“Valuating the capital structure under incomplete information”

AUTHORS	Dong Meng Ren Yunmin Chen Alex Maynard Sergiy Pysarenko

ARTICLE INFO	Dong Meng Ren, Yunmin Chen, Alex Maynard and Sergiy Pysarenko (2023). Valuating the capital structure under incomplete information. *Investment Management and Financial Innovations*, *20*(3), 48-67. doi:10.21511/imfi.20(3).2023.05
DOI	http://dx.doi.org/10.21511/imfi.20(3).2023.05
RELEASED ON	Tuesday, 18 July 2023
RECEIVED ON	Sunday, 04 June 2023
ACCEPTED ON	Wednesday, 05 July 2023

JOURNAL	"Investment Management and Financial Innovations"
ISSN PRINT	1810-4967
ISSN ONLINE	1812-9358
PUBLISHER	LLC “Consulting Publishing Company “Business Perspectives”
FOUNDER	LLC “Consulting Publishing Company “Business Perspectives”

NUMBER OF REFERENCES	NUMBER OF FIGURES	NUMBER OF TABLES
61	6	0

BUSINESS PERSPECTIVES

LLC "CPC "Business Perspectives"
Hryhorii Skovoroda lane, 10,
Sumy, 40022, Ukraine

www.businessperspectives.org

Dong Meng Ren (Canada), **Yunmin Chen** (Taiwan), **Alex Maynard** (Canada), **Sergiy Pysarenko** (Canada)

VALUATING THE CAPITAL STRUCTURE UNDER INCOMPLETE INFORMATION

Received on: 4th of June, 2023
Accepted on: 5th of July, 2023
Published on: 18th of July, 2023

Dong Meng Ren, Doctor, Cape Breton University, Canada. (Corresponding author)

Yunmin Chen, Doctor, National Chiayi University, Taiwan.

Alex Maynard, Doctor, University of Guelph, Canada.

Sergiy Pysarenko, Doctor, Cape Breton University, University of Guelph, Canada.

Conflict of interest statement:
Author(s) reported no conflict of interest

Abstract

Can higher uncertainty increase the valuation (market-to-book value) of young firms compared to more established ones? As the current market shows higher levels of uncertainty about companies' expected cash flows and changes in firm value, the question of the fundamental convex relationship between the two becomes more relevant. This paper aims to study how cash flow uncertainty affects the capital structure/leverage of a firm over time. A simple Bayesian learning framework is employed to assess leverage ratios in the presence of parameter uncertainty about expected cash flow. This study provides an analytical solution for leverage as a function of firm age and explores the implications using numerical results. The model links market leverage with expected cash flow volatility and firm age. Young firms face uncertainty about their expected cash flows and hence their firm value. Managers continuously update their evaluation of leverage ratios when they observe realized cash flow until firms reach maturity. Therefore, the paper provides a novel explanation of why the leverage ratio for many start-ups increases over time: the resolution of uncertainty decreases upside shock expectations as the firm ages. This result is useful both for academics, who can test the formulas derived in this paper for various industries, countries, and conditions, and for practitioners, who can use them to calibrate algorithmic trading models when linking uncertainty and firm valuation.

Keywords leverage ratio, Bayesian learning, life cycle, cash flow, parameter uncertainty

JEL Classification G12, G13, G31, G32

INTRODUCTION

The level of uncertainty about future cash flows for new firms is typically high even in "good times". Brexit, COVID-19 pandemic, war in Ukraine, increase in inflation and interest rates all showcase that the time of great moderation is long gone, and uncertainty is the new norm. Higher overall levels of uncertainty increase the uncertainty of cash flows for individual firms even more. This paper introduces Bayesian uncertainty into the structural continuous time trade-off model of debts (Merton, 1974; Black & Cox, 1976; Duffie & Lando, 2001; Morellec, 2005; Streblaev, 2007). It allows us to trace out the implication for the behavior of the leverage ratio over the firm life cycle. The model provides a new explanation for why leverage ratios increase with firm age, which emphasizes the evolution of firm value across the life cycle.

Much of previous research has been devoted to modeling parameter uncertainty in asset pricing. In almost all its subfields, such as return predictability, equity premium, options markets, credit spreads, the term structure of interest rates, and long-run risk, Bayesian learning theory is utilized actively, with contributions, including Pastor and Veronesi (2003), Brennan and Xia (2001), Xia (2001), Dufrense and Goldstein (2001), Cremers and Yan (2016). Recently, there has been a growing interest in incorporating uncertainty into corporate finance. Duffie and Lando (2001) study the effect of uncertain accounting information on firms' credit spreads. Liu et al. (2017) have applied Bayesian learning in a two-state dynamic model with low and high cash flow states. They find that both optimal debt and leverage ratios are higher in the presence of uncertainty.

The aim of this paper is to incorporate Bayesian learning into a dynamic model of capital structure with a continuum of firm types. The model presented in this paper provides an alternative complementary explanation for why firm leverage may rise with firm age.[1] Hence, it incorporates the insights of Pastor and Veronesi (2003) and Ju and Ou (2006) into a model of corporate leverage. Although uncertainty is usually associated with risks that lower firm value, Pastor and Veronesi (2003) show that right-tailed uncertainty can have a positive effect on firm value. This is because downside risks to investors are normally capped by bankruptcy protection, whereas upside potential is virtually unlimited. To capture the asymmetry, this paper models the unknown expected cash flow via a lognormal distribution. Then, as this right-tailed uncertainty is resolved, the effect on firm leverage is studied. The model shows the dynamics of the corporate leverage ratio across the firm life stages. Specifically, the aim of this paper is to prove that uncertainty can increase the valuation of a firm and to model how uncertainty resolution explains an increase in leverage ratios increase with firm age.

1. THEORETICAL BASIS

This paper fits into a voluminous prior study on both firm life cycle theory and leverage. One strand of literature mainly concerns the research on firm life cycle. Since the 1970s, much research on firm life stage has been done in order to define and categorize firm stages in the field of organization, including Bulan and Yan (2010) and Adizes (1999). Another strand of research focuses purely on the determinants of capital structure without linking them to the financial life cycle directly. Four factors have been identified as the key drivers of capital structure: firm size, the market-to-book ratio, profitability and tangibility (Titman & Wessels, 1988; Rajan & Zingales, 1995; Mackay & Phillips, 2005; Lemmon et al., 2008). In particular, these studies show that profitability is inversely related to the leverage ratio in line with the pecking order theory. On the other hand, Frank and Goyal (2009) compare empirical tests of the pecking order model against the trade-off model, finding evidence of mean reversion in the leverage ratio as predicted by the trade-off model.

Yet another related stream of research studies the theoretical evolution of firm leverage over firm age under the guidance of classic capital structure theories, pecking order, trade-off theory, or agency cost theory.2 To test these implied relations between leverage and life cycle, many novel empirical methods have been developed. There is also insightful empirical literature of financial life cycles. Miller and Friesen (1984) put forward five distinct firm life cycle stages, and Dickinson (2011) provides empirical evidence that profitability and growth indeed vary across firm life stage. Many subsequent papers utilize a firm's fundamentals to construct proxies for firm life stages. These proxies include investments (Wernerfelt, 1985), product efficiency (Spence, 1977), retained earnings (DeAngelo et al., 2006; Kim & Suh, 2009), and joint factors (Dickinson, 2011).

This paper also highlights the role of cash flow volatilities in corporate finance. Larrain and Yogo (2008) find that expected changes in a future cash flow explain much of the variation in firm value, supporting the key role of cash flow volatility on firm value in the current model. Recently, Dudley and James (2015) argue that the role of cash flow volatility on firm leverage is prominent, especially when firms are financially constrained. They find that debts are issued in response to the low volatility of constrained firms, but that rising volatility causes them to have difficulties in diminishing

1 The leverage ratio is defined as debt over debt plus equity and can be re expressed as debt book ratio divided relative to firm value, by the sum of the debt book ratio and the market book ratio. In the article, market book value ratio is abbreviated as market book ratio, the same as debt book value ratio.

2 See Kim and Suh (2009), Castro, Tascon and Tapia (2014), and DeHan (2014).

their debt ratios. Bradley, Jarrell and Kim (1984) show that volatility of cash flow or firm earnings negatively correlates with firm leverage, implicitly indicating lower leverage for younger firms. Also, Whited and Riddick (2009) and DeAngelo et al. (2011) argue that cash flow volatility enables firms to meet their need to maintain future financial flexibility or reduce financial tightness to cope with financial uncertainty. Gorbenko and Strebulaev (2010) suggest that permanent cash flow volatility is more important than temporary volatility in impacting leverage ratios. However, their empirical effect on corporate leverage ratios is small. The current paper enriches understanding of this strand of literature by elaborating on the relation between leverage ratios, mean cash flow volatility, and firm age from the perspective of Bayesian learning.

This paper fills a void in the corporate finance literature in two ways. First, it provides a novel explanation for why the leverage ratio experiences a rise from firm infancy to its maturity. In this way, the presented model complements classic capital structure theory, such as the pecking order, the trade-off theory, and the agency cost theory, while providing an additional novel explanation for the dynamics of firm leverage ratios across age using Bayesian updating. Secondly, it bridges the gap between corporate finance and models of Bayesian learning using the dynamic continuous trade-off model. Bayesian learning has been used widely in the asset pricing literature but has been underutilized in corporate finance.

Liu et al. (2017) is the only paper known to other authors that applies Bayesian learning into dynamic capital structure modeling. The similarities between the presented model and their model are that investors and mangers have same information on future cash flow and that they use Bayesian updating to learn about the unknown cash flow process, and finally that both approaches are aiming at an optimal capital structure. However, their paper differs from this one in several key respects. First, they do not employ a stochastic discount factor. Secondly, they do not study the evolution of firm leverage across firm age. Thirdly, and most importantly, their model allows for only two states. Consequently, their model does not allow for extreme outcomes, so convexity is not captured in their model. Liu et al. (2017) is quite new and published in 2017. The initial version of the paper is Ren's 2016 dissertation and is enhanced with the effect of uncertainty of future cash flow on leverage. By contrast, this paper allows a continuum of firm types, allowing researchers to model investor learning about right-tailed uncertainty and incorporate it into a model of firm leverage. As a result, the two papers make related but distinct contributions.

