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Proposing a Pure Binary Linear Programming (PBLP) Model to Discover Eulerian Circuits in Complete Graphs

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ABSTRACT

Known as a branch of Discrete Mathematics (DM), the Graph Theory (GT) describes and solves problems of discrete nature through nodes (i.e., vertices) and arcs (i.e., edges). In this regard, a prominent problem is to find the Eulerian circuits. This paper indicates that the problem can be analyzed through operations research methods. In more general terms, finding the Eulerian circuits could be considered a pathfinding problem. Hence, this paper proposes a pure binary mathematical model to describe the relationship between the variables employed to find the Eulerian circuits. All the analyses in this paper were performed in MATLAB. The proposed model can be solved by many optimization software applications. Finally, several numerical examples are presented and solved through the proposed method. All the analyses in this paper were performed in MATLAB. This paper indicated that the problem(Eulerian Circuits in Complete Graphs) could be studied and solved from the perspective of operations research.

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1. Introduction

The graph theory (GT) is a branch of discrete mathematics that analyzes graphs [1]. It is a branch of topology associated closely with algebra and the matrix theory [2]. Unlike other branches of mathematics, the GT has a specific beginning, i.e., the publication of Leonard Euler's paper on how to solve the Seven Bridges of Königsberg in 1736 [3]. Recent advances in mathematics, particularly in its application, have substantially developed the GT [4].It is now a potent research tool in various fields such as coding, operation research, statistics, electrical circuits, computer science, chemistry, biology, and sociology [5]. Euler introduced his theorem for planar graphs (PGs) in 1752. However, the field became fairly dormant for the next hundred years [6]. Gustav Kirchhoff studied trees, a special type of graphs, in 1847. He used that concept to generalize Ohm's current law in electrical circuits [7]. Ten years later, Arthur Cale applied the same kind of graph to count the number of distinct isomers in saturated hydrocarbons [8]. Two other significant ideas came into view around the same time. The four-color theorem (FCT) was first studied by Francis Guthrie around 1850 [9]. It was finally proved by Kenneth Oppel and Wolfgang Haken through a sophisticated computer analysis [10]. The Hamiltonian cycle (HC) was named after Sir William Rowan Hamilton [11]. He came up with the idea (i.e., the Hamiltonian graph) while solving an interesting puzzle involving the edges of a dodecahedron. Finding a solution to that puzzle is not particularly difficult Nevertheless, mathematicians have not yet found necessary and sufficient conditions that identify undirected graphs with Hamiltonian paths (cycles) [12]. No noteworthy breakthroughs had been made in the field until 1920. Kazimir Kuratovsky, a Polish mathematician, solved the problem of the characterization of PGs in 1930 [13].The first book on the GT was published in 1936. The author was Dénes Kőnig, an outstanding researcher in the field. A myriad of relevant activities followed. In the last four decades, the research became computer-assisted [14].

Many situations and states in the world around us can be described by diagrams consisting of points and lines. A graph is a set of vertices (nodes) and their connections [15].More specifically, a graph includes some vertices connected through some ordered pairs (edges) [16]. Edges are either simple or directed. Each type has its specific applications [17]. As mentioned earlier, the GT began in the eighteenth century. Euler, the eminent mathematician, invented the concept of the graph to solve the Königsberg bridges problem. However, the current growth and vibrancy in the GT are due to the growth of informatics in the last half century [18]. The problem has now evolved into what is called finding the Eulerian circuit. Simply put, it means finding the Eulerian circuits in an existing complete graph. There are different approaches to solving this problem. It is analyzed here from the perspective of operations research. In other words, the problem is modeled mathematically and then solved in relevant software applications.

Pure binary linear programming (PBLP) is a branch of mathematical optimization. Unlike similar linear optimization problems, the values of all its variables are equal to either zero or one [19].The approach is described later.

2. Literature Review

Somme et al. [20] employed a Voronoi dual graph (VDG) to approximate the shortest-path algorithms based on the distance criterion. They demonstrated that an approximate solution to the distance-based shortest path problem could be computed in VDG as the resultant graph had fewer nodes than the initial graph [20]. AmirTeimori [21] tried to find the shortest path (SP) in accordance with different indices for routing in a network in 2012. He proposed integrating indices through the data envelopment analysis (DEA) mathematical method. His results indicated the success of his method in determining optimal efficient paths. Xu et al. [22] analyzed counting the number of spanning trees (STs) in directed graphs in 2015.Their proposed mathematical approach was really powerful.

