Summer 6-14-2023

High School Student Narratives of Error Analysis: A Qualitative Investigation into Literacy for Geometry Learning

Dean Petti
dpetti@students.kennesaw.edu

High School Student Narratives of Error Analysis:

A Qualitative Investigation into Literacy for Geometry Learning

by

Dean P. Petti

Kennesaw State University

June 14, 2023

Dissertation

Dr. Megan Adams, Chairperson

Dr. Darren Crovitz, Committee Member

Dr. David Glassmeyer, Committee Member

Table of Contents

Acknowledgments

Thank you to my dissertation committee for supporting me throughout this research. Thank you, Dr. Megan Adams, for being my committee chairperson and providing all of your guidance and advice about narrative inquiry and qualitative research, particularly with awareness of the dynamics and tensions that are involved and need to be managed. Thank you, Dr. David Glassmeyer, for taking me to extended levels of mathematics research, teaching, and learning that have helped me to grow more professionally as a Geometry teacher and have helped me to connect narrative inquiry to the ways students perform and think mathematically. Thank you, Dr. Darren Crovitz, for your publication *Sudden Possibilities: Porpoises, Eggcorns, and Error* which gave me insight of how narrative inquiry is connected to an explanation for students committing errors in their Geometry in a similar manner as they do in English.

I have much appreciation for my dissertation committee for providing a comprehensive approach that brings together the realms of narrative inquiry, mathematics, literacy, and error analysis. I am also grateful for the support that my parents, Patrick and Lorna Petti, who have supported me throughout the entire dissertation process. The motivation and encouragement my parents have provided are commendable and priceless. I would also like to thank my Paulding County High School principal, Mr. Craig Wilcox, and administrator, Ms. Heather DeJohn-Mathis, for introducing me and our entire school of the importance for implementing literacy into the Geometry classroom and every classroom. Mr. Wilcox and Ms. DeJohn-Mathis provided professional development that has taken me to new heights of Geometry learning that I never thought would be possible and has inspired me to conduct this study with implementing error analysis as literacy for the learning of Geometry.

Abstract

This narrative inquiry aimed to relate student narratives to error analysis of Geometry problems as a form of literacy implementation. Teachers need to develop a deep understanding and application of mathematics content through qualitative research to find connections to practice (Enderson et al., 2010). Geometry is a worthwhile subject to understand our world from various perspectives and to spur our imagination with constructing objects based on properties undergoing dynamic changes. However, teachers and researchers often wonder why most secondary students need help with the significance of learning and applying Geometry concepts. Literacy implementation bridges students' narrative experiences and relevance to Geometry concepts (Draper, 2002; Ratnaningsih & Hidayat, 2020).

This research involved four high school Geometry students revealing narratives about errors experienced with Geometry error analysis problems. Under a blend of a cognitive apprenticeship (Collins et al., 1991) and anchoring examples (Fast & Hankes, 2010), students responded to error analysis problems related to a Geometry concept and a field of employment. Prior research revealed how narratives about errant situations subconsciously affect errors from real-life experiences to problems in a course like Geometry (Clandinin & Connelly, 2000; Mertova & Webster, 2020). The errors included a wrong strategy needing comprehension, a miscomputation needing process skills, and faulty algorithms needing focus in the midst (transformational) or end (encoded) of a problem (Riastuti et al., 2017; Pomalato et al., 2020; Ratnaningsih & Hidayat, 2020).

This research’s findings reveal that students experience narratives involving errors related to Geometry problems. The narratives implement behaviors and actions by students that are analogous to the standards for mathematical practice. The findings support implications for

teachers having a model for literacy implementation based on the standards of mathematical teaching practices (Boston et al., 2017).

Keywords: Geometry, literacy, narrative inquiry, qualitative, error analysis, cognitive apprenticeship, anchoring, standards for mathematical practice, standards of mathematical teaching practices

Chapter 1: Introduction

The etymology of the word “Geometry” comes from the Greek words “geo,” meaning “earth,” and “metry,” meaning “measure.” Geometry resembles a blueprint needed to understand our world from various perspectives. Realistically, Geometry consists of the tools needed to measure structures and create processes that society needs to live comfortably and function properly. Within the realm of solving problems, Geometry is influential in spurring awareness of multiple pathways and relationships between concepts by exploring the comparisons and contrasts of changing shape properties. Moreover, aesthetically, Geometry opens imaginations to describe behaviors, relationships, and attributes of phenomena witnessed in daily experiences.

Experiences teaching Geometry generate questions concerning why conflict exists overall about the worthiness of the subject’s contributions. Personal epistemology inquires why most secondary students, including colleagues and the population in general, have struggled with the significance of learning and applying Geometry concepts. After considerable dialogue with students and colleagues, there needs to be some bridge, some unknown factor, that helps students understand, make sense of, and apply Geometry meaningfully.

Professional development sessions at my school of employment is a blessing to help discover and apply this hidden connection as literacy. The origin of the word “literacy” derives from the Latin words “littera,” meaning “letter,” and “acy,” meaning “quality, state, or condition.” The composition together forms a fundamental definition of literacy as a level of ability to read and write a letter. There is critical importance in defining literacy to accommodate all students with “extensive needs of support” that include standards of acquisition capabilities, social interaction expectations, and multimodal communications to read and write (Keefe & Copeland, 2011, p. 92). Regarding learning mathematics, literacy refers to a form of

constructivism where students make sense of the text by applying prior knowledge, learning what they want to know, and having a pathway to learn more knowledge (Draper, 2002). Mathematical literacy is the student's prowess to devise, apply, and explain mathematics in various contexts (Ratnaningsih & Hidayat, 2020).

Inquiry, debate, and skepticism exist concerning the relationship between literacy implementation and Geometry learning based on concrete form, appearance, and strategy (Ratnaningsih & Hidayat, 2020). Over the years, error analysis is one pathway that has proven to be an effective form of blending literacy with Geometry learning (Ratnaningsih & Hidayat). Although considered unorthodox, strong habits of mind that match mathematical practice standards develop when errors are included intentionally within Geometry problems for students to correct that relate to real-life situations (Rushton, 2018; Ratnaningsih & Hidayat). There is reason to believe that the errors existing within high school Geometry students' problem-solving serve as practical literacy lessons for what Borasi (1987) describes as "springboards to inquiry."

A Narrative Inquiry of Error Analysis

This qualitative research intends to analyze a narrative inquiry approach where high school students share personal stories about subconscious motives (Mertova & Webster, 2020) that spur some understanding of errors transpiring within responses and potential solutions to Geometry problems. Numerous assessment experiences of high school student responses to Geometry problems are persuasive to believe that underlying stories exist that are suspect for the presence of errors in problems that are in the form of a selected wrong strategy, a miscalculation, or a faulty sequence of operations (Riastuti et al., 2017; Pomalato et al., 2020; Ratnaningsih & Hidayat, 2020). Clandinin and Connelly (2000) refer John Dewey's work concerning how these types of experiences lead to inquiry opportunities for a teacher-researcher as a way to help

improve the understanding of a continuous role that student narrative based on one experience emerging out of prior experiences and leading to future experiences.

Research shows metaphors inform the mind to engender or enact current embodied and embedded actions without realization based on knowledge acquired from previous experiences (Lakoff & Johnson, 1980). The nature of choosing a wrong strategy is due to the presentation of Geometry as a rote appearance of rigorous proof that still needs to have some bearing on the endeavors and goals a student wishes to achieve (Simons & Gold, 2011). Maybe narratives of what a student has learned to think of as important subconsciously explain why the notion of acquiring a passing Geometry grade overrides the importance of noticing and correcting miscomputations to make sense of Geometry learning on a personal level (İlhan et al., 2021). Perhaps narratives involving teacher-led instruction are a root depiction of why students seek teacher confirmation for being correct rather than developing the self-confidence needed to drive their own doing of Geometry to detect and avoid faulty algorithms (Erickson & Herbst, 2018).

Enderson et al. (2010) assert the need for teachers to develop an in-depth understanding, reasoning, and meaning of mathematics content through qualitative research to find connections to practice. After considerable dialogue with secondary students, I have determined there needs to be some temporal bridge from a narrative perspective between past lived experiences as theory and present error analysis as a reality that helps students with some understanding of their behavior and performance for writing responses to Geometry problems (Polkinghorne, 1988).

This study is worth undertaking because of presenting a most beneficial opportunity for students to discover and build on their identity as effective, efficient Geometry problem solvers through the sifting of understanding behaviors and actions that are subconsciously occurring from past narrative stories (Riessman, 2008). This study is important to Geometry education and

is sufficiently original because of how narrative inquiry drives the researcher and participants to deal with existing tensions that include temporality, people, action, certainty, and context (Clandinin & Connelly, 2000). These tensions elicit and reveal information that is diverse and genuine within a natural setting of student learning (Creswell and Poth, 2018). The tensions also reveal student learning differences from the perspectives of a Geometry teacher, which are beneficial for a teacher to "think outside the box" by improvising and adapting to the differences, along with the teacher continuing to learn as a narrator (Bateson, 1994).

This research offers practice on how to handle misconceptions via the narrative stories of students that have somehow intertwined a repertoire of interpretations that influence how students respond to Geometry problems and how responses evolve for the future (Geertz, 1995). This study also provides an environment of role reversal, where each student as a participant becomes a teacher, and the teacher, as a researcher, becomes a listener for a student to have a voice that matters and to learn how to teach lessons like a teacher through narration (Coles, 1989). The stories are all intended to relate to why students either choose a wrong strategy, commit some form of miscalculation, or write an algorithm containing errors in their problem solving trials (Riastuti et al., 2017; Pomalato et al., 2020; Ratnaningsih & Hidayat, 2020). These studies all lead to the following research questions:

1) What stories can students tell about using a wrong strategy in a Geometry error analysis problem?
2) What stories can students tell about using a miscalculation within a Geometry error analysis problem?
3) What stories can students tell about using a faulty algorithm in a Geometry error analysis problem?

An Overview of the Narrative Inquiry

A cognitive apprenticeship learning model applies as a means for students to become informed about the relationship between learned knowledge and skills from narratives as it pertains to error analysis of Geometry problems related to real-life employment fields (Collins et al., 1991). An anchoring example learning model also applies as an aim for students to connect and construct their learned knowledge from narratives for meaning to error analysis and real-life contexts (Fast & Hankes, 2010). The blend of these models, along with the aforementioned narrative inquiry studies and research questions, drive the beginning of this study to implement weekly tasks about various fields of employment that apply matching concepts of Geometry using error analysis problems. The tasks are trivial with the sole purpose to promote student awareness of how concepts of Geometry are relevant in various contexts.

As participants, leadership opportunity are afforded to students to apply shared narratives as a bridge to understand why certain errors are occurring and how to correct them based on the mentalities learned from their narratives. Documents, semi-structured interviews, and observations gather from participant responses to acquire data as field texts. The field texts base on narrative stories and projected metaphors that relate to the sample erroneous responses that participants read on the tasks and the sample responses they write for correction. Analysis of field texts share results with Geometry teachers and stakeholders as attempts to explain how and why potential past experiences carry out latent root causes for students committing errors on Geometry problems. Only the relationships between narrative stories and error analysis performances occur in this study as a means of implementing literacy with Geometry learning. Future studies may focus on how students correct problems.

Narrative inquiry applies as the methodology for this research. Polkinghorne (1988) explains how narrative inquiry is descriptive leading to the meaning of narrative stories and how

narrative inquiry is explanatory by connecting narrative stories to current behaviors and actions. Both types of inquiry pose unique ways potentially to link how student stories from the past are, in some way, causing the constant, subliminal errors that students commit in writing responses to Geometry problems. This study is not designed to demonstrate how to overcome these misconceptions; rather, this research seeks to provide student mentalities to a Geometry teacher. This is important to highlight student narratives that are embedded within their minds and to illuminate how these narratives have concealed roles in hindering appropriate student learning of Geometry. Finally, this research may help students become cognizant of those hidden roles that prevent proper learning of Geometry. Coles (1989) upholds how the learning of a subject like Geometry becomes narrative through interactive storytelling by participants and active listening by the researcher. For the discovery and awareness of narrative stories as possible culprits to student errors in Geometry problems, the teacher as a researcher learns by being that listener and taking a looking inward-outward stance of participant voices that narrative inquirers possess (Clandinin & Connelly, 2000). Schön (1983) emphasizes the power of narrative to restory what participants share through member checking that helps teachers go beyond from indoctrinations of required practices and into a realm of reflective, applicable, and meaningful practices based on student experiences.

This dissertation includes four additional chapters. Chapter 2 contains a literature review that analyzes existing research about the error performances found in Geometry problems as it relates to narrative inquiry. Chapter 3 focuses on methodology by explaining how and why this study's research will be carried out, focusing on the need for data collection in the form of field texts containing responses from semi-structured interviews, observations, and documents. Chapter 4 describes the results of this research by transforming the field texts into research texts

and includes analyses and interpretations of the results. Chapter 5 responds to the aforementioned research questions.

Definition of Terms

Error Analysis–the implementation of erroneous examples related to at least one mathematical concept that allows students to gain more insight of the concepts through correction of the examples (Borasi, 1996).

Mathematical Literacy–the sense-making of mathematical text by using prior knowledge to verify mathematics in various contexts (Draper, 2002; Ratnaningsih & Hidayat, 2020).

Wrong Strategy–an erroneous way to solve a problem based on a need for comprehension.

Miscomputation, Miscalculation, or Misestimation–an erroneous way to solve a problem based on a need for process skills.

Faulty Algorithm–an erroneous way to solve a problem based on a need for correct transformation during the problem or a need for correct encoding at the end of the problem.

Lexile–a measure of an individual's reading ability or a text's readability (or difficulty) represented by a number followed by an "L."

Chapter 2: Literature Review

There are sound, historical reasons for applying error analysis as literacy to learning Geometry. Since its conception, learning Geometry concepts has undergone several narratives of overcoming errors through iterative reading and writing from many notable geometers. Mathematicians learned to be cognizant of errors leading to their correction in a very similar way to how students need practice and experiences to make sense of Geometry (Borasi, 1996). For mathematicians, the context of noting and overcoming errors perceives as meaningful. For our students, the context of detecting and correcting errors needs a meaningful perception (Collins et al., 1991; Fast & Hankes, 2010). This chapter begins by exploring some of the origins of error analysis for supporting its implementation for learning Geometry. The exploration leads to segments on how error analysis is a form of mathematical literacy and how error analysis is a part of math education in general. There is a concentration on the types of errors committed by students in the form of a selected wrong process, a miscalculation, and faulty operations (Riastuti et al., 2017; Pomalato et al., 2020; Ratnaningsih & Hidayat, 2020) that narratives explain (Clandinin & Connelly, 2000). Finally, metaphorical learning models focus on establishing meaning to connecting narratives for awareness and corrections of errors in Geometry problems.

Origins of Error Analysis in Mathematics

The foundation of all types of Geometry has its roots in Euclid's Parallel Postulate:

> That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. (Heath, 1956, p. 353)

Historical explorations have shown how equivalent versions of this postulate have been formed that were considered true for all types of geometries (Scriba & Schreiber, 2015). In 1796,

Scottish mathematician John Playfair rejuvenated the ideas of the Egyptian mathematician Ptolemy by conjecturing:

> Through a given point only one parallel can be drawn to a given straight line or, Two straight lines which intersect one another cannot both be parallel to one and the same straight line. (Heath, 1956, p. 387)

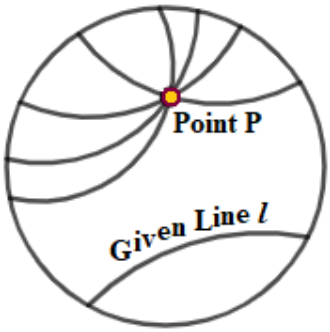
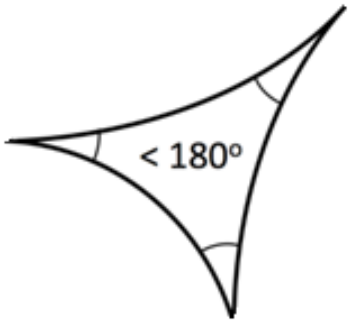
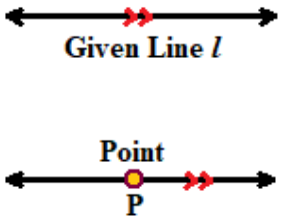
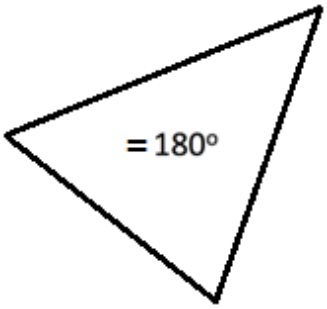
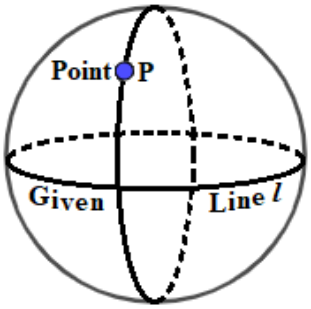
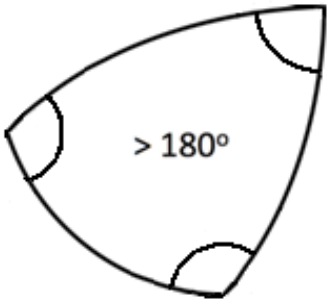
In 1823, French mathematician Adrien-Marie Legendre extended on the Parallel Postulate to visualize how the sum of three angles of a triangle is 180°, known today as the Triangle Sum Theorem:

> (3) There exists a triangle in which the sum of the three angles is equal to two right angles (Legendre). (Heath, 1956, p. 388)

Scriba and Schreiber (2015) claim that mathematicians initiated the first non-Euclidean models of spherical and hyperbolic geometry by perceiving our world as three-dimensional rather than a two-dimensional flat surface: the outcome of the perception disproved Euclid's postulate, Playfair's axiom, and Legendre's Triangle Sum (see Figure 1).

Figure 1

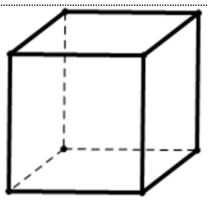
Non-Euclidean Geometries with Conflicting Characteristics to Euclidean Geometry as a Result of Error Analysis of the Parallel Postulate Applying to all Geometries

GEOMETRY	PARALLELISM Given a line and a point NOT on the line, there … the given point and is PARALLEL to the given line.		TRIANGLE SUM The sum of the angles a triangle is …
Non-Euclidean Hyperbolic (Negative Curvature)		… are INFINITELY MANY LINES that pass through …	
Euclidean (Zero Curvature)		… is EXACTLY ONE LINE that passes through …	
Non-Euclidean Spherical (Positive Curvature)		… is NO LINE that passes through …	

Borasi (1996) mentioned the errors performed by 19th century mathematicians who strove to disprove Leonhard Euler's polyhedron formula (see Figure 2). Lakatos (1976) cited the attempted polyhedron sketches from Swiss mathematician Simon L'Huilier in 1812 (see Figure 3) and German physician Johann Hessel in 1832 (see Figure 4). However, Borasi claimed French mathematician Ernest de Jonquières clarified the definition of a polyhedron to disprove L'Huilier (see Figure 5). To illustrate L'Huiler's error, Lakatos (1976) used the metaphor of a child in a mother's womb as two human beings with two different heads. Similarly, Borasi stated how German mathematician August Ferdinand Möbius provided another specific definition to disclaim Hessel's contributions using an example versus non-example approach (see Figure 6).

Figure 2

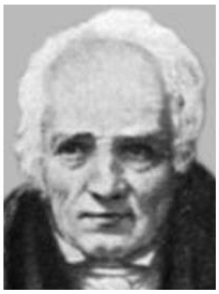
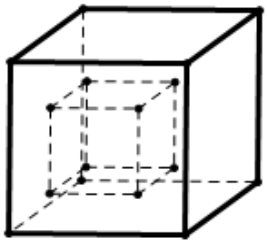
Leonhard Euler's Formula

 Leonhard Euler (1707-1783)	Polyhedron	
	# Faces (f)	6
	# Vertices (v)	8
	# Edges (e)	12
	Characteristic f + v - e	6 + 8 - 12 = 2

Note: For any three-dimensional shape called a polyhedron, the Euler characteristic is defined as the number of faces plus the number of vertices minus the number of edges, which is equal to 2.

Figure 3

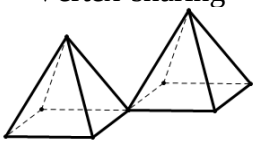
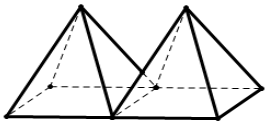
L'Huilier's Attempted "Cavity" Polyhedron to Disprove Euler's Formula

 Simon Antoine Jean L'Huilier (1750-1840)	Attempted Polyhedron	
	# Faces (f)	12
	# Vertices (v)	16
	# Edges (e)	24
	Characteristic f + v - e	12 + 16 - 24 = 4

Note: The characteristic equals 4 instead of 2.

Figure 4

Hessel's Attempted Vertex- and Edge-Sharing Polyhedron to Disprove Euler's Formula

 Johann F. C. Hessel (1796-1872)	Attempted Polyhedron	Vertex-sharing 	Edge-sharing 
	# Faces (f)	10	10
	# Vertices (v)	9	8
	# Edges (e)	16	15
	Characteristic f + v - e	10 + 9 - 16 = 3	10 + 8 - 15 = 3

Note: The characteristic equals 3 instead of 2.

Figure 5

De Jonquières' Polyhedron from a Solid to a Surface of Connected Faces to Disprove L'Huilier

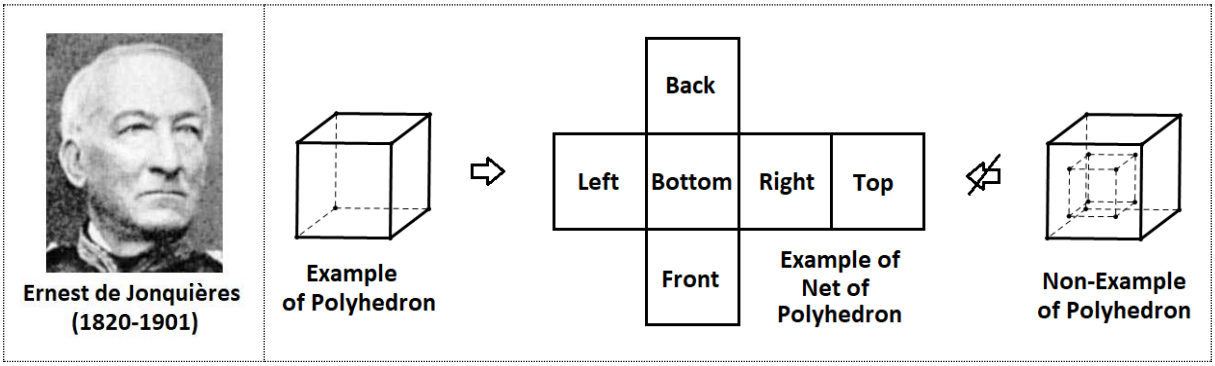
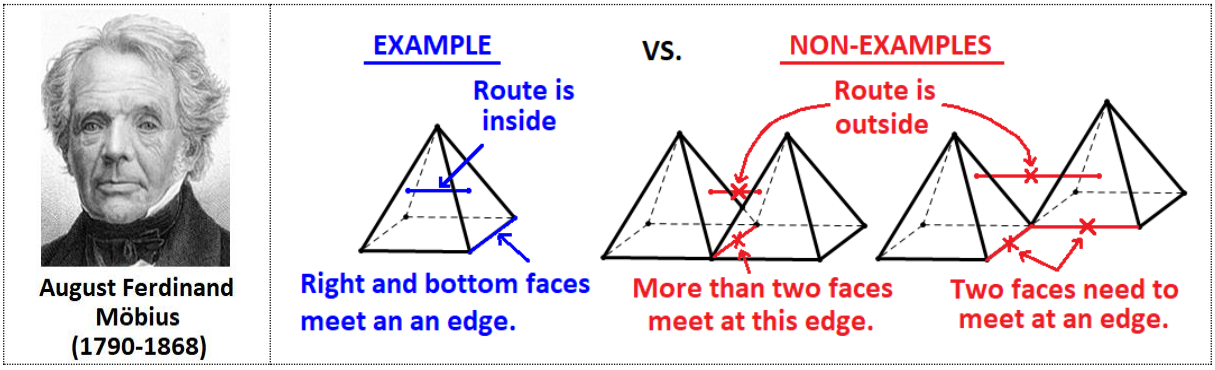


Figure 6

Möbius's Polyhedron as Edges Connecting Faces and Interior Routes to Disprove Hessel

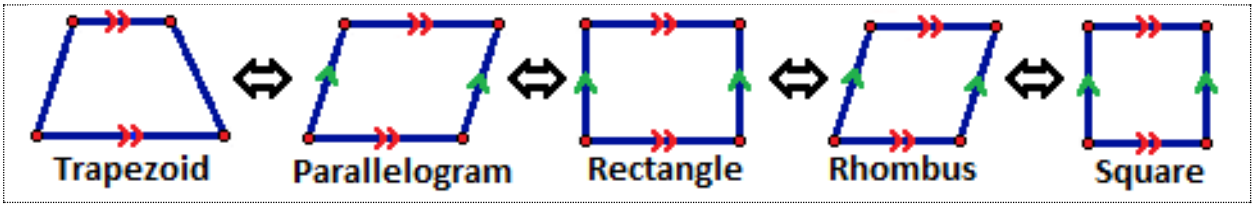


The debates resulting from narrative histories involved states of mind between persons going from tensions of absolute truths as theories to tensions of wonders as realities (Polkinghorne, 1988; Clandinin & Connelly, 2000). The debates caused tensions among stakeholders on definitions, particularly with those indoctrinated to think one way versus those inclined to be more accepting and open-minded (Clandinin & Connelly; Usiskin, 2008). Narratives encompassed whether prior knowledge can be proven untrue or remained true based on personal experiences (Polkinghorne; Clandinin & Connelly). For Geometry, recent examples included all parallelograms as trapezoids (see Figure 7) and all kites as rhombi (see Figure 8).

Stakeholders are allowed either to choose one or accept both definitions of these quadrilaterals depending on experiences (Clandinin & Connelly, 2000; Usiskin, 2008).

Figure 7

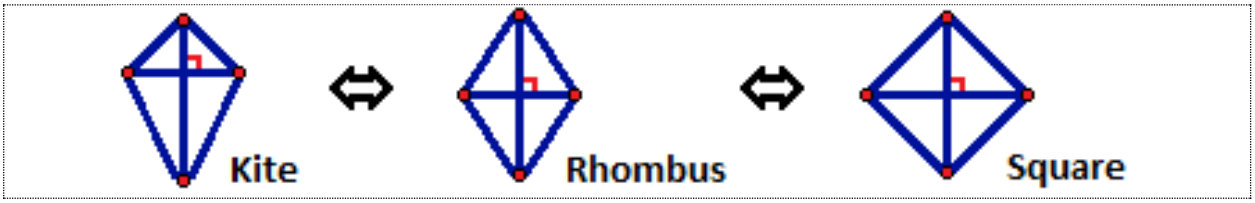
The Transformation of a Trapezoid into Parallelograms



Note: The transformation conflicts with the Euclidean definition of a trapezoid having exactly one pair of opposite parallel sides.

Figure 8

The Transformation of a Kite into Rhombuses



Note: The transformation conflicts with the Euclidean definition of a kite having exactly two pairs of consecutive congruent sides.

Error Analysis as Mathematical Literacy

The history of error analysis in mathematics suggests the learning practices mathematicians have undergone are congruent to the standards of mathematical practices (see Figure 9) that our students need to undergo as well for mathematical literacy (Hillman, 2014). Past mathematicians demonstrated mathematical literacy by making sense of the text and using prior knowledge to explain mathematics in various contexts (Draper, 2002; Ratnaningsih & Hidayat, 2020). In addition, mathematicians have communicated mathematical literacy through

discourse with each other (Sfard, 2007; Hillman). One of the features of the discourse has been familiar narratives in the form of their common language and prior knowledge developed from each other (Sfard; Hillman). With error analysis over extended periods, mathematicians have made sense of and persevered with mathematical literacy.

Figure 9

The Standards of Mathematical Practice

1. Make sense of problems and persevere in solving them.	2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.	4. Model with mathematics.
5. Use appropriate tools strategically.	6. Attend to precision.
7. Look for and make use of structure.	8. Look for and express regularity in repeated reasoning.