The way this paper models right-tailed uncertainty is closely related to and motivated by Pastor and Veronesi (2003) who propose a parsimonious model to evaluate the market value of young firms' stocks by means of market-to-book ratios. However, their model only allows for equity financing and is thus silent on the implications for firm leverage. This paper provides a non-trivial extension of their model to characterize the effect of uncertainty of mean log cash flow on leverage ratios through its impact on market equity. In addition, this paper extends Ju and Ou (2006) to include parameter uncertainty into his model. In an extended framework of their model, the current model allows for the effect of mean cash flow on both market equity and debt value.

The current model is also motivated by empirical findings of Welch (2004), which suggest that stock returns solely determine cross sectional changes in leverage ratios and that all other firm specific factors, even if they have some explanatory powers on leverage ratios, will only propagate their influences through stock returns. Their paper concludes that leverage ratio valuation is made mainly in response to the changes in firms' equity value. Similarly, in the current model, uncertainty affects leverage through its effect on firm equity value.

This paper is also related to Lemmon et al. (2008) and Graham et al. (2014), although they do not consider learning in their model. In particular, Lemmon et al. (2008) find that after their initial IPOs, firms' leverage ratios increase gradually. The current paper provides a Bayesian explanation of why there is such an increase in leverage ratios after an initial IPO. Moreover, the empirical analysis of the current paper uses their identified firm-specific determinants as control variates and partly validates some of their empirical results.

http://dx.doi.org/10.21511/imfi.20(3).2023.05

The explanation provided in this paper sheds new light on the behavior of leverage over firm life cycle. Its focus on incomplete information complements the predictions of the pecking order and agency cost models that focus instead on asymmetric information. Beginning with trade-off theory as considered by Modigliani and Miller (1963), and agency costs (see Jensen & Meckling, 1976), their proponents hold that a firm's leverage is determined with an aim to strike a balance between benefits and costs of debts and therefore the firm needs to establish a target leverage ratio. Introduced subsequently, the pecking order model implies that due to adverse selection, the firm's financing follows an order: internal funding comes first, and then debt is preferred to equity to raise external funding (see Myers & Majluf, 1984). Thus, the pecking order theory implies a positive relationship between a firm's leverage and its life stage. Alternatively, the static trade-off theory holds that firms in infancy cannot raise more debt due to high bankruptcy costs, whereas higher earnings in their stable stage make debt financing affordable and beneficial. Agency cost theory suggests that as firms proceed to the mature stage, more debt will be used to reduce agency costs. Agency cost theory predicts that a firm's leverage ratio should follow an inverted U shape over its life cycle (see Castro et al., 2014; DeHan, 2014; Frielinghaus et al., 2005). A common aspect of these theories is their focus on managerial decisions on the issuance of new debt and equity. However, the leverage ratio is also driven by firm equity valuation, which enters its denominator. The current model complements the existing literature by studying the impact of learning on leverage that channels through equity valuation. Similar to pecking and agency cost theory, this model also predicts an initial fall in firm leverage with firm age, but due to a different causal mechanism.

In reality, it is reasonable to expect all of these effects to be present. The relative importance of cash flow uncertainty may be greatest for startups and growth firms, which have intangible assets with high growth potentials, but also high uncertainty. This paper also suggests a conjecture that cash flow (and hence firm value) uncertainty plays a more important role for high-tech and growth firms, which is supported by existing literature, which finds that these firms rely more heavily on equity financing. Opler and Titman (2001) suggest that firms should use relatively more debt to finance assets in place and relatively more equity to finance growth opportunities, which is supported by the view of Damodaran (2001) that high-growth firms would prefer equity financing, while mature firms favor debt finance. Morgan and Abetti (2004) argue that high-tech firms will finance with more equity than debt. Graham (2000) concludes that firms with large future growth opportunities, presumably at introduction or growth stages, incline to display low leverage.

2. ASSUMPTIONS

To provide a tractable and interpretable model of the median firm across its life cycle, the same principle of parsimony as suggested by Pastor and Veronesi (2003) is applied. In the current framework, uncertainty about firm prospects is modeled via parameter uncertainty rather than firm heterogeneity (see Morellec et al., 2013). A single representative agent learns about parameter uncertainty regarding the cash flow process for one single representative firm. This results in both a parsimonious model and a tractable analytical solution. The advantages of such an approach are explained at length in the survey by Pastor and Veronesi (2009). The novelty of the presented model is to incorporate the key assumption of expected cash flow uncertainty along the lines of Pastor and Veronesi (2009). The presented set up is otherwise standard and widely used and allows obtaining familiar model predictions about the leverage ratio when shutting down the expected cash flow uncertainty.

To provide a clear basis of comparison, first a detailed analysis of a firm's value and leverage ratios is derived. To maintain tractability, it is assumed that there are no transaction costs incurred in the business operation in this section, and this study models firm debt exogenously, since, in practice, firms do not rebalance their leverage ratios frequently. The representative firm lives for T periods, where T is finite and exogenous. In particular, a firm's cash flow is assumed not to hit default or restructuring boundaries. In each period, the firm issues a constant coupon C and a flat term structure with constant interest rate r is assumed.

Investors and borrowers finance their projects with this interest rate. Net operation incomes are used to reinvest in the firm's operation and to pay taxes. The firm's log cash flow follows a stochastic process. There are no default, restructuring, or bankruptcy costs. Following Pastor and Veronesi (2003), a young firm's cash flow will continue for a relatively long period so that expected cash flows can be used to repay debts without a probability of default. Also, market structure in this paper is not in an equilibrium. Infrequent balancing of financial structure due to transaction costs suggested by Strebulaev (2007) and changes in capital structure mainly attributed to variations in stock returns (Welch, 2004) motivates this assumption.

Firm cash flow, denoted by X_t, is defined here as EBIT, the instantaneous earnings before income and tax. A firms' log cash flow is modeled using the following mean reverting process

$$d\log\left(X_t\right)=\varphi\left(\bar{u}-\log\left(X_t\right)\right)dt+ \\ +\sigma_{x,1}dZ_1+\sigma_{x,2}dZ_2, \qquad (1)$$

where $\sigma_{x,1}$ is the standard variance of log cash flow driven by systematic shocks dZ_1, and $\sigma_{x,2}$ is the standard deviation of log cash flow driven by idiosyncratic shocks $dZ_{2,}\cdot\phi$ is the parameter controlling the mean reversion speed.

Two information environments are considered. To provide a point of comparison, the case in which the mean of log cash flow $\bar{u}$ is known is considered first. Under this full information assumption, which is a standard assumption in the traditional dynamic trade off model (see Broadie et al., 2007), the proposed model reproduces familiar predictions regarding firm leverage ratios. Then, this is compared to a more realistic assumption under which managers, acting in the interests of stockholders, cannot observe the mean of log cash flow $\bar{u}$. Instead, $\bar{u}$ is assumed to be pre-distributed as $N(\hat{u}_0, \hat{\sigma_0})$. Under this assumption, managers or investors can only learn about $\bar{u}$ through the firms' past realized cash flow.[3] The logged cash flow is denoted by $x_t = \log(X_t)$.

A firms' total levered value at time t is given by

$$V_t = EQ_t + D_t, \text{ where}$$

$$EQ_t=\begin{pmatrix} E_t\int_t^T\dfrac{\pi_w\left(1-\tau\right)\left(\delta X_w-C\right)}{\pi_t}dw+ \\ +\left[\dfrac{\pi_T\left(1-\tau\right)\left(\delta X_T-C\right)}{\pi_t}\right]\end{pmatrix} \text{ and} \qquad (2)$$

$$D_t=\left(E_t\int_t^T\pi_w/\pi_t\tau Cdw+\pi_T/\pi_t\tau C\right),$$

conditional on the information set F_t: $\{H_t = (v_t, \log(X_t))\}$, with $v_t = \log X_t\pi_t$.

The first term in EQ_t is the equity value or the value stemming from net operating income subtracting debt payment, while the second term is the firm's terminal equity value at time T. Debt financing enhances firm value by providing a tax deduction. The first term in Dt denotes the present value of tax savings associated with the firm's tax-deductible debt repayments and the second term is the tax savings due to the debt's terminal value. π_t is the stochastic discount factor, δ represents the value added by firm managers, and τ refers to corporate income tax rate. The coupon C is exogenous and constant.[4]

The stochastic discount factor is assumed to follow the log normal diffusion process:

$$d\pi_t=-\pi_t rdt-\pi_t\sigma_{\pi,1}dZ_1, \qquad (3)$$

in which the dynamics of π_t are driven by the systematic shock dZ_1. In this set-up, $\sigma_{x,1}$, $\sigma_{x,2}$, $\sigma_{\pi,1}$, r, τ, τ_i, C are all scalars, which do not vary with time.

3 Investors are uncertain about the expected cash flows of young firms which are denoted by $\bar{u}$. It also represents the investor's uncertainty about the prospects of young firms. Investors can obtain prior information about expected cash flow from past cross-sectional experience, where some young firms become very successful and others do not. Given this prior information of expected cash flow, investors can form their posterior inference about expected cash flow.

4 The parsimonious model abstracts from the long-run steady growth level of cash flow by assuming it is zero. An upward growth trend for firm cash flow would imply a higher initial present discount cash flow value and hence a higher firm value that would be equivalent to a higher mean cash flow in a model without cash flow growth. Provided that investors discount future consumption, trend growth would also imply an expected growth in firm value in the absence of the Bayesian learning captured by the presented model. Intuitively, firms would be expected to grow at the stochastic interest rate adjusted by a risk premium. This would imply a tendency of firm value to grow with firm age, which is the opposite prediction provided by the Bayesian learning effect. To be more precise, this paper effectively models deviations from expected firm value relative to the long-term trend. The Bayesian learning effects of the current model are most heavily pronounced during the first several years of a firm's life, during which time substantial cash flow uncertainty is resolved. Within this period the drop in expected firm value due to the resolution of uncertainty is likely to dominate the upward trend in firm value that would otherwise be observed.

http://dx.doi.org/10.21511/imfi.20(3).2023.05

There are two features distinguishing this model from the dynamic continuous time trade off model in the previous literature on capital finance. The first different feature is the assumption of parameter uncertainty due to mean log cash flow. This is contrary to the known mean cash flow assumption in most continuous time capital structure literature. The second feature distinguishing this model from previous continuous time trade off models is that it employs a stochastic discount factor, instead of a constant discount factor. The stochastic discount factor also follows a stochastic diffusion process.