Bach et al. [23] studied pathfinding in a graph subject to the contrast of limited arch movement in 2016.They proposed a mixed integer linear programming (MILP) and used a branch and price algorithm (BCA) algorithm as the solution. According to their results, the proposed approach solved the small-scale problems properly. Yaghobi and Akrami [24] modeled and solved the pathfinding problem for transporting perishable raw materials in 2019. Due to the high complexity of the problem, they used ant colony optimization (ACO) and particle swarm optimization (PSO) algorithms. Adamo et al. [25] developed a method to determine the lower bound for the time-dependent traveling salesman problem (TSP) in 2020.The time-based TSP is a generalization of TSP that depends on anode's position in the Hamiltonian path. Computational results indicated that the new mechanism in the branch and bound algorithm (BBA) improved outcomes. Jafari and Sheykhan [26] studied the transport path for semi-finished goods under fuzzy conditions in 2021.They employed a genetic algorithm (GA) to optimize machinery layouts in different stations. Optimizing the layout reduced the path costs to a bare minimum. Mohammadi et al. [27] analyzed the shortest Hamiltonian cycle (SHC) problem in the same year. They believed that the large-scale SHC problem had no precise solutions in 2021. Hence, they recommended using meta-heuristic algorithms such as particle swarm optimization (PSO) to solve the problem. Their evaluation outcomes indicated that PSO was highly effective in solving that problem. Pourasghar et al. [28] analyzed the clustering problem in 2022. They proposed a partitioning-based algorithm to cluster large-scale software systems. For this purpose, they employed the GA to deal with the complexity of the problem.

Operations research (OR) gained in popularity in Iran first in 1973 when it was taught in a management program at Tehran University. It is now taught in thirty-five doctoral, postgraduate, and undergraduate programs. Despite its fairly long history in human sciences, OR has not found its due place as opposed to other academic fields. Industry, agriculture, and the service sector do not practically use OR as they should. The demand for analysis in this field is roping in the Iranian higher education community [29]. This necessitates new studies in this area. The authors believe in the significance of such studies.

Optimization is the constant human quest to find the best solutions to both natural and engineering problems [30, 31]. Mathematical modeling was employed in this paper to find the Eulerian circuits in complete graphs. This is the first study of its kind. For this purpose, a method is proposed to turn "finding the Eulerian circuit" into an optimization mathematical model (MM).The model uses a pathfinding approach (i.e., the longest spanning path) to solve the problem.

3. Methodology

This study adopts mathematical modeling methodology. In this method, the relationships of variables are expressed through mathematical rules. This gives more precise control over decision variables. Using such control will lead to the optimization of dependent variables in decision-making [32].The optimization occurs because the variables could be connected as a chain, a hierarchy, or a network. Understanding and processing all those relationships might be difficult [33]. Mathematical modeling is a well-known method to handle such studies. If the designed models are assumed to be appropriate and efficient, they can typically be processed by most of the standard relevant software applications.

3.1. Eulerian Circuit

The GT defines the Eulerian Circuit (EC) as a path that begins from a vertex, visits all the edges (only once), and returns to the same vertex [34]. An EC exist when the graph is connected and the degree of each vertex, that is the total number of its connected edges connected, is even. A graph is Eulerian if and only if it is connected and the degrees of all its vertices are even [35].

3.2. Complete Graph

A complete graph is a simple graph where each vertex is connected to all other vertices by one edge. Usually, a complete graph with n vertices is denoted by k_n [36].

3.3. Adjacency Matrix

The adjacency matrix of a graph G with n vertices (named v_1 to $n \times n$ respectively) is an M matrix of zeros and ones denoted by $m_{i,j}$. Its $m_{i,j}$ element is 1 if there is an edge from v_j to v_j and zero if there is none [37].