Concerning the instances of the correction of Euclid's Parallel Postulate and the upholding of Euler's polyhedron formula, mathematicians explored the relevance and application of vocabulary, notation, and quantities; these actions produced abstract and quantitative reasoning. Mathematicians' error analysis have included searching for structure, attending to precision, constructing viable arguments, and critiquing the reasoning of others. These standards are particularly true with how de Jonquières and Möbius used revised polyhedron definitions to disprove L'Huilier and Hessel's reasoning, respectfully (Lakatos, 1976; Borasi, 1996). Both de Jonquières and Möbius used an example and non-example approach to their error analysis, strengthening their viable arguments with their mathematical literacy (Loibl & Leuders, 2019). Mathematicians' error analysis also included modeling mathematics and strategically using appropriate tools. Here, the prime example is the visualization and connection of Euclidean Geometry with hyperbolic and spherical Geometry (see Figure 1). Regularity in repeated

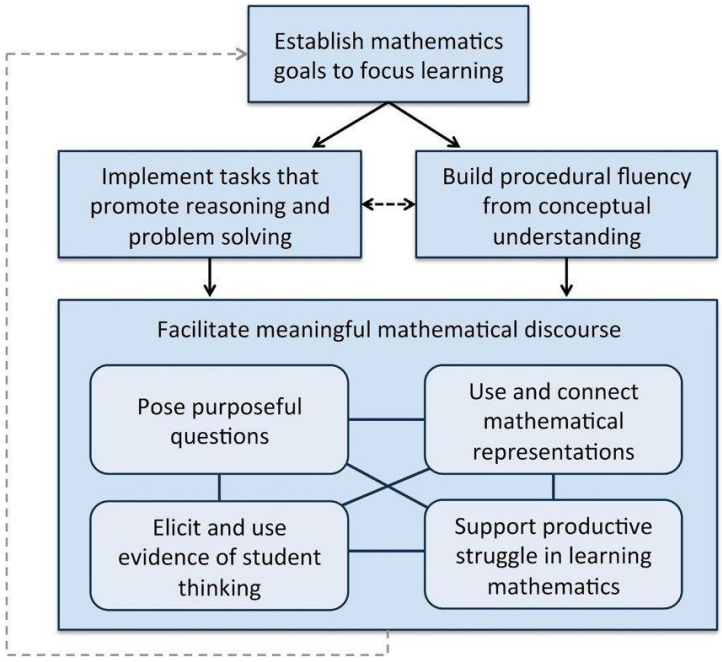
reasoning is present with error analysis, especially with the repeated ways trapezoids and kites transform into parallelograms and rhombi, respectively (Usiskin, 2008).

Error Analysis in Math Education

The ways of thinking exhibited by past mathematicians serve as the foundation of relating error analysis as mathematical literacy to the standards of mathematical teaching practices (see Figure 10). The primary goal to focus on learning with error analysis has been to raise inquiry about the validity of mathematical statements that are traditionally true (Borasi, 1996). Real-life erroneous examples have been implemented for students to rewrite corrected responses to apply more understanding of prior knowledge and to promote reasoning and problem-solving tasks (Große & Renkl, 2007). Simultaneously, reshaping prior knowledge based on discovering truths and fallacies of concepts within the erroneous examples builds procedural fluency (Tsovaltzi et al., 2010). For mathematicians and students, error analysis in its beginning stages has been influential in triggering discourse for sense-making and explanations of conceptual understanding (Tsovaltzi et al.; Borasi; Große & Renkl). The discourse is evident in the error analysis cases of Euclid's Parallel Postulate and Euler's Polyhedron Formula, as mathematicians strove for sense-making and explanation of truths and fallacies over time (Lakatos, 1976; Scriba & Schreiber, 2015; Borasi). In each case, purposeful questions pose as "dialogic" discourse, where an environment of open-ended thoughts and various perceptions are efforts to confirm truths (Trninic et al., 2018).

Figure 10

The Standards of Mathematical Teaching Practices



Note: This figure was produced by Boston et al. (2017, pg. 215) showing the fluidity of relationships between implementation of the standards of mathematical teaching practices.

In error analysis, "visual mediators" constitute mathematical representations in such displays as formulas, diagrams, and drawings that apply as an attribute of discourse for sense-making (Sfard, 2007; Hillman, 2014). For mathematicians of the past, representations were present for discourse through the adaptation of Euclid's Parallel Postulate to other geometries (see Figure 1) and the adaptations of the definition of a polyhedron (see Figures 5 and 6). Like our mathematical predecessors, students progress through a productive struggle with error analysis discourse by capturing, processing, and assessing information from reading passages to model more specific truth in mathematics learning (Pomalato et al., 2020).

Through the discourse of error analysis, evidence of student thinking applies in various ways. Students assess their prior narrative experiences of what they consider valid and expand

beyond their narratives into sense-making to discover what is true (Kline, 1980). As a strategy, students have learned how to revise and modify the information they read to defy deviations from the norm (Borasi, 1996). Students generally learned to question the customary truths in math and other contexts outside the math classroom through reasoning sensibly to correct erroneous problems (Große & Renkl, 2007). Students detected and repaired personal errors by rewriting corrected versions of bad examples, particularly when the bad examples matched the misconceptions arising from their narratives (Kawasaki, 2010). Error analysis motivated students to make sense of and correct personal fallacies that expand prior knowledge from narratives (Durkin & Rittle-Johnson, 2012). Using erroneous examples applied student narratives and built more comprehension of math content over time (McLaren et al., 2012). During the discourse, learning gaps revealed and led to additional inquiries as efforts to maximize sense-making and minimize gaps (Loibl & Rummel, 2014). Learning gaps showed with the contrasting definitions of polyhedron between L'Huilier and Hessel versus de Jonquières and Möbius, respectively (Borasi, 1996). Over time, discourse decreased the learning gap, with de Jonquières and Möbius clarifying examples and non-examples of polyhedra (see Figures 5 and 6).

With error analysis, students related personal narratives to mathematical practice standards, specifically with sense-making and perseverance in finding precise solutions, building successful arguments, and critiquing peers (Rushton, 2018). In addition, mathematical errors allowed the creation of new criteria to establish the truth of results by citing how a concept changes through discourse (Borasi, 1996). Non-Euclidean geometries elicited through the discourse of disproving the Parallel Postulate true for all geometries (Sfard, 2007; Hillman, 2014). Discourse facilitated stakeholders to uphold relationships using updated or new information, like how the faces, vertices, and edges of a polyhedron sustain using a definition

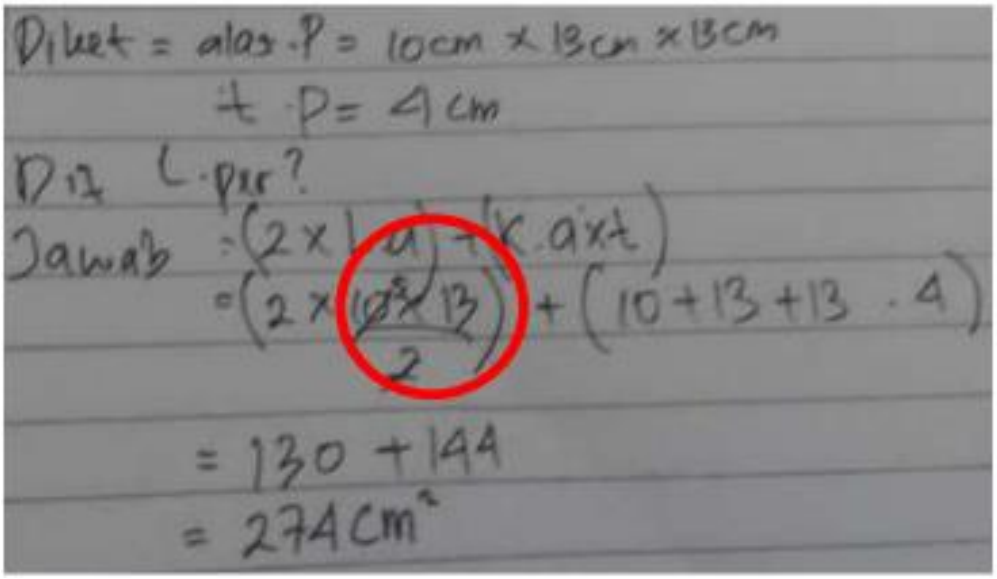
based on surfaces rather than solids (Borasi; Sfard; Hillman). The discourse has continued to aid stakeholders in discovering new representations, like those of quadrilaterals that go against accepted properties (Usiskin, 2008; Sfard; Hillman). With error analysis, inquiry for truth elicits through discourse, and learning outcomes resulting from personal narratives have the potential to change (Sfard; Hillman).

The Wrong Strategy

A standard error students commit on Geometry problems is choosing the wrong strategy (see Figure 11). Riastuti et al. (2017) asserted that the cause of this error is the student needing comprehension to make sense of what the problem is asking for and to know an appropriate strategy for solving the problem. Mertova and Webster (2020) implied at least one story exists in students' lives that emphasize learning as a way of knowing what to do to be sensible. Clandinin and Connelly (2000) implied such a story is valuable in potentially helping the student connect to a realm of perseverance needed to understand how to solve the problem and devote a genuine investment of time for knowing and practicing strategies for appropriate selection of solving the problem. Lakoff and Johnson (1980) suggested a metaphor for comprehension can be effective for students to relate a story of success or determination with solving a real-life problem to the same mentality needed for solving a Geometry problem. Polkinghorne (1988) stressed how the temporal connections of experience with a context like comprehension in various realms can foster the application of the narrative that is meaningful to the student to a similar application of finding an appropriate process for solving a Geometry problem.

Figure 11

The Wrong Strategy Error Based on Need for Comprehension



Note: This figure was produced by Riastuti et al. (2017, pg. 3) showing how 13 was used to find the surface area of a right triangular prism instead of 12 that should have been derived from the Pythagorean Theorem.

Riessman (2008) offers how narrative can help a student discover their identity for their ability to comprehend and solve a Geometry problem by recalling their persona of comprehending and solving a problem they have experienced in real life. Bateson (1994) suggests how teachers can gain more insight into how students acquire a comprehension of a Geometry problem through their narratives involving comprehension. Geertz (1995) implies that a narrative involving comprehension and what a student interprets as facts explain how and why comprehension occurs in its nature and how it continues to progress at a subconscious level. Coles (1989) encourages the Geometry teacher to become the narrator and listener of narratives that students share as an exchange to understand the nature of student comprehension of problems. Schön (1983) claims that restorying by students of a narrative of comprehension can help the teacher as a researcher better interpret how and why students are

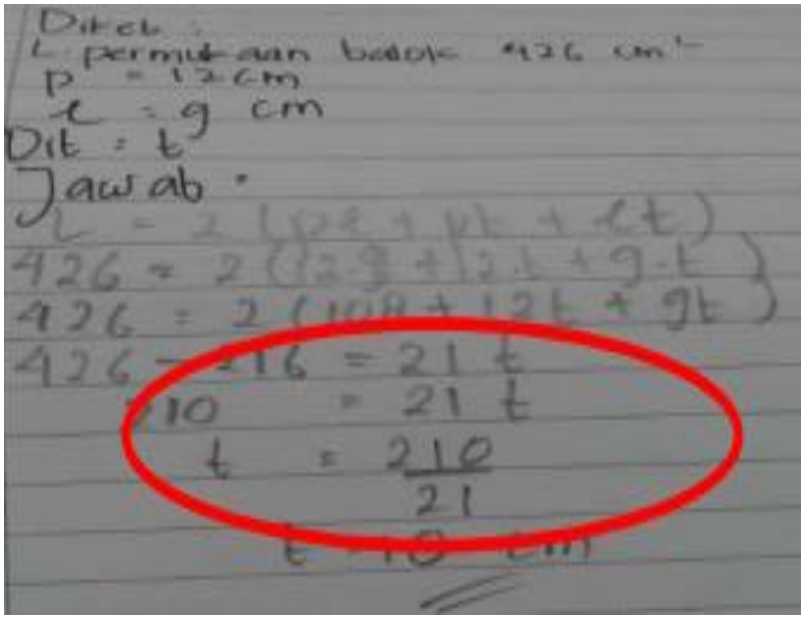
committing errors involving comprehension and adopt designs of lessons that help students understand and correct the error as reflective practices. McLaren et al. (2012) suggest that erroneous examples can provide a more profound learning experience that can help students build upon their initial understanding of math strategies, leading to a deeper understanding over time. Große and Renkl (2007) imply that since errors are always present in real life, it is beneficial for teachers to implement situations of how to manage comprehension errors that encourage students to have conflicting perceptions with prior, practical knowledge based on narratives and to construct more coherent, thorough strategies of solutions.

The Miscomputation

Another frequent error that transpires with student trials on responding to Geometry problems is in the form of miscomputation (see Figure 12). Riastuti et al. (2017) characterize this error as a "process skill," with a need for the student to know how to appropriately apply a process within their solution to a problem so that results make sense in context. Research shows how prior knowledge most likely formed by narratives of students reshape by using an erroneous solution of a problem as a scaffold to derive a corrected version of that solution (Große & Renkl, 2007; Durkin & Rittle-Johnson, 2012; Loibl & Rummel, 2014; Loibl & Leuders, 2019; Barbieri & Booth, 2020). Kawasaki (2010) reports how teachers help students be more likely to repair their errors by comparing flawed examples with their corrected ones, particularly when the flawed examples match the misconceptions of students. Borasi (1987) encourages teachers to implement a form of literacy where a process is presented correctly in at least one case that stimulates reflective thought by students on whether the process is to be true for all cases in what is termed as the "degree of wrongness."

Figure 12

The Miscalculation Error Based on Need for Process Skills



Note: This figure was produced by Riastuti et al. (2017, p. 4) showing how the sum of 12t and 9t equaling 21t should have been multiplied by 2.

Loibl and Leuders (2019) encouraged teachers to generate student awareness and reshape misconceptions through the comparison of incorrect prompts that represent the problem with their corrected version. Mertova and Webster (2020) implied how a narrative can be involved with process skills that indirectly make students believe the inclusion of this error is needed because of a hunch or as a product that an interested stakeholder would like to see. Clandinin and Connelly (2000) asserted this narrative can assist the student to understand why this miscomputation is occurring and why it would make sense to perform in the context of the narrative but not necessarily in the context of solving a Geometry problem.

Lakoff and Johnson (1980) suggested a metaphor for process skills can benefit students to reflect how the miscomputation can make sense in certain situations but not in others. Polkinghorne (1988) emphasized comparisons of temporality with procedural

experiences as a way to understand the effects of performing computations in different contexts. Riessman (2008) stressed the role of narrative on a topic like process skills to help students understand their persona of how to manage and foresee consequences of performing miscomputations. Studies recommended teachers should be listeners of process skill narratives as a way to interject intermittently a showcase to students how their process can be correct within that narrative and to offer students the challenge of how they can correct the process so that it makes sense to the solution of a Geometry problem (Coles, 1989; Bateson, 1994). Ingram et al. (2015) mentioned how the inclusion of erroneous examples has had a positive impact on student narrative and teacher feedback on the learning math process than with just correct examples alone. Geertz (1995) suggested the combination of narrative about the process and some appealing features for including the error can offer a motive as to why it occurs continuously by instinct. Schön (1983) supported the restorying of a narrative-like process skill for the teacher to delve deeper into the mentality of students for designing lessons that engage how misconceptions can be reshaped in a sensible direction.

The Faulty Algorithm

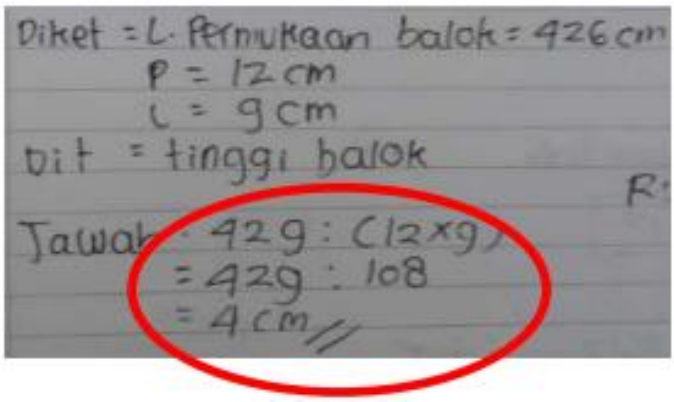
Recurring errors that are observed within student responses to Geometry problems are known as faulty operations (Riastuti et al., 2017). These errors either occur in the midst or at the end (see Figures 13 and 14, respectively) of a problem solving solution. Riastuti et al. perceived these errors as “transformational” during the process and “encoding” at its conclusion. Mertova and Webster (2020) suggested that a narrative can represent the occurrence of transformational/encoding errors that uphold students to believe what is being experienced makes sense in their minds. Clandinin and Connelly (2000) implied this narrative can be used to help the student to understand the need to question and pay attention to detail

and sense-making in all facets of problem solving. Rushton (2018) implied to teachers how this relationship between narrative and error analysis can be essential for students to engage with standards of mathematical practice, specifically with sense-making and perseverance in solving problems and attending to precision.

Lakoff and Johnson (1980) suggested that a metaphor for transformational/encoding errors can benefit students to reflect and check on awareness of the possible existence of faultiness. Polkinghorne (1988) stressed how comparisons of temporality with faultiness can serve as reminders of how these prior experiences can be unintentionally happening again currently and in the future with a need for student awareness. Riessman (2008) viewed how the narrative of transformational/encoding errors can spur moments where students recall their identity for handling the detection and correction of these errors when they occur.

Figure 13

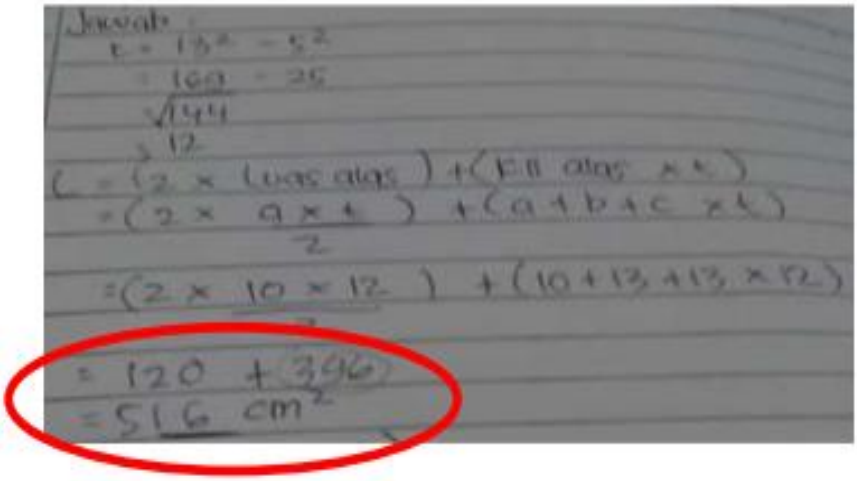
The Faulty Algorithm Error Based on Need for Transformation During the Response



Note: This figure was produced by Riastuti et al. (2017, p. 4), showing how a volume formula was used instead of surface area.

Figure 14

The Faulty Algorithm Error Based on Need for Encoding at the End of the Response



Note: This figure was produced by Riastuti et al. (2017, pg. 4) showing how the problem needed to be solved correctly by adding 120 plus 179 at the end of the problem.

Research suggested teachers listen to narratives of transformational/encoding errors as a means for students to check several times for their possible existence and for determining the ways they provide modifications for correction (Coles, 1989; Bateson, 1994). Geertz (1995) conjectured the bond between narrative and problem solving solutions concerning these faulty operation errors can provide insight to students into what needs inspection within an algorithm of operations. Schön (1983) upheld the restorying of a narrative like transformational/encoding errors for teachers to offer lesson opportunities for students to detect and modify these errors. Große and Renkl (2007) stated how teachers can help students learn more deeply by engaging with a discourse on erroneous problems that trigger students' algorithmic reasoning. Durkin and Rittle-Johnson (2012) encouraged teachers to apply transformational/encoding errors to increase the chances of students wanting to correct their fallacies to expand upon limited prior knowledge.

Metaphorical Learning Models for Meaning

Metaphors can govern our interests and performances based on knowledge learned from narratives (Lakoff & Johnson, 1980). Models of learning that focus on connections of meaning can be represented by topics that lead to metaphors. Cognitive apprenticeship is one of those models that can be viewed as "the vehicle for transmitting the knowledge required for expert practice" (Collins et al., 1991, p. 1). An anchoring example is another learning model that "clearly illustrates a target concept that is not well understood or misunderstood" (Fast & Hankes, 2010, p. 331). Cognitive apprenticeship is a means for students to become informed about the relationship between learned knowledge and skills needed for applications (Collins et al.). Anchoring examples are ways students can connect and construct their learned knowledge to real-life situations (Fast & Hankes). Together, cognitive apprenticeship and anchoring can interpret how prior knowledge learned from narratives can relate to error analysis of Geometry problems that involve skills in various fields of employment (Collins et al.; Fast & Hankes). Benefits include a more in-depth comprehension of narrative to error analysis performances of a Geometry concept, increased confidence to apply that concept, and the development of a more positive attitude about the importance of learning Geometry in general (Fast & Hankes).

In conjunction, Trninic et al. (2018) implied how an understanding of abstract math concepts can be optimized with error analysis tasks that focus on the relationship between content and application. Collins et al. (1991) suggested how the connection between knowledge from narratives and the same knowledge needing to be modified correctly to address sensibility to solve real-life problems become visible. The modeling blend of cognitive apprenticeship and anchoring allowed students to take ownership of modifying the knowledge learned from their narratives into a leadership role of reading for errors and writing corrected solutions (Collins et al.; Fast & Hankes, 2010; İlhan et al., 2021).

Chapter 3: Methodology

This section reviews the selected theory of social constructivism for this research. A description of a conceptual framework leads to the formation of the research questions. A discussion of narrative inquiry as the research design ensues, emphasizing tensions caused by temporality, people, action, certainty, and context (Clandinin & Connelly, 2000). The phrasing of research questions follows, based on the findings of methodologies in prior research. Data collection describes participants' experiences. Data analysis procedures focuses on the use of codes for emerging themes.

Social Constructivism Theory

Social constructivism (SC) is the paradigm that drove the design of this research. Creswell and Poth (2018) summarized its methodology as an elaborate way of writing descriptions for themes to emerge from codes and categories of information based on participant narratives. Clandinin and Connelly (2000) asserted how these experiences extracted from field texts that describe a three-dimensional inquiry space in the form of participant observations, researcher-participant interviews, written responses to documents, and field notes. Clandinin and Connelly implied how a paradigm like SC allows the researcher to manage the place and balance of theory by having a more comprehensive focus on exploring narratives. SC served as the guide to goals for investigating experiences through the field texts mentioned above to evaluate responses and interpret outcomes to inquiries. Glesne (2016) mentioned how SC helped to understand better my subjectivities as a Geometry teacher by progressing through SC to help me put together interpretations that can be significant to the Geometry learning field.

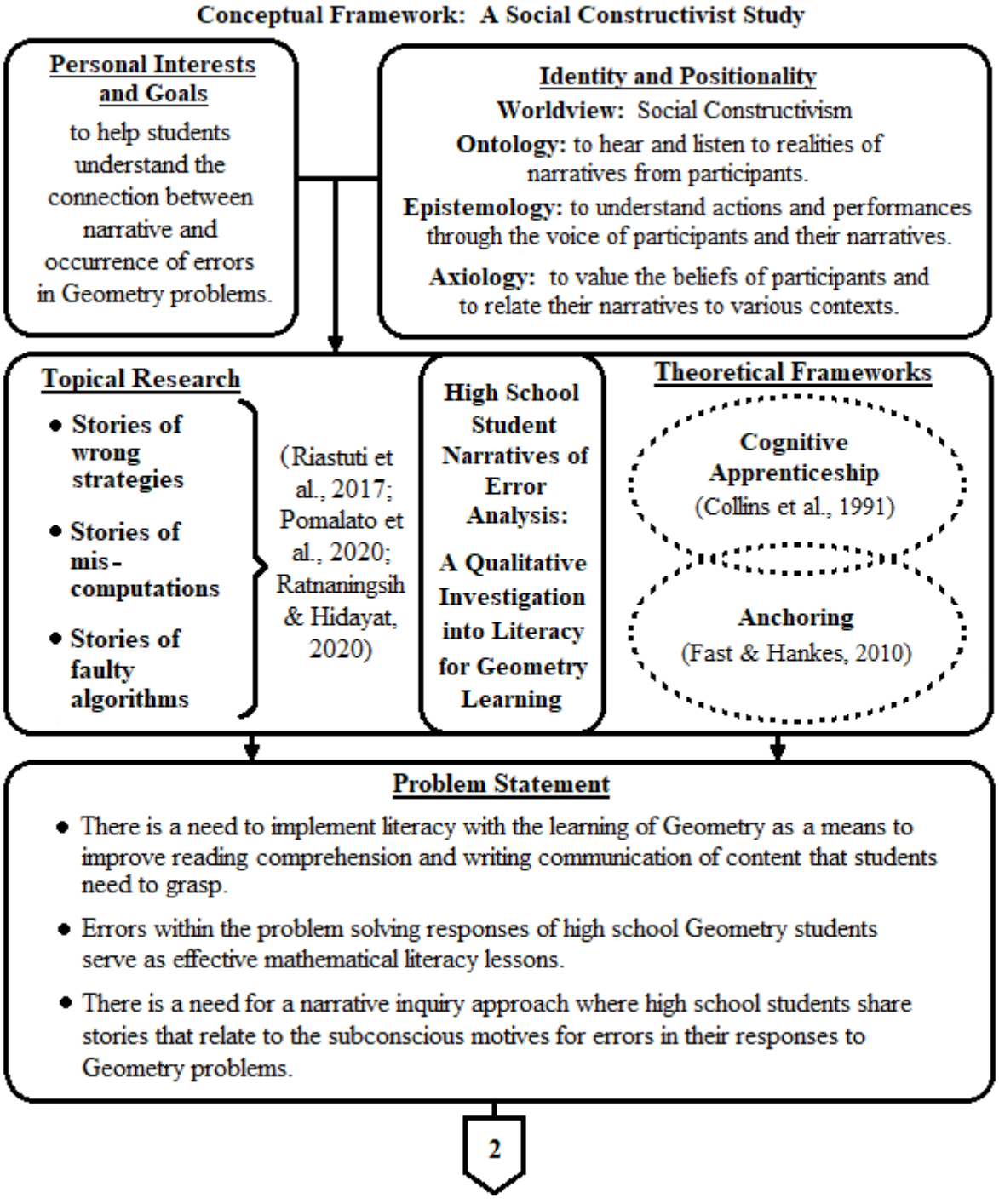
Conceptual Framework and Research Questions

For a conceptual framework, SC was the interpretive paradigm that best fit the worldview of this research. The ontology matched participants as secondary students who entered a Geometry classroom for the first time with unique subjective meanings about experiences with how to learn math from previous classes (Creswell & Poth, 2018). The epistemology involved the synchronous, negotiated learning of interpretations between the teacher as a researcher and students as participants. The teacher explored making sense of student subjectivity through theories or patterns while students attempted to understand their lived experiences (Creswell & Poth). The axiology focused on valuing the participants' beliefs, acknowledging their situated subjective experiences, and sharing the researcher's positionality (Creswell & Poth).

The initial conceptual framework (see Figure 15) showed how the interests/goals and SC identity/positionality led to a literature review that was thematic for topical research and theoretical for frameworks. The former involved the study's focal points based on participants' responses in the form of narratives about errors involving wrong strategies, miscomputations, and faulty algorithms. The latter represented the blueprint of the framework as the blending of two instructional design theoretical frameworks called anchoring (Fast & Hankes, 2010) and cognitive apprenticeship (Collins et al., 1991). The literature review was the foundation for the problem statement, research questions, and narrative inquiry as research design and methodology (see Figure 16). SC drove the development of relationships that led to the ending stages of the conceptual framework (see Figure 17). Under narrative inquiry, interpretation occurred concerning the nature of perplexity involved with the subjectivity each participant shares about learned knowledge through narratives that are possibly causing errors in their work (Stake, 2019).

Figure 15

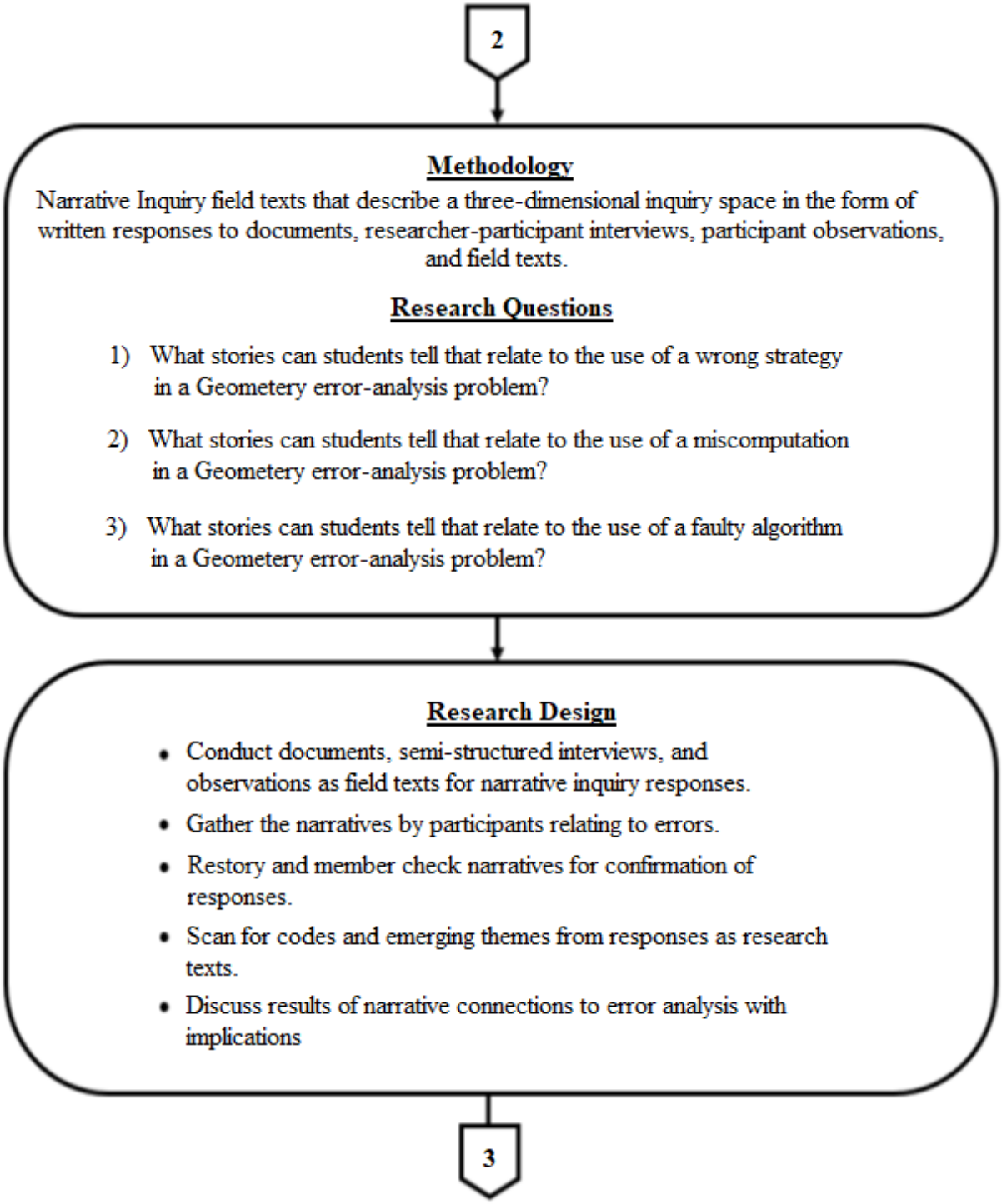
Conceptual Framework Part 1 of 3



Note: A graphical representation of the initial stages of the conceptual framework.