To maintain analytic tractability, leverage ratios in a non-optimal environment are evaluated. That is, this model does not derive the optimal leverage ratio.[5] Instead, the debt-to-book ratio is modeled exogenously, as an approximately constant ratio, because this ratio is infrequently rebalanced in practice. Also, it is assumed that C is an exogenous constant, which results in a value of D that is relatively fixed over time. This is an empirically realistic simplifying assumption that enables derivation of closed form solutions both with and without learning. As a result, this model produces testable implied leverage ratios which are not optimally derived. This is supported by Myers (1984), who holds that any empirical tests of financial leverage ratios should be specified to see whether their variations are ascribed to different optimal values or due to the dispersion of actually observed ratios from optimal leverage ratios, as also suggested by Strebulaev (2007). He argues that in most cases, the leverage ratios of most firms deviate from optimal leverage in a dynamic model with frictions. (See Strebulaev (2007), in any cross section or panel data regression analysis, firms are not at the phase of optimizing their financial policy).

3. RESULTS

3.1. A simple model with no uncertainty about cash flow

As a basis of comparison, this section provides an analysis of leverage ratios without learning, assuming that the mean log cash flow is known. First, closed-form solutions for the equity value, debt value and the leverage ratio are derived. Then, comparative statics is performed to assess the impact of mean log cash flow, current cash flow, systematic and idiosyncratic shocks to log cash flow, the stochastic discount factor, and the tax rate on equity value, the debt value and the leverage ratio respectively. In this model, firm book value, B_t is normalized to one, since B_t will be canceled as it appears in both numerator and denominator of the leverage formula.

Proposition 1. *Levered Firm Value V_t takes the following form, which includes two parts: firm equity EQ_t and debt D_t.*

$$V_t = EQ_t + D_t, \tag{4}$$

where $\varepsilon = T - t$ denotes the time to maturity. The leverage ratio at time t is given by

$$Leverage_t = \frac{D_t / B_t}{EQ_t / B_t + D_t / B_t}. \tag{5}$$

The equity value is given by

$$\begin{aligned} EQ_t = {} & \int_0^{\varepsilon} \delta(1-\tau) A_1(\bar{u}, x_t, k)\,dk - \\ & - \int_0^{\varepsilon} C(1-\tau) A_2(k)\,dk + \\ & + \delta(1-\tau) A_1(\bar{u}, x_t, \varepsilon) - C(1-\tau) A_2(\varepsilon). \end{aligned} \tag{6}$$

The debt value is given by

$$D_t = \int_0^{\varepsilon} C A_2(k)\,dk + C A_2(\varepsilon), \tag{7}$$

and A_1 and A_2 are exponential functions given by

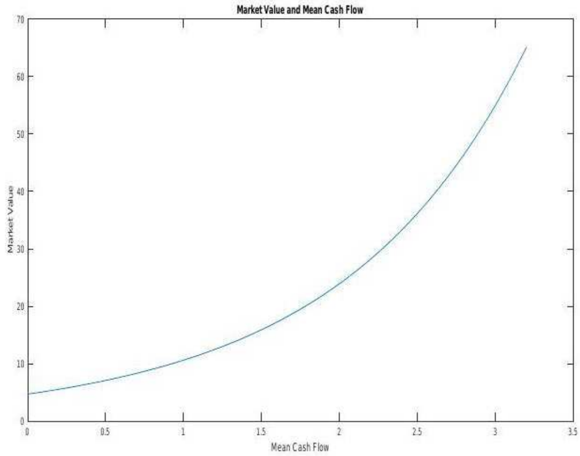
$$A_1 = \exp\left\{ x_t - \left(r + \frac{1}{2}\sigma_\pi \sigma_\pi' \right)\varepsilon + \left(1 - e^{-\varphi\varepsilon}\right)(\bar{u} - x_t) + \frac{\sigma_{x,1}^2 - \sigma_{x,2}^2 + 2\sigma_{x,2}^2 e^{-\varphi\varepsilon} + 2e^{-\varphi\varepsilon}\left(\sigma_{x,1}^2 + \sigma_{x,2}^2\right)}{2\varphi} \right\}, \tag{8}$$

$$A_2 = \exp\left\{ -\left(r + \frac{1}{2}\sigma_\pi \sigma_\pi' \right)\varepsilon + \frac{3\sigma_{x,2}^2\left(e^{-\varphi\varepsilon} - 1\right) - 2\left(1 - e^{-\varphi\varepsilon}\right)\left(\sigma_{x,1}^2 + \sigma_{x,1}\sigma_{\pi,1}\right)}{2\varphi} \right\}. \tag{9}$$

Proof is given in the Appendix.

The closed form solution for the leverage ratio given in Proposition 1 above characterizes a convex relationship between mean log cash flow and firm

5 Usually, in continuous time framework, optimal coupon rates are derived to get an optimal leverage ratio at time zero or at the refinancing point. When the dynamic trade off model are too complicated, a simulation will be performed to study the financing behaviors of firms.



Note: The vertical axis shows market value. The horizontal axis shows mean log cash flow.

Figure 1. Market value is convex in mean cash flow

value. Mean log cash flow mainly affects the leverage ratio via its effect on market equity to book value, while this model abstracts from any effect of it on the debt to book value ratio over age. In the corollary below, standard comparative static results from this model are provided in the absence of mean cash flow uncertainty, thus establishing that it is a reasonable benchmark to compare to later, after incorporating uncertainty.

Corollary 1. *Mean log cash flow $\bar{u}$ has an increasing effect on firms' equity value. Equity value rises in current log cash flow* $\log(X_t)$*. Interest rate and σ_π, σ'_π 's effect on equity value is indeterminate. The more managerial value, the higher equity value. Equity value is decreasing in the coupon rate. The effect of mean reversion speed on equity value cannot be determined. The cash flow shock variances $\sigma^2_{x,1}$, $\sigma^2_{x,2}$ have a positive effect on firm equity value. Firm market value declines over the corporate tax rate.*

Proof is given in the Appendix.

When either the expected cash flow or the current level of cash flow increases, the firm's stock price will rise and thus the valuation of a firm's market value will increase. This also confirms the results of most empirical finding about the determinants of capital structure (Roberts, 2008). The interest rate's effect is ambiguous. This is because the marginal effect of the interest rate on equity value is two-fold: on the one hand, its effect on net cash flow is negative or ambiguous; on the other hand, its effect on the tax value of firm debt is ambiguous. Therefore, it is difficult to judge which effect will dominate. The larger the cash flow shock variances, the higher the market value. This reflects the convex relationship between firm cash flow and firm value.

Corollary 2. *Firm equity value is convex in mean log cash flow $\bar{u}$.*

Proof is given in the Appendix.

Figures 1 and 2 employ the same parameters as Pastor and Veronesi (2003) and Goldstein et al. (2001): $\phi = 0.39$, $\sigma_{x,1} = 0.0584$, $\sigma_{x,2} = 0.0596$, r = 0.03, $\tau = 12$, $\tau_i = 0.40$. The figures confirm their finding that the larger the mean log cash flow or cash flow growth rate, the more pronounced the convexity. One interesting feature is the role of mean reversion speed in affecting convexity, which will be discussed in the subsequent learning section.

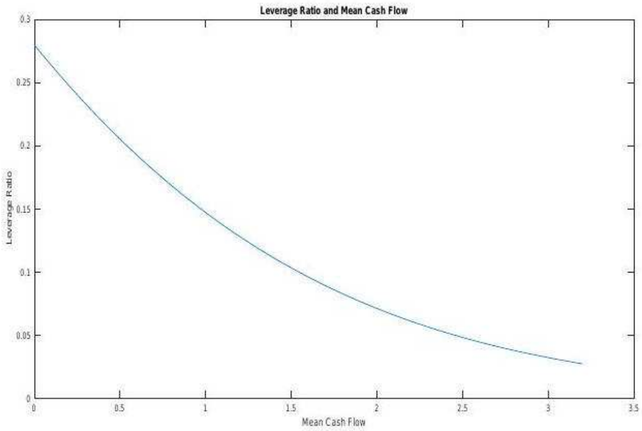
The debt value addressed in Corollary 3 below reflects the tax advantage of holding debt.

Corollary 3. *Debt value rises with the coupon rate. The interest rate has a negative effect on firm debt value. The effect of cash flow shocks on debt is ambiguous.*

Proof is given in the Appendix.

If the interest rate goes up, this means that firms face a larger external financing constraint, and so the debt available to the firm will decrease. $\bar{u}$ impacts the leverage ratio mainly through equity value.

http://dx.doi.org/10.21511/imfi.20(3).2023.05



Note: The vertical axis shows leverage ratio. The horizontal axis shows mean log cash flow.

Figure 2. Leverage ratio declines over mean log cash flow

The results above establish the model without cash flow uncertainty as a reasonable baseline model. Moving closer to the primary research question, next the relationship between mean log cash flow, current cash flow and the leverage rate in this baseline model is determined. This provides a point of comparison to the results in the subsequent section.

Corollary 4. *Without learning, the leverage ratio is decreasing in* $\bar{u}$ *and current log cash flow level* x_t.

Proof is given in the Appendix.