4. Proposed Mathematical Model for Finding EC

Assume matrix $M = \left(m_{i,j}\right)_{n\times n}$ to be the adjacency matrix of a complete and Eulerian graph with distinct vertices. The EC beginning at s can be found through the following mathematical model:

$$Max\ Z = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{i,j} \tag{1}$$

Subject to:

$$\sum_{k=1}^{n} x_{k,j} = \sum_{k=1}^{n} x_{j,k} \quad ; \quad j = 1,2, \ldots n \tag{2}$$

$$x_{i,j} \le m_{i,j} \quad ; \quad i = 1,2, \ldots n \ \ and\ j = 1,2, \ldots n \tag{3}$$

$$x_{i,j} + x_{j,i} = m_{i,j} \ ; \ i = 1,2, \ldots n-1 \ \ and\ j = i+1, i+1, \ldots, n \tag{4}$$

$$x_{i,j} = \begin{cases} 0 \\ 1 \end{cases} \ ; \ i = 1,2,3, \ldots n\ and\ j = 1,2,3, \ldots n \tag{5}$$

In the above mathematical model, the objective function is to cover maximum number of edges in designing a path from the vertex i to the vertex j.

The first constraint means that the number of outgoing and ingoing edges connected to the vertex j must be the same.

The second one states that vertex i can connect to vertex j only if $m_{i,j} > 0$.

Constraint three expresses that each edge must be traversed exactly once.

The fourth constraint means $x_{i,j} = 1$ if we visit vertex i after vertex j otherwise $x_{i,j} = 0$.

This is a pure binary linear programming mathematical model.

4.1. Traversing the Solution Matrix Found by the Proposed Mathematical Model

Let us regard the current vertex as the source (s) and start traversing edges from this vertex. Select one of the edges to connect (postpone selecting the source vertex (s) as much as possible).Once the edge is traversed, delete it from the graph (set the corresponding element in the solution matrix

$(X_{n\times n})$ to zero). Replace the current vertex with the one at the end of the traversed edge. Repeat until all the elements of $X_{n\times n}$ matrix become zero.

Here are some numerical examples to demonstrate the efficacy of the proposed method.

- **Numerical Example (1)**

Consider graph k_3 shown in Figure 1. The EC discovered by our method in MATLAB is demonstrated in Figure 2.

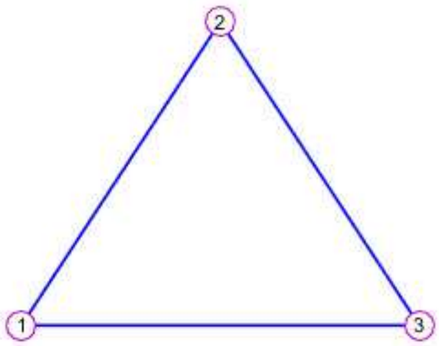

Fig. 1. Graph k_3.

Figure 2 depicts that the traversing of graph k_3 begins at vertice one. It traverses all the graph edges and returns to the source vertex in the end.

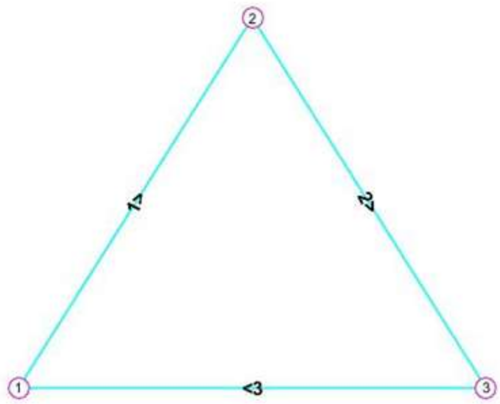

Fig. 2. Eulerian circuit for graph k_3.

The solution matrix is $X_{3\times 3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

The following sequence is a more accurate presentation of the EC.

1-->2-->3-->1

- **Numerical Example (2)**

Consider graph k_5 shown in Figure 3. The EC discovered by our method in MATLAB is demonstrated in Figure 3.

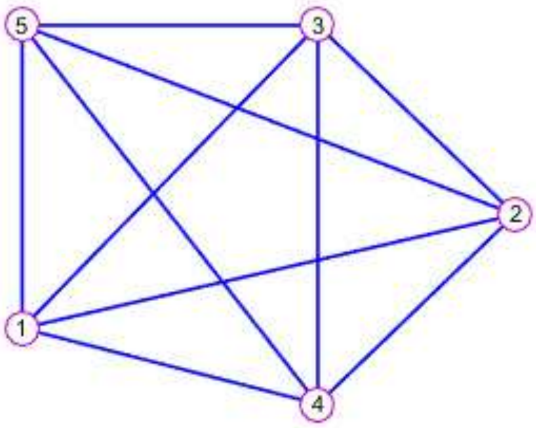

Fig. 3. Graph k_5.