Figure 16

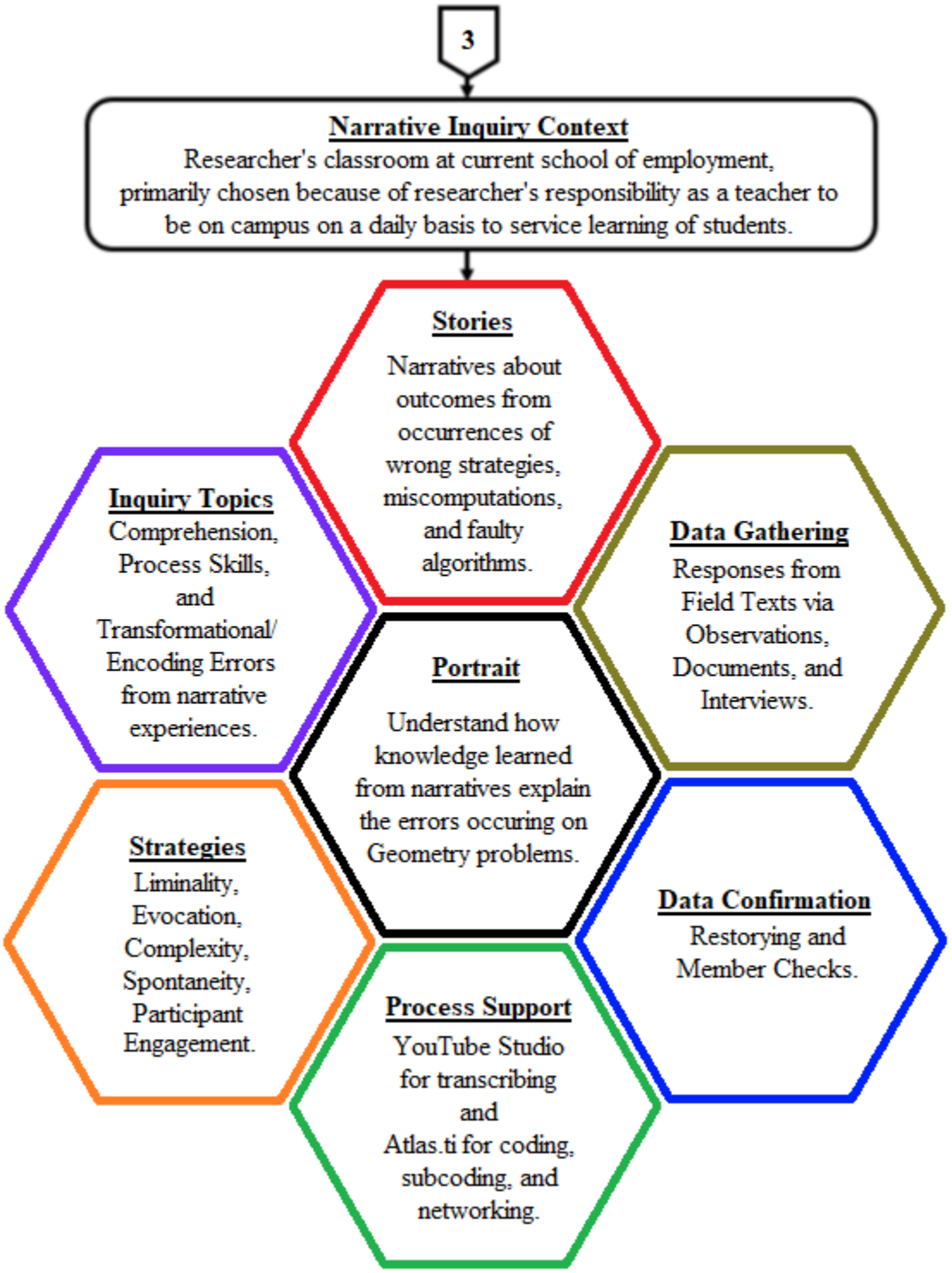
Conceptual Framework Part 2 of 3



Note: A graphical representation of methodology, research questions, and research design.

Figure 17

Conceptual Framework Part 3 of 3



Note: Ending stages of the conceptual framework adapted from the Hopscotch model.

This study's research questions addressed participants' narrative stories that account for the occurrence of errors in Geometry problems. The development of research questions matched previous studies with current research to inform the researcher's thoughts about what data to collect and how to analyze that data (Ravitch & Riggan, 2017). As stated in the introduction, the research questions are:

1) What stories can students tell about using a wrong strategy in a Geometry error analysis problem?
2) What stories can students tell about using a miscalculation in a Geometry error analysis problem?
3) What stories can students tell about using a faulty algorithm in a Geometry error analysis problem?

Research Design

Narrative inquiry matched this research ambition to gather and analyze narratives to research questions based on errors in Geometry problems. Collections, descriptions, and analyses of the narratives shared by participants involve errors in participants' lived experiences. There needed to be awareness and management of tensions that emanated from the narratives, namely temporality, people, action, certainty, and context (Clandinin & Connelly, 2000). Any information acquired through these narratives' tensions was more diverse and genuine (Creswell & Poth, 2018).

Temporality

For participants, a narrative inquiry applied events existing at present with narratives and events as they existed over time with narratives (Clandinin & Connelly, 2000). Temporality connects subconscious knowledge from past narratives as a theory to error analysis of Geometry problems as a reality (Polkinghorne, 1988). Temporality allowed participants to connect their past narratives to the presence of errors in the present and future (Glesne, 2016). In this study, participants underwent temporality, describing how some narrative involving error related to an error they experienced or witnessed with a Geometry problem. Participants described how the error in narratives can be corrected, which related to correcting an error in a Geometry problem.

An example is how a participant overcame the error in the narrative to overcome the error in the Geometry problem. For this reason, participants' verbal and nonverbal communication over time described their temporality. Verbal included such examples as interpersonal interaction or peer discourse while nonverbal were gestures, posture, or facial expressions. Like productive struggle, these states of temporality were cues on whether to interject via scaffolding and inquiry to get participants to describe a meaningful relationship of their narratives to the error analysis tasks they performed (Zeybek, 2016). The descriptions involving past knowledge acquired from the narratives and current performances of error analysis problems served as data responses that considered temporality's tension.

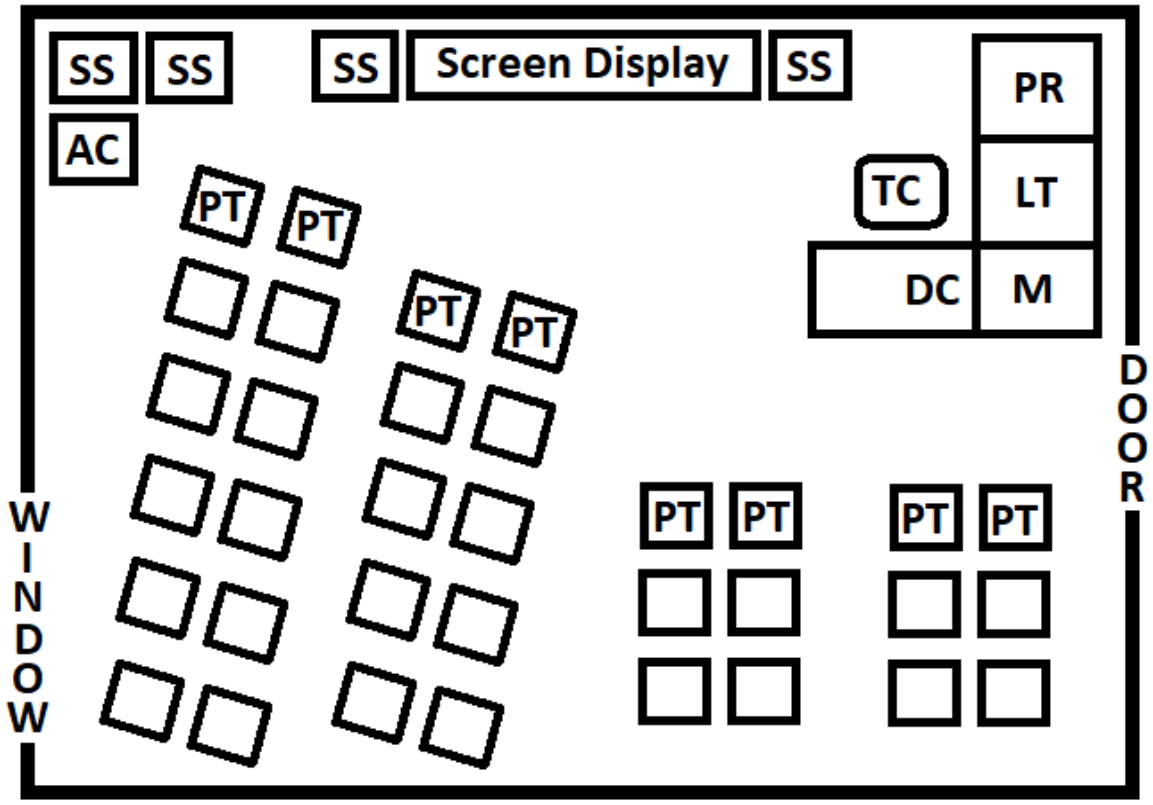
People

Clandinin and Connelly (2000) stressed the importance of knowing the educational history of a participant regarding curriculum, standards, and assessment. For fundamentals, each participant of this study ranged from 9th to 10th grade and had earned credit for a full year of Algebra 1, a prerequisite for Geometry. Each participant had the willingness to endure the

learning and application of a Geometry concept through appropriate literacy channels, instruction, guidance, narrative-sharing, and productive struggle based on what made sense to them. Out of 34 possible students in the classroom, there were four participants (but at most

Figure 18

Seating Assignments of Participants (PT) in the Three-Dimensional Inquiry Space



Note: KEY: AC = Anchor Charts, DC = Document Camera, LT = Laptop, M = Monitor, PR = Printer, PT = Participant Seats, SS = Supply Shelf, TC = Teacher Chair

eight) who sat in the front seats of the class (see Figure 18) to acquire field texts on the three-dimensional inquiry space based on observations, verbal exchanges, and written responses.

With more than eight participants, Creswell and Poth (2018) suggested using purposive sampling to decide who should be part of the study. For official selection, each participant submitted a student assent (see Appendix A) and a parental consent form (see

Appendix B) that contained information about the purpose, procedures, benefits, risks, confidentiality, and voluntary rights. The plan was to explain and express gratitude to those students who would not participate because of purposive sampling. Each participant partook in recorded observations of the three-dimensional inquiry space, verbal exchanges, interviews, written responses, and member checks throughout the research. The intention was to have "maximum variation" (Glesne, 2016, p. 51), where analysis occurred with commonalities of data and with as many various ethnic and cultural backgrounds as possible.

Action

Clandinin and Connelly (2000) perceived the action as an indicator of where a participant's narrative history expressed performance on a certain level. Content-wise, the intent was to have an Honors Geometry class as part of the participant pool because the history of prior classes was more rigorous, detailed, and comprehensive than an on-level class. However, there was flexibility to include students from any Geometry class because many 9th to 10th graders within this pool had a history of on-level Lexile scores ranging from the 25th to 75th percentile (see Table 1). In addition, the school's initiative was to include a blend of heterogeneous students within on-level classes by gender, culture, ethnicity, socioeconomic status, and Lexile level scores. One of the potential seating arrangements that the school required was in pairs or triads of similar and relatively close Lexile level scores. The ideal plan for maximum variation was to select pairs of students within a low, medium, and high Lexile level range to establish an equitable, diverse population of participants to account for diverse representation. In a case of an odd number of participants or perhaps participants wanting to work as a triad, an alternative, secondary plan for maximum variation have a combination of pairs and triads or just triads alone within a low, medium, and high Lexile level range.

Table 1

Lexile Ranges by Grade

Grade	BOY - Fall		MOY - Winter		EOY - Spring	
	25th	75th	25th	75th	25th	75th
9	1025L	1335L	1040L	1345L	1050L	1360L
10	1075L	1385L	1085L	1395L	1095L	1410L
11	1115L	1425L	1130L	1440L	1140L	1450L

Note: From "Lexile tools: Grade level charts" by the Lexile Framework for Reading. (https://hub.Lexile.com/Lexile-grade-level-charts).

Certainty

Clandinin and Connelly (2000) noted how the interpretation of events can be chosen from several options of interpretation and can be at least one that is not selected. Implementing "turning inward watching outward" (Clandinin & Connelly, p. 86) provided facilitation as a place for researcher input and elicitation for participant response. This approach intended to balance changing the distance and closeness of sharing personal connections as an effort for participants to discover and share their own experiences (Clandinin & Connelly). To take certainty into account, restorying and member checks were conducted with participants to clarify that interpretations were based on the participants' personal experiences and not on those of others, especially the researcher.

Context

Clandinin and Connelly (2000) implied how context is affected by observations of a three-dimensional inquiry space. In this study, the focus on context was to establish and maintain

a classroom with a safe, positive environment through written expectations for students to freely express and share their narratives, thoughts, and ideas with others and to explain their prior knowledge acquired from narratives. The site for this study was a classroom at the researcher's current school of employment. This school has been classified as rural with nearly half of the student body on free-reduced lunch and part of a minority population. This location is convenient because the researcher was responsible as a teacher to be on campus and in class daily to service students' learning.

Clandinin and Connelly (2000) implied how context can be perceptions of temporality, interpreted as moments when participants are engaged in a lesson versus moments when they needed to be engaged. As the primary researcher, intervention occurs on observations of the degree of productive struggle based on verbal and nonverbal behaviors between participants. Verbal entailed information exchanges based on the lesson, while nonverbal involved gestures, facial expressions, hand movements, and voice tone. The focus was to guide all participants on a verbal and nonverbal production pathway. The plan was to demonstrate models of productive engagement by the researcher or with cooperating teams of participants. Such models included reading the passage, questioning others about what the passage was saying, highlighting information, analyzing errors, thinking of a related narrative, and searching for sensible processes as corrected responses. In the case of any struggle, the scaffolding of thoughts and ideas of the lesson through continuous inquiry engaged students upon a productive course of learning (Große & Renkl, 2007; Durkin & Rittle-Johnson, 2012; Loibl & Rummel, 2014; Loibl & Leuders, 2019; Barbieri & Booth, 2020).

Clandinin and Connelly (2000) implied how other people can also affected context. Participants may want to escape the lesson by saying they have permission to see a coach,

counselor, administrator, or another teacher. According to Clandinin and Connelly, these types of tensions must be addressed and negotiated with participants at the study's beginning. The plan was to conduct this negotiation verbally with a meeting and nonverbally through signed assent and consent forms (see Appendices A and B). The concern about these tensions was that they may contribute to temporal context through patterns of engagement and disengagement, which will need to be explained to participants initially (Ingram et al., 2015).

Data Collection

The participants' narratives took center stage as field texts in documents, recorded interviews, and observations, focusing on subjective meanings to explain why errors occur (Stake, 2019). The field texts were interpretive notes about participant narratives entering and being amidst the research (Clandinin & Connelly, 2000). This section will discuss field texts as written documents in lessons and scripts, visually recorded semi-structured interviews, and recorded observations. All field texts will focus on narratives involving errors. A timeline for the collection will be provided (see Figure 19). Field texts occurred during five class periods, one each week over the minimum one-and-a-half month span, lasting about 15 to 30 minutes.

Field Texts of Written Documents

Clandinin and Connelly (2000) imply that written documents are relevant for matching verbal with nonverbal. In this study, written documents will be lessons and scripts. Lessons brainstormed narratives that related to the errors witnessed within the context of a Geometry concept and a field of work. The scripts will plan for tangible support for semi-structured intervals and recorded observations of three-dimensional inquiry space. The plan was first to implement one lessons per week, taking approximately 30 minutes of class time, for five weeks.

Figure 19

Proposed Timeline

Date	Researcher or Participant	Action
December 2022	Researcher	Apply and obtain IRB approval
December 2022	Researcher	Develop error-analysis literacy lessons on … • distance, midpoint, and slope with baseball/softball diamonds, • perimeter/area of rectangles using the Ishango bone, • counting points and segments of airplane travel routes, • finding transformations of Mars rovers, • finding rigid motions of tetris tetrominoes
January 3-6, 2023	Researcher, Participants	Identify participants, request permission to record, communicate with parents, collect consent forms, build relationships
January 6-12, 2023	Researcher, Participants	Administer error-analysis literacy lessons on distance, midpoint, and slope with baseball/softball diamonds. Collect written responses as documents. Conduct and record interviews. Collect descriptive and reflective field notes.
January 12-20, 2023	Researcher, Participants	Administer error-analysis literacy lessons on perimeter/area of rectangles using the Ishango bone. Collect written responses as documents. Conduct and record interviews. Collect descriptive and reflective field notes.
January 20-27, 2023	Researcher, Participants	Administer error-analysis literacy lessons on counting points and segments of airplane travel routes. Collect written responses as documents. Conduct and record interviews. Collect descriptive and reflective field notes.
January 27-February 3, 2023	Researcher, Participants	Administer error-analysis literacy lessons on finding transformations of Mars rovers. Collect written responses as documents. Conduct and record interviews. Collect descriptive and reflective field notes.
February 3-10, 2023	Researcher, Participants	Administer error-analysis literacy lessons on finding rigid motions of tetris tetrominoes. Collect written responses as documents. Conduct and record interviews. Collect descriptive and reflective field notes.
February 10-March 1, 2023	Researcher	Transcribe verbal responses, analyze field texts, and code for themes.
March 1 - May 1, 2023	Researcher	Write Chapter 4 results and Chapter 5 discussion and implications.

Each lesson described a field of employment, related the field to Geometry concepts, and provided directions for a leadership role to novice employees as prescribed from prior research connecting content to context (Fast & Hankes, 2010; Collins et al., 1991). Each lesson's reading passage (see Appendices C-G) applied non-published sources. Based on response repetition (Rapp et al., 2011), eight tasks contained different erroneous solutions (see Appendices C-G). These types will either be completely wrong, contain a miscalculation, or have some defect in their algorithm, as seen in prior studies (Borasi, 1987; Riastuti et al., 2017; Pomalato et al., 2020; Ratnaningsih & Hidayat, 2020). Participants read the passage, probed the error analysis problems, inspected details of given visual diagrams and wrote modified, corrected solutions to problems. The goal was to assist participants with a structured foundation to think of narratives that relate to the errors in the context of Geometry.

The lessons led to responses on scripted documents about sharing narratives related to the error types witnessed in the lessons (see Figures 20, 21, and 22). Each script contained probes for error analysis from prior studies showing participants' effectiveness in detecting and describing the kinds of errors they encountered (Borasi, 1987; Riastuti et al., 2017; Pomalato et al., 2020; Ratnaningsih & Hidayat, 2020). Each script had participants share a lived story that was similar to the errors in the lesson. Participants shared practices of experiencing the context of the narrative again but with more appealing outcomes. Participants also responded regarding what practices helped them understand how to avoid errors on Geometry problems. Students thought of a metaphor that described the error in the hopes of providing self-awareness of how the error can subconsciously occur again (Lakoff & Johnson, 1980).

Figure 20

Sample Script on Wrong Strategies as Documents for Interviews and Observations

What stories in your life relate to the use of a wrong strategy in a Geometry error analysis problem ?

Probe: Tell me about a time in your life when you did something wrong thinking you did something right.

Probe: Tell me about a time in your life of how you made a situation right after approaching it wrong.

Probe: Tell me about how your story of doing something wrong relates to a Geometry problem of choosing a wrong strategy.

Probe: Suppose the situation of that time in your life when you did something wrong were to happen again. Tell me what practices can be done that would help you to avoid the same outcome.

Probe: Tell me how these practices can be applied with a Geometry problem showing up again where a wrong strategy is applied.

Probe: Think of a metaphor that describes a wrong strategy.

Figure 21

Sample Script on Miscomputations as Documents for Interviews and Observations

What stories in your life relate to the use of a miscomputation in a Geometry error analysis problem ?

Probe: Tell me about a time in your life when you made a miscalculation of something.

Probe: Tell me about a time in your life of how you calculated correctly after calculating wrong.

Probe: Tell me about how your story of miscomputation relates to a Geometry problem of miscomputation.

Probe: Suppose the situation of that time in your life when you did a miscomputation were to happen again. Tell me what practices can be done that would help you to avoid the same outcome.

Probe: Tell me how these practices can be applied with a Geometry problem showing up again where there is a miscomputation.

Probe: Think of a metaphor that describes miscomputation.

Figure 22

Sample Script on Faulty Algorithms as Documents for Interviews and Observations

What stories in your life relate to the use of faulty algorithm in a Geometry error analysis problem ?

Probe: Tell me about a time in your life when you did something with a faulty algorithm.

Probe: Tell me about a time in your life of how you modified the algorithm after noticing it was faulty.

Probe: Tell me about how your story of faulty algorithm relates to a Geometry problem of a faulty algorithm.

Probe: Suppose the situation of that time in your life when you did a faulty algorithm were to happen again. Tell me what practices can be done that would help you to avoid the same outcome.

Probe: Tell me how these practices can be applied with a Geometry problem showing up again where there is a faulty algorithm.

Probe: Think of a metaphor that describes a faulty algorithm.

Field Texts of Semi-Structured Interviews

Clandinin and Connelly (2000) asserted that responses to the inquiry as field text can be in the form of interviews to promote viability with informal exchanges. Participants verbally expressed their reflections from their scripted documents to provide more input on their narrative stories with their connections to a wrong strategy, miscomputation, and faulty algorithm (see

Figures 20, 21, and 22). The goal was to acquire responses with the informal language of participants in a nonchalant mode of exchange where the researcher listened to what was said (Coles, 1989; Bateson, 1994). Verbal responses through inquiry probed on the scripted document to the interview emanated elaboration and rich detail of the narrative from the participant (Clandinin & Connelly). Interpretations and descriptions of the content of the exchange provided participants the opportunity for participants to restory and member check their responses for more clarification. Restorying and member checking allowed proposed changes to reshape appropriate descriptions by participants (Clandinin & Connelly).

As prior studies suggested, there was an inquiry as to how prior knowledge acquired from narratives was related to the existence of errors in Geometry problems (Große & Renkl, 2007; Durkin & Rittle-Johnson, 2012; Loibl & Rummel, 2014; Loibl & Leuders, 2019; Barbieri & Booth, 2020). The one significant difference in this study was that narrative stories related to errors were participants' channels to discuss connections to errors. Interview statements spurred participants to share their stories about errors from the scripts in more detail and related them to the errors witnessed in Geometry problems. In addition, participants elaborated on a metaphor that described and represented the existence of the error within the narrative and the Geometry problem (Lakoff & Johnson, 1980).

Field Texts of Observations

There was a "problem of the influence of the observer on the observed" when recording and writing field texts on observations (Clandinin & Connelly, 2000, p. 87). As mentioned earlier, with the tension of certainty, a "turning inward watching outward" (Clandinin & Connelly, p. 86) approach balanced researcher facilitation and input with the participant's voice on narratives. The plan for taking observations began by using recording

devices during lesson-script implementations and semi-structured interviews. Notes contained details about the observations of ideas and thoughts of participants' experiences of narratives related to a problem on error analysis. Nonverbal and verbal behaviors corresponded to "existential, outward events" (Clandinin & Connelly, p. 86). Personal reflective notes matched "inner responses" (Clandinin & Connelly, p. 86) to the realities. The ambition was to provide an organization of participants' thoughts and ideas regarding their narratives to separate personal interpretations of the meaning of their experiences.

It is important to note that field texts in this realm contain ambiguity regarding knowing what notes referred to as temporality, space, and context (Clandinin & Connelly). This study focused on organizing field texts on the learning status of participants. The texts showed engagement (conversations about content), descriptions of the environment supportive of learning (transitions, sense-making of the lesson), and descriptions of the types of discussions that were taking place (i.e., inquiry of wrong strategies, miscomputations, or faulty algorithms).

Data Analysis Procedures

Ingram et al. (2015) warned that tensions exist between affective and cognitive states of mind when interacting with problems involving error analysis and connecting with narratives. Clandinin and Connelly (2000) mentioned the importance of negotiating tensions when composing research texts. There was a minimizing role of memory reliance in interpreting field texts that conducted static and changing information acquired from field texts. Uncertainty was managed based on voice, signature, and audience tensions. Finally, a balance of tensions occurred when defining the shape of the narrative form.

Management of Boundaries

Field texts were organized based on the genuineness of written responses on documents, verbal responses from interviews, and outward-inward observations of the three-dimensional inquiry space to manage boundaries. Field texts focused on finding meanings of experiences from participants about the blending of narratives with Geometry error analysis. However, there was concern about how participants' other foci can include traditional, motivational upbringings of acquiring excellent grades or simply passing Geometry with a satisfactory grade with a need to concentrate on providing genuine responses for the aim of interpretation (Clandinin & Connelly, 2000; İlhan et al., 2021).

Clandinin and Connelly also warned how reliance on memory of interpreting ambiguity or lack of substantiality of field texts caused skepticism by readers on whether these learning outcomes were genuinely happening. Pseudonyms replaced the names of participants on instruments to protect identities. Participants' pseudonyms were written on the top right-hand corner of any document to establish accountability, legitimacy, and organization of field texts. Instruments were in a secure binder. Restorying and member checks took place to respond to ambiguity and validate interpretations of responses. Each part elicited responses about connections with participant experiences of narratives to the errors of incorrect strategies, miscomputations, and processes that needed to make sense. All documents were organized in a folder called "Written Response Documents" by pseudonym, date, and time. The YouTube Studio transcription tool transcribed all recorded interviews and observations. A folder named "Verbal Inquiry Response Transcripts" organized transcriptions by pseudonyms, date, and title. All transcripts uploaded into the qualitative data software Atlas.ti for emerging codes and themes.

Minimization of Memory Reliance

Clandinin and Connelly (2000) implied how field texts transformed memories of experience existing in one form or another. For the trustworthiness of data collection based on memories, the field texts were read several times in detail to question and determine the ownership of a narrative (Pinnegar & Daynes, 2007). This analysis process of decision-making, data triangulation, interpreting, restorying, and collaboration was iterative to tell and retell the rich individual experiences of participants as an effort to avoid ambiguity and reliance on memory (Creswell & Poth, 2018). Inquiry responses from interviews allowed for spontaneity and the inclusion of unexpected ideas for participants to own, tell, and change the narrative. There was a focus on finding connections between participants' narratives to error performances on Geometry problems. There was a plan to minimize the role of memory from the researcher contained in traces of information within field texts (Clandinin & Connelly). Conflicting interpretations from notes of narratives occurred from relying on memory. Here, restorying and member checks occurred to rewrite interpretations chronologically so that compiled information minimized conflict and promoted clarity (Creswell & Poth). Restorying and member checks transpired by feelings of inconsistency with interpretation within the narratives based on categories related to understanding the use of a wrong strategy, a miscalculation, or an incorrect procedure. In addition, collaboration with a team of professional researchers occurred for recommendations for re-assessments and modifications. This process occurred cyclically.

Management of Uncertainty

Multiple voices transpired to tell stories of experiences to establish a diversity of acquired interpretations (Clandinin & Connelly, 2000). The intent was to listen, absorb, and accept as much knowledge and experiences as possible from participants in verbal and nonverbal

exchanges. The information shared is transcribed for meaning and clarified through restorying and member checks. There was an inquiry about the positionality and persona of participants as signatures to owning the shared experiences (Clandinin & Connelly). Restorying and member checks transpired as efforts for participants to take pride in sharing their experiences and letting stakeholders know who they were and how they wished to represent themselves as practitioners for correcting mistakes from error analysis.

The hope was for participants to enrich their experiences with more details. There were anticipations of reactions from the audience as tension along with the struggle to represent multiple voices and signatures of participants (Clandinin & Connelly). Skepticism was expected, particularly with how narratives and error analysis related and whether results occurred and could duplicate. There was a discussion to inform readers that this study contributed to the research of blending narratives and error analysis of Geometry problems as a means for awareness and appropriate response. The overarching goal was to continue improving and revising the lessons to impact positively the learning of Geometry concepts and literacy skills in a subject like Geometry.