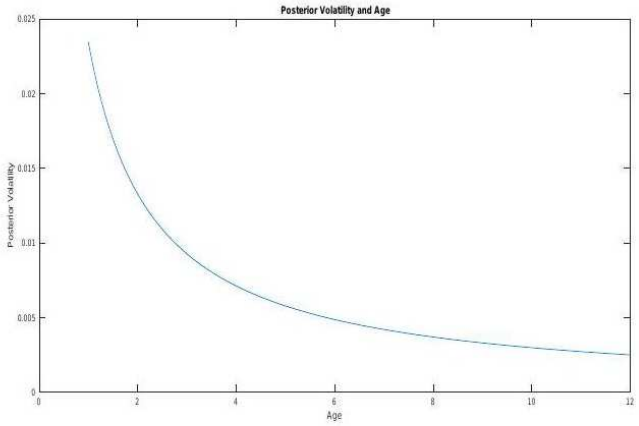
Note that the mean log cash flow only affects the leverage ratio through its effect on equity value, and according to Corollary 1, equity value is increasing in mean cash flow. Thus, the leverage ratio is decreasing with $\bar{u}$, ceteris paribus. The same holds true for x_t. This prediction is in line with the pecking order theory, in which more profitable firms will reduce their reliance on external debt financing especially when firm are mature. Meanwhile, the result is also consistent with Strebulaev (2007) whose dynamic trade off model predicts that if firms do not refinance frequently, then there is a negative relation between future cash flow and the leverage ratio. Another explanation is that expected net cash flow increases a firm's retained earning which can be reinvested to positive cash flow projects instead of issuing more debt externally to satisfy the financing needs. This again confirms many of the empirical findings.[6]

The model implied leverage ratio is plotted in Figure 2 using the same parameters as before. Interestingly, when firms' expected cash flows are very low, their leverage ratio appears to be high (about 0.27). As the cash flow growth rate increases, firms become less levered. Even if the leverage ratio diminishes, it does so very slowly towards its steady state value. It is noteworthy that the leverage ratio fluctuates between 0.25 to 0.16 before firms mature. The slow mean reversion speed of the leverage ratio has been observed by Lemmon et al. (2008) and Dufrense and Goldstein (2001). The presented results are consistent with these empirical results.

3.2. Learning about cash flow

In this section, Bayesian learning is incorporated to ask how leverage ratios change with firm age as the uncertainty resolves. Mean log cash flow $\bar{u}$ is no longer assumed to be known. Instead, managers and investors have a common prior distribution on $\bar{u}$, specified as $N(\hat{u}_0, \hat{\sigma}_0)$, and use Bayesian rules to update their posterior belief based on realized cash flow. They must therefore assess the influences exerted by uncertainty of mean cash flow upon firm equity value. Since firm debt value is not affected by cash flow uncertainty, they then revise the leverage ratio, whose changes are due to uncertainty of expected cash flow only. The uncertainty, as discussed before, affects firm value via the convex relationship between the cash flow growth rate and firm equity value.

6 Refer to section 5.



Note: The vertical axis shows posterior volatility. The horizontal axis shows age.

Figure 3. Posterior volatility declines over age

Since $\bar{u}$ has a normal prior distribution at time $t = 0$, the posterior expectation of $\hat{u}_t$ will also follow normal distribution as demonstrated by Lipster and Shiryayev (1977). To specify this distribution, it is needed only to solve for the posterior mean and variance.

Lemma 1. *Managers/investors update their posterior mean following the rule*

$$d\hat{u}_t = \hat{\sigma}_t^2 \frac{\varphi}{\sigma_{x,2}} d\tilde{Z}_{2,t}, \tag{10}$$

where $\tilde{Z}_{2,t}$ captures the idiosyncratic shock. [7]

Proof is given in the Appendix.

This equation provides the updating formula for the posterior mean log cash flow as a function of its posterior variance. The smaller the posterior variance of mean log cash flow, the smaller the mean log cash flow. This is due to the bankruptcy protection informally modelled via the log-normal distribution for cash flow. Since the log-normal distribution is right-skewed, a decrease in variance reduces right-tailed outliers more than left-tailed outliers, thus reducing the expected value.

After tedious calculations, the posterior variance $\hat{\sigma}_t^2$ of mean log cash flow is equal to

$$\hat{\sigma}_t^2 = 1 / \left(\frac{1}{\hat{\sigma}_0^2} + \frac{\varphi^2}{\sigma_{x,2}^2} t \right). \tag{11}$$

The posterior variance of mean log cash flow is a decreasing function of firm age (t). As the firm matures, managers and investors learn about mean cash flow from repeated observation of actual cash flow. This reduces the investors' uncertainty regarding mean cash flow. There are three other parameters in formula (11): mean reversion speed ϕ, prior variance of mean cash flow $\hat{\sigma}_0^2$ and cash flow shock $\hat{\sigma}_{x,2}{}^2$. The posterior variance falls with mean reversion speed but has a positive correlation with both the idiosyncratic shocks and prior variance of mean cash flow, holding other parameters fixed. Using the same parameters as before, Figure 3 is obtained, which indeed verifies the negative relation between cash flow uncertainty and firm age.

The levered firm value may be expressed as an implicit function of the mean (log) cash flow, $\bar{u}$, as

$$\begin{aligned} V_t^L(\bar{u}) = E_t \Big\{ & \int_t^T \pi_w (1-\tau)(X_w - C)/\pi_t dw + \\ & + \left[\pi_T (1-\tau)(X_T - C)/\pi_t \right] + \\ & + \int_t^T \pi_w / \pi_t C dw + \pi_T / \pi_t C \,|\, \bar{u}, \Big\} \end{aligned} \tag{12}$$

where E_t denotes the expectation with respect to the information set given by F_t: $\{H_t = (v_t, \log(X_t)) : 0 \le \omega \le t\}$. Since the mean (log) cash flow is unknown, managers value the firm by taking the expectation of $V_t^L(\bar{u})$ with respect to the posterior distribution of $\bar{u}$ to solve for $E_t[V_t^L(\bar{u})]$. In the proposition below this solution is provided.

7 Its definition can be found in the section of Proof of Proposition 2 of Appendix A.

http://dx.doi.org/10.21511/imfi.20(3).2023.05

Proposition 2. *With unknown $\bar{u}$, the solution of the levered firm value is expressed as follows:*

$$E_t[V_t^L(\bar{u})] = \int_0^{\varepsilon}(1-\tau)\widetilde{A_1}(\hat{u}_t, x_t, k)\,dk + +(1-\tau)\widetilde{A_1}(\hat{u}_t, x_t, \varepsilon) + +\int_0^{\varepsilon}\tau C A_2(k)\,dk + \tau C A_2(\varepsilon), \quad (13)$$

where ε denotes the time to maturity, A_2 remains the same as equation (9) and

$$\widetilde{A_1}(\hat{u}_t, x_t, \varepsilon) = A_1 * exp\left\{\left(1-e^{-\psi\varepsilon}\right)^2 \hat{\sigma}_t^2\right\}$$

is where A_1 defined in equation (8).

Proof is given in the Appendix.

The proof of Proposition 2 uses the properties of the log-normal distribution to arrive at a solution that depends on both $\hat{u}_t$, the posterior expectation of $\bar{u}$, and $\hat{\sigma}_t^2$ the posterior variance of $\hat{u}_t$. The following corollary shows that both $\hat{u}_t$ and $\hat{\sigma}_t^2$ positive effect firm value.

Corollary 5. *Levered firm value, as defined by $E_t[V_t^L(\bar{u})]$, is increasing in both the posterior mean, $\hat{u}_t$, and posterior variance $\hat{\sigma}_t^2$ of log cash flow.*

It may not be surprising that firm value is increasing in the posterior mean $\hat{u}_t$ Importantly, firm value also increases with posterior variance, due to the right-skewed properties of the lognormal distribution. An increase in posterior variance increases the probability of both negative and positive outliers, but the effect of the positive outliers dominates due to positive skewness. Intuitively, the possible gain in value due to the firm being unusually successful outweighs the possible loss to firm value due failure. While the possible gains are essentially unbounded, potential losses are limited by the bankruptcy protection implicit in the use of the log-normal distribution.

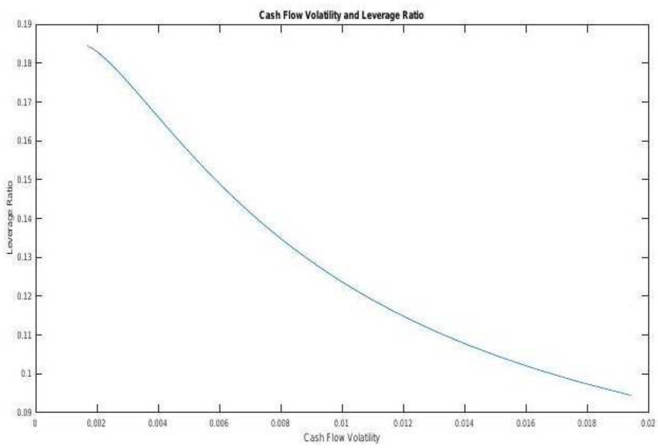
Next corollary investigates how equity value varies with the posterior variance.

Corollary 6. *The leverage ratio decreases with the posterior variance.*

Proof is given in the Appendix.

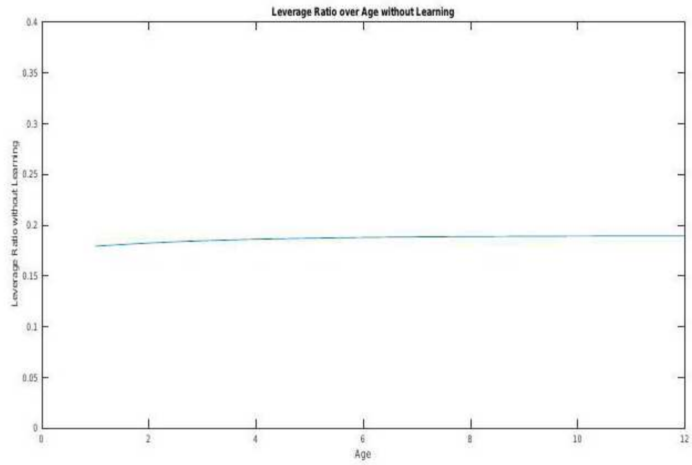
Recall from Corollary 4 that firm value is increasing with the posterior variance. Therefore, firm value will go up with the rise in the posterior variance, while the leverage ratio will decrease since debt value is unaffected by it. Again, this is because changes in debt ratios do not depend on managers' expectation of future cash flow. Figure 4 illustrates this positive relationship between posterior variance and the leverage ratio.

This in turn has interesting implications for the evolution of firm leverage over the firm life cycle. Since the leverage ratio decreases with the posterior variance of log cash flow (Figure 4) and since the posterior variance decreases over time (Figure 3), this implies that the firm leverage ratio increases with firm age.



Note: The vertical axis shows leverage ratio. The horizontal axis shows posterior variance of log cash flow.