Figure 4 depicts that the traversing of graph k_5 begins at vertice one. It traverses all the graph edges and returns to the source vertex in the end.

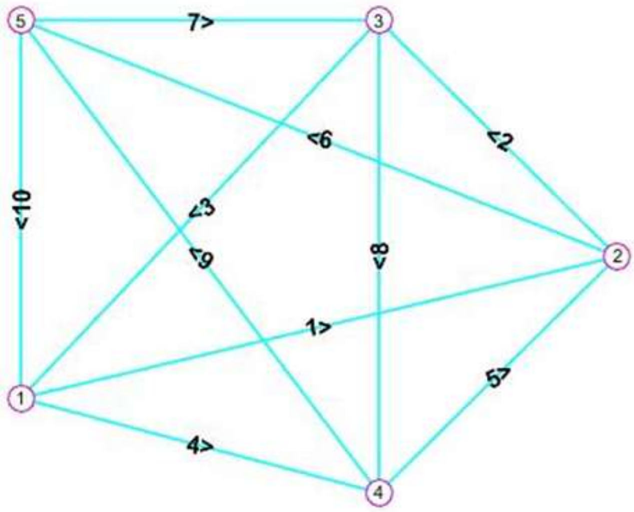

Fig. 4. Eulerian circuit for graph k_5.

The solution matrix is $X_{5\times 5} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$.

The following sequence is a more accurate presentation of the EC.

1-->2-->3-->1-->4-->2-->5-->3-->4-->5-->1

- **Numerical Example (3)**

Consider graph k_7 shown in Figure 5. The EC discovered by our method in MATLAB is demonstrated in Figure 5.

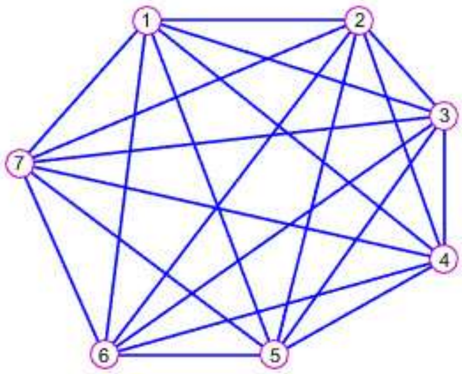

Fig. 5. Graph k_7.

Figure 6 depicts that the traversing of graph k_7 begins at vertice one. It traverses all the graph edges and returns to the source vertex in the end.

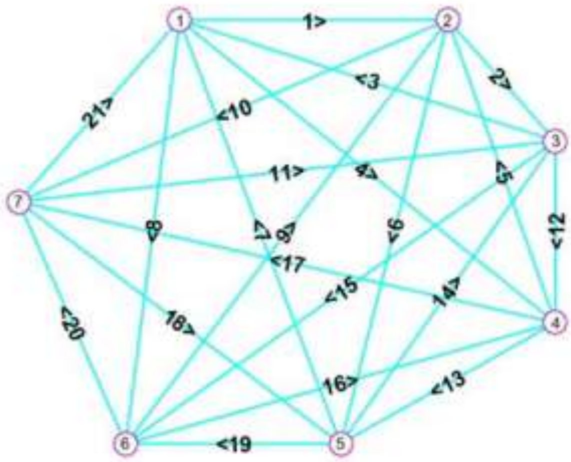

Fig. 6. Eulerian circuit for graph k_7.

The solution matrix is $X_{9\times 9} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$.

The following sequence is a more accurate presentation of the EC.

1-->2-->3-->1-->4-->2-->5-->1-->6-->2-->7-->3-->4-->5-->3-->6-->4-->7-->5-->6-->7-->1

5. Conclusion

Path finding has received great attention in operations research studies. It aims to optimize (*i.e.*, optimally design) paths through existing techniques in operations research. Finding the Eulerian circuit as a graph theory problem can be formulated as an optimization problem. This paper indicated that the problem could be studied from the perspective of operations research. Broadly speaking, finding the Eulerian circuit is a path finding problem. This paper proposed a pure integer mathematical model to describe the relationships between the variables in finding the Eulerian circuit. Finally, the results indicated that the modeled problem could be readily solved in certain programs such as MATLAB.

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Conflicts of Interest

The authors declare no conflict of interest related to this publication.

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