Narrative Form

The narrative form of this research endeavored to use a metaphor to help narrative researchers think about their work (Lakoff & Johnson, 1980; Clandinin & Connelly, 2000). Two metaphors resonate … “this study is a mission to explore new learning worlds of experiences out of comfort zones” and “to boldly go where Geometry learning has little gone before.” These metaphors reminded the purpose and intent of narrative form for an overall plan to describe and interpret information from field texts. Field texts coded narratively to describe meanings of experience that transform into research texts (Clandinin & Connelly). Coding is a cyclic,

systematic order of qualitative data into part of a general classification called a category. This process intended to combine data into more coherent groups to develop an explanation (Saldaña, 2021). A plan was to use inductive or open coding where codes are descriptive about what participants are talking about in vivo from the actual language used or process based on the conceptual thoughts shared by participants (Saldaña). Recoding and recategorizing transpired after the inductive coding of qualitative data to detail the patterns and interpretations of the participants' experiences (Saldaña). The plan initially was to sort the field texts into smaller samples by emerging categories. After reading one of these samples multiple times, codes formed for the representation of that sample. Recoding and recategorizing transpired after the inductive coding of qualitative data to detail the patterns and interpretations of the participants' experiences (Saldaña).

Another sample checked for codes from the previous sample and created additional codes that characterized other information in the sample. This process was iterative until all data within the field texts were coded. Changes to the set of codes based on additions, splits, and re-descriptions matched up with a single code (Saldaña, 2021). Atlas.ti applied in the coding process for saved storage, organization, and display of networks. Two initial coding cycles occurred as analysis and synthesis (Saldaña, 2021). The former involved identifying and placing similar patterns of information together; the latter regrouped codes based on meaning. Using Atlas.ti, a codebook and network of proposed codes and categories were created. The expectation was that these codes and categories most likely change drastically once actual qualitative data was acquired. The categories focused on the topical research of the literature review, representing the facets of relating narratives to error analysis.

Chapter 4: Results

Over a span of five weeks, the narratives of four students offered experiences and insight about wrong strategies, miscomputations, and faulty algorithms. For each week, the triangulation of data involved the completion of a literacy document containing error analysis problems, semi-structured interviews to extract narratives, the collection of field notes, and restorying of events.

Initially, each of the four participants provided a completed literacy document before interviews. The document involved Geometry content, a real-life topic, and a managerial, supervisor role (see Figure 23). The document contained a Georgia performance standard for Geometry learning, vocabulary, a reading passage about the topic and content, and erroneous problems for correction. The role emphasized the writing of corrections for erroneous problems related to the content and topic. The plan for the erroneous problems was to provide insight and experience for the appearance of a wrong strategy, miscomputation, and faulty algorithm.

Figure 23

Timeline of Document by Content, Real-Life Topic, and Application

Week	Geometry Content	Real-Life Topic	Error Analysis Job Manager Application
1	Distance, Slope, Midpoint	Baseball/Softball Diamonds	Recreation Supervisor
2	Perimeter and Area	Ishango Bone	Archaeologist
3	Counting Points and Segments	Airport Routes	Air Traffic Controller
4	Sequence of Transformations	Mars Rover Opportunity	Robotic Engineer
5	Rigid Motions and Congruence	Tetris	Video Game Designer

Semi-structured interviews subsequentially took place. Participants shared and described stories related to outcomes that involved a wrong strategy, miscomputation, and faulty algorithm. For each story, participants shared alternatives to produce more favorable outcomes. Then, participants described how these alternatives were applied in their literacy document. As a summary, attempts at metaphors were shared that represented their experiences.

The interviews were designed and geared for participants to generate responses to the research questions, which did not all follow the order of wrong strategy, miscalculation, and faulty algorithm of operations. Instead, there were times when participants shared the story and then classified them based on what they thought was the error. In some cases, participants asked for clarification and reminders of descriptions for each error type. There were times where I questioned myself about providing clear distinctions between error types. Each participant had the script ahead of time to prepare for the interview. However, given the workload from other classes, each participant felt it was more work to fill out the script as a journal. For more feasibility, the journals changed to field notes taken during the interviews for each participant. For more comfort and negotiation, each participant received examples of story topics and wait time during the interview to brainstorm and think of stories to share. When a story was shared, repetition of keywords and paraphrasing of ideas occurred to confirm a summary of the story and the practices applied in their narratives and their written documents. A series of wrong strategies, miscalculations, and faulty algorithms overlapped for some of the stories.

The following sections described what was shared from each participant. It was challenging to interpret where a sentence began and where it ended. Frequently, sentences were interpreted to begin with words like “and” or “because.” However, it was open for debate where a sentence should begin and end.

There was also debate between descriptions as metaphors that turned out to be descriptions as similes. The word “like” transpired in the responses of all participants when comparing the entities of the Geometry topics to real life experiences. Most often, the comparisons of two entities were similes. However, the intention were to compare entities in a more direct manner as metaphors.

Athena’s Stories: Paying Attention to Detail

Athena was a freshman at Paulding County High School. Athena excelled at learning concepts in her Honors Geometry class. She had a Lexile reading score of 1351, indicating she was at an advanced reading comprehension level for her grade. She generally enjoyed reading, always taking the time to read a book. Her extracurricular activity was the marching band, and her hobbies included cooking and baking. She had a mild-mannered, articulated demeanor about herself that served as a role model for her peers in general.

Athena’s Stories of Wrong Strategies

There were various stories shared by Athena that involved the choice of a wrong strategy. However, they each shared a common theme of her being lost and relying on others for decision-making. In Athena’s view, the outcomes of these experiences occurred because of a need to focus on her own abilities for inspecting details.

Week 1 - What Wrong Strategy Story did Athena Relate to Distance, Midpoint, or Slope?

Athena mentioned a story that related to applications of distance and slope in Geometry. She described being unable to find her mom in a store because she needed to read posted signs to help her with directions and distances around a grocery store:

> Okay, so I remember one time when I was around 8 or so. And I was told to go find something in a grocery store. And I went and found it. But then I didn't know which direction to go. So I was like looking around, trying to find my mom. And she was like I'm gonna be over here. Like she told me she was gonna be, I think near the clothes aisle. And so I was looking around trying to like find my way and I was looking at signs and different things. And well I walked for a minute and well I knew my sister was with her and all my sister was pretty wild too. So I kept looking around for my sister or my mom. I finally stopped my sister and then I was, well, she can't be too far from my mom so I came to her, I was like … do you know where Mom is? And so she was that way so I took the thing that I had. I don't remember what it was specifically but I eventually found her and so that was that and so that's one memory I have from distance because I remember really getting lost.

Athena expressed that she wished she looked at signs instead of just looking around:

> I feel I kind of made a mistake with how I did it because they were signs and originally I was wandering around. I was looking for something, looking for her but I didn't know where she was and so I really do think it was I started out wrong. I didn't use the correct strategy to find her. I think I probably… would have immediately looked at the signs but little me would have been…. I don't know that they had signs until I looked up. So I think I would have immediately looked up at the signs and read what they said and then I would have went from there.

When Athena mentioned direction in her story, I questioned if slope played a role:

> I think slope could because … slope kind of tracks a pathway and so slope is basically you're going a certain path and it's a very straight path. There's not really a lot of it's not another graph where you're tracking different variables how you win science it's more of like a straight dead-end thing. There's only one answer and the slope is really an efficient way to find the one answer.

Athena related the correction of her strategy with getting lost in the grocery store to the literacy she performed by describing her mentality:

> I feel like I look at every single little detail. I examine every single little thing so I understand it and then I think I try and understand it in my head before start writing things down. So I look at the numbers. I look how that would work. I examine every single little detail as well. Personally, I'm a person who reads directions and I do them by a fine line. That's how I do things.

Athena reiterated her rectification of being lost:

> I think I would have not gone lost if I would have immediately knew to like look around where I was and then also eventually look at the signs but also to examine my surroundings and see where I was.

Week 2 - What Wrong Strategy Story did Athena Relate to Perimeter and Area?

Athena found herself astray on a math test, where she questioned her understanding of perimeter and area:

> Okay earlier on the math test I was looking at it was the same square but you wanted us to use perimeter on one and an area on the next one and I kept like overthinking it because I was like I kept getting the same exact names of return getting 41 for perimeter and 41 for area and I'm like, am I using the wrong strategy? Am I doing am I using perimeter for area? Am I using like area? Like perimeter for area? Like I kept like freaking out. I was like and I tried doing it the other way and then I was like I don't know what to do. I was like just confused and I just eventually was like … I have to go with my gut and think they're both the same and so that's what I went with.

She continued describing her productive struggle to help make sense of her intuition, using given choices that she had:

> I was sort of calm because, well, I was like I over thought it a bunch because like I'm just naturally like that. I was like just looking at it and I kept just staring at it and I'm like this is freaking me out. Is it right or is it wrong? And I kept like looking at both of them and then I kept looking at like the multiple choice answers since they're both related to those. I was like okay what are the choices on the multiple choice?… Are they right are they wrong? Yeah I was like something must be wrong here like and I don't know what question I messed up on but I think I must have been a lot of those.

Athena described overcoming her uncertainty through reviewing information and memorizing:

> I feel like if I feel like there's something wrong or if I feel like I can sense something that's off I will just go back to it and I'll look it over and I'll try and memorize like what I see is going wrong and then try and put that and try and just look at the little details of all like the perimeter and area like the lengths and like just try and figure out how to fix it by like looking at what I did wrong.

Athena also shared her understanding of the difference between the values of perimeter and area, relating to the outcome of her narrative:

> I feel like perimeter like outside it can seem smaller. Like say you walk into a house like on the outside it can seem smaller than it actually is. But once you walk inside it's bigger than you think. I feel area is going to be larger than perimeter because perimeter is really just the outline of the place. Like if you're walking around a house and it's so many square feet around, once you walk inside you have to take it apart. Every single bit of the area is like inside of that house and it's a lot bigger than the outside.

Week 3 - What Wrong Strategy Story did Athena Relate to Counting Points and Segments?

A narrative was shared where Athena walked from one point to another point as a route on a dance floor to find her missing friend:

> Okay so I think wrong strategy wise. So around homecoming, I was going with a bunch of my friends to homecoming because well I just didn't have like anyone else to go with and my friend was supposed to spend the night with me after the dance and I was like okay I have to keep track of her. I can't lose her or else I'm gonna freak myself out because then I don't know where she is. All right, I step out for a minute and I come back in and she is nowhere to be seen in the gym so I run down the bleachers. I'm on the dance floor. I'm looking around and when it comes to like different points, I'm really thinking of like the four different corners of the room because there's the two corners that are really just used for seating, the one that's used for dancing, and there's the photo booth. And so I felt like I was kind of going from point to point just looking for her. I was like everyone's just sitting and then I went out to the dance floor and I felt like every person that I ran into was kind of a point because I was like looking at them, checking to see if it was her and then eventually I end up at the photo booth and that's where I find her and so I really felt like I used a wrong strategy there because well the one thing about that was four different points of the room is that I probably should have just texted her automatically and I just could have automatically just went to that point where she was but I just didn't think of that at the time.

She shared how some the of the routes repeated and that how all the people were similar to finding the right point in a set of points on a geometrical network:

> I did feel like I did [repeated routes] because I would check at the dance floor and then because it was a party like people were moving around constantly. So I felt like I retraced my steps multiple times and looking for her yeah they're like going from like a bunch of little points everywhere to just eventually going straight to one.

Athena wished to use phone texts rather than thinking of herself and her lost friend as points on a homecoming dance floor, with their repeating paths representing line segments going back and forth. Athena mentioned how texting is a more efficient way of finding her friend over physically exerting energy walking the routes:

> Basically with texting I think it's more it's an easy way to get in touch very quickly because as points too it's being sent from one your phone to other persons so that tech is basically traveling that distance to the person's phone.

I clarified by asking if the text is like traveling the distance for you:

> Yeah and so they can basically reach back out to you and tell you where they are so that you can go and find them.

Week 4 - What Wrong Strategy Story did Athena Relate to a Sequence of Transformations?

Athena related transformations by experiencing a feeling of disorientation with her choices of stacking pieces to forming a pyramid in a puzzle game. She felt her choices were part of a wrong strategy:

> When I was gonna be like fifth grade I got a gift for Christmas. And it was this little puzzle game and like you had to basically take all the pieces out and it would give you a little diagram. And then it would have like some pieces placed but then you had to fill in the rest of the pieces. And so I was on the hard levels which is basically where they're not flat anymore but you build them into a pyramid. And so I um was stacking them and I was like shifting things around to basically like doing transformations. Like trying to figure out what goes where and how to position each one. And I ended up placing one wrong puzzle piece and the entire thing fell.

When questioning what should could have done differently for a better outcome, she talked about more comparison of the pieces and more visualization of how the pyramid should be:

> I think I could compare the puzzle pieces and kind of see like how they would look in the pyramid. Because, like by the end of it, the pyramid is always going to look the same. There is always going to be a pyramid. There's not going to be anything funky. And there was something funky about it. And I guess I didn't notice it and so it caused it to fall.

She ensued by mentioning how stacking of the pieces is similar to following steps from the beginning to end of the construction:

> I feel, like related to stacking, I feel like each little part is kind of like a stack because you're starting at the bottom which is basically the beginning and then eventually you reach the top, which is where you end and basically where you get your final point. They're both pretty easy to look at but really just seeing them as more of like steps is easier … yeah I feel like it's easier to do like once you're started I think it gets easier as you do it more over time.

Week 5 - What Wrong Strategy Story did Athena Relate to Rigid Motions and Congruence?

As a correspondence to rigid motions, Athena shared how using egg whites to bake macaroons did not turn out very well:

> I recently made macaroons over the weekend for the first time and I was using a box recipe. So, it obviously probably wasn't going to be perfect. So, my icing that's supposed to go in the middle was too thin. And my aunt was over and she saw that my icing was too thin to be piped. Because, well, it has to be perfect because that's how it dries. And so she was like to thicken it, why don't we add egg white? And so we added the egg white. It didn't thicken it. And so then we were like how about cornstarch? So, we added cornstarch and that ended up doing the trick. But once we piped it and then it cooled and then we ate it, the macaroons ended up tasting like eggs.

When asking what could she have done different to make the macaroons taste better:

> Probably dividing the egg white and cornstarch immediately because the egg white messed it up very badly and made it taste like eggs. I mean they're still good but … I feel like things would have gone better because I went with what my aunt told me to do because she was there and she's ordered an egg and so I was like okay and at first I thought of adding flour but because I didn't know if we had anything to like help thicken it but I ended up with the cornstarch after the egg because well the egg didn't work so we had decent cornstarch but now I know to use the cornstarch.

When asking if listening to herself would have turned out a better outcome with the macaroons, Athena replied, "I feel like it would have maybe." I rephrased it to basically just listen to herself, "Listen to what my gut tells me basically yeah."

Athena's Stories of Miscomputations

Athena talked about erroneous stories during an extracurricular activity or event that either she participated in or witnessed from another person. Each story had the common occurrence of a misestimation caused by repetition of a task. Although the error is bound to happen, Athena still found it worthwhile to think of ways to prevent the error from happening.

Week 1 - What Miscomputation Story did Athena Relate to Distance, Midpoint, or Slope?

Athena described a volleyball game where she cost her team a point due to her misestimation of passing the ball rather than setting it for her teammate to spike it over the net:

> So I played volleyball for about one and a half years. And so in volleyball there's a lot of, it's not as simple other sports. There are there's different moves, you have settings, you have spiking, you have passing. And the one thing is though, you have to be very aware of like you have to be aware of every single little detail. Because if you go over the line, that's an error. If you go over four touches, if you go over three touches, that's an error and it can cause you to lose a point. So you have to be so aware of what you're doing. And so I remember one time, and especially with different moves too, because your other player, like once you touch the ball you can hit it again. Like that's how it works when the ball comes over the net. So one time, I was I think I was back middle and I there was the ball heading straight for me. And it would have been an easy set. It should have been an easy set but instead I passed it. And it ended up causing the ball to not go as high as it needed to. And so it hit the ground which caused us to lose the point.

She sounded overwhelmed describing this event. She agreed that her use of the word "high" related to distance. I asked if the appearance of distance formula is overwhelming for her as well and whether the formula is something that needs to be broken down:

> I feel it is something you have to break down because especially with a lot of long formulas that have a lot of steps to them. Immediately looking at them, you can immediately get a sense of like overwhelmed. Like just because there's so many steps but like once you break it down and focus on every single little thing, it becomes so much easier.

Week 2 - What Miscomputation Story did Athena Relate to Perimeter and Area?

Athena described a narrative about underestimating the size of a garden space for an overabundance of plants:

> We moved into a new house this past spring and [with] my grandparents. And since they're both very big gardeners and my younger brother is very like an outdoorsy kid, they were like we'll make a garden. And so we were building the base of the garden and it turns out we didn't make it big enough. We had bought too many plants to fit in there and so everything was on top of each other and things were overgrowing other things. And so it was just this huge mess. And so eventually we have decided this next, I think, yeah, this next spring, we're going to expand it to make it thicker because well too many plants and then we also want to add more plants. I think [this is a] miscomputation

> because we underestimated how big [the garden] needed to be and we just didn't like know that all the plants were gonna fit in there.

Athena reiterated how the miscomputation came to be with the garden:

> We bought the plants separately. We … bought too many because well, apparently, my grandparents caved to my younger brother. They were like, he was like ... I want this one and this one and this one because well he was trying to grow like salad gardens. He's like growing with lettuce and tomatoes rather than staying with other things. And we bought too many and so we had leftover ones that my grandparents had to take home because as well [where are] they are going to fit?

I asked if the garden could be made bigger and if one is able to walk around it:

> Yes [we were able to make the garden bigger] because it's just pieces of wood that have little, I'm pretty sure there's slots in them and basically they just kind of slide down into them so it basically just gonna buy more plants and just expand…yeah [able to walk around the garden].

I also asked if you have to be careful where you step in the garden:

> Oh yeah so very careful [where you step] especially because I grew flowers in there and he grew vegetables. So like we each had our own side. And so I was always very like … Don't touch the flowers! It was like a 50-50 thing yeah my brother had his half which was on the right and I had my half so on the left.

Athena commented about the perimeter and area of the garden:

> The perimeter I guess can kind of make it things seem bigger in reality… that's like the garden … the edges of the garden take up space in like the area too.

She attributed memorization to be a factor to overcoming miscomputation:

> I do feel like that happens a lot [memorization overcomes miscomputation] especially like even when I'm like in lit like reading. I feel like in a lot of subjects memorization is a huge thing. It's something that is a big skill you need like to learn and to further yourself. Like if you're reading a book and if you have a test on that book later you have to memorize what's happened in the book in the first place or else you won't be able like you finish the book and let's say it's 176 pages and you're on page 176 you have to be able to remember what happened on page 1.

She continued by claiming that memorization really depends on the situation:

> I feel like it's different with certain things like if my mom asks me to like write down this code because she needs it. Like T-Mobile's always sending me like those little codes like these like punch in to like do whatever. And so she's always like okay remember this and

> so I'll remember it for that time but then once it's once it's already finished like once she's done with the task, I'll basically just kind of like throw it out of my brain I guess. But when it comes to stuff like math and reading and school and stuff that I might need for my future, I try and remember most of it as much as I can. Like I can still basically tell you the plot of what I read for lit last semester. I feel like I sort of used memorization for error analysis. Because well along with just looking at it, you kind of have to also like see the errors, remember what went wrong with them until you can like look at it and remember and go like fix it as you're going along.

Week 3 - What Miscomputation Story did Athena Relate to Counting Points and Segments?

Athena described a situation of playing soccer, where she envisioned players as points and passes as segments:

> Okay so I played soccer when I was a kid and I was probably in like the fourth grade. So I was about nine… I was in the nine-year-old all girls soccer team and I was a front middle player. And the one thing about sports is that everyone has their own area and so the one thing with I kind of see all the players as points and whenever you're running with soccer it's kind of like a line because you're getting from here to here because you're trying to get across the field to go to the other goal and so I made the mistake of going into… uh she was this girl was a defense player … so I was going not where I was supposed to be and we basically collided. So I kind of think of that as like two points meeting because we were both trying to get the ball … because I accidentally went into her space.

I asked if some of the points in the literacy come toward each other. Athena imagined several points going towards one point and vice-versa:

> I feel like they are because they're all coming from somewhere else but then they all meet at one point like as you can see here [See Figure 23] it's like a little they eventually meet at the one point in the center.

However, she claimed that she is able to see point to point in two directions depending on the situation,

> I feel like I look at it both ways [in to out, out to in]. It depends on how with this one it's more of like the story that it goes along with it or like what you immediately think of that goes with it. I think of it as the same way because it's a really the same path because either way going in or out it's still the same like basically it's the same direction you're going opposite or forward.

Week 4 – What Miscomputation Story did Athena Relate to Sequences of Transformations?

Athena depicted a feeling of exhaustion when overdoing her walking in a neighborhood that her family recently moved into:

> So as I told you a couple years ago that I moved recently in about May. And so I have like just now like been like trying to learn the roots of like the neighborhood and how it's mapped out. Um and this goes along with miscomputation but I was walking in my around my neighborhood just like to figure out what things were, and like where all the houses were, where the neighborhood ends, where it starts. And I walked too far to the point and I had forgotten water so I was getting tired. And so I kind of count this as a miscomputation then because I felt like I overdid my walking the point where I was too tired especially because there was a hill and I had to get back up that hill to get back to my house. And so by the end of it I was just exhausted.

When questioning for a more conducive outcome of this walk, "Um I feel like I could have like thought it out more and like went at like a shorter distance per time like more just smaller distances for like each area instead of going just like all the way the back of the neighborhood, moving all the way back to the front."

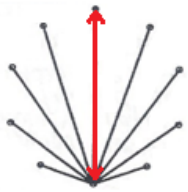
Athena agreed that it's just walking smaller bits and pieces instead of walking the entire whole. I asked if these bits and pieces helped her single out the transformations occurring from the pre-image to image of the literacy problems (see Figure 24):

> Yes, because seeing like, just like avoiding those two (pre-image and image) but it's going from P to A, it really does help because it's just like you're focused on that one thing and you can get the answer easier than having to look at the whole image.

Figure 24

Athena's Corrected Response with the Ability to See One Route as Two Pathways

Air Traffic Controller 2 wishes to find the total amount of one-way routes between 10 airports.

Amount of Routes from one airport	Miscalculation Process Skills	Corrected Calculation Process Skills
	There are eight segments from one point. So, the total amount of one-way routes is $10 \times 8 / 2 = 40$.	There are 9 segments from one point. So the total amount of routes is $10 \times 9/2 = 45$.

I questioned whether other points could be distractors to her as a way to learn about her level of focus:

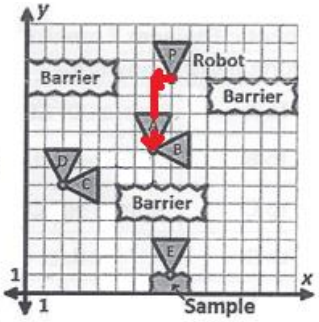
> I mean, they (other points) can be. Like if I'm looking from P to A [see Figure 25] and since B is right next to A, I can sometimes get them mixed up and I'm like no that's wrong you gotta go here.

Figure 25

Athena's Ability to Focus from P to A Without Allowing B to be a Distractor

Robotic Engineer 2 monitors movement of the robot

Transformation	Miscalculation Process Skills	Corrected Calculation Skills
P to A	Translate right 1, down 5	translate left 1, down 4.
A to B	Rotate 90° CW about (8,7)	rotate 90° cw about (7,8).
B to C	Translate left 6, up 1	translate down 2, left 5.
C to D	Rotate 90° CCW about (2,8)	rotate 90° ccw about (2,6)
D to E	Translate right 1, down 7	translate down 5, right 6.



Week 5 – What Miscomputation Story did Athena Relate to Rigid Motions and Congruence?

Athena construed a narrative that associated an erroneous line placement for one child out of ten during a face painting activity:

> Oh I was face painting, I think it was last Halloween for a booth at my church. And, well we had like a few designs the kids could choose from like the Spider-Man design, a few ones. But the one thing about it was that I did the one Spider-Man design on one kid and then I messed up the next time with the other kid. I was like it like, I did it [the line placement) right the first time, then I messed up the next time and so that was bad.

When questioning how she could have prevented this error from happening:

> I feel like to prevent it, um, I would have done my line placement better. Like I feel like because I did it the first time I got maybe a little bit careless. Because I did it the first time and it was fine but then the second time around, I messed up. So it was like oh that's not right.

When stating to describe the line placement:

> So it's kind of like with the Spider-Man mask it's kind of like a dot and then you go outwards like a spider web would be.

I questioned if she would consider the line placement to be a translation, reflection, or rotation:

> Rotation because they're going like… I mean it's like a circle shape so it's um just from… yeah point to a point … it's not like they're all connected somehow like there's no line going oh look at that line

Athena agreed that the line placement is like a curvature, "Yeah right and then it's like you're trying to keep the straight line straight. Athena also concurred that the line placement is also like a rotation. She described how she applied placement with literacy, "With placement, it's more of a like a place that basically dictates the entire thing like how it moves, how it rotates, like that's how you get from the pre-image of the image is the placement of the lines." Athena confirmed that her mentality of placement is being able to see the connection of points as translation, reflection, or rotation.

Athena's Stories of Faulty Algorithms

Athena shared stories involving alternatives or replacements. Although the narratives differed in context, Athena felt a more favorable outcome could have resulted by doing something else other than what occurred originally in the story. She paid attention to structure, regularity, and repeated reasoning.

Week 1 - What Faulty Algorithm Story did Athena Relate to Distance, Midpoint, or Slope?

Relative to distance, Athena shared how her sister broke her finger while playing softball as a catcher. She was surprised at how distance played a role:

> So I grew, like my sister of course, grew up playing (softball). So, however, she was, um, I think she was catching, yeah, So there was someone, the pitcher was throwing and I stirred one of she caught with her hand or of course in her glove. And she caught the ball the wrong way and it ended up breaking her finger. And so I remember that very distinctly. And like the amount of distance that it takes, especially like, my cousins of course played baseball too, and like the amount of force there is, like with the ball. Because people like teenagers can get up to 120 fastballs like throwing. So it was very

> crazy to see how even through such a long distance that could impact so hard and by doing one little thing that could end up breaking her finger.

In the interview, Athena did not formally identify the preceding story as a faulty algorithm. However, she confirmed later that it was faulty because her sister relatively caught well during practice but the lack of experience during the game was a factor to the error occurring. I inquired if Athena's sister did anything differently:

> She was definitely a lot more careful and she ended up moving to the outfield where she felt more comfortable. Because well she, she, it was her first time I think catching because she had practiced of course during practice. But I think it was her first time during a game. But she moved to the outfield and I feel like, she probably adjusted something because well I don't know, I don't know that much about softball.

Her sister moving to the outfield supported the theme of playing somewhere else other than catcher. When Athena mentioned "adjusted" and "comfortable," I asked if these components were felt with literacy:

> I feel like the adjustment makes me comfortable because deep down in my head I know there's something wrong by seeing the mistake. You can tell it's there's no really other way to go around it because it's a wrong answer. I feel like if you're doing a math problem, your main goal is to get the answer. So I guess it kind of makes me comfortable to know I have the right answer.

Week 2 - What Faulty Algorithm Story did Athena Relate to Perimeter and Area?

Athena described a reorganization of her room. Everything was going well except for one particular problem:

> So this one kind of portrays to area, sort of I think. And, I'm going back to how we moved this past Spring. I have my new bedroom and well since my room like I have…. Originally, I got rid of like this bedroom set I had, which I had a dresser and a bed frame whatever. And I got rid of it so that I could have more room. And so my grandmother bought me a new bed frame. And so I was, we recently moved in. And so I had my new dresser on one side, which used to be my dad's, and then I had my bookshelf, my desk, and my bed. One thing I figured out is that I'm always constantly hitting my foot on the side of my bed frame because I don't really see it because it's covered up by my bedspread. And so, I'm always like underestimating like where the corner is. That's one thing and also like just one thing where like area is that I wanted to … I originally wanted an L-shaped desk to go into my room but it wouldn't be able to fit because I have my

> bookcase, my record stands, everything else in there. And so we ended up going with the corner desk. And so I think we kind of underestimated like how the room was planned out because I wanted to put all these things in there but I couldn't.

I reiterated that she kept hitting her foot, "Yes, I actually have a pretty short bruise right now on my foot right now too." Since Athena implied that her room was divided into sections for various furnitures, I questioned whether she sees semi-perimeters in her room. I actually meant to say sub-perimeters here:

> I do. I feel like I see like the little gaps in between my room, like the little, like spaces where there's like open space. And like, because, well my room's like this, my bed is in the center of the room between the two windows. Because, well, my room's like the fire escape room because, well, it's like close to the roof. And like you need to slide down. So my desk is next to my bed. And then my bookshelf is next to my desk. And then the record stands in the corner over here in my dresser. And then I have my mirror next to my two closets. And so I think they're kind of always like examining like the little spaces in between. Like you can kind of see like the rectangles and like triangles that are in my room.

I inquired whether she gets perimeter and area confused sometimes when she is setting up her room:

> I think I have a real distinction because with perimeter you can basically see like the outline of things but you really with area it's just the space it takes up. So like you can look at the base of something like the outline. But in reality you won't know how much space it takes up until you have like the actual area. Like when I originally had like got my desk for Christmas, I was very nervous because it looked bigger than it actually was. Because I didn't see it next to anything else in my room so I was like that's gonna be too big. I was like stressing but once I moved it in there, I was like looking and I actually do have some space left. And it's just like you have to like look at the area of it first because the perimeter I guess can kind of make it things seem bigger in reality as well usually like that's like the garden the edges of the garden take up space in like the area too.

Week 3 - What Faulty Algorithm Story did Athena Relate to Counting Points and Segments?

Athena shared how a stream of Coca-Cola, representing a line, was a substitute for eggs, representing points, to bake a cake at a party:

> This was a while back. I was having some party with a bunch of friends and we were making cakes. And we were.. We had the batter out. The eggs were already in the batter. Wait no no no … we didn't have the eggs. We had the milk in there already and it was already in the mixer. I go to check my fridge. We didn't have any eggs. And so we're like we can't waste all this batter that we've just made. So we were looking online trying to find out where to like what you can just do eggs for. And we found out you can substitute eggs, I'm pretty sure, for like Coca-Cola like the drink.