Figure 4. Cash flow volatility is inversely related to leverage ratios



Note: The vertical axis shows leverage ratio. The horizontal axis shows age.

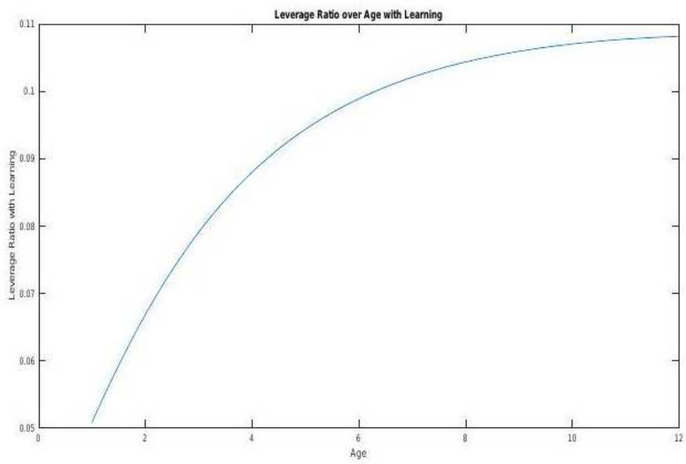
Figure 5. Leverage ratio varies over age without learning

The decrease in firm leverage with firm age is a direct implication of learning. To illustrate this, Figures 5 and 6 provide the contrast of the relationship between the leverage ratio and firm age without learning and with learning. Figure 5 shows the case without learning in which mean (log) cash flow is known. The figure shows a nearly horizontal relationship between leverage and firm age, with approximately constant leverage rates. By contrast the case with learning shown in Figure 6 shows a clear increase in leverage with firm age. This increase is steepest for young firms, when both the uncertainty and learning are greatest and gradually tapers off with firm age, as the uncertainty resolves and the learning slows.

In additional results, omitted to save space, the impact of varying the mean reversion speed, ϕ is studied. The larger the value of ϕ, the quicker cash flow reverts to its mean value and the faster managers learn. This results in a quicker contraction in firm value and a more rapid increase in firm equity.

4. DISCUSSION

The presented results are consistent with and potentially explain some previous empirical findings. Changes in debt ratios are independent of managers' expectation of future cash flow. Managers hesitantly decide to take financing actions and



Note: The vertical axis shows leverage ratio. The horizontal axis shows age.

Figure 6. Leverage ratio varies over age with learning

http://dx.doi.org/10.21511/imfi.20(3).2023.05

just update their belief about mean cash flow. The changes in the belief are manifested in posterior variance of expected cash flow. This is in accordance with Welch (2004). Dufresne and Goldstein (2001) propose that the leverage ratio is mean-reverting. Lemmon et al. (2008) point out that serial correlation of the leverage ratio is a major feature of leverage ratio and also if a firm initially has a low leverage ratio, then the leverage ratio gradually increases until it reaches a steady level.

The proposed model has an interesting prediction for the leverage ratio over the firm life cycle. In particular, it suggests that the leverage ratio starts low and rises over age. This prediction is consistent with the empirical findings of Kim and Suh (2009), who find an inverted U-shape of leverage as a function of returns to earning which they employ as a proxy for firm age. Pastor and Veronesi (2003) also report, but do not explain, an interesting pronounced increase in median firm leverage ratios with firm age.

As Kim and Suh (2009) and Frielinghaus, Mostert and Firer (2005) suggest, these findings may also be consistent with the static trade-off theory, capital structure life theory, and agency cost theory. The leverage ratio is intended primarily as a measure of debt financing, the above expression makes clear that it is also sensitive to the market value of firm equity, as measured, for example, by the firm's market-to-book ratio.

Building on insights from the asset pricing literature, particularly Pastor and Veronesi (2003) and Ju and Ou (2006), this paper focuses on the life cycle dynamics of the market valuation when there is uncertainty about firm prospects, as measured by expected cash flow. Uncertainty in this model arises because managers/investors have only a prior distribution of beliefs about expected cash flow. Therefore, they will form their beliefs about expected cash flow to inform their assessment of market equity value, which in turn affects firm leverage ratios. This uncertainty reflects the fact that investors do not know which young firms or startups will be successful and which will not. Since downside losses are limited by bankruptcy protection,[8] whereas the upside potential is virtually unlimited, risk-neutral investors will gamble on the upside potential when firms are young. As the firm ages, investors observe repeated realizations of the firm's cash flow, from which they learn about the firm's potential. Since most startups will not be next Meta (Facebook) or Alphabet (Google), this reduces the upside potential for the median firm, thereby lowering the firm's market value relative to its book value. For a given value of a firm's debt-to-book ratio, this lowers firm leverage.

The resolution of uncertainty captured by the proposed model may be most relevant to high-tech firms with patents, especially when they are very young or when they just finish Initial Public Offerings (IPOs). Investors or managers in this sector gamble on the new technology adopted by young firms. Over the first several years of the median firm's life, investors learn that its new technology will not lead it to be the next technology giant and trim their assessment of firm value accordingly. The proposed model captures this effect and its implication for firm leverage. Gradually, as these firms age and their future prospects become more predictable, the normal dynamics of stable growth can be expected to replace the initial decline in firm value due to the resolution of uncertainty that is captured succinctly by the model.

It is worth noting that the suggested model does not imply a fall in the market-to-book ratio for *all* young firms. Rather, it explains the fall in value for the *median* young firm, which turns out not to be the tremendous success that investors had hoped for. For these typical firms, as well as for the low-performing firms, the resolution of uncertainty disappoints investors by reducing the upside potential that they might have become exceptional investments. Of course, some firms do become usually successful, and, in their case, the resolution of uncertainty can lead to very large gains in firm values. Indeed, it is exactly these out-sized returns in a few very lucky cases that induce investors to place bets on all new firms, thus driving up their initial firm price and driving down their initial firm leverage.

The incorporation of Bayesian learning in continuous time capital structure models complicates the modelling process. Benzoni et al. (2022) is the on-

8 The asymmetry is captured via lognormal distribution without formal modelling bankruptcy.

ly paper known to the authors that develops a new model to find a closed form solution for the optimal capital structure under incomplete information. In fact, even without learning, previous capital structure models appear complex, often requiring numerical solutions for an interpretable result. Yang (2013) uses Bayesian uncertainty to explain corporate structure. In his model, changes are induced by differences of opinion between external investors and managers who hold divergent beliefs. His model requires the use of numerical analysis to simulate the effect of difference of beliefs on the debt ratio. Even the model of Liu et al. (2017) requires very restrictive parameter assumptions on the cash flow process in order to obtain an analytical solution. While this paper obtains an analytical solution for a continuum of investors without restrictive parameter assumptions, it does not require strong assumptions on firm debts and bankruptcy.

To provide a tractable analytic solution, this paper models firm debt exogenously and focuses strictly on the effects of right-tailed uncertainty on leverage that flow through the equity valuation channel.[9] This simplifying assumption may be justified given the focus on right-tailed uncertainty, which is far more important for equity valuation than for bond pricing. Simply put, equity investors benefit from, and therefore price, right-tailed risk, whereas only default, or left-tailed risk is of primary concern for bondholders. Although the proposed framework indirectly captures the bankruptcy protection afforded to equity holders via the log-normality assumption on cash flow, it does not formally model the default risks faced by bondholders.

There are still several directions for future research. The model is possibly generalized so that managers can choose the optimal leverage ratio in the presence of learning. It may also be an interesting topic to study the behavior of the debt ratio in an equilibrium environment in which the stochastic discount factor is to be endogenized. Learning from peer firms' leverage ratios possibly constitutes another theoretical research area of capital structure, as recent papers have empirically found that managers are motivated to learn from peer firms' leverage ratio when making financing decisions.

CONCLUSION

The aim of this paper was to incorporate Bayesian learning into a dynamic model of capital structure with a continuum of firm types to explain. This paper proposes a simple dynamic model to explore the relationship among the leverage ratio, expected cash flow volatility and firm age. It shows that younger firms/start-ups with higher uncertainty can have higher valuation (market-to-book value) compared to more established firms with lower uncertainty. It also shows that leverage ratios grow over firm age due to learning. Initially, when uncertainty about expected cash flow is high, firm market value is high, and this in turn implies a low leverage ratio holding all other factors fixed. However, as cash flow uncertainty decreases with firm age, market values are found to decline slowly. Consequently, the leverage ratio increases gradually.

The secondary findings in this paper compared two cases: with no uncertainty about cash flow and with uncertainty (and consequently learning) of the cash flows.

For the no uncertainty case, the firm value increases with the mean log cash flow (which is expected), but also the firm equity value is convex in mean log cash flow, showing that firm value accelerates with increases in mean log cash flows. Without learning, the leverage ratio is decreasing in mean log cash flows, and in current cash flow level.

9 The lognormal distribution is used to keep equity from going negative and debt is infrequently rebalanced and debt-to-book ratios are relatively constant. Also, bankruptcy is added to the model. Incorporating bankruptcy caps downside risk further. If firms pick debt according to trade-off model, then their rebalancing may offset equity effects, particularly holding bankruptcy cost fixed. However, allowing both bankruptcy and rebalancing, initial greater uncertainty will mean a higher probability of bankruptcy, which will be weakened by the high firm value. This discourages debt holding suggesting that at most they only partially rebalance to offset the equity effect. The key point is that equity holders and bondholders are affected differently by uncertainty. Equity can take advantage of upside potential, while bond holders face only the increased probability of bankruptcy.

http://dx.doi.org/10.21511/imfi.20(3).2023.05

With learning, the leverage ratio decreases with the posterior variance. In this case, the firm value increases in both posterior mean log cash flows and, in posterior variance of cash flows. What makes this finding interesting is that, contrary to conventional logic, higher uncertainty (variance of cash flows) can be associated with an increased firm value and decreased leverage.

