P_{veR}(J_n) \leq \frac{3(n+1)}{2}$ for n odd. To show that $\gamma^P_{veR}(J_n) = \frac{3n}{2}$ for n even and $\gamma^P_{veR}(J_n) = \frac{3(n+1)}{2}$ for n odd, then we consider $\gamma^P_{veR}(J_n) \geq \frac{3n}{2}$ for n even and $\gamma^P_{veR}(J_n) \geq \frac{3(n+1)}{2}$ for n odd. To do this we assume that $\gamma^P_{veR}(J_n) < \frac{3n}{2}$ for n even and $\gamma^P_{veR}(J_n) < \frac{3(n+1)}{2}$ for n odd. Then, we have the following subcases:

Subcase 1. n even.

If $f(x^i) < 2$, for i odd, then the edges between J^{i-1} and J^i will not be *ve*-perfect Roman dominated. If $f(x^i) < 1$, for i even, the edges in J^i will not be *ve*-perfect Roman dominated. Hence, $\gamma^P_{veR}(J_n) \geq \frac{3n}{2}$. Therefore, $\gamma^P_{veR}(J_n) = \frac{3n}{2}$

Subcase 2. n odd.

If $f(x^i) < 2$, for i odd or $f(x^i) < 1$, for i even, then subcase 1 above applies. Also, if for $i = n$, $f(w^i) < 1$ or $f(y^i) < 1$ or $f(z^i) < 1$, then the edges between J^{n-1} and J^n will not be *ve*-perfect Roman dominated. Therefore, $\gamma^P_{veR}(J_n) \geq \frac{3(n+1)}{2}$. Hence $\gamma^P_{veR}(J_n) = \frac{3(n+1)}{2}$. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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