She continued by relating points to locations and explaining why she considered this narrative as a faulty algorithm:

> And so we went rummaging through everywhere, like which is what I kind of connect the points with is also like doing the steps but also going from place to place. Like, because, first we looked in my fridge and then we looked in the pantry and like all the cupboards and everything. Eventually, we found some (Coca-Cola) in the pantry. And we eventually got the cakes made. Then, it got too small. Um, I just felt like that was a faulty algorithm because we didn't look at all the points and made sure we had everything before we started.

I asked Athena if she thought of the eggs as "points:"

> I feel like the eggs could be points, kind of because I feel like the different ingredients could be points. Because it's kind of like a step. I see it as you add like you go from here like you have milk and batter but then you add eggs to the mix which makes it a different point in the way of baking.

When inquiring a substitute of eggs is the same as a substitute for points:

> I feel like whenever I think of points in my mind I kind of think of them as … like just places in general … Because the most general thing I think of when it comes to points is point A to point B, which is … like getting from here to here.

I also questioned whether something like a line can replace the eggs as points:

> Hmm I mean I don't know. With the eggs, it was more of like you had to figure out like the best way what was the best thing to replace it but I don't really know.

We were so engrossed with our conversation that I forgot that Athena replaced the eggs with Coca-Cola. I asked if there was something she did replace the eggs with. She reminded:

> We did. We replaced it um with Coca Cola like the drink. We looked it up and we were like okay. So we don't have that but we do have this so we're gonna use this.

I then inquired how much Coca-Cola:

> We're talking like the whole entire can because it has to substitute for about two or four eggs… So I was pretty sure it was the whole can.

At this part of the interview, I realized how Athena was indirectly using ingredients to represent points and line segments. I asked if Coca-Cola can be thought of as a line:

> Yeah it's like a stream. It's getting from point A to point B. It's like the … actual physical coke, connects with the bowl in the batter.

Overall, Athena agreed that the egg as a point replaced Coca-Cola as a stream. I questioned whether this replacement could have been avoided, "I feel like I probably would have asked one of my parents to go to the store and buy eggs." I interpreted this as help from another person,

> Yes but we didn't … we weren't able to do that because we were making cakes like very late at the night.

Week 4 - What Faulty Algorithm Story did Athena Relate to Sequences of Transformations?

Athena told a narrative about a faulty move during a marching band drill:

> Um so I'm a marching band student. I'm in marching band. And whenever we were first learning our first piece at the drill, it's basically just transformations and reflections and everything like that. Because it's really just moving in lines and like that's basically all it is. And I remember, [sighs] I accidentally went to the wrong dot, which is like basically where you're supposed to end up by the end of the movement. And I ended up setting everyone else off track by the end of it. Because everyone else is not on their dot just because I was not on my dot. So, the curve was very wrong.

I questioned if things were going smoothly:

> Things were going smoothly until like I was marching. I was doing what I should have been, but then I accidentally miscalculated my steps and I went too far.

Athena mentioned a practice that could have helped her with a better outcome of the drill:

> Um, I feel like I could have looked at my peripherals more because with marching band, you rely a lot on peripheral vision because the entire time your head is facing straight towards the stadium. Like your head cannot move. And so um I feel like I could have looked at my peripherals more instead of just like looking dead ahead.

I inquired if peripheral vision is a common practice of marching band:

> Yes, it's actually a very big thing because we have this thing called, um, focalizing. And basically, it's where you look at your peripheral vision and make sure you're lined up with the two people beside you. It's like and then even with when we make curves, we still have to look at people beside us because if you're too close to this person too or too far from that person, it throws off the entire thing.

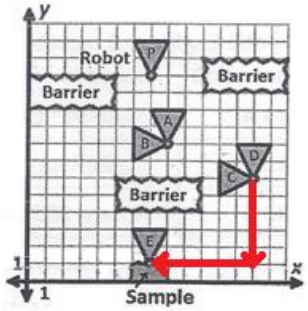
At this juncture of the interview, I wished of asking Athena more of how peripheral vision aided her with making modifications of certain rules that some geometric concepts impose on students. I reminded Athena about a rule about writing translations where the horizontal component is written first followed by the vertical. She agreed that it's easy to make a transition of going against this rule (see Figure 26). I wished of asking the role of peripheral vision with this transition but did not think of the question at this time. I inquired to describe some of the transformations she does with marching band relating to this transition:

Figure 26

Athena's Transition to Move Downward First, Then Left to Avoid the Barrier

Robotic Engineer 3 monitors movement of the robot

Transformation	Transformed Faulty Algorithm	Corrected Algorithm
P to A	Translate left 1, down 4	translate down 4, right 1
A to B	Rotate 90° CW about (7, 8)	rotate 90° CCW about (8,8)
B to C	Translate left 5, down 2	translate down 2, right 5
C to D	Rotate 90° CCW about (2, 6)	rotate 90° CW about (3,6)
D to E	Translate down 5, right 6	translate down 5, left 6



> Um, well you have the entire field, the entire football field that we use. So it's basically, every single section is divided into how many feet there are like a yard line to a yard line is eight steps in marching bandwise. So it's kind of really like a graph and then like … I don't know how wide the buffer is because I don't really pay attention. We usually use just the hash lines as like our things. And we just count off of those which is basically like graphing too. But, you're basically using those markers as like translating because you go by a certain amount of steps. Like every single time you're moving eight steps here, left, right, left two step like two to the left, backwards 12, it's just it's basically just like this [the transition].

I asked if there are any rotations:

> Uh yeah because you can sometimes like, at least movement wise, you have to shift your body to go like if they said go left two, you have to shift your body like this [she demonstrated a rotation of her body] but keep your head straight and then keep going that way.

Week 5 - What Faulty Algorithm Story did Athena Relate to Rigid Motions?

After some hesitation and reflection, Athena told a story about her friend who forgot an important component while developing a quiz for their French class:

> Oh yesterday, I was helping my friend doing her work in French. And we were, you had to take the vocab words and you had to basically write them down. And like different, you had like maybe like nine options you could do like each one a different way. And so I went with the quiz because well it was the easiest one you could do. And I was like I can get this done quickly. My friend saw me doing the same thing she did the same. And I was looking at her thing [quiz]. And she was basically done and then she messed up with her…. It was supposed to be a quiz and an answer key. She messed up by not making an answer key. And so I had to remind her to make one. And so then she had to restart another quiz. Because while she had done the answer key basically like she'd filled in the answer so she shouldn't have. But she just made another quiz.

When asking Athena what could have been done differently:

> I feel like I probably should have like warned her because like [the teacher] didn't really like specifically tell us. Like I had to ask her myself to make an answer, if I need to make an answer key. She [the teacher] didn't like announce it. So I was like I probably should have just told her before like she had started that. But I forgot to. So I probably should have, like once I found out she was doing that same thing what I was doing I should have told her.

At this part of the interview, the camera got hot and unexpectedly shut down. We summarized our conversation the next day. I questioned what Athena would have warned her friend:

> Um, I probably would have told her that the answer key had to be separate from the quiz that she made.

Lyssa's Stories: Focusing on Making Sense

Lyssa was a sophomore at Paulding County High School. She had a Lexile reading score of 1130, meaning she was proficient in reading comprehension for her grade. She enjoyed being involved with athletic activities, particularly as an ardent golfer for the high school girls' team and a manager for boys' basketball. Lyssa also excelled at learning concepts in her Honors Geometry class. Her favorite class, though, was a course called Law, Public Safety, Corrections, and Security. This course introduced Lyssa to the pathways of the forensic science field, which she took a liking to as a potential career. Lyssa had a candid, fluid, and fluent nature about herself, eliciting responses that focused on sense-making. She also was unique with sharing narratives as a blend of wrong strategy, miscomputation, and faulty algorithm.

Lyssa's Stories of Wrong Strategies

Lyssa uncovered wrong strategy stories about rethinking and considering choices rather than just doing. In Lyssa's perception, the outcomes of her narratives occurred because of a need to make the right decision based on what made sense the most. Lyssa shared that she needed to slow down and think of options rather than make hasty decisions.

Week 1 - What Wrong Strategy Story did Lyssa Relate to Distance, Midpoint, or Slope?

As a golfer, Lyssa described multiple trajectories of a golf ball toward the greens. She described the challenges of forwarding a golf ball, particularly out of a downward to upward contour. She referred to the golf ball as "it" and a pit before the green as a "hole":

> Okay so I play golf right … and … we start out on like a flat surface, like where we hit off of, it's always on a flat surface. But the thing is sometimes the course curves around. But a lot of times, if there's a hill, it goes down on the sides. So, like last year, when we played what I would do is I wouldn't really like look at their direction first, and I would go too much to the left. And then it would it would go straight down on the sides. And then because that would happen a lot. And then there's also like the divots in the ground

> that have the sand and stuff in them. So that's another thing is they [golf balls] go down in the middle and then you have to get it back up out again. So … if I hit it in a little, … you have to hit it back up out. So I have to kind of like look around and see how far up I have to go to get it out of the hole [pit before the green]. Because if you don't hit it hard enough, then it's not going to go out, just gonna roll back down.

I questioned what topics from distance, midpoint, and slope that she was relating to:

> I mean, distance mainly, and then kind of when you're like on the green. Because normally the hole is in the middle. So, I mean, technically that could be the midpoint depending upon where you hit it on the outside. Because you have to get it straight and make sure it hits perfect or else it's not going to go in. But then, I mean, I don't know, I feel like slope kind of applies with the hill thing because it's like they're falling down on the sides. When I think of slope, I think of stairs because that's all it's been in algebra. So like that and then distance like you have to like you have to make sure you're hitting it far enough. Because you only get like, you've only technically get so many hits. Like you have to make sure you're hitting it farther enough and like process that first before you go for it. So it's kind of like … make sure you know what's going on first before you go.

I started to think that Lyssa was really describing a case of miscomputation. When inquiring about the difficulty of this estimation, Lyssa shared:

> Oh yeah, especially when we go somewhere that's not our home course that we're used to.… Like say we're playing North and we got 30 minutes away, it's a whole different thing that we're not used to. So it's kind of like you have to like step back and take all of that and then think here we go.

I questioned whether if location comfort is similar to working inside a classroom versus outside:

> I mean kind of because it's like it's much different. It's more my bedroom than my house because it's a whole different thing especially when like semesters and stuff change and you get so accustomed to everything else and then you come back. It's like a brand new thing. So it's kind of like a weird feeling.

Lyssa agreed that she is trying to avoid a miscomputation. This made me wonder how this story could be a wrong strategy. I continued my inquiry of what she could do differently to minimize miscomputation:

> Well you have to angle it the right way because depending … upon what club you use like the thickness. Like every club is meant for like how high or how far you want to go. So you have to first you have to pick out your right club to make sure. Like because if I want to go like a short period of time but I have to go really high up in the air, then obviously I have to get like an eight to make sure it goes up. But if I want to go super far,

> then I have to use a driver so it goes super far. Like it all depends upon … kind of like the process where you have to figure out which strategy you have to use to like get what you're going. You have to use the right club to get where you're going to go.

It was at this point of the interview that I began to realize why Lyssa chose this narrative as a wrong strategy. I questioned whether using the wrong club is like using the wrong strategy:

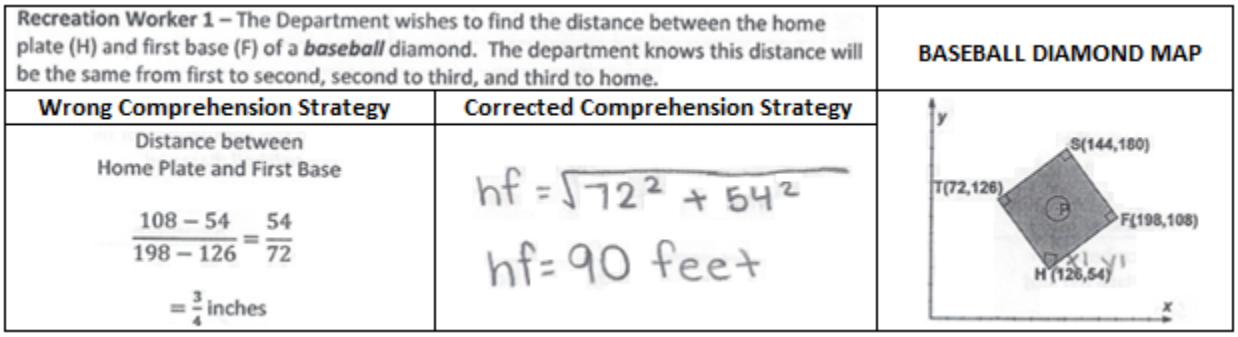
> Yeah, because it messes everything up and then you kind of gotta like redo it all and reset yourself.

Through more inquiry, Lyssa described the relationship between choices as a golfer and choices for error analysis correction. She shared why she believed the choice of an alternative strategy is right:

> Sometimes, like if I don't like assess it enough, or like if I just take a quick glance at it, like even like I messed up somewhere here. But it was like right here [see Figure 27], I think I didn't read this through all the way and I kind of just looked at this and I was like I'll just use this one and then I went back and I was like this literally makes no sense. So then, I had to like step back especially the one where it had the wrong strategy I was like okay so it's not that one but there's two other options so maybe it's this one. It wasn't kind of like a whole read through thing and like if I'm on the course and I kind of just look at it and I'm like okay well this might look right maybe and then I just go for it and then it's like oh well that clearly wasn't right and then I have to redo it all. So it's kind of like the hole. You gotta look at the stuff before you do it.

Figure 27

Lyssa Almost Agreed to the Wrong Comprehension Strategy When Realizing it Made no Sense



Lyssa elaborated:

> Because you gotta like, because even like, you can mess up on this because if you pick the wrong strategy because you don't know what's going on. Or where you just like, you don't fully read the problem because like I have to reread things a couple of times to like actually get it stuck in my brain. So like, if I don't reread it a couple times, then I might not completely understand what's going on.

She commented on the level of error analysis literacy that she performed:

> This [see Figure 27] was pretty easy because it gave like, it told you like straight up what was happening.

I questioned whether the error analysis literacy should be more challenging:

> I think it's good because it's like it's math and we're doing all that stuff right now. So like, I don't know, I feel like it ties in just fine because I don't want to be too much. Because if it's too much, then it's gonna like, it's two different subjects mixing together and then that's too much.

She elaborated:

> Because it's like, it's a good balance to where it's like it gives you the base of what's going on and then it tells you like it gives you real life situation so it makes more sense.

Out of curiosity, I asked if this activity could be converted into golf:

> Honestly, probably, if you think about it enough because this was measuring the outside of the diamond of the thing. And obviously, like golf course isn't a perfect shape. But normally like so when you get there, it tells you like, so you get to a new hole, it tells you how long it is. So like it tells you the base and how long you're gonna have to go around and like how many hits you have and all that stuff. So, I mean technically yeah, because this was all about a lot of this was about distance and the distance between all the stuff in the diamond. So I mean the distance from where you're heading off of to the hole. Oh yeah.

Week 2 - What Wrong Strategy Story did Lyssa Relate to Perimeter and Area?

Lyssa shared a narrative about when she needed to consider the perimeter and area measurements of furniture to fit them appropriately in her room:

> Okay so my room, I like decorating right. So, like my whole room all of my walls are covered. And when you walk in, it's kind of like, so you walk in, and then it's like a walk right here. And then you open it and it's another, like a square, no it's a rectangle. Okay

> and then I had a lot of furniture so I had a vanity in my desk and I had a dresser in my bed. And my room is like, since it's set up like that, I was like okay I can put all this stuff in a certain way. But, I had, there was a point in time like I didn't. Because I didn't measure anything, I just kind of let it, I was like you know this will work. So when I put everything. I like you walked in and it felt super crowded. So then I actually used my brain and I was like okay. And I measured the desk and then I measured the vanity in my bed and all of that. And I found like a corner that I could put everything in. So when you walk in, it doesn't feel like everything's suffocating in. And then it's the same thing with my closet. Because when you walk into my closet and it's got, it's a rectangle. And um, I have like one of those wrap things that hang and it like you hang clothes on it. And that takes up a lot of space. So I had to like, when I was putting my dresser and I had to make sure that nothing was overlapping on each other.

She agreed that this was a story of a wrong strategy:

> Yeah, it was like same thing as last time where I didn't think before I did stuff. Yeah because I didn't measure anything I kind of just went with it. I didn't like, I just put everything in corners. I didn't measure my bed. I didn't measure my desk. I didn't measure my vanity. Because then it was like oh everything was too close together and then there wasn't enough space for everything either.

She agreed organization played a role and described her vision of organization:

> A lot of times, I have it like planned out in my head. But then, when I put stuff, everything together, it like sometimes it doesn't work out the way I want it to or sometimes it does. It just depends.

I questioned what makes her decide that the plan needs to change:

> If I put it there and it like feels too enclosed, it doesn't look right, it doesn't fit like I can't fit everything and it's gotta go.

After inquiring if it has to be pleasing to the eye:

> Yeah it has to look nice. It has to look like ordinary because I don't like when things are like all over the place. Everything has to look like perfectly clean.

I asked whether the problems that she did for literacy problems had to look nice. She took her literacy document, borrowed my pen, and began demonstrating what she was sharing:

> Well yeah because these things [see Figure 28] were throwing me off, because so like, can I see your pen real fast? [I gave her my pen and replied sure.] Okay so you see how it's like this, so I was counting them, I was just counting these one two three four five whatever [see Figure 28]. I didn't and then I was like because I did this and you know how like it obviously has to like kind of blend with that, I was like literally this makes no sense. So I was like okay and then I drew like little boxes [see Figure 29]. Because I was

like, okay maybe they're all boxed in like that. And then that's how I got a lot of this stuff was because I boxed everything in. And I didn't realize that they were all multiplied by two because it just, the lines didn't make sense to me.

Figure 28

The Intervals, Referred to as "Things," that Threw Lyssa off Because of not Looking Nice to Her

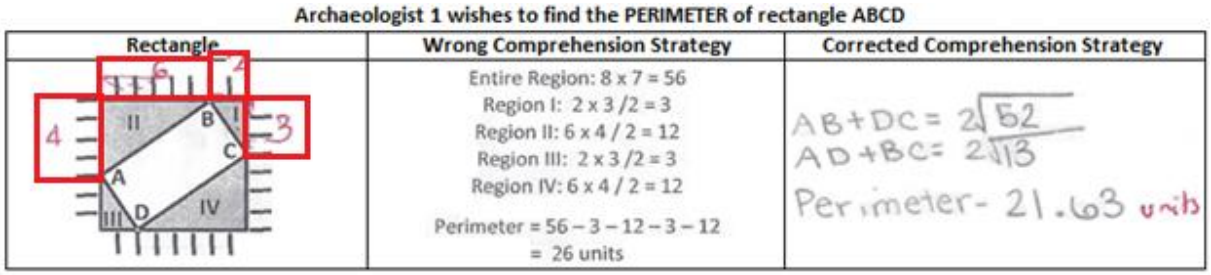
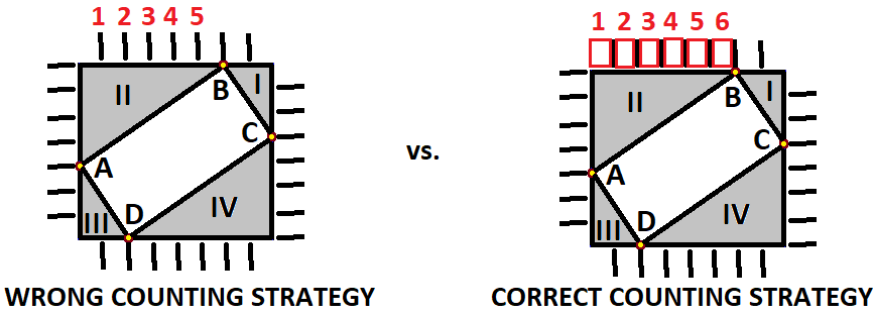


Figure 29

Lyssa's Transition from Counting Marks to Counting "Little Boxes"



Lyssa continued her narrative by implying how a different representation of perimeter and area can be interpreted as something that still needs to be covered before a literacy assignment:

> Because like I look at this and then I'm like okay. And I think about everything else we've done, I was like, this looks like nothing we've done. So I was like super confused. So I was just trying to make this look like what I knew and like the graphs and stuff. Like make this look like a graph by making everything boxes because then it made more sense in my head. And I think that's why I messed up a lot because that's not what we're supposed to be at … first.

Lyssa agreed that she was trying to organize what she was seeing and trying to make it a better picture:

> Yeah that's how I do a lot of my stuff. Like this is all like my handwriting is super clean because I make sure. Like if everything's all messy, then like it doesn't make sense to me at all. And it will make it all a mess and it won't like click in my brain because like it was like it was something. I think it was algebra you know how an algebra you have like a lot of different numbers all in one space. Like I would, I had [a teacher] last year. And I specifically highlight every different section. Because it couldn't like, if it was all blended together, I would just have no idea what was going on. Because then I would get everything mixed up. So if it's not like completely clean and like I know what's going on and it confuses me.

Lyssa concurred that the same way she arranges her room is the same way she arranges the problem for sense-making. She elaborated:

> Yeah, because if I can, like our test today, like I could look at it and I could tell what was going on. So it was just, like it was super easy, because my brain could like process everything. But like even like reading like but like there's so much stuff going on all at once that it kind of like confuses me. And, like I think that the thing is, like if there's too much happening at once, then my brain can't process it all. So I have to like reread it like three or four times. And that's what I did with this. Because if I read this through, I can tell you that I read it through but I can't tell you what it's about. So I have to like keep on rereading it and it just has to like make sure I know what's going on first before I do anything else.

Lyssa continued by describing her process of rereading:

> Like when you first read, it it's kind of like it goes in one ear and out the other. And then I read it a second time and I go through and try and get main points. And then I read a third time so it's like all in there in case I missed anything from the second one.

Lyssa agreed that the first time she reads something, it is similar to brainstorming:

> Yeah kind of it's like so I know what's going on and then the second and third time it's more like intricate and it's like okay now I need to like actually pay attention like I weren't intricate.

Lyssa concurred that she fine tunes information for minimizing a wrong strategy, "I break it down into a lot of things because I do process by process."

Week 3 - What Wrong Strategy Story did Lyssa Relate to Counting Points and Segments?

Lyssa discussed a narrative about what entails to be a boys' basketball manager:

> Okay so let's talk basketball.… So um I manage boys basketball right. So when you have … every shot that they make, I have to count down. And I have to write down and they have their own like a little section where I write it down. But there's when there's a lot of people on the court, you can't like really see what's going on and you kind of have to just look at their feet. So there's … a three point so everything outside of that one line is all three points and then everything inside and then there's the free throw line which is only one point. So it all goes into the book and then at the end of the like at the end of the game you have to add everything up together and there's been like more times than I can count that I've messed up and not added all right. And then I look at the scoreboard and I'm like uh-oh where did I mess up? Or I have to go back and recount everything.

She continued from points to fouls:

> And then we have player fouls and each player has their own little section of fouls. Because you can only get five and they have like they're just like little square boxes that you have to fill in and that represents the player's certain amount of points that they can go up to or else they're done. So like if somebody gets a foul you have to cross it out and then like you just keep going up until they're done like sometimes when there's like a lot going on in the court and there's a lot of people there you can't really see where everybody's shooting from. So you kind of just have to like really stare at the line on the court for a good second to see what it's like how much it is because like um it's outside certain it's three and inside two blah blah blah blah. So from each point that they can shoot from on the court it is a different amount that you have to put down for them.

Lyssa thought of the players as points and their passes to each other as lines. She related this analogy to her literacy document:

> When they pass to each other there's multiple different ways that they can do it and like when we were doing this [the literacy document] there's multiple different ways that you can connect everything yeah like when you're doing the rays and stuff like this can connect all the way to that but this one can also connect this one and this one can connect all the way to that this one also kind of just that one.

When inquiring what can be done differently:

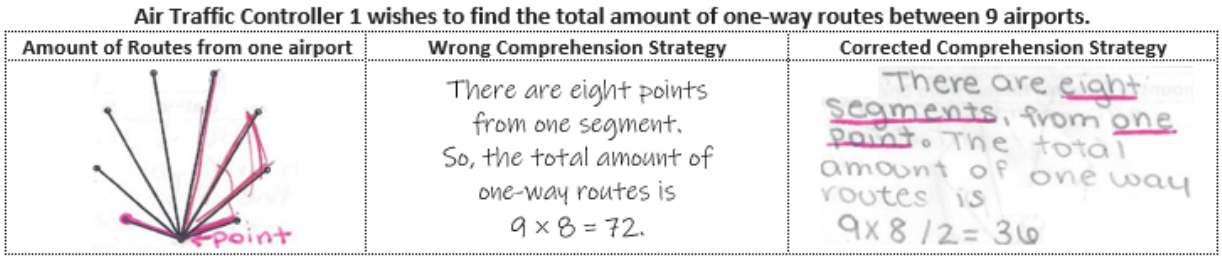
> Um a lot of the times if they like if they play something wrong they just need to play it right the next time. They just need to like take what they messed up on and fix it. And then when I mess up I have to go back and fix what I did wrong. Like when I'm reading it wrong or I'm reading it too fast, you're like there's too much stuff going on. And I have to like take a second and then redo it. Because it's all about like … if there's too much going on and you read too much too quickly times then like you have to like go back again like triple check.

When inquiring about the way she triple checked her literacy:

I underline stuff to make sure I know what's going on so like this [see Figure 30] is all underlined. And I draw my little lines to make sure I know everything's counted right because especially when we're doing graphs and stuff. If I can't miscount, like I have to mark it off. So I know that I already did it. So like these are all marked off and then I like um like when they did it wrong on here I underlined what they did wrong so I knew that I had fixed something in the problem because like this it says nine and I fix it to make it 10 and I underlined it counted all those and they're all pink. Same thing with this one, these were all getting counted. These already had lines so I didn't really need to worry about it but like the ones without lines I made my own lines. This is messed up so I fixed it and it's kind of just like the whole making sure like dividing into sections. So like I did this section first and then I fixed this. And I was like okay I know that this is wrong. So if this is wrong something else has to be wrong because they just didn't they didn't divide this part. Yeah so I was like you have to go through divide it up into a little sections so it makes sense yeah.

Figure 30

Underlining by Lyssa to Determine Correct Strategy from a Wrong One



Week 4 - What Wrong Strategy Story did Lyssa Relate to Sequences of Transformations?

Initially, Lyssa perceived the following story as a miscomputation. The narrative involved one of the many challenging tasks of a basketball manager:

> Oh I was trying to unlock a lock yesterday and like twist it. So we have, like they have all the balls locked up in like a little container thing. And like we had the code for the lock and we kept on rotating the wrong way. Do you know like how like those little spinny locks and you have to like spin them like four times to clear them? We weren't doing that. So we had to keep on rotating it backwards like 40 times. And then we had to do it forward and then we had to do it backwards again. So it was like you have to rotate it on the right angle or it wouldn't unlock. So, that was a miscomputation because we did it all wrong. We did not understand what was going on with the lock.

She agreed that it was like she was turning it clockwise when she had to turn it counterclockwise:

> Yes, because okay, so it was like, you know it was like, I don't even know what they're called, the master locks where they're like little and they're like at the little thing just spinning around. So we were spinning it because like you have to spin it to clear it. And there were so many of us trying to do it. So we had to spin it to clear and set it to zero. And we're like okay cool. And then we had to turn it one way. And we were turning it clockwise because we were like okay this let's make sense. But then we forgot that you have to like turn it, then you have to turn it all the way counterclockwise before you can do it again. So it was kind of just like a, we didn't really understand what was going on. But we kept on rotating the little circle thingy like this mini thingy.

I inquired whether in some cases, she was turning too many times:

> Yeah because we didn't find the perfect, we didn't find the right angle, like we weren't sitting at the right angle on it. And we were like overdoing it.

I questioned if this narrative was a miscomputation or wrong strategy. Lyssa asked for clarification of a miscomputation. I shared that is was like a miscalculation, "It was a wrong strategy. I thought miscomputation meant you didn't understand." This was a moment that I wished of being more transparent with the difference between a miscomputation and wrong strategy.

As the interview progressed, I asked what could be done differently to avoid turning the lock too many times in a wrong manner:

> Um, I don't really know how to avoid it. But like sometimes like, you know how we have like our wax paper and we have the little triangles and thingies like when we're flipping them around.

Lyssa was referring to the use of wax paper that was used in our class to demonstrate slides for translations, flips for reflections, and turns for rotations. She continued:

> For some reason, flipping them [the wax paper] around and like getting perfect 90 to like 180 and 270 is like kind of hard for me. So like um just kind of like thinking it through and like instead of like looking at it, physically thinking of it in my head. Like okay so if it's going this way and then we got to go this way. And then like dividing it up into little parts. Like if I had looked at, if I had looked at the lock before yesterday, I'm like okay,

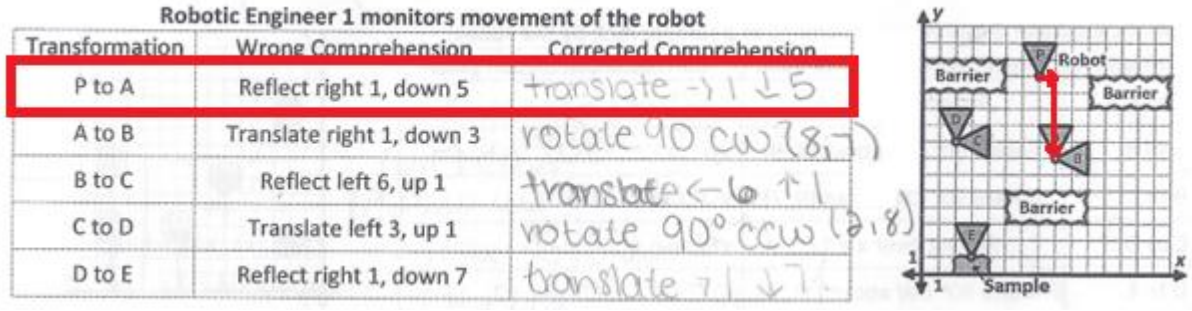
> well first I have to reset it and then I have to go to the first number and then I have to go the opposite way to get the second number and then back the opposite way to get to the third number. Kind of like when we're doing like the things where it's like first rotate it this way and then translate it this way. So it's kind of like, um, you have to like do it by parts and then make it a whole process. Because kind of like when we're rotating stuff. Like, um, I can't like, like if we're rotating something like from one section and it's going like 270 to the other, I have to do one and then make it go to this and then draw the 90 degree out and then go from 90 degree to the 180 degree and then draw the 180 and then go from the 180 to 270 so then it can match up in my brain. It's bits and pieces.