AUTHOR CONTRIBUTIONS

Conceptualization: Dong Meng Ren, Yunmin Chen, Alex Maynard.
Data curation: Dong Meng Ren.
Formal analysis: Dong Meng Ren, Alex Maynard.
Funding acquisition: Dong Meng Ren, Sergiy Pysarenko.
Investigation: Yunmin Chen, Alex Maynard, Sergiy Pysarenko.
Methodology: Dong Meng Ren.
Resources: Sergiy Pysarenko.
Software: Dong Meng Ren, Yunmin Chen.
Supervision: Alex Maynard.
Validation: Yunmin Chen, Alex Maynard, Sergiy Pysarenko.
Visualization: Dong Meng Ren.
Writing – original draft: Dong Meng Ren.
Writing – review & editing: Yunmin Chen, Alex Maynard, Sergiy Pysarenko.

REFERENCES

1. Benzoni, L., Garlappi, L., & Goldstein, R. S. (2022). Incomplete Information, Debt Issuance, and the Term Structure of Credit Spreads. *Management Science*. https://doi.org/10.1287/mnsc.2022.4529
2. Black, F., & Cox, J. C. (1976). Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance, 31*(2), 351-367. https://doi.org/10.2307/2326607
3. Bradley, M., Jarrell, G. A., & Kim, E. H. (1984). On the existence of an optimal capital structure: Theory and evidence. *Journal of Finance, 39*(3), 857-878. https://doi.org/10.2307/2327950
4. Brennan, M. J., & Xia, Y. (2001). Stock return volatility and equity premium. *Journal of Monetary Economics, 47*(2), 249-283. http://dx.doi.org/10.1016/S0304-3932(01)00042-3
5. Broadie, M., Chernov, M., & Sundaresan, S. (2007). Optimal debt and equity values in the presence of chapter 7 and chapter 11. *Journal of Finance, 62*(3), 1341-1377. https://doi.org/10.1111/j.1540-6261.2007.01238.x
6. Bulana, L., & Yanb, Z. (2010). Firm maturity and the pecking order theory. *International Journal of Business and Economics, 9*(3), 179-200. Retrieved from https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1760505
7. Collin-Dufresne, P., & Goldstein, R. S. (2001). Do credit spreads reflect stationary leverage ratios? *Journal of Finance, 56*(5), 1929-1957. https://doi.org/10.1111/0022-1082.00395
8. Cremers, M., & Yan, H. (2016). Uncertainty and Valuations. *Critical Finance Review, 5,* 85-128. Retrieved from https://cfr.pub/published/papers/cfr-020.pdf
9. Damodaran, A. (2001). *Corporate finance: Theory and practice*. New York: John Wiley and Sons.
10. DeAngelo, H., DeAngelo, L., & Stulz, R. M. (2006). Dividend policy and the earned/contributed capital mix: a test of the life cycle theory. *Journal of Financial Economics, 81*(2), 227-254. https://doi.org/10.1016/j.jfineco.2005.07.005
11. DeAngelo, H., DeAngelo, L., & Whited, T. M. (2011). Capital structure dynamics and transitory debt. *Journal of Financial Economics, 99*(2), 235-261. https://doi.org/10.1016/j.jfineco.2010.09.005
12. DeHan, C. P. (2014). Capital structure over the life cycle. *Advances in Business Research, 5,* 16-32.
13. Dickinson, V. (1984). Cash flow patterns as a proxy for firm life cycle. *The Accounting Review, 86*(6), 1969-1994. Retrieved from https://papers.ssrn.com/sol3/papers.cfm?abstract_id=755804
14. Dudley, E., & James, C. (2015). *Cash flow volatility and capital structure choice.* Retrieved from http://www.econ.queensu.ca/cea2001/papers/carrasco-cea.pdf
15. Duffie, D., & Lando, D. (2001). Term structure of credit spreads with incomplete accounting information. *Econometrica, 69,* 633-64. https://doi.org/10.1111/1468-0262.00208
16. Duffie, D. (1996). *Dynamic Asset Pricing Theory*. Princeton University Press.
17. Feltham, G. A., & Ohlson, J. A. (1995). Valuation and

clean surplus accounting for operating and financial activities. *Contemporary Accounting Research, 11*(2), 689-731. https://doi.org/10.1111/j.1911-3846.1995.tb00462.x

18. Feltham, G. A., & Ohlson, J. A. (1999). Residual earnings valuation with risk and stochastic interest rates. *The Accounting Review, 74*(2), 165-183. https://doi.org/10.2308/accr.1999.74.2.165
19. Frank, M., & Goyal, V. (2003). Testing the pecking order theory of capital structure. *Journal of Financial Economics, 67*(2), 217-248. https://doi.org/10.1016/S0304-405X(02)00252-0
20. Frank, M. Z., & Goyal, V. K. (2009). Capital structure decisions: Which factors are reliably important? *Financial Management, 38*(1), 1-37. https://doi.org/10.1111/j.1755-053X.2009.01026.x
21. Frielinghaus, A., Mostert, B., & Firer, C. (2005). Capital structure and the firm's life stage. *South African Journal of Business Management, 36*(4), 9-18.
22. Goldstein, R., Ju, N., & Leland, H. (2001). An EBIT-based model of dynamic capital structure. *Journal of Business, 74*(4), 483-512. https://doi.org/10.1086/322893
23. Gorbenko, A., & Strebulaev, I. (2010). Temporary versus permanent shocks: Explaining corporate financial policies. *Review of Financial Studies, 23*(7), 2591-2647. https://doi.org/10.1093/rfs/hhq039
24. Graham, J., & Harvey, C. (2001). The theory and practice of capital structure: Evidence from field. *Journal of Financial Economics, 60*(2-3), 187-243. https://doi.org/10.1016/S0304-405X(01)00044-7
25. Graham, J., & Harvey, C. (2011). A review of empirical capital structure research and directions for the future. *Annual Review of Financial Economics, 3,* 309-345. Retrieved from https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1729388
26. Graham, J. R. (2000). How big are the tax benefits of debt? *Journal of Finance, 55*(5), 1901-1941. https://doi.org/10.1111/0022-1082.00277
27. Graham, J. R., Leary, M. T., & Roberts, M. R. (2014). A century of corporate capital structure: The leveraging of corporate America. *Journal of Financial Economics, 118*(3), 658-683. https://doi.org/10.1016/j.jfineco.2014.08.005
28. Jensen, M., & Meckling, W. H. (1976). Theory of the firm: Managerial behavior, agency costs and ownership structure. *Journal of Financial Economics, 3*(4), 305-360. https://doi.org/10.1016/0304-405X(76)90026-X
29. Ju, N., & Ou- Yang, H. (2006). Capital structure, debt maturity, and stochastic interest rates. *The Journal of Business, 79,* 2469-2502. https://doi.org/10.1086/505241
30. Larrain, B., & Yogo. M. (2008). Does firm value move too much to be justified by subsequent changes in cash flow? *Journal of Financial Economics, 87,* 200-226. https://doi.org/10.1016/j.jfineco.2007.01.002
31. Leary, M. T., & Roberts, M. R. (2005). Do firms rebalance their capital structures? *Journal of Finance, 60*(6), 2575-2619. https://doi.org/10.1111/j.1540-6261.2005.00811.x
32. Leland, H. (1994). Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance, 49*(4), 1213-1252. https://doi.org/10.2307/2329184
33. Leland, H. (1998). Agency costs, risk management, and capital structure. *Journal of Finance, 53*(4), 1213-1243. https://doi.org/10.1111/0022-1082.00051
34. Leland, H., & Toft, K. (1996). Optimal capital structure, endogenous bankruptcy and the term structure of credit spreads. *Journal of Finance, 51*(3), 987-1019. https://doi.org/10.2307/2329229
35. Liu, B., Liu, Y., Pen, J., & Yang, J. (2017). Optimal capital structure and credit spread under incomplete information. *International Review of Economics and Finance, 49,* 596-611. https://doi.org/10.1016/j.iref.2017.03.020
36. Mackay, P., & Gorden, P. M. (2005). How does industry affect firm financial structure? *Review of Financial Studies, 18*(4), 1433-1466. http://dx.doi.org/10.1093/rfs/hhi032
37. Martin, T. B., Sakoulis, G., & Henriksson, R. (2016). Bubbles, anti-bubbles, and equity expected returns. *NBER.*
38. Merton, R. C. (1974). On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance, 29*(2), 449-470. https://doi.org/10.2307/2978814
39. Miller, D., & Friesen, P. (1984). A longitudinal study of the corporate life cycle. *Management Science, 30*(10), 1161-1183. https://doi.org/10.1287/mnsc.30.10.1161
40. Modigliani, F., & Miller, M. (1958). The cost of capital, corporation finance, and the theory of investment. *American Economic Review, 48*(3), 261-297.
41. Modigliani, F., & Miller, M. (1963). Corporate income taxes and the cost of capital: a correction. *American Economic Review, 53*(3), 433-443.
42. Morellec, E., Nikolov, B., & Schuerhoff, N. (2012). Corporate governance and capital structure dynamics. *Journal of Finance, 67*(3), 803-848. https://doi.org/10.1111/j.1540-6261.2012.01735.x
43. Morgan, I. W., & Abetti, P. A. (2004). Private and public 'cradle to maturity' financing pattern of U.S. biotech ventures. *Journal of Private Equity, 7*(2), 9-25. http://dx.doi.org/10.3905/jpe.2004.391047
44. Morrelec, E. (2004). Can managerial discretion explain observed leverage ratios? *Review of Financial Studies, 17*(1), 257-294. https://doi.org/10.1093/rfs/hhg036
45. Mueller, D. (1972). A life cycle theory of firms. *Journal of Industrial Economics, 20*(3), 199-219. https://doi.org/10.2307/2098055
46. Myers, S. C. (1984). The capital structure puzzle. *Journal of Finance, 39*(3), 575-592. https://

http://dx.doi.org/10.21511/imfi.20(3).2023.05

doi.org/10.1111/j.1540-6261.1984.tb03646.x

47. Opler, T. C., & Sheridan, T. (1994). Financial distress and corporate performance. *Journal of Finance, 49*(3), 1015-1040. https://doi.org/10.2307/2329214

48. Pástor, Ľ., & Veronesi, P. (2003). Stock valuation and learning about profitability. *Journal of Finance, 58*(5), 1749-1789. https://doi.org/10.1111/1540-6261.00587

49. Pástor, Ľ., & Veronesi, P. (2006). Was there a Nasdaq bubble in the late 1990s? *Journal of Financial Economics, 81*(1), 61-100. https://doi.org/10.1016/j.jfineco.2005.05.009

50. Pástor, Ľ., & Veronesi, P. (2009a). Learning in financial markets. *Annual Review of Financial Economics, 1*(1), 361-381. Retrieved from https://www.nber.org/papers/w14646

51. Pástor, Ľ., & Veronesi, P. (2009b). Technological revolutions and stock prices. *American Economic Review, 99*(4), 1451-1483. https://papers.ssrn.com/sol3/papers.cfm?abstract_id=875727

52. Pástor, Ľ. P., Veronesi, L., & Taylor (2009). Entrepreneurial learning, the IPO decision, and the post-IPO drop in firm profitability. *Review of Financial Studies, 22*(8), 3005-3046. https://doi.org/10.1093/rfs/hhn082

53. Spence, M. (1977). Entry, capacity, investment, and oligopolistic pricing. *Bell Journal of Economics, 8*(2), 534-544.

54. Strebulaev, I. A. (2007). Do tests of capital structure theory mean what they say? *Journal of Finance, 62*(4), 1747-1787. https://doi.org/10.1111/j.1540-6261.2007.01256.x

55. Strebulaev, I., & Yang, B. (2013). The mystery of zero-leverage firms. *Journal of Financial Economics, 109*(1), 1-23. https://doi.org/10.1016/j.jfineco.2013.02.001

56. Titman, S., & Wessels, R. (1988). The determinants of capital structure. *Journal of Finance, 43*(1), 1-19. https://doi.org/10.2307/2328319

57. Welch, I. (2004). Capital structure and stock returns. *Journal of Financial Economics, 112*(1), 106-131. https://doi.org/10.1086/379933

58. Wernerfelt, B. (1985). The dynamics of prices and market shares over the product life cycle. *Management Science, 31*(8), 928-939. https://doi.org/10.1287/mnsc.31.8.928

59. Whited, T., & Riddick, L. (2009). The corporate propensity to save. *Journal of Finance, 64*(4), 1729-1766. https://doi.org/10.1111/j.1540-6261.2009.01478.x

60. Xia, Y. (2001). Learning about predictability: the effects of parameter uncertainty on dynamic asset allocation. *Journal of Finance, 56*(1), 205-246. https://doi.org/10.1111/0022-1082.00323

61. Yang, B. (2013). Dynamic capital structure with heterogeneous beliefs and market timing. *Journal of Corporate Finance, 22,* 254-277. https://doi.org/10.1016/j.jcorpfin.2013.05.003

APPENDIX A

PROOF OF PROPOSITION 1

The instantaneous cash flow (cash flows and EBIT will be alternatively used over the whole article) is given by $f(X_t) = \delta X_t$, where X_t is the firms' EBIT, and $f(X_t)$ is considered to be firms' net earnings.

Firms' $\log$ cash flow is assumed to follow the process below:

$d\log(X_t) = \varphi(\bar{u} - \log(X_t)dt + \sigma_{x,1}dZ_1 + \sigma_{x,2}dZ_2$, where $\sigma_{x,1}$ is the standard variance of $\log$ EBIT and $\sigma_{x,2}$ is the standard deviation of $\log$ cash flow. In this set up, managers, who are in the interests of stockholders, cannot observe the mean of log cash flow and can only learn through past realized counterparts. That is, $\bar{u}$ is assumed to be priorly distributed as $\left(\widehat{u_0}, \widehat{\sigma_0}\right)$.

The firms' values are given by

$$V_t = E_t\left\{\int_t^T \pi_w (1-\tau)(\delta X_w - c)/\pi_t dw + \left[\pi_T (1-\tau)(\delta X_T - C\right.\right.$$
$$\left.\left./\pi_t\right] + \int_t^T \pi_w (1-\tau_i)/\pi_t C dt + \pi_T (1-\tau_i)/\pi_t C\right\},$$

conditional on the information set $F_t : \left\{H_t = \left(v_t, \log(X_t)\right) : 0 \le \omega \le t\right\}$.

The first term and second term of the right-hand side denote the equity value or cash flow values, while the third and fourth value refer to firms' tax shielding function of debts. π_t is the stochastic discount factor. The stochastic discount factor follows $\log$ normal Brownian process: $d\pi_t = -\pi_t r dt - \sigma_{\pi,1} dZ_1$, and the dynamics of π_t is driven by the same systematic shock as firms' cash flows and is correlated with cash flow shock.

V_t can be first evaluated by Fubini's theorem, by which expectation operator moves inside the integration. Therefore, it is needed to evaluate $E(\pi_t x_t)$.

Let $v_t = \log(\pi_t X_t)$, then $\pi_t X_t = e^{v_t}$ and also $dv_t = d\log\pi_t + d\log X_t$.

It is possible to substitute the process of $d\log\pi_t$, after Ito's lemma is used with respect to $dlog\pi_t$ and $d\log X_t$ defined above, and then:

$$dv_t = \left(-r - \frac{1}{2}\sigma_\pi \sigma'_\pi\right)dt - \sigma_{\pi,1}dZ_1 + \varphi\left(\bar{u} - \log(X_t)dt + \sigma_{x,1}dZ_1 + \sigma_{x,2}dZ_2,\right.$$
$$dv_t = \left(-r - \frac{1}{2}\sigma_\pi \sigma'_\pi + \varphi\left(\bar{u} - \log(X_t)\right)dt + \left(-\sigma_{\pi,1} + \sigma_{x,1}\right)dZ_1 + \sigma_{x,2}dZ_2,\right.$$

where $\sigma_\pi = \left[-\sigma_{\pi,1} \quad 0\right]$.

Let $F_t = \left(v_t, \log X_t\right)', \left(v_t, \log X_t\right)'$ is a vector of state variables, a matrix representation of it is given by:

This is a standard multi-dimensional linear process (Duffie, 1996) whose solution is a known closed form solution $F_T | F_0 \sim N\left(\mu(F_0, T), \Sigma_F(T)\right)$, where

$$\mu(F_0, T) = \psi(T)F_0 + \int_0^T \psi(T-t)A dt$$

$$\Sigma_F(T) = \int_0^T \psi(T-t)\Sigma\Sigma'\psi(T-t)'dt$$

http://dx.doi.org/10.21511/imfi.20(3).2023.05

$$d\begin{pmatrix} \breve{\upsilon}_t \\ \log Xt \end{pmatrix} = \left[\begin{pmatrix} \varphi\bar{u} - r - \frac{1}{2}\sigma_\pi\sigma'_\pi \\ \varphi u \end{pmatrix} + \begin{pmatrix} 0 & -\varphi \\ 0 & -\varphi \end{pmatrix}\begin{pmatrix} \upsilon_t \\ \log Xt \end{pmatrix} dt\right] + \begin{bmatrix} -\sigma_{\pi,1} + \sigma_{x,1} & \sigma_{x,2} \\ \sigma_{x,1} & \sigma_{x,2} \end{bmatrix}\begin{bmatrix} dZ_1 \\ dZ_2 \end{bmatrix}.$$

In this vector form, $A = \begin{pmatrix} \varphi\bar{u} - r - \frac{1}{2}\sigma_\pi\sigma'_\pi \\ \varphi u \end{pmatrix}, B = \begin{pmatrix} 0 & -\varphi \\ 0 & -\varphi \end{pmatrix}, \Sigma = \begin{bmatrix} -\sigma_{\pi,1} + \sigma_{x,1} & \sigma_{x,2} \\ \sigma_{x,1} & \sigma_{x,2} \end{bmatrix},$

$\psi(T) = \mathrm{w}\exp(\Lambda \cdot T) w^{-1}$, where Ë is a diagonal matrix with eigenvalues of the matrix B along the principal diagonal. W is the matrix of the associated eigenvectors. $\exp(\Lambda \cdot T)$ denotes the diagonal matrix with $e^{\lambda_i T}$ in the diagonal position. After matrix computations, given eigenvalues of $\psi(T)$ is $\begin{bmatrix} 0 & -\varphi \end{bmatrix}$ and its associated eigenvectors are

$$w = \begin{bmatrix} & 1 \\ 0 & 1 \end{bmatrix},\ \exp(\Lambda \cdot T) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\varphi T} \end{bmatrix}, \psi(T) = \begin{bmatrix} 1 & -1 + e^{-\varphi T} \\ 0 & e^{-\varphi T} \end{bmatrix}.$$

The normality of F_T implies that $v_T = e_1 F_T$ with $e_1 = (1,0), v_T$ is also normally distributed $v_T \mid F_0 \sim N\left(e_1\mu(F_0,T), e_1\Sigma_F(T)e'_1\right)$, then one can use the properties for lognormal random variables

$$E(\pi_t x_t) = E\left(e^{v_t}\right) = E\left(\exp^{(e_1 F_T)}\right) = \exp\left(e_1\mu(F_0,T) + \frac{1}{2}e_1\Sigma_F(T)e'_1\right).$$

It possible to back out conditional variance of π_t conditional on the information set $F_t : \{H_t = (v_t, \log(X_t)) : 0 \le \omega \le t\}$, using the above derived formula.