I asked Lyssa to elaborate on bits and pieces on her literacy document. As she looked at the first wrong strategy problem [see Figure 31], she began talking to herself:

> Ok so, like this one, so it said. This is the one that I think they did wrong, right, okay. So, oh okay, so this one, reflect right, so P and A are like, where are they? they're right here.

Figure 31

The Productive Struggle by Lyssa to Understand why a Reflection was Used Instead of a Translation



Robotic Engineer 1 monitors movement of the robot

Transformation	Wrong Comprehension	Corrected Comprehension
P to A	Reflect right 1, down 5	translate → 1 ↓ 5
A to B	Translate right 1, down 3	rotate 90 cw (8,7)
B to C	Reflect left 6, up 1	translate ← 6 ↑ 1
C to D	Translate left 3, up 1	rotate 90° ccw (2,8)
D to E	Reflect right 1, down 7	translate 7 ↓ 7

She continued:

> So, this one is going, it said, reflecting is when you go across something. So I was like, this does not make sense. Because if you were reflecting something, first of all you have to have X and Y in there because if you don't have X and Y and then it's not talking about anything. So I'm, first in my mind, I was like okay well maybe this automatically means Y and this is why because down means Y and left and right are X. So, I was like, okay cool, so into my mind I was like, you know, what whatever. So I was trying to understand where they were getting this from first because like I already knew how to do that. But I was like where and then what did they get this from. So I kind of drew out like a little line. And it was like, so why one is right here okay and then down five would be one. (She hesitated and studied the problem). So, I was like, okay right here and then I

> kind of looked at it. And I was like well if you go down five, this is right here. But it wasn't like and I was like you can't. That's not the process you're supposed to do because it would have flipped it off and then it [the Mars robot Opportunity] would have gone off the ground. So I was like okay maybe it's the opposite way. So I was like, if I count five, I can't count up five. So I was like you know what they just did it all wrong. And you have to translate it.

It was here that Lyssa helped me to understand that she had to progress through a productive struggle that made sense to her but not necessarily others. She needed this time to sort without teacher or peer intervention. She continued:

> And that's what happened with a lot of these is because when I do the literacy, because it always has what they did wrong, and I kind of sit here like, and I kind of try and understand where they messed up on. So, like I can understand what they did wrong and um like with the whole like this one with the whole reflecting things and it's got it going over and stuff um oh I didn't even answer this one. Okay, anyways um, because they didn't mess up on it.

Week 5 - What Wrong Strategy Story did Lyssa Relate to Rigid Motions?

Lyssa shared a story about the choices her family made in an escape room:

> Okay so wrong strategy at least I think this is a wrong strategy. I might have got this in multi algorithm stuff but you can tell me. Okay so me and my family we do escape rooms for fun right, you know what those things are? [I replied yes.] Okay, so we were doing one of those and normally like so they give you clues before you go in right so you know what's going on. So they give you like the premise of the situation and they tell you what's going on. So you can like use those to like you know escape with the whole point of the thing. So basically me and my family we had taken those clues but they gave them to us in a certain order. But we didn't think to keep them in the same order. So we kind of moved them out of order and did each one like in a different section. So like instead of doing like one, two, three, four, we did like four, three, one, two. Like it was all messed up. So since we did it in the wrong order, we couldn't like everything else after that got messed up. So, like we were supposed to like, move like, one thing and then that was supposed to move another thing and then that was supposed to move another thing. Like it was all supposed to fall together. But since we did it out of order, nothing was really able to fall together. And it all kind of just like I would kind of just like fell apart basically because we used like the wrong strategy to um follow the steps of the clues.

I questioned the form of the clues:

> They were kind of, they were, so there was a bunch of different kinds. So originally like they give you, um, like a list of three, like just three written down and they show them

> like on a screen and then there's clues around the room. So like, if you like unlock one box, that box might have another clue in it to help with the next step. So like it's kind of just like you start off with three and then you have to keep going along using what you already have.

I clarified with Lyssa that the clues were basically on a given list and the walls of the room:

> Yeah so they give you. It's kind of like what we're doing right now. [Here, she referred to anchor charts that are posted around the room for students to use as references for responses to Geometry problems in general.] We have to like find the missing one, you know. So, like they give you a certain thing and you have to find everything else that goes with it.

When inquired on what Lyssa and her family could do differently:

> Um, if we had read the room more and like really taken take. Because it gave us a certain process. So if we had taken the process that it gave us, read the room more and then moved on from there, all the pieces would have fallen together better instead of us looking for what we needed first before using what we had. So, like if we had used what we had first and then got the extra instead of looking for the extra and then using what we had. We did it backwards basically.

I confirmed with Lyssa to basically don't mess with the given information:

> Yeah keep, like keep the givens first. And make, they, the givens always stay the same. And then find out what is extra.

Lyssa's Stories of Miscomputations

Lyssa unpacked miscomputation narratives that entailed the results of an experiment or task. For Lyssa, it was important that she did her part of the task correctly to avoid affecting others. However, the following narratives indicate that how thinking too fast resulted with misestimations on her part that also affected others.

Week 1 - What Miscomputation Story did Lyssa Relate to Distance, Midpoint, and Slope?

Lyssa recounted a law class simulation involving the distance of a fake gun and two chairs, where a miscalculation prevented her team from acquiring sensible results:

> Okay so like I have law for fourth period. And we're measuring crime scenes right now. And when you do it, you have to like, so like if you have, it's like a square on the outside right. And say like, there's like a gun in the middle, you have to like triangulate it. So like yesterday, we were doing one and we had a gun and there was two chairs. And what because like he partners us up. And you have to measure the outside first. And what we did is, you know how on the tape measure it starts at zero but like it has the things that sometimes cover it? So yesterday we started at what we thought was zero but it wasn't that. So we messed that up. So like that was like the distance of the outside which messed our whole thing up. Because we watched a video that said we sort of started at 10. And then just like, so like, if we started at 10, like the whole thing was 65. Then minus 10 from 65 would never get the whole thing. So we're doing that and then so like, if it was because it was like a square was what we were looking at. And then the gun was kind of in the middle and they had like two chairs on the side. So the gun was kind of like it was the middle of it so it's kind of like the midpoint, I don't know, whatever you want to call it. So it was in the middle and you had to try and triangulate it so you had to like go from the corner. And then go inside and what we did was we didn't start at like the legitimate corner. We started more like to the wall instead of the actual corner of the square. So that messed our whole thing up too. So we had to go over and redo all of that yesterday.

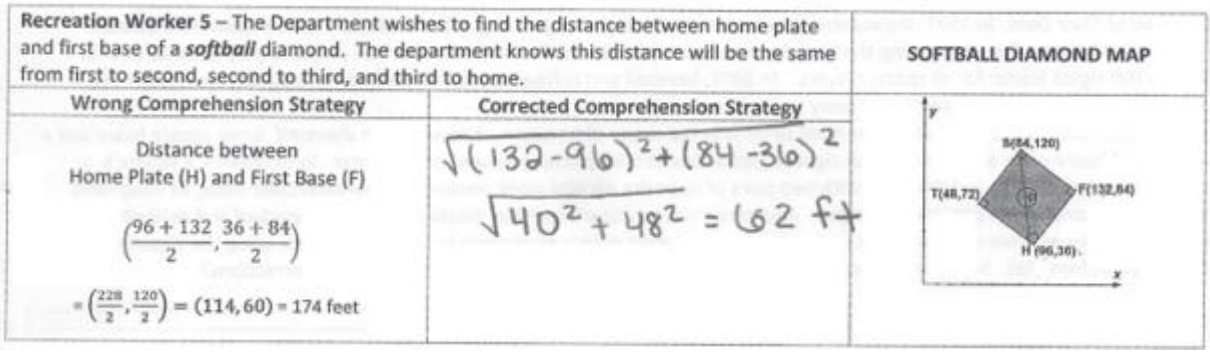
Lyssa spoke of this simulation as a cross between a miscomputation and faulty algorithm.

I asked if this blended type of error was noticed in her literacy document:

> Yeah because um yeah, there was one of them what they like subtracted the wrong thing. I forget I think it was distance because it was when we were doing this where we were finding out the overall, yeah like where they did all the math wrong [see Figure 32].

Figure 32

Lyssa Chose a Wrong Strategy Problem to Relate to a Miscomputation



I asked what could be done differently to produce a different result to the simulation:

> We didn't really like, because it was like, it was right in front of us. And we didn't really look at it long enough to, like because it like, we thought it was a rectangle at first. So

> like, you know we measured every single size. It really wasn't so. We looked like if we had took more time and like processed what was happening and then done everything instead of just going like step by step. Like plan the steps out ahead of time instead of going, like, okay, we do this. And then oh no what do we do next? Oh, we do this next. And da, da, da, da, da. But if we had like written it down or like even just in our minds, we're like okay so we're gonna do this and then do this and then do this. Yeah.

I iterated it was important for her to know the content first before a literacy assignment:

> Yeah, like right. I'm like even like in like we were doing other day like the Lexile thing that you have to take. When I read something, I read the questions first and then go back. Because it just, I don't know, because if that's in my brain, then I know what I'm looking for. Because I have a hard time with like reading stuff and then retaining it. But if I know that I'm looking for something, it makes it easy.

I confirmed if she knew she had to be looking for something, "Yeah, like having everything planned out ahead of time." After inquiring on whether the literacy assignment involved planning ahead of time:

> I mean yeah because I had to know what process I was going to use for each one before I read through everything.

Week 2 - What Miscomputation Story did Lyssa Relate to Perimeter and Area?

Lyssa elaborated on a prior narrative where she miscalculated perimeter with area on a Geometry test:

> Oh yeah [perimeter] is supposed to be the outsides added together right. I understand what's going on but I think I flip-flop them by accident.

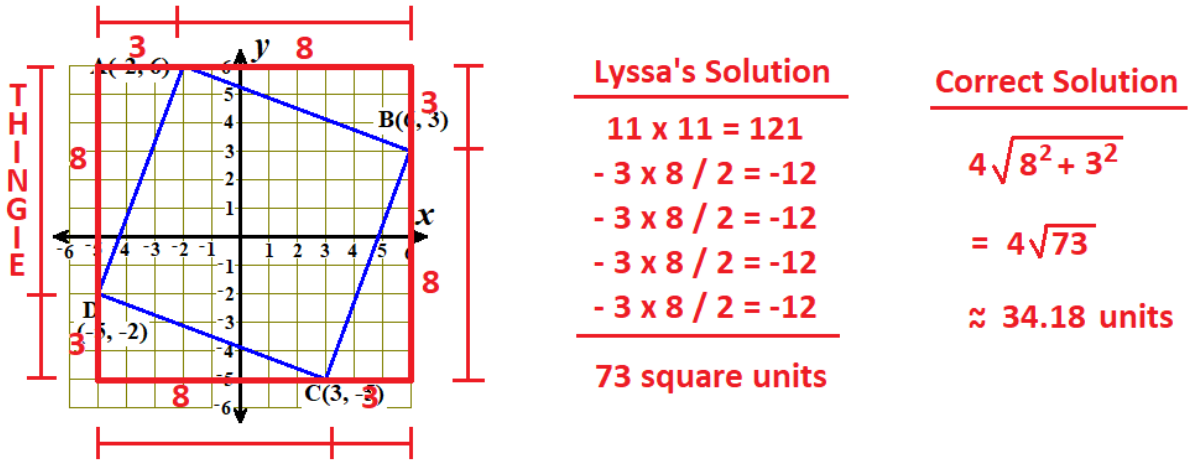
I questioned what's the reason for the flip-flop and why does it happen by accident. She referred to horizontal and vertical distances as "thingies" (see Figure 33):

> My brain's not really good at holding processes, like keeping, like retaining information, especially to tests, I'm really bad at taking tests. But um retaining information when there's a lot of formulas and stuff going on, it just flip-flops. And since they're [perimeter and area] so close together and they're closely related, it's kind of just like a [there was hesitation here]. And this made, I don't know why but this made more sense to me to be perimeter because you were taking. I think it was like you know how we did the thing where you drew out the little thingies [see Figure 32] on there. And I think that's what I was doing with this because you drew out the thingies on there. And I thought we were

doing perimeter for that or whatever. I don't know because I counted these boxes because I looked at this I was like oh this looks like the thing that we were doing the other day. So I counted the boxes in here and then I did that and then I just subtracted the outsides and I just thought that was right.

Figure 33

Lyssa's Thinking of Horizontal and Vertical Distances as "Thingies" and Lyssa's Flip-Flopping of Perimeter with Area



I questioned whether the appearance of the distance formula interfered with her thinking of perimeter and area:

> Kinda sort of sometimes. It depends upon which ones. Like this makes sense, this formula makes sense where you're doing the next one of the things do that make sense and then so does this one like the slopes make sense. I think it just since area and perimeter are literally like related to each other, that's when it flip flops.

I questioned whether when area and perimeter being similar to each other leads to a miscomputation being involved. I really meant to say wrong strategy. In any case, Lyssa replied:

> Yeah because they're [perimeter and area] like, they're so, they're in like the same section of my mind, so they're super similar to me, so yeah.

I asked how would she overcome the difference in terminology of both perimeter and area, or, in Lyssa's words, how would she "fix it"? There was some drama detected in her voice when she responded:

> I probably just like keeping redoing problems until I die. Because that's what counts the most is rereading and redoing stuff. Because if I keep redoing it and then it clicks, like this [pointing to her literacy document], it was not always there, so something on here but it was like I couldn't remember what was going on. And I had to like read the problem like a good four times and then it was like oh this is what's supposed to be happening. So then it made sense. So I think if I just practiced it more. then like I would get the sense of it. But I think since like it was like perimeter and then area and then perimeter then area just like so much was happening that it just all blended.

I asked whether a literacy lesson should be more organized so that perimeter and area are not overwhelming: "Kind of yeah or just more separated so they're not like overlapping each other."

Week 3 - What Miscomputation Story did Lyssa Relate to Counting Points and Segments?

As noted in an earlier narrative about Lyssa's experience as a basketball manager, she felt a miscomputation occurred on her part as a result of the players following a wrong strategy:

> Well a lot of times not even when I mess up. When they mess up if they use the wrong play and then they mess it up and pass it to somebody that they're not supposed to, then it gets all messed up. Like if they pass it to somebody and he's not in the position to shoot at that moment and he goes to shoot then they're all messed up and they have to reset it again. But sometimes even if they have so much going on and I look at the court at the wrong time and I think it's a number and it's actually not, then it's I mess up. That's more miscomputation though because that's like I didn't really make it through.

She reiterated that it's miscomputation on her part resulting from wrong strategy on the part of the players:

> Yes because it all falls together. Because what they do I just have to write down and make sure I know everything that's happening.

I inquired if the miscomputation is occurring because of overwhelmingness:

> It is a lot of times. Because there is a lot going on and you have to make sure you are watching every single thing. Because so much happens at once like make sure everything is connected together.

When asking what she can do to avoid miscomputation, she suggested an alternative way of reading that related to the movement of her eyes:

> Um, not like, the way I read it read it is a lot of times I read it across. And I should probably start reading it up and down by quarter. So just change the way I read it or even have someone else triple count for me to make sure that everything's right on there. Because, a lot of the times, there's so much filled out in one area that you have to like take a second, and like okay three plus two plus this plus this plus this and like let it all go one by one.

I asked if it's almost like she needs a partner in a way:

> Yeah like another set of eyes just to make sure everything's good and like let someone else help.

When relating how she corrects miscomputations in her literacy:

> It all depends upon what the person messed up on. So like, if I like, if I can read this and look at that and I'm like okay that's there's something wrong. And then I go through and check it. I'm like, okay now that I know that they mess up, I need to figure out how to fix it. So it's like finding out what went wrong first and then fixing everything else and then I'm like okay well they did this right. So that's less important because it's already good.

I questioned whether she felt of supporting the mistake first:

> Yeah it's when it's like I went through, found the mistake, fix a mistake, and then finish the problem.

Since Lyssa previously mentioned about changing her reading from across to up and down, I was curious how she thought her eyes moved during the reading of an error analysis problem. I inquired if her eyes moved left to right or up to down. She implied a combination:

> Oh yeah sure so like this I count as one [referring to a segment between points]. So this is up and down and this is up and down [referring to other segments] because they're all, because reading them left to right is like okay one two three four but then there's the little sections in between them. So I count the lines up and down first and then do the little like point.

She agreed that this method of reading is similar to reading sections of basketball information by quarter in an up and down manner:

> Yeah it all just depends upon how it's easier to like comprehend in my brain. Because I know that I'm counting these up and down then I know that I can count these the other way because there's not too much going in one area.

Week 4 - What Miscomputation Story did Lyssa Relate to Sequences of Transformations?

Lyssa shared a narrative about the struggle of placing a golf ball at the right rotation on a tee:

> It's more angles and stuff. I have to make sure like, when I'm putting my ball on the thingie [tee], I have to make sure like it's rotating because it's flat. Because I put in the circle on the little holes it's kind of hard. So you have to make sure that it's like it's rotated to the right angle so it'll sit flat. Because then it'll roll off and you can't handle it's wrong.

I was confused and asked if she was talking about the ball rolling off the green or the tee:

> Thank you I forgot what they [the golf tees] were called. And even when you're on the green and you have to make sure that it's in the perfect like lineup because if it's not lined up with the hole, then you're not going to get the ball into the hole and it's going to be a mess.

When asking if she used divots on the green, "Oh yeah a lot of times when we're playing as a team we do." I asked if she rotated the golf ball wrong:

> Yeah if you don't have it rotated the right way it's not gonna, you're not gonna be able to hit it right.

I clarified by stating that she has to rotate the ball the right way in order to hit it right:

> Hmmm, hmmm [meaning yes]. Because it's a circle. So you have to, you got to make sure.

I questioned whether she ever rotated the ball wrong:

> Oh yeah one time I just threw it on the floor. And I didn't like, I didn't been trying to do anything. I just smacked it in there. And it didn't went very far. I put my ball in the lake that day. Because like we were on the green. And I kind of just tossed it down. And I was like okay, this looks fine. Like, I didn't sit there and try and like angle it out. And I hit it right, like I was supposed to hit it. And since it wasn't in the right lineup with it and it wasn't rotated right. Because on the side of golf balls you know how like they have the little dot that you're supposed to hit on the side of them? It wasn't rotated to where I could see the dot. So I kind of just hit it. And it did not go in the hole but fall in the water. It was really bad. Yeah.

When asking what gives her confidence that a golf ball is correctly rotated on a tee:

> Um, a lot of times, because in golf you only have so many strokes especially when you're playing a match. Like driving range okay cool whatever. But when you're in a match, you only like that's when things get serious. You have to like be like, okay I only I can only hit this once. Because if I hit this more than once, then that's like counting down. Because you know how golf scores right? [I nodded.] So like if I have, if it's not on there right, and it's not on there to match up with my club when I hit it, then it's not like it's not gonna go as far as I wanted to. It might just fall off the tee and then that automatically counts as one (stroke). So just sitting there kind of and making sure it's rotated so it's perfectly sitting on the tee, it's not going to fall off. And the dots are lining up with where my club is going to hit it. I just have it so like when it all goes through and it all goes smoothly. It's like kind of like the background process for the whole entire thing. It's like setting it up and planning it.

I inquired about describing the lining up and background process on her literacy document:

> It's like yeah it makes sense, like it all goes together. Because, like especially when you're moving from like see like this one's moving from P to A [see Figure 30] and then this one's going from A to B and then this one's going from B to C and C to D and D to E. Like you have to make sure P to A is right in order for A to B to be right and then from B to C. Whatever you understand what I'm trying to say. You have to like, every process has to be right. Because, like if I mess up, like this one, then they're all, then the rest of them are going to be wrong. Everything following suit is gonna be wrong. Like if I mess up the first one, everything after that is going to be wrong. So you have to like make it, make sure to do it, make sure it's right. Double check and then move on because everything kind of it all falls together. So it's like bits and pieces okay.

Week 5 - What Miscomputation Story did Lyssa Relate to Rigid Motions?

Lyssa talked about a translation error in Spanish that led to a different meaning than what she intended:

> Okay sure. In Spanish right now, we had to, we have to like write like the ares and stuff. And it was basically just like he gives us a prompt and we have to like answer it in Spanish. But in Spanish, you have personal pronouns so like he, she, they, me blah blah blah blah blah. And then you have to conjugate a verb. So if you have a verb and you have the personal pronoun, you have to like conjugate the verb to fit the pronoun. And certain, um, personal pronouns don't need the verb conjugated. So I was just you know I was just on a roll and I was doing my whole thing. Everything was going great. But then I, um, I conjugated a verb that wasn't supposed to be conjugated. And it was supposed to stay the same. So when I conjugated it, it kind of like messed up my whole sentence because it ended up meaning something completely different in Spanish than I wanted it to mean. Because I, um, I changed it when it was supposed to stay the same. Because

> like, I think it's like they or they are we in Spanish isn't supposed to change. Like you're not supposed to conjugate it with something else. So when you do that, it messes up the whole word. And it messes up the whole sentence. So basically me messing up that one sentence kind of made my whole story confusing because it didn't make sense anymore.

Lyssa briefly described a relationship, "The pronoun and the word kind of go together." When inquiring what the prompt was about in Spanish:

> Um, it was, I think it was, um. The one that I messed up on it was like describing um, how the school day is like, if I think it should be shorter or longer, and how it affects the people.

She elaborated:

> Yeah it's like describe the schedule, like the school schedule and do you like it, do you not like it, is it too long, is it too short like what can make it better and all that? Yeah and I wrote about how um if the school day like started earlier and ended earlier, kids would have more. Because we start we end really late and we start pretty late too. Like if we started at like maybe like seven, then we could end at like two. And then people could have time for jobs. People could have time for sports. Because most sports like, if you so, like if you play a varsity sport, normally you don't get out until six. And then you have no time for a job or any of that because no one's gonna hire a kid who can only work from like six to eight.

She claimed how she misused the pronoun "they" over "we:"

> Well I was supposed to [she stopped her sentence and started a new one]. So "they," when you put a verb next to "they", it's not supposed to get conjugated. But I thought it was supposed to be. Because everything else does. So it was kind of like well everything else has this happen. So why doesn't this one have this happen?

She described an analogy of the hypotenuse-leg theorem with triangle congruence criteria:

> And it's kind of like when you're doing angle side and a side effects. So you know how like there's angle side angle and then angle blah blah blah all that stuff. And then there's just the HL [hypotenuse leg] thing that doesn't match up with anything else, it's kind of like that is where they is literally the only one that doesn't have three letters in it.

When inquiring what she could have done differently:

> Um I didn't, so I think the thing I messed up on was I just didn't fully look at like the sentence. I was writing first and I didn't look at the pronoun beforehand. kind of like if you don't look at the angles enough, then it looks like it could be side angle side when it's really um HL [hypotenuse-leg]. So like, if I had looked at the whole thing first and then like been like oh yeah this literally doesn't make sense. Then it would have the

> sentence would have made sense instead of me messing up the entire thing [so fully look at the sentence first] and trying to like make it make sense in my brain.

I questioned if fully looking at the information in her literacy document was applied particularly with clockwise and counterclockwise problems, "Sometimes I just forget which way is clockwise and counterclockwise." When questioning what causes this forgetfulness:

> Um, if I think too fast and I have like and if I think I know what's going on so, I just go go go go go go go. Because like it seems it seems so easy in my brain that I already know what's happening so I just take it as it is. And I don't really like take it back and take every single step as important. I'm just like oh okay well I already know what's going on. So it makes sense but then I mess up on little things like this.

I questioned whether slowing down depended on the level of rigor of the problem:

> I think it [the literacy lesson on rigid motions] needs to be harder because I think since it was it looked it looked so simple to me. I was like oh okay, this is easy, like there's nothing complicated about it and I would just like completely missed a bunch of parts.

I then inquired how the literacy lesson can be more rigorous:

> Um I like, … see these are just rotating like instead of making it … like rotate and translate like multiple rigid motions in one instead of just one rigid motion.

Lyssa's Stories of Faulty Algorithms

Lyssa revealed narratives involving tasks that needed better measurements for more favorable outcomes. From her perspective, an investment of time to focus on necessary components for each situation would have made a difference. Sense-making was again a significant factor for progressing in a positive manner.

Week 1 - What Faulty Algorithm Story did Lyssa Relate to Distance, Midpoint, or Slope?

Lyssa did not specifically tell a story that related only to a faulty algorithm. As previously mentioned, Lyssa felt that the law simulation activity was really a blend of a miscomputation and faulty algorithm that related to distance:

> Yeah because we didn't like. We were also like, we were doing it so quick that we were like, we kept on messing it up. So it was, um, because then when we redid it, everything was right we just had to like go back.

She agreed that there were components of the miscomputation and the faulty algorithm in the wrong strategy of playing golf, implying that sometimes these errors occur simultaneously. However, I felt a need to provide more clarity with the distinctions of the error types. Overall, Lyssa concurred that this experience is really like operating a camera, where the focus is on making sense:

> Yeah you have to like focus in on the subject [information] because when you're taking pictures, you have to like make sure that's exactly what you want.

Week 2 - What Faulty Algorithm Story did Lyssa Relate to Perimeter and Area?

Lyssa unfolded a narrative about the challenge her family endured while placing a couch in their home:

> We were moving a couch the other day because it didn't fit in my house. Because my mom measured the house wrong. Because we had, she was like well actually thinking I measured it wrong. Because she was like I didn't decide to this side. So we went too far and like we were like okay this makes sense to fit. So we got the couch and we ordered it. And then it came in. And we're like okay now this doesn't fit in our house. So the perimeter of the couch is too big to fit in my living room. Because we didn't like, um, we didn't measure it right so.

I began to think of this as a wrong strategy but then she convinced me of experiencing an encoded faulty algorithm:

> At first it seemed great. Because we were like oh my goodness this couch looks great. It'll look great in the living room. And then it gets there we're like hey this is way too big to fit in the living room. And we like spun it around and everything. So like, it just like, the ending of it messed up the whole beginning.

She related this situation to her literacy:

> So like even like if you're doing a problem sometimes and like everything's going great then you miss you have to redo the whole thing.

I inquired what she would have done differently with the measurement of the couch's perimeter:

> We would have had to like, because how we measured it like measured the living room to measure the living room differently. So like we just we just messed up how we measured it. And even that was like I messed up how I was counting [pointing to intervals on her literacy document].

I asked about the appearance of the visual diagram, "I think since it was unboxed, it didn't make sense." Lyssa elaborated:

> Yeah because like how I drew boxes here [see Figure 28]. Because these were just lines, it confused me. I'm used to seeing boxes so like it changed it and it made sense.

After questioning what kind of object reminds her of the narratives with perimeter and area, she thought about a while. I prompted for what is perimeter and area to her. She delineated:

> Perimeter is like the outside of a shelving area and then area is the box that you put in the shelf. Have you ever seen those cupboard things that you have, like it's like the outside of the bookshelf and then you have like the box that you put inside of them you know what I'm talking about? [I replied yes.] Yeah [that's what she was talking about].

Week 3 - What Faulty Algorithm Story did Lyssa Relate to Counting Points and Segments?

Lyssa previously mentioned a narrative about her task as boys' basketball manager that related to wrong strategy plays by the boys and miscomputations by her. She also depicted faulty algorithms as a manager:

> Like sometimes, like because, um, our book is flat. And then when you read all like the players then I have to add them all up at the end of the game. And sometimes I'll double count, I'll miss something. Because when they do free throws, you fill it in if they make it and you don't fill it in if they don't make it. But you have to write that they took one. So sometimes I'll accidentally count a free throw and they didn't actually make it or I won't count one and they did. So, sometimes it's just a whole reading it once and just going through and checking everything off.

Lyssa agreed that this situation involved all three error types. I mentioned that being a manager in this situation does involve an encoded faulty algorithm:

> Yeah because then when I like, if I add it all up and it's like say, we had 65 and I only have 63 now. And I'm like uh-oh and I have to go back and triple check.

When inquiring a way to overcome the double counting that Lyssa mentioned, a process of sectioning was referred to again:

> Double counting is just like, um, well obviously like I can't tell when I messed up because like if I look at it and they're not even. So figuring out when I messed up and then going through and doing them by sections. So like sectioning out first quarter and then doing that and sectioning out second quarter and doing that. And then seeing like which one it's not right, like which one isn't right. Because if I recount them by sections, then I add them all up at the end. Then I might be like oh okay so this is the one that I messed up in before. So like taking it and dividing it into little like separate parts.