Then without learning, the closed form solution for firm value is given by

$$V_t = \int_0^\varepsilon \delta(1-\tau) A_1\left(\bar{u}, x_t, k\right) dk + \delta(1-\tau) A_1\left(\bar{u}, x_t, \varepsilon\right) + \int_0^\varepsilon \delta(\tau - \tau_i) CA_2(k) dk + (\tau - \tau_i) CA_2(\varepsilon),$$

where ε denotes the time to maturity, T is known terminal valuation time. The algebra involved in the derivation is tedious and they are available upon request.

$$A_1 = \exp\left\{x_t - \left(r + \frac{1}{2}\sigma_\pi\sigma'_\pi\right)\varepsilon + \left(1 - e^{-\psi\varepsilon}\right)(\bar{u} - x_t) + \frac{\sigma_{x,1}^2 - \sigma_{x,2}^2 + 2\sigma_{x,2}^2 e^{-\psi\varepsilon} + 2e^{-\psi\varepsilon}\left(\sigma_{x,1}^2 + \sigma_{x,2}^2\right)}{2\psi}\right\},$$

$$A_2 = \exp\left\{-\left(r + \frac{1}{2}\sigma_\pi\sigma'_\pi\right)\varepsilon + \frac{3\sigma_{x,2}^2\left(e^{-\psi\varepsilon} - 1\right) - 2\left(1 - e^{-\psi\varepsilon}\right)\left(\sigma_{x,1}^2 + \sigma_{x,1}\sigma_{\pi,1}\right)}{2\psi}\right\},$$

$$E_t = \int_0^\varepsilon \delta(1-\tau) A_1(\bar{u}, x_t, k) dk - \int_0^\varepsilon C(1-\tau) A_2(k) dk + \delta(1-\tau) A_1(\bar{u}, x_t, \varepsilon) - C(1-\tau) A_2(\varepsilon),$$

$$D_t = \int_0^s (1-\tau_i) CA_2(k) dk + C(1-\tau_i) A_2(\varepsilon).$$

As a result, leverage ratio L_t can be evaluated at time t, which is equal to

$$\frac{\int_0^s (1-\tau_i) sA_2(k) dk + s(1-\tau_i) A_2(\varepsilon)}{\int_0^\varepsilon \delta(1-\tau) A_1\left(\bar{u}, x_t, k\right) dk - \int_0^\varepsilon s(1-\tau) A_2(k) dk + \delta(1-\tau) A_1\left(\bar{u}, x_t, \varepsilon\right) - s(1-\tau) A_2(\varepsilon) + \int_0^s (1-\tau_i) sA_2(k) dk + s(1-\tau_i) A_2(\varepsilon)}.$$

This formula is actually equivalent to $L_t = \frac{D_t}{E_t + D_t}$, substituting E_t and D_t generates the result. The proof of Proposition one is finished.

Proof of Corollary 1

$$\frac{\partial E_t}{\partial \overline{u}} = \delta(1-\tau)\left(1-e^{-\psi\varepsilon}\right)\int_0^{\varepsilon}\delta(1-\tau)A_1(\overline{u},x_t,k)\,dk+\delta(1-\tau)\left(1-e^{-\psi\varepsilon}\right)A_1(\overline{u},x_t,\varepsilon)>0$$

$$\frac{\partial E_t}{\partial x_t} = \delta(1-\tau)e^{-\psi\varepsilon}\int_0^{\varepsilon}\delta(1-\tau)A_1(\overline{u},x_t,k)\,dk+\delta(1-\tau)e^{-\psi\varepsilon}A(\overline{u},x_t,\varepsilon)>0$$

Other proof can be done in the same way. In particular, it is hard to decide the sign of interest rate on the equity value and debt value, because the derivative is too complicated. The sign is not straightforward to be derived immediately.

Proof of Corollary 2

$$\frac{\partial^2 E_t}{\partial^2 \overline{u}} = \delta^2(1-\tau)^2\left(1-e^{-\psi\varepsilon}\right)^2\int_0^{\varepsilon}\delta(1-\tau)A_1(\overline{u},x_t,k)\,dk+\delta^2(1-\tau)^2\left(1-e^{-\psi\varepsilon}\right)^2A_1(\overline{u},x_t,\varepsilon)>0$$

Proof of Corollary 3

$$\frac{\partial D_t}{\partial r} = -\int_0^{S} CA_2(k)\,dk - C\tau_i A(\varepsilon)<0$$

$$\frac{\partial D_t}{\partial C} = \int_0^{S}(1-\tau_i)A(k)\,dk+(1-\tau_i)A_2(\varepsilon)>0$$

The effect of cash flow shocks on the debt value is also ambiguous, since the resulting derivative is too complicated to decide the sign immediately.

Proof of Corollary 4

From the proof of Corollary 1, one can see that Corollary 4 holds.

PROOF OF PROPOSITION 2

If mean log cash flow is unknown, the firm value can be written as

$$E_t[V_t^L(\overline{u})] = E_t\left\{E_t\left\{\int_t^T \pi_w(1-\tau)(\delta X_w - C)/\pi_t dw + \left[\pi_T(1-\tau)(\delta X_T - C)/\pi_t\right]+\right.\right.$$

$$\left.\left.+\int_t^T \pi_w(1-\tau_i)/\pi_t C dt + \pi_T(1-\tau_i)/\pi_t C \mid \overline{u}\right\}\right\} = X_t E_t\left[A\left(\overline{u},x_t,\varepsilon\right)\right],$$

where $A\left(\overline{u},x_t,\varepsilon\right)$ is already defined as A_1 and A_2, based on previous of derived results for them:

$$E_t\left[V_t^L(\overline{u})\right] = \int_0^{\varepsilon}\delta(1-\tau)E_t\left\{A_1\left(\hat{u},x_t,k\right)\right\}dk+\delta(1-\tau)E_t\left\{A_1\left(\hat{u},x_t,\varepsilon\right)\right\}+$$

$$+\int_0^{\varepsilon}\delta(\tau-\tau_i)CA_2(k)\,dk+(\tau-\tau_i)CA_2(\varepsilon),$$

As a result,

$$A_1 = \exp\left\{x_t - \left(r+\frac{1}{2}\sigma_\pi\sigma_\pi'\right)\varepsilon+\left(1-e^{-\psi\varepsilon}\right)\left(\hat{u}-x_t\right)+\right.$$

$$\left.+\frac{\sigma_{x,1}^2-\sigma_{x,2}^2+2\sigma_{x,2}^2e^{-\psi\varepsilon}+2e^{-\psi\varepsilon}\left(\sigma_{x,1}^2+\sigma_{x,2}^2\right)}{2\psi}\right\}\cdot\exp\left\{\left(1-e^{-\psi\varepsilon}\right)^2\hat{\sigma}_t^2\right\},$$

http://dx.doi.org/10.21511/imfi.20(3).2023.05

$$A_2 = \exp\left\{-\left(r+\frac{1}{2}\sigma_\pi\sigma_\pi^{'}\right)\varepsilon+\frac{3\sigma_{x,2}^2\left(e^{-\psi\varepsilon}-1\right)-2\left(1-e^{-\psi\varepsilon}\right)\left(\sigma_{x,1}^2+\sigma_{x,1}\sigma_{\pi,1}\right)}{2\psi}\right\}.$$

Now leverage ratio $\tilde{L}_t$ with learning is defined as

$$\frac{\int_0^\varepsilon \delta\left(\tau-\tau_i\right)CA\left(k\right)dk+\left(\tau-\tau_i\right)CA\left(\varepsilon\right)}{\int_0^\varepsilon \delta\left(1-\tau\right)\left\{A_1\left(,x_t,k\right)\right\}dk+\delta\left(1-\tau\right)\left\{A_1\left(\hat{u},x_t,\varepsilon\right)\right\}+\int_0^\varepsilon \delta\left(\tau-\tau_i\right)CA_2\left(k\right)dk+\left(\tau-\tau_i\right)CA\left(\varepsilon\right)}.$$

To derive the updating process for expected log cash flow, let

$$dF_t = d\begin{pmatrix} v_t \\ \log X_t\end{pmatrix} = \left[\begin{pmatrix} -r-\frac{1}{2}\sigma_\pi\sigma_\pi' \\ 0\end{pmatrix}+\begin{pmatrix}\varphi \\ \varphi\end{pmatrix}\bar{u}+\begin{pmatrix}0 & -\varphi \\ 0 & -\varphi\end{pmatrix}\begin{pmatrix} v_t \\ \log X_t\end{pmatrix}dt\right]+\begin{bmatrix}-\sigma_{\pi,1}+\sigma_{x,1} & \sigma_{x,2} \\ \sigma_{x,1} & \sigma_{x,2}\end{bmatrix}\begin{bmatrix} dZ_1 \\ dZ_2\end{bmatrix}.$$

Now denote $\widehat{u_t} = E\left(\bar{x}\right) = \bar{u}$, this is the expectation of $\bar{x}$ conditional on the information set $F_t:\left\{H_t = \left(v_t, \log\left(X_t\right)\right): 0 \le \omega \le t\right\}$, following Pástor and Veronesi (2003),

define $d\tilde{z}_t = \Sigma^{-1}\left(dF_t - E\left(dF_t\right)\right) = \Sigma^{-1}\left(dF_t - \left(\kappa_0+\kappa_1\bar{x}+\kappa_2 F_t\right)dt, \tilde{z}_t\right.$ is a standard Weiner process with respect to F_t. Given a prior distribution at time $t = 0$,

$\bar{u} \sim N\left(\bar{u}_0, \bar{\sigma}_0^2\right)$, the conditional $\hat{u}$ satisfies the SDE $d\hat{u}_t = \hat{\sigma}_t^2\kappa_1'\ \left(\Lambda\right)^{-1}d\tilde{z}_t = \hat{\sigma}_t^2\dfrac{\varphi}{\sigma_{x,2}}d\tilde{z}_{2,t}$

Finally, $d\hat{u}_t = \hat{\sigma}_t^2\dfrac{\varphi}{\sigma_{x,2}}d\tilde{z}_{2,t}$. Then posterior variance of mean log cash flow satisfies the Riccati differentiation

$$\frac{d\hat{\sigma}_t^2}{dt} = -\hat{\sigma}_t^2\kappa_1'\left(\Sigma\Sigma'\right)^{-1}\kappa_1 = -\hat{\sigma}_t^2\begin{pmatrix}\varphi\dfrac{\varphi}{\varphi}\end{pmatrix}\left\{\begin{bmatrix}-\sigma_{\pi,1}+\sigma_{x,1} & \sigma_{x,2} \\ \sigma_{x,1} & \sigma_{x,2}\end{bmatrix}\begin{bmatrix}-\sigma_{\pi,1}+\sigma_{x,1} & \sigma_{x,2} \\ \sigma_{x,1} & \sigma_{x,2}\end{bmatrix}^{-1}\begin{pmatrix}\varphi \\ \varphi\end{pmatrix}\right\},$$

$$\hat{\sigma}_t^2 = 1/\left(\frac{1}{\sigma_0^2}+\frac{\varphi^2}{\sigma_{x,2}^2}t\right).$$

Proof of Corollary 5

The proof of this part is similar to Corollary 1.

Proof of Corollary 6

If the derivative of firm value with respect to σ_t^2 is taken, it is obvious to see that firm value increases with respect to σ_t^2 and, therefore, for leverage ratios with learning, it is straightforward to see that they decrease with posterior variance.