Lyssa nodded that this process is similar to a scan of the sections to be certain that numbers balance out. I questioned about the sectioning with her literacy:

> Oh yeah like what I said, where I find out what's wrong first and then go like, I section it out into different like steps. So it's like first you read it and then you so like okay. So like if I were to do this again, I would read this first and then look at this. And then so I would go through count all of these and then look at this. And be like okay where did they mess up here and then fix what they messed up on and then write it all down. So like writing it all down is like the final part.

Week 4 - What Faulty algorithm story did Lyssa relate to sequences of transformations?

Lyssa used hand motions to depict a challenging repertoire of driving tasks as a prelude to a faulty algorithm. She described leaving her school first:

> Okay, so finally, I'm driving. Say like we're going from here to a QuikTrip [gas station] down the road right. So you go like, you go out of the school right. But when you're driving, you gotta have, like you got to make sure you're going straight. Because obviously like if you're gonna pull out of school, you make a left and you're not getting a QuikTrip, you're going to like to make sure everyone is down the road.

Lyssa then mentioned a role of memorization for what she considered trivial driving tasks:

> But you have to make sure it's like, it's like, since like my parents don't like let me use a map, if it's like places I know where I'm going, you have to make sure you have this like the sequence of places that you're going to memorize. So it's like you have to go out and you have to make a right and then you can, like for that, you can just go straight for a little bit and then you make it right in there.

Lyssa transitioned to memorization for more complex driving tasks:

> But like for more complicated places, like if you're trying to get to the golf course down the road from here you would have to go out and you have to go left and then just keep going straight. So you have to memorize where you're going. Because even on a map you have to like connect from like where you are to one place to another place to another place.

As a result of memorization, she described the creation of her own map as a process:

> So it's like a whole little, like, like you make a little map and a little path for yourself. So you have to like, it's moving, you're moving from one place to another to another to another. Because you can't just like teleport. It doesn't work. You have to, it's like a process. Because you have to like drive down the road and then you stop and then you turn them. You drive to the other room and you stop. It's like, it's a much more slower process than driving things because like obviously driving is like, it's not easy but like it's kind of easy.

She talked about how the mapping process as a division of procedures to prevent a waste of gas:

> You have to like break it into like little parts so you can roll in one place to another place or like if you're going multiple places at once. So like if you're going here to Kroger [grocery store] to Target [grocery store]. And you're going back here. Then you're going to QuikTrip. Like, it's like, you have to make, you have to divide it. Like when I'm running errands with my mom and she lets me drive I have to like, I have to like map out in my head which would be the smartest way to go by like connecting them all first. And then being like okay, I can go here to go here to go here then come back here without wasting 600 gallons of gas.

The description of this process led to her faulty algorithm:

> So the other day I was going to QuikTrip and I was … going for my house. Okay so, it's like my house is here and the QuikTrip is this way [as she pointed in opposite directions on table]. And the Chevron is this way. So I had gone out and I had gone to go this way and I was driving for a little bit. And then I made a turn down the wrong road.

She continued by describing counter-alternatives to rectify the drive:

> So I was like oh okay maybe I can turn out this way and then it'll bring me out to the same. It did not bring me out to the same road. So then I was like okay and then it was a dead end. That wasn't a dead end, which one what was those roads called like an end but then there's like another road? It's like a little intersection.

For some reason, I envisioned and mentioned a cul-de-sac:

> Not really, like there's a dead end here but then there's a road that goes out this way [I responded with "oh" but could not think of the word]. Cul-de-sacs are the circle things. [I

> agreed, “yes okay”.] It was kind of like a little V connector thing. So whatever so I’m driving this way. And then I was like well obviously I can’t go this way.

She related this error to analogy of going off a grid as she continued with alternatives:

> So that’s like when you’re doing problems and you go off the grid. And then obviously you cannot go off the grid because that’s not possible. So I was like well I can’t go this way. So then I turned around. Then I was like okay maybe like back up.

Lyssa related this process to a Geometry assessment:

> So then that was like today when I was taking my test. And I messed up on the thing that I showed you where I had to redo the whole thing. And then I thought I did it right and I did it wrong. So you have to like take a step back. Then you have to kind of back up. And you’re like, okay, where did I mess up at?

She transitioned back to her narrative:

> And I was like, well I wasn’t supposed to turn down this road. So I go back down the road and then I go back down the other road, get out, and I’m like oh QuikTrip is this way. So kind of I went off my path and then I was like okay well this isn’t possible anymore. Like this is like the stopping point of it. And then I backtracked a little bit and then went forward and then it worked.

After asking Lyssa what she could have done to take the right road:

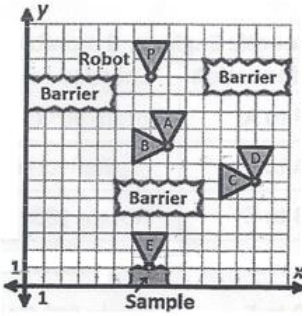
> Um if I had like paid more attention to where. Because I was pulling into a neighborhood. And most of the time it was like you can’t. A lot of neighborhoods don’t have a way in and out. It’s moment, it’s basically like it’s just one way in and one way out. Like you can’t go in the front and then go out the back. So it’s kind of like the same thing when you’re on a graph. Like if I were to, can I see my thing (literacy document)? (I replied “oh yeah sure sure”) Thank you. Anyways, so like if I was like right here [see Figure 34] and I would go like down, that’s literally like pulling into a neighborhood with no outside. Because where are you going there’s no plots there. But like if I was going like up here, that would have been like okay cool. Like there’s more space for me to move around. Like this was Main Road and this is the neighborhood, the neighborhood was not even on the map. Shouldn’t have even been there. So it was kind of like um it was like if I had read it first and like thought more common sense like. And was like okay so this doesn’t make sense. Like a lot of times when I get numbers and I go to plot them, and I’m like there’s literally no space for me to plot them. And then I have to like take a back track I’m like okay so where did I mess up where I got a negative number and there’s not even a negative on the graph. So it’s kind of like you have to like back track and make sure it makes sense. That’s like my whole thing, making sure things make sense. Because things don’t make sense when it’s not gonna work.

Figure 34

Lyssa Felt her Responses to a Faulty Algorithm of a Robotic Engineer Related to her Faulty Algorithm of Driving Through a Neighborhood

Robotic Engineer 3 monitors movement of the robot

Transformation	Transformed Faulty Algorithm	Corrected Algorithm
P to A	Translate left 1, down 4	translate down 4, right 1.
A to B	Rotate 90° CW about (7, 8)	rotate 90° CCW about (8,8).
B to C	Translate left 5, down 2	translate down 2, right 5.
C to D	Rotate 90° CCW about (2, 6)	rotate 90° CW about (13,6)
D to E	Translate down 5, right 6	translate down 5, left 6.



She agreed that she is all about remaining on a grid, "Yeah basically I don't want to go off the grid." After inquiring if she used GPS (global positioning system):

> I had like Apple Maps pulled up. And my mom was like okay go from here to here. So I was like okay cool. I looked at it for a second and I was like yeah whatever I got this. (She shook her head left to right). I did not got it.

Lyssa compared the way she looked at Apple Maps with the say she looked at tests:

> So it's kind of like when you look at something. Like when I looked at my test like real quickly. And I was like you know what, this question looks fine. Like I went through the question. And then I got it [a response] wrong. Because I looked at it too quickly and didn't like re-read the whole thing fully. So I had to like, if I had taken like one more minute instead of being like yeah don't worry Mom I got this. If I'd taken like one more minute and then gone through and proceeded with it [the Apple Maps], it [her route] would have been good. Like that's what I did with a lot of these [her error analysis literacy problems] is I have to like look at it real quick, make it make sense. and then do it.

As a summary, she described some attempts at metaphors of transformations:

> I said translating is like cars right yeah. So translating is like driving a car. Um, because you have to make sure you're driving down the right road and then rotating is like putting your golf, putting your golf ball on your tee before hitting it.

She struggled with reflections but thought of the following:

> Okay what was the other one? reflecting? Um what did I say for reflecting? Reflecting is kind of like when you cut a sandwich in half. Yeah because most of the time like if the bread is ready, you cut them in half, they're the same size. Yeah.

Week 5 - What Faulty Algorithm Story did Lyssa Relate to Rigid Motions?

> Lyssa delineated a narrative about how cookies did not turn out so well:
>
> So I was making cookies on Saturday. And I had like looked up a recipe on Google. And me and my sister were making cookies in the kitchen. And it said that you needed a third of a cup of milk. And I didn't have a third of a cup measuring thing because I broke it like last week. So I only had a fourth and a half. And I was like, you know what, I can guesstimate this. And it didn't really work out so well. Because I tried to like use a whole fourth of a cup and then use like a little bit of a half and then mix it together and make it make sense. And, basically, when I did that, I put way too much milk in. So it was, um, like my cookie batter wouldn't stay together. So it was all watered down and then they didn't cook right. It was just like everything was all messed up because I tried to mix. Like I tried to use two things that literally didn't even go with a third of a cup. Because a fourth and a half don't even like there. A third is in between them but you can't mix them together to make equal them and I found that out at the hard way.

I prompted by saying "So everything was going smooth until…"

> Well it happened in the beginning. Yeah, because I was mixing all the stuff together and I put too much milk in and I was like you know it'll be fine and then at the end my cookies were just, it didn't taste good.

I inquired whether things looked like they were going well during the middle:

> Yeah I thought like, so I messed up at the front. And then I was like, oh okay, it's fine now. And then I was like, actually, it wasn't fine at all. I messed up again.

When questioning what she could have done differently:

> I think if I had found a different way to make a third. Like maybe like looked up how many tablespoons are in a third of a cup, even though that would have taken forever. Instead of just trying to guess and just keep guessing if I had like taken the back and been like well I know that this these can make a perfect amount of a third of a cup instead of like kind of guesstimating in between and then my cookies would not have melted so much.

She agreed that it's like a step back to get a step forward:

> Yeah it's like you have to like since I didn't have the third of a cup, I kind of had to like figure out how to get that before I could move on with the process.

She also agreed that a mentality of thinking too fast went on with the baking of her cookies:

> Yeah, because I thought yeah, because I was like, it made sense in my brain to just like mix those two together. But it didn't work out on the end. Just like in these [error analysis problems], it made sense for it to like go a certain way.

I questioned taking a step back to get a step forward was applied in her literacy document. She reiterated on the level of rigor of the problems that she discussed with miscomputations:

> Um when I was doing, when I was doing this one [a faulty algorithm problem] like if I had taken a step back and like counted these two [intervals] and added that then it would have been six. And a lot with the clockwise and counterclockwise, sometimes I just forget that it's spinning. And you have to write which way it's spinning. So if I had like taken a back and thought about which way it was going first instead of just saying that it was going away.

Lyssa elaborated:

> Because like a lot of times I can just tell which like if it's going 90 degrees or 180 or blah blah blah blah first but whether it's going clockwise or counterclockwise is sometimes what I forget about.

She concurred that if the literacy problems appear too easy, then she gets a little bored, "Yeah it (relevant information) kind of just slips my mind."

Xana's Stories: Modeling Experience over Theory

Xana was a sophomore at Paulding County High School. Xana excelled at learning concepts in her on-level Geometry class. She had a Lexile reading score of 1180, meaning she was at a proficient reading comprehension level for her grade. She generally enjoyed being social with friends but always focused on learning concepts and completing assignments with quality. Xana was eloquent and gesticulated when telling her stories, demonstrating poise in her speech and considering changes in "size" and "fit" from real-life experiences to deviate from

theoretical calculations. Xana embraced errors as opportunities to learn from experience rather than to accept the theory. She did not particularly specify any extracurricular school activities but looked forward to long road trips and baking with her family.

Xana's Stories of Wrong Strategies

Xana uncovered wrong strategy stories about accidents with tasks or activities. She considered a measurement or perspective of size to be a factor to emanating a more favorable outcome. Xana implied that these wrong strategies had to be endured to acquire insight of the role of size that a theoretical calculation could not provide.

Week 1 - What Wrong Strategy Story did Xana Relate to Distance, Midpoint, or Slope?

Xana depicted a narrative about riding her bike with her dad:

> There was one time when I was riding bikes with my dad. And we were supposed to be going back to the car from the trail. And we took a steeper hill down to the car instead of the less steep one. Because the one that was less steep had less branches and roots. And it looked less dangerous. But, when we took the steeper one, our bikes went too fast. And we ended up speeding past the car and crashing into a bush. Because like, I was riding right behind my dad. And so when I was riding downhill, I ended up pushing us further into that bush so we ended up both crashing.

Since Xana demonstrated hand motions frequently, I was curious to see how she used her motions to depict the slope of the hill. I asked Xana to estimate the slope and provided examples like down 1, over 2 or down 2, over 3:

> I think maybe it was like down two over like three or four maybe. Because it was very steep. It was like, it was almost like a really big hill. Um and it just like kept going down, like for a while. It felt like it was always gonna go down.

She positioned her hands in a motion of down 2, over 4. I commented that was like a 50 percent grade. I questioned what she would have done differently:

> I think I may have taken the less steeper one. It might have been harder to ride through but we still wouldn't have ended up passing the car and then crashing into the bush.

When inquiring an estimation of the less steeper slope:

> The less steep one was kind of like, like maybe like down 1, like over 2. It was, it was not that steep.

Clarification was needed as she described the same slope value for the less steep hill as the steeper hill. I questioned whether she meant like down 1 and maybe past 2:

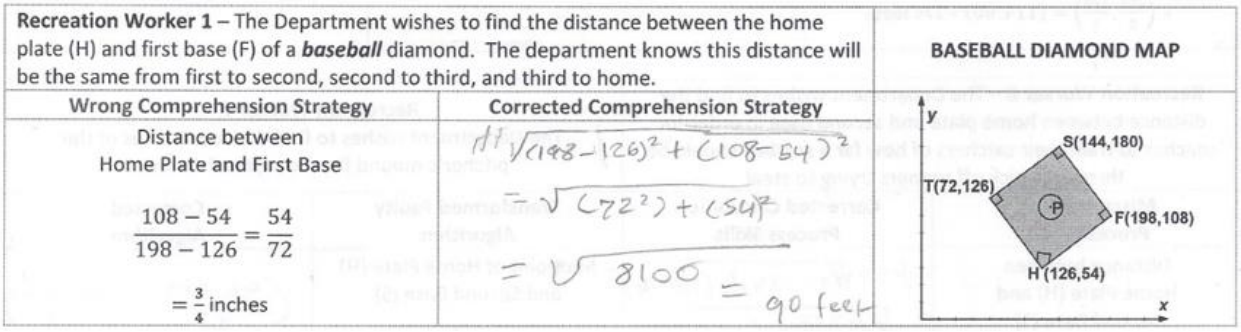
> Yeah, I think I think past two. It was like it went down like maybe for a few seconds but then it like gradually evened out.

Xana agreed that choosing the less steep slope was a case of making a better choice. When questioning about how a better choice as a practice was used in her literacy:

> Um it's like when you're thinking about like the distance when it asks for the distance between the home plate and the first base [see Figure 35], um you would assume that it would be in feet rather than inches because counting from inches would be a lot harder to do than it would just to just say like 60 feet or 90 feet or something.

Figure 35

Xana Knew it was Much More Feasible to Use Feet Between Bases Than Inches



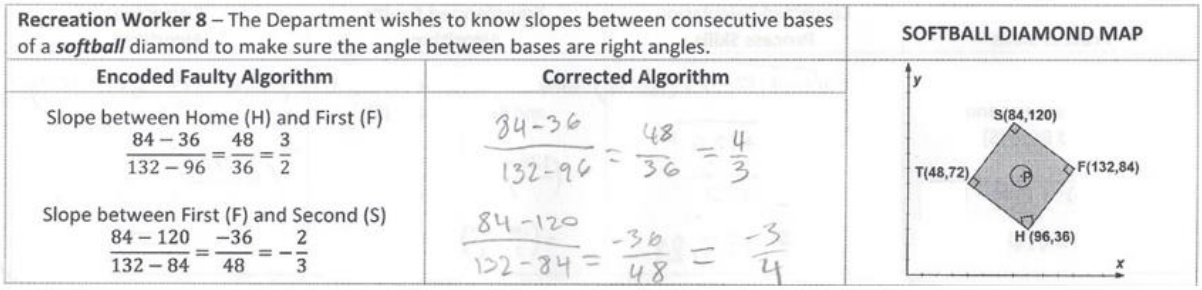
When inquiring whether her estimation of slope in her literacy relates to estimation of the bike's slope, Xana emphasized a role of experience with situation to determine the truth:

> I think a little bit. I think um because when I was going down that path, I assumed it would be easier. Because it looked easier just because it didn't have all the complicated like steps and like the roots and stuff and it wasn't as crowded. Um, but because kind of like with this one [see Figure 36], I assume that it's correct because it gives you a fraction because the other ones didn't get fractions. But this one did so I assumed it was correct

but like when you actually test it out and you like put it into the calculator, put it to work, it didn't work out. Because it's not what it equals. Just like how when I went down on that slope it didn't end up working out until you know I actually went through it and experienced it.

Figure 36

Xana Assumed the Given Response as a Fraction was Correct



Metaphorically, Xana agreed that this type of error discovered through experience is like riding a bike:

I think it can. I think it could compared to the to the bike thing. Because you know like something can be, like from viewing it, something can be like fine in theory. But when you test it and you experience it, it could go completely different from what you thought could happen. Like … a road could be clear or you could um but you could like go through it and it could like take you way too fast or way too far or way too slow or not far enough. Things like that.

Week 2 - What Wrong Strategy Story did Xana Relate to Perimeter and Area?

Xana described a story where her family tried to get a new refrigerator in their kitchen:

I think there was a time where my dad bought a new fridge and … we were trying to figure out how to get it into the kitchen and through the doors and everything. But when we were trying to get it into the kitchen, we didn't think about like the space in between the kitchen, the stove and the wall and the space to enter the kitchen. And so we tried seven different ways and we still didn't get it right. And it was mostly because we weren't … actually thinking about the height and the length of the fridge and the kitchen. So we had to hire someone else to do it.

When she mentioned the word “space,” I questioned if she thought of perimeter or area:

> I think I’m thinking about perimeter. I think it’s there was a lot of, there was a lot of space around the everything in the kitchen. There is yeah.

She agreed that the dimensions of everything in the kitchen turned out to be problematic. I questioned if one of the seven ways would work currently or whether another strategy would be needed:

> I think I’d have to like think about a different strategy. As I think we weren’t considering like the pace in the kitchen. But I think … maybe if we approached it and with a different perspective, I guess, it might have worked.

When asking to describe that perspective:

> I think we definitely needed to think more about how we were going to get it in the kitchen. And you know if it would fit in between where we were putting it. And we definitely needed to think about where we were in terms of like when we were holding it. Because we were also, it was also difficult to get us and the fridge in the kitchen at the same time. Because we also took up space that needed to be there for us to be able to get the fridge in.

Xana concurred that she had to count herself as part of the perimeter and area to get the refrigerator in the kitchen.

Week 3 - What Wrong Strategy Story did Xana Relate to Counting Points and Segments?

Xana depicted a story about building a model pyramid:

> There was one time in elementary school where we had this project where we had to build a pyramid out of toothpicks and marshmallows. Me and my team decided we would try to determine a ratio of how many toothpicks we should use to marshmallows. And we thought that the amount of toothpicks would be better to use. Like a greater amount of toothpicks would be better to use than marshmallows because they provided structure. But we didn’t draw a blueprint or anything. So it ended up failing what my teacher called the earthquake test. Because there were more toothpicks than there were marshmallows. And they were not evenly stacked on top of each other.

Xana acknowledged that toothpicks serving as lines and the marshmallows as points. She also thought of this narrative as being both a wrong strategy and miscomputation:

> I think I used a little bit of both when it comes to wrong strategy and miscomputation because we didn't… draw a blueprint or anything. We didn't consider the amount of the mass between the toothpicks and the marshmallows. We didn't try to evenly distribute them. So yeah, I think it was kind of a little bit of both (wrong strategy and miscomputation).

I asked Xana about a correction to her pyramid construction with toothpicks and marshmallows by explaining blueprint:

> I think definitely we could have drawn out what we wanted the structure of our pyramid to be. And kind of try to use some sort of formula to try to see what exactly you should go where and how much of it should go where and try to make it even distribution so that it wouldn't fall or crash or whatever.

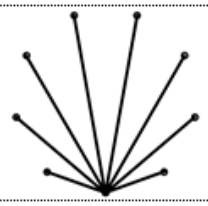
The wrong strategy (see Figure 37) served as an example to help students in general understand the double count of routes occurring between any two points. Xana admitted to understanding the double count within the wrong strategy problem. I questioned whether a problem like correcting the wrong strategy, as an example, was necessary for understanding how to correct problems with miscomputations and faulty algorithms:

> I think the errors and stuff, I think they helped me to see, you know, like … what I could be doing wrong. But I think, you know, like with enough time I would be able to like figure out how to do it without it [an example].

Figure 37

Xana's Response to Consider the Double Count that was Ignored in the Wrong Comprehension Strategy

Air Traffic Controller 1 wishes to find the total amount of one-way routes between 9 airports.

Amount of Routes from one airport	Wrong Comprehension Strategy	Corrected Comprehension Strategy
	There are eight points from one segment. So, the total amount of one-way routes is $9 \times 8 = 72$.	There are 8 segments from one point. total: $9 \times 8 / 2 = 36$.

I questioned what she would have to do to figure the errors out:

> I think I'd have to analyze it a bit more. Like actually take them, like I think I'd have to like really think about it long and hard.

Week 4 - What Wrong Strategy Story did Xana Relate to Sequences of Transformations?

> Xana chronicled a narrative about one of her first experiences with driving:
>
> There was a time where my dad was trying to teach me how to drive. And he let me use his truck. And we were in … my parents driveway. And he was trying to teach me how to back out of a driveway. And it wasn't the best space to do it because there was a lot of stuff surrounding the car. And there were other cars in the driveway as well. So when I tried to back out of the driveway, I didn't know how to properly turn the wheel. So I ended up turning like a little too far and I backed into a trash can the first time. And the second time when I tried to move back into the driveway, I kind of pushed the car into the other cars in the driveway. So from then, I had to … back out and let my dad drive for the rest of it.

When exploring what Xana could do differently to avoid this situation:

> I wouldn't have done it in the area that I did it in. Because there were too many cars and too many things besides where I was moving. And I wouldn't have turned so much. I think I turned a little. I turned the car way too far.

She confirmed of a need to turn in a different location where there was more space. She agreed that having a more open area would have made her feel more comfortable. When Xana mentioned "different location," interest triggered on whether her environment played a role with her performance on a literacy document:

> I think environments are very important to me because wherever I am, when I'm trying to figure something out, like whatever has happening around me influences me. And kind of, it can either distract me or like or whatever. Like if I'm in a really loud area it's really hard for me to think. Or if there's a lot of, if there's a lot of things going on, I try to, I tend to almost like pay attention to that rather than what I'm doing. But if I'm in a quieter area and I'm more comfortable and it's like, I guess, more calm, I can think about it [literacy] more. And I feel less compelled to go and try to like fix whatever is happening around me or pay attention to that.

She agreed that the more comfortable environment would have made her see that sequences of transformations better in her literacy. When asking Xana to think of a metaphor for translations,

similar examples to the forward and backward movements taken in her driving narrative emerged:

> I think … getting from my home to school is kind of like translating. I think a lot of things to me could be compared to translating. I think any time anyone moves, it's kind of like translating somewhere like going from like my desk to the front of the classroom to get like paper or something. Or going on the bus and then going home. I think things like that they're very similar.

Week 5 - What Wrong Strategy Story did Xana Relate to Rigid Motions?

> Xana described the challenge of drawing between landscape and portrait orientations: I do a lot of art stuff. So I like, I need references a lot of the times for my pictures. And there was one time where I had the image reference next to my screen where I was drawing. And I tried to replicate it. But the placement of the person in the picture was very difficult to follow. Because my canvas was a lot wider than the photo I was copying. It was also like positioned differently. So I … had to try to trace over it almost. But that also didn't work because they were both different sizes of images. So… my references image was vertical [portrait] and my canvas was horizontal [landscape]. So I ended up having to change the size of the canvas. Because whenever I would trace over the picture, some parts didn't fit on the canvas because they would get cut off.

She agreed that the canvas used to draw the reference image needed to be a different size. When asking the type of rigid motions she experienced:

> Yeah … I had to rotate and move the canvas a lot, to try to get the image to fit on my canvas. Because the reference was a vertical image and my canvas was horizontal. So, I had to try to like rotate the character a little bit and like make her smaller so that she could fit on the canvas.

When asking Xana what could have been done to get a better duplication, she talked about the property of congruence, within the definition of rigid motion:

> I think I would have tried to choose a better reference image that had the same dimensions and sizes of my canvas.

When Xana said the words "choose" and "better," I requested she describe how a better choice was applied in her literacy problem involving an erroneous reflection of a tetromino in Tetris:

> The way it [the tetromino] reflected [see Figure 39], it didn't … make sense because when Tetris, when the things fall, tetrominoes, when they fall they're supposed to go like

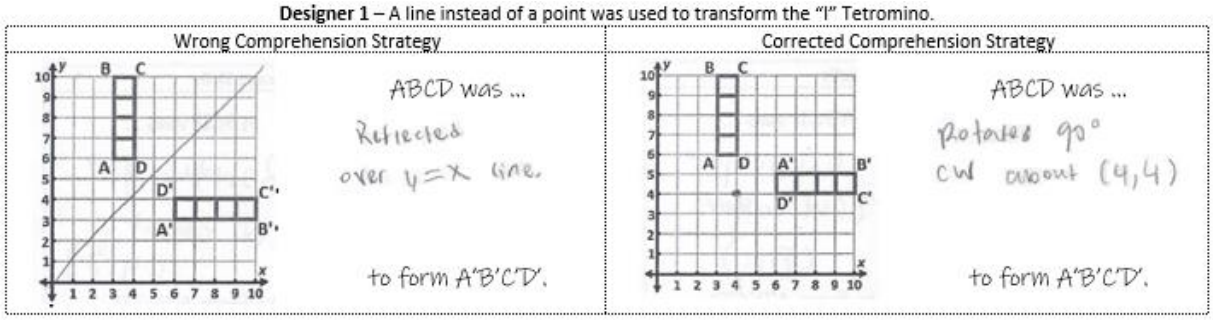
> straight down and, you know, fit perfectly. But this, the way that it [the tetromino] fell here wouldn't have made sense because it was, it reflected over the y equals x line. So, it didn't… quite make sense the way that…the way it was moving. Rotating it [the tetromino] 90 degrees, I think would have made more sense because it doesn't, [she hesitated], I'm trying to figure out how to word this.

I thought of a way to help Xana express what she wanted to say. She acknowledged having played Tetris before, noticed the movements of tetrominoes within Tetris, and knew the reflection of a tetromino. I reiterated by asking what she had to write in place of a reflection to make sense of how the tetromino is supposed to be moving:

> It's a rotation. So it had to be, it had to change, from like, it had to change going clockwise because if… it fell that way it wouldn't have fit with everything else [She pointed to correspondence of points A to A′, B to B′, etc. (see Figure 39)]. And it would be because they have, they have more… how's this? I'm still trying to figure out how to word this. I lost my train of thought.

Figure 39

Zana Knew a Tetromino Rotated Instead of Reflecting but had Difficulty with Expressing the Correction Based on Correspondence of Points



Xana struggled to express the correction from a reflection to a rotation. When she mentioned "fit," she agreed that the rotation fits more than the reflection. Similarly, she also concurred that choosing a better reference image was a better fit for her duplication.

I switched contexts with her from the tetromino rotation back to her art picture duplicate. I thought the switch was a more comfortable way to express herself. When inquired to describe the feeling of choosing a better fit for the duplication:

> When there's like a better reference image that matches a canvas better, I know exactly where to put things and then more. I'm more likely to get the exact proportions of the character's body and stuff with the… picture. If I'm… copying it exactly because if it's too wide, then I'm drawing the character too big and if it's *too small then I'm drawing it on a canvas that's it's bigger. So it'll look way smaller in there. So…*

The change of context allowed her to transition back to knowing how to express herself. She agreed of considering "size" like she did with the bed frame.

Xana's Stories of Miscomputations

Similar to Lyssa, Xana uncovered miscomputation stories about rethinking and considering choices rather than just doing. Xana also believed that the outcomes of her narratives occurred because of a need to make the right decision based on what made sense the most. Xana shared that she needed to slow down and think of options rather than make hasty decisions.

Week 1 - What Miscomputation Story did Xana Relate to Distance, Midpoint, or Slope?

Using hand and arm motions to express herself, Xana shared a narrative about gymnastics:

> One time when I was seven my dad signed me up for gymnastic classes. And when I was being tested for how well I could like flip and dive and stuff like that with the foam pit. There was, the pit was really big. So I was, when I was on this top platform, I was kind of just like more swinging back and forth. Because it was a swing that I was supposed to be going in from. And I when I would swing back and forth, I couldn't tell which side of the pit I was on. And I was supposed to land in the middle. So I was directed to dive in the center. But I couldn't tell where it was, where I was landing. So I ended up like trying to swing myself into the center but instead like I propelled myself further into the left part of the pit.

I questioned if that's like an application of the midpoint formula in a way:

> Yeah, um, it's kind of. It's kind of like I tried to land in the center but I ended up going way farther out.

When inquiring what Xana could have done differently:

> Um, I think I should have like um when because I jumped off when it was swinging forward, which propelled me past the center. So I think maybe I should have waited till I swung back and then jumped off into the center. Because if I was farther back, I could propel myself more accurately into the center rather than waiting for me to pass it and then jump off.

Xana concurred that a better estimation and number sense was needed for a more accurate and precise jump towards the center:

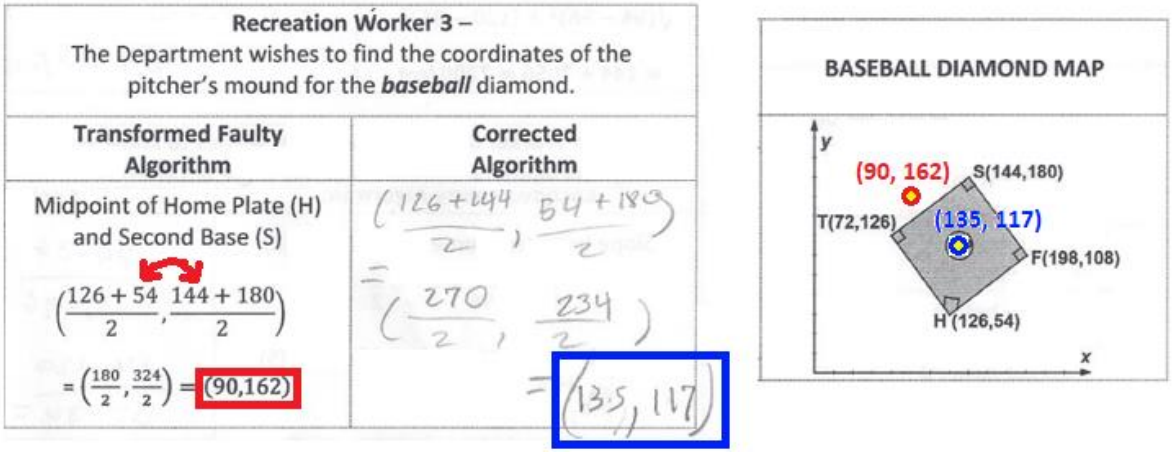
> Um, I think they're really because when you look at them like the image with the coordinates and everything. It helps you better to like know what you're going to place and where. Because the, if you were just like estimating just by like looking at like an actual plate like a baseball diamond thing, it would be difficult not to know exactly how much is in between and what the coordinates were.

On the literacy assignment, there was no miscalculation problem that involved a midpoint. I referred to a faulty algorithm problem that did involve the coordinates of a midpoint. When asking her describe her understanding of coordinates based on estimation:

> Um, it's like it with the with the faulty, the bad room, the wrong one. It's looking at it [see Figure 40], you know there's something wrong because it looks right at first but then you realize that they're swapped. And it's like I knew where I was supposed to be going. And I knew it looked right because when I reached that point, I knew that I was close. And I would go I was going to go down and land there but instead halfway through jumping I realized I was not there. I was past it. So it's like looking at this, I knew that it was right. But when I was writing it out, when I was like trying to match it up with the answer. It didn't. I realized that it was wrong because two of them were in the wrong place. And it would make better sense if it was if it was swapped around.

Figure 40

Xana Knew Coordinates had to be Swapped Because the Erroneous Result Showed a Point not in the Center Compared to her Corrected Result



Week 2 - What Miscomputation Story did Xana Relate to Perimeter and Area?

Xana described a narrative about placing furniture in her family's home:

> Um there was a time where, when we were first, me and my family, were first moving into our house. We had a lot of furniture from our previous house that we used because we didn't have any money to buy any new furniture. So a lot of the furniture we had were really old and borrowed from our family in the house that we were previously living in. And we didn't think about the size of the rooms that we were going to be staying in for the living room. And so we kind of just tried to make things work as they were with the sizes and everything and the feet in the room. And we had … two beds that we couldn't fit into our rooms because they were … way bigger than they needed to be. And we didn't also have any space that they were convenient to put in. So it was like we placed it in front of like the window or something but then it would be blocking the door to the closet. And we had about six pieces of furniture that we had to throw away because we were, we had to donate at least because we didn't have the right sizes to go in the in the rooms that we had.

I questioned her mention of the word "size" as perimeter or area:

> I'm thinking an area because the rooms were a lot bigger and had more space in the living room and the bedrooms were smaller but. [She abruptly ended her response here.]

When asking if the formulas for perimeter and area have to be followed or changed:

> I think they could have been followed if we if we thought about it differently and we carefully calculated the area and like the width of the room and the things we were putting in the room.

When inquiring what could have been done differently:

> I think I definitely would have tried to consider the area and like the width of the room and the stuff that I was putting in the room. And tried to figure out what things could and couldn't go in there or which way I could put it that it would fit without blocking anything.

When asking how she would consider the area and the width, "I'd have to measure everything and try to calculate what could go where." Curiosity struck as to what Xana would measure with. "I think either like a tape measure or whatever measure." There was a transition from the tape measure to how Xana counted on her literacy document:

> I think [counting in the literacy] was I think it was a little challenging. I think it was just because you know there's … a lot of different areas that I had to consider. And I … kind of … take a little more time counting things because I have to like keep my eyes in place for it but yeah.

I questioned whether it is easier to do measuring by counting over a tape measure:

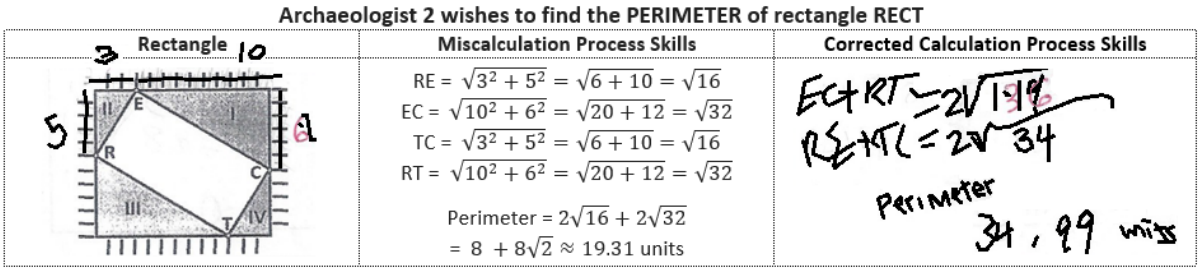
> I think it's easier for me to do [measuring] by counting [over a tape measure] because I don't confuse myself by like looking at all the different lines and numbers and stuff.

I inquired if whether she was able to see a doubling pattern in the problems of the literacy (see Figure 42) document or if the doubling pattern was hidden in the problem:

> It was a little hard [to see the doubling pattern in the literacy]. I had to like think about it because I didn't think it at first. I didn't really notice it. But when I thought about it more, I definitely noticed it later.

Figure 42

Xana Did not Initially Notice the Doubling Pattern



She agreed that not noticing the doubling pattern was the same mentality occurring as with the wrong strategy refrigerator narrative and the miscalculation furniture narrative. When asking for a metaphor that represented her experience with perimeter and area:

> I think with the room thing, this situation where we were trying to fit furniture all in one thing, I think it can kind of be compared to like making a building. But like using the wrong, I guess, using the wrong blueprint. Because we were trying to, we were trying to fit all these things in the room that wouldn't be able to fit in there. And wouldn't be convenient next to each other because they wouldn't take up too much space and things like that.

When Xana said the word "fit," she agreed that perimeter and area were similar to fit of dimensions in space. I asked how she thought of this "fit" in her mind:

> I think … of it like ... I guess like the same way like I would a shoe or something like that. Like … you know if you get the wrong size for your feet or something or you know like if you if you buy a coat that's too big for you or too small.

Week 3 - What Miscomputation Story did Xana Relate to Counting Points and Segments?

> Xana shared a story about a driving trip with her cousin:
>
> There was a time where a cousin was visiting the country of mine. And she agreed to take me you know because she could drive at the time. She agreed to take me to wherever places that I wanted to go for the day that she hadn't seen around here as long as I knew the directions and stuff. And being 12 at the time, I didn't, or not 12, 11 [she corrected her age as 11 instead of 12]. I was I didn't have a phone yet. So I was going purely off of memory. So, any time we went from one place, I was kind of just trying to vaguely remember how to like what turns to take to get to one place or another. And at some

> point, we were … trying to find a place to eat. But when we got to the place to eat, we realized that we couldn't go back home yet. Because we needed gas. And I couldn't quite remember where the nearest gas station was but I remembered all the places around the gas station. So I tried to guess how far the nearest gas station would be by using what I knew and how to get to the places around the ones I knew. So I did a very bad estimation. And the nearest one near a place that we drove to ended up being too far. So she ran out of gas. And her car broke down before we got to the gas station. And so because I was lost and guessing every direction, we, she had to use her flip phone to call my dad. And we had to, you know, be taken home.

Xana conceded the gas stations to be points and paths being taken as line segments. I questioned her correction for a better outcome:

> I think I should have went from a point that I knew every direction from and try to determine how long it would take to get to there with the amount of gas that she had to whatever nearest gas station was, you know, visible.

When inquiring about a metaphor that described her experience working with errors that involved the counting of points and line segments:

> I think like with the with the car situation, I think I could compare us to like, I guess like, ants in a colony. Like … we were the ants in every, you know, location was the colony. Like just like how like ants use a specific trail and follow each other to get to one point to another, that's how we were trying to get from one point to another.

She agreed that the colony was like a set of lines and the ants were the points. I commented that ants move and questioned whether the points were moving in her mind as well:

> A little bit. Yeah. I think it can in both ways. I think it could be like back and forth, or up and down. I think, you know, like when you're looking at like segments, it's like it can be kind of back and forth, and you know up and down like traveling from one place to another. Like, or going up and down a ramp or something like that. Like I think … it could be depending on the situation, you know, It could be either way.

Week 4 - What Miscomputation Story did Xana Relate to Sequences of Transformations?

> Xana imparted a story of a struggle she had with a virtual construction on a video game:
>
> There is a game that I play … Minecraft, from time to time. And there were times where I was building something like … a fort or something like that. And … I tried to build it so that … whenever I wanted to return to my fort, I could like rotate it, sort of. So, … I could have it facing a different way so that whenever something tried to come in, it

> wouldn't be able to. But, I couldn't properly figure out how to how to rotate the building exactly. Because … you need to use a lot of technology to be able to rotate. But when I did, it only rotated halfway because whatever I was doing, … I didn't … use the right materials to try to rotate the building.

When probing for what Xana could do differently:

> I think I would have tried to figure out which materials I could use. And what … I needed to use to make it fully rotate. And how exactly I would use the technology to do that.

I asked Xana to describe some of the technology tools on Minecraft:

> There are tools like pistols and … switches and stuff. And there's also something called redstone. So, if you needed to push something or move something in a certain direction, you could use pistols to like shove it one way. And you could put another object in front of it so whatever was in front of the pistol would move whenever you flipped a lever.

She concurred that the pistol tool was like a translation in a way. I questioned about what a switch does:

> If you flip a switch like one way or another, if you have redstone connected to it and it's connected to something with a pistol or… some sort of object, it can like, it kind of helps trigger whatever is happening. So if you put just a pistol and an object, it'll stay… in that one place, but it'll still be out. But if you have a lever attached to it, then you can change whether or not it moves.

Xana confirmed that the lever is coming from the redstones. I inquired more about the redstones:

> It's like a powder. So, when you like place it on the floor, it connects to whatever, it can connect two objects together like a lever and a pistol.

Xana confirmed redstones is like glue. I still had trouble understanding what the switch did.

When asking again:

> So like if you had it [a switch] connected to one object, like if you were trying to, if you placed it like next to a door or something, then whatever way you would turn, it [the switch] would like open or close the door. Or if you had it [a switch] connected to a pistol, it would move an object left and right.

After Xana described the tools of her Minecraft game, I questioned about the type of tools that she uses for her literacy:

> I think the wax paper really helps me look at it. Because I like, I'm kind of a visual learner. And seeing the, what is actually happening and how it's turning and rotating is helpful for me. Because it's kind of hard for me to like just eyeball it I guess.

Xana agreed that it is hard for her to see the transformations with just her eyes alone and that the use of wax paper is more of a match to help her with visualization. I questioned whether the wax paper as a tool was similar to pistols, switches, and redstones in the Minecraft game:

> Well without the levers and the pistols, it's difficult to complete the task. Like actually follow through with it. Like you don't necessarily need it but it helps out a lot more. And it, you know, it's kind of, it kind of pushes you to what you need to do.

She agreed that the wax is not really needed as well but it does help with completing a task. When she talked about "flipping" of switches in Minecraft, she thought of a common metaphor for reflections:

> I think … of mirrors whenever I think of reflecting. I think that's the main one that I always think about. I think that's the only one I can really think about.

Week 5 - What Miscomputation Story did Xana Relate to Rigid Motions?

> Xana recalled a quandary endured at a bake sale for her school:
>
> One time, me and my sister, we had … this bake sale thing going on at our school. And we wanted to make lemon squares. And we had everything we needed except for … it was some sort of special flour we needed for it. And so we had substituted it [the special flour [with something else]. I can't exactly remember what it [the replacement ingredient] was. But … when we tried that, it didn't work. So we had to ask our dad to go like buy some flour so we could … try it again and do it the right way. But when we did, we didn't use a recipe. So we used the wrong amounts of flour and the wrong amounts of sugar. So they [the lemon squares] turned out watery and they weren't … fully cooked. And the proportions were wrong and everything. And so every time that we tried to cook it and we'd retry, we just guessed on the amount that we would need for … each ingredient. So it would always end up being wrong or undercooked or things like that.

When I inquired the types of rigid motions she applied:

> I think, I'm trying to think, we did have to like, when you make lemon squares, apparently you're supposed to like shake the thing back and forth like up and down. And like the directions that you move it, even it out, and keep it flat and so that it cooks

> evenly. But we shook it at some point and the batter moved to the left of the … pan. So the only that side got cooked and the rest of it was kind of uneven and raw.

When inquiring about a different strategy for the lemon squares:

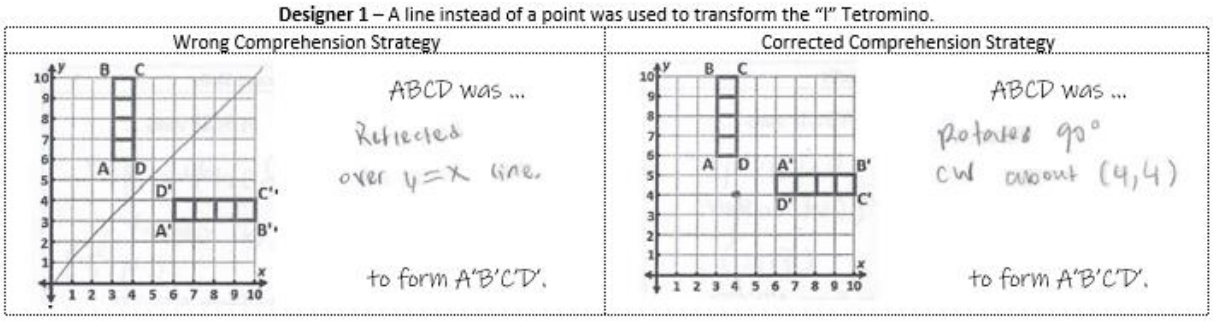
> I would have … definitely looked at an exact recipe that had like the exact amounts of and measurements of the ingredients. And I think we definitely should have shook it [the batter in the pan] evenly, tried to place it to where it would all get cooked.

From the wrong strategy of rigid motions, Xana needed help finding a way to express the error of choosing a better reference image for her art picture duplication. When mentioning the phrase "exact recipe," she agreed that vocabulary was a substitute for recipe in the context of rigid motions of tetrominoes. Coupled with her explanation of choosing a better image and the use of vocabulary, she transitioned to an expression of the tetromino rotation that she had struggled with previously. Listening to her narratives allowed a rephrasing of an inquiry to describe what made her decide to know to go from a reflection to a rotation of a tetromino:

> Here [see Figure 43], it is different because going, when you, if you try to rotate this [she pointed to the reflected tetromino] in some sort of way, it wouldn't make sense because it's more going over this line that it is moving in some sort of direction. But here [she pointed to the rotated tetromino] it would make more sense that it rotates because if you… try to move it clockwise, then it actually fits like perfectly into… the other tetromino [she pointed to the correspondence of points].

Figure 43

Xana Compared the Difference between Reflecting and Rotating based on a Correspondence of Points



Xana's Stories of Faulty Algorithms

Xana unpacked faulty algorithm stories that involved primarily family situations that occurred either inside or outside the home. Xana tended toward a need of prior knowledge to improve the outcome of each narrative. Although each narrative led to an unfavorable outcome, she believed that acquisition of experience helped with building knowledge for similar, future circumstances.

Week 1 - What Faulty Algorithm Story did Xana Relate to Distance, Midpoint, or Slope?

Xana delineated a narrative about a time when she gave herself a haircut:

> Um when I was younger, I wanted to be a hair stylist. And there was one time when I tried to cut my own hair. And I thought if I measured how long my hair was. And then how I measured my neck. And my face like marking it. And stuff I could cut my hair at the desired length. But both sides of my hair ended up being uneven. Because I miscalculated and my bangs were shorter than I wanted. Because I didn't consider that my hair was not perfectly straight and that they would spring back up whenever I cut it. Because I was pulling it down when I was cutting it.

I noticed she mentioned "miscalculated," which made me think a narrative about miscalculation was transpiring. When describing how cutting of her hair was going well initially:

> Um it was going relatively well because my hair from the back part is I need to pull it down to cut it. I could just cut it from where I was looking. And it looked okay. Because I had my hair pulled out to the front. But when I pulled it to the back, everything looked choppy. Because obviously when I push it to the front, the hair is positioned in different pieces and stuff.

She agreed that the pulling of the back of the hair is when the faulty algorithm occurred. I questioned what she could have done differently:

> I think I should have um tried to use the correct strategies for um. Because I didn't have any prior knowledge on how to cut hair and like what to look at. So I should have measured everything right. Like I should have figured out how much exactly in inches that I was going to be cutting off. Because I wasn't actually like trying to take off a certain amount I was just cutting.

When questioning what exactly the coordinates tried to help her figure out:

> I don't know where I was placing what like, um if I needed to know the distance I know to look at the coordinates because the coordinates imply how much was um in between one place or another.

An encoded faulty algorithm involving slope was referred to. I forgot to mention the application of prior knowledge in my inquiry. I asked why her corrected response made more sense than the erroneous response:

> Um, because when you calculate this [see Figure 41], like when you when you put it like if you were to type it in the calculator and like look at what it actually equals to, it's like all of this, like except for the end part, is correct. So something happened at the end. And it's because like looking at it I wouldn't assume anything if I wasn't trying to figure it out. But if I was, I would think two-thirds like 54 over 72 does not, like it doesn't look like it would equal two-thirds. So it's like it's like I knew I had to like fix something about it but I didn't quite know what so I had to try to like um piece together what went wrong.

Figure 41

Zana Discovered that all of this Encoded Faulty Algorithm was Correct Except for the End

MAIN IDEAS

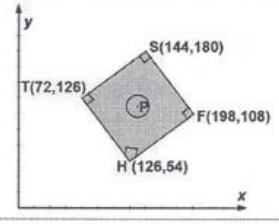
Given two points A(x_1, y_1) and B(x_2, y_2),

The distance formula is ...	The slope formula is ...	The midpoint formula is ...
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	$m = \frac{y_2 - y_1}{x_2 - x_1}$	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Recreation Worker 4 – The Department wishes to know slopes between consecutive bases of a ***baseball*** diamond to make sure the angle between bases are right angles.

BASEBALL DIAMOND MAP

Encoded Faulty Algorithm	Corrected Algorithm
Slope between Home (H) and First (F) $\frac{108 - 54}{198 - 126} = \frac{54}{72} = \frac{2}{3}$	$\frac{108 - 54}{198 - 126} = \frac{54}{72} = \frac{3}{4}$
Slope between First (F) and Second (S) $\frac{108 - 180}{198 - 144} = \frac{-72}{54} = -\frac{3}{2}$	$\frac{108 - 180}{198 - 144} = \frac{-72}{54} = \frac{-4}{3}$



When asked to think of a metaphor that deals with these types of errors:

> Um I think … it kind of can be um compared to like driving on a really bumpy road or something like that or maybe going on a roller coaster. Because you know you start off smooth but then you end up going like all different directions and you never really know what you're gonna get into because it just always changes.

Week 2 - What Faulty Algorithm Story did Xana Relate to Perimeter and Area?

Xana shared a story about biking on a trail:

> Yeah … there was a time where me and my dad went biking somewhere on the trail and … the bikes that we had were way bigger than normal bikes. Because I think they were made for a different type of biking. But the trail that we were on was a lot thinner and it was more made for like smaller bikes. And like you know fewer people. It was a small park. So when we all would, [I] mean my sister and my dad, we all went trying to bike there. We ended up being halfway off the trail. We couldn't all together like ride next to each other or at the same time because the trail was too small. And we were also taking up other people's spaces and yeah.

She concurred of working with perimeter. I inquired how she could improve the outcome of the biking trail:

> I think we … should have taken smaller bikes. And considered the size of the trail that we were on. And I think we should have definitely lined ourselves up differently so we weren't riding off of the trail.

When Xana mentioned taking smaller bikes on her trip, I was curious as to whether she related smaller counts to helping her take larger counts. An example was provided from that pattern of a 3 by 5 unit dimensions to 6 by 10. I questioned whether she trusted this pattern or if she still felt like she had to count 6 by 10:

> I think counting for me is easier [than applying a pattern]. I think I like to see it visually and be able to try to group it together on my own.

Xana agreed that counting instead of following the pattern was much more trustworthy to her.

Week 3 - What Faulty Algorithm Story did Xana Relate to Counting Points and Segments?

Xana described a narrative about a family member in need of help:

> Yeah there was a point where me and my family… we had this family member of ours that she was not financially stable. And so we invited her to come stay with us. And for a while we thought it was okay because you know we had a pretty steady income. And both my grandparents were working jobs and my dad. So we thought we had enough money to feed everyone which at the time we were living in this big house where like all of our family lived in. So everything was fine for a while because well he had a place to sleep. He had, you know, we both generally ate like the same amount. Like my mom

> cooked a lot. So we all had a good amount of food to eat for all of us. But at some point, it became to where, you know, he couldn't just like, it was … hard to, you know, keep it going, keep it steady. Because you know he had, he would, he's different from our family because he has a certain condition which causes him to need to use the bathroom a lot more. And he was very, I can't remember the name, he was very needy. And he also required a certain amount of medications for his condition. And at some point, our family was running out of money because we were paying for the conditions, the medications and the stuff that he needed for his conditions. And so it became harder for us to pay for things like groceries. Because, you know, we were spending a lot of money focusing on him. At some point, it just kind of crashed and burned because everyone was trying too hard to try to focus on him. And we kind of all forgot about us. So we ended up having to try to all work jobs to find a home for him. And no matter what we did, it was always like there was not enough money for a lot of us.

Although the family member in the story changed from a "she" to a "he," I decided not to question that part for personal reasons. I felt the gist of this story as a faulty algorithm was justifiable enough. Xana thought of her family members as points. Since resources to support the family became limited, Xana also perceived the allocation of resources from an endless line to a limited segment. After inquiring how she would correct this situation:

> I think we should have budgeted for everything that we had, you know, like the food. I think we definitely should have tried to evenly distribute food and things like that. And definitely should have like limited how much we're spending on him.

I inquired about whether she applied the practices of even distribution in her literacy:

> Yeah I think … with all of them, I could, you know try, to kind of, I guess, compare them in my head to those situations and see it … kind of like through distance and distribution and things like that.

Week 4 - What Faulty Algorithm Story did Xana Relate to Sequences of Transformations?

> There was a time where me and my dad, where we were taking a very fragile vase out of my mom's room. And the vase was too big for us to … pick it up, like just one of us. So we had to like, me and my dad had to like team up, to do it. But the thing is we had to like turn it upside down to try to get it through the doorway. Because the way that it was, it wouldn't … fit if we had it like straightforward. And when we flipped it, it still didn't fit through the doorway. So we had to like … find a way that it could fit through the frame of the door without anything like pushing or like knocking against the frame. When we turned it upside down for the last time, … it knocked against the frame. And we dropped it and it broke.

When probing what Xana would have done differently:

> I think we could have tried to … use, like turn it a different direction that wasn't like up or down. I think we could have fed it through the door if we tried it vertically or maybe if we'd … wrap something around it, it wouldn't have broken.

She agreed that if something was wrapped around the vase, that would make the vase thicker and more difficult to move through the door:

> Yeah I think so. I think that would have made it a little harder. But I think then it the [vase] wouldn't break but also like it wouldn't fit through the frame. But. [She stopped here.]

Xana acknowledged a feeling that the vase, being wrapped, would have just gone through the door without being broken just by turning the vase a different amount of degrees. When Xana mentioned "different direction," I asked her to describe the feeling when she's reading her literacy and encounters a problem where an object needs to be turn clockwise instead of counterclockwise, or vice-versa:

> I think at first I believe it because when I don't notice little details when I look at things like that like counterclockwise. But when I … recognize the way that it was turning, I was, I kind of … thought about it a bit more. I was like, oh yeah the counterclockwise is this way [she moved her hand and arm in a counterclockwise direction] and then clockwise is this way [she moved her hand and arm in a clockwise direction]. And I had to … think about a clock for a minute.

Xana thought of a couple of metaphors that related to the rotations taken with the vase in her narrative:

> I think of rotating … I think of a train on those platforms where it like spins them into what tracks they need to go to … or like a ballerina like how they spin and stuff like that.

Week 5 - What Faulty Algorithm Story did Xana Relate to Rigid Motions?

> Xana talked about the construction of a frame for a bed:
>
> Me and my dad, we needed, when we were moving, we needed like a bed frame for my … mattress. Because the old one that we had was starting to break apart and it was very

> flimsy. It was growing mildew and stuff on it. But we needed to get a new one. But we didn't have the money at the time to buy another one. So we tried to make one ourselves with the stuff that the wood that my dad or my grandpa had in his closet. And when we tried to make the frame it was … more according to like the size of the mattress. So we tried to fit it around the mattress. And for the most part it was good because it did fit the mattress in it when we were building it. But when we … actually started to continue and like build the rest of the bed frame, the legs were too short. They were too thin so they couldn't hold up the rest of the bed. And it would end up breaking apart and like falling. And it couldn't hold my mattress and everything. So the size of the bed frame was right but it wasn't proportionate. So it couldn't hold up my bed.

When questioning the rigid motions used:

> There was, we had the actually, we had to rotate the mattress to try to figure out how to get it in the bed frame. Yeah.

When inquiring about a different outcome with the bed frame construction:

> I think we should have considered the size and the weight of the bed that we were making or the bed that we had for the bed frame. And the support for the bed frame and the legs and stuff.

She mentioned "size" once more. Size turned out to be a theme in an attempt to describe her overall experience with a metaphor:

> I think like … whenever you're buying clothes and stuff, you always have to consider, you know, like … your body size, like your weight. And like … sometimes you have to measure like around your waist and stuff, your pants, things like that. Or whenever you're like buying gloves to like, you have to make sure, you have like the right size hands for the gloves because if they're too small, it's not gonna go on easily.

Xana agreed that when items like clothing and gloves are purchased, there are the rigid motions of translations, reflections, and rotations transpiring for a match.

Icis's Stories: Constructing Arguments and Critiquing Reason

Icis was a sophomore at Paulding County High School. There were times when Icis both excelled and struggled at learning concepts in her on-level Geometry class. She had a Lexile reading score of 1073, meaning she was at a basic reading comprehension level for her grade. Like Xana, Icis also generally enjoyed being social with friends but always needed some type of

video to watch on her phone to help with focus on learning concepts and completion of assignments. Icis was exuberant and voluble with her narratives, providing invigorating consultation of "taking a step back to go forward" with subtle traces of humor to inspirit the content of her responses. She posed argument with positive intent and provided commentary on reason as forms of sharing her perceptions of life in general. Icis did not specify any extracurricular activities but looked forward to any activity that was adventurous and fun to her.

Icis's Stories of Wrong Strategies

Icis shared stories that involved choices and change. She preferred autonomy of decision-making based on experimentation to accommodate the sense of her adventurous spirit. She embraced trusting her instincts and deviating from the norm because she claimed that's how her brain operates. She was jocular about making wrong strategy errors and posed an aspect of positivity around their occurrence.

Week 1 - What Wrong Strategy Story did Icis Relate to Distance, Midpoint, or Slope?

> Uh distance I would say, uh we used to take long trips with me, and my mom, my sister to go visit my father. And they [her GPS that she called Sheila] gave us a certain, like it was like a certain amount of way like we had to go. It was like a 12-hour trip and we thought we were gonna take like I-95. And like our GPS was saying that was the best route. But turns out like there was a really bad wreck. And it wasn't the best route in the end. Like we probably should have gone the back way like usual. I'd say it was probably a wrong strategy… [she humored] never listen to Sheila [the name of her GPS].

After asking whether she should have trusted the GPS:

> Yeah we probably should have gone like, we probably shouldn't have listened to the GPS. And that kind of since we should have gone to the route that we knew was going to be faster than. Oh well my GPS says this is way faster than what we usually do. So because usually the back roads are like empty. So it's like you can go a little faster than the speed limit. Kind of. Uh but when it comes to like the main roads, like when you get in the interstate, there's you know, there's a stop and go or sometimes it's really going to go and go. And then sometime it's like you're at a dead stop.

Icis agreed the importance to trust yourself over the technology, "Yeah trust your instincts, trust your gut [versus the technology]. Yeah." When probing whether the technology could have worked in a different way:

> Oh yeah like it gives you alternate routes you could take. So like the route that we used to take was on there. It was just saying that it was going to be like an hour or two longer than this route. So instead, if we go I-95, we were going … to be there by at least like honestly it was like midnight and if we were going to go the way that we were going, … we were going to be there like one two three in the morning. Like yeah like [the route] was on there but my mom was like oh well this one says it's gonna be faster. So let's go this way. Let's trust … Sheila. And we're gonna go this way. And we probably should have just trusted our instincts and gone the other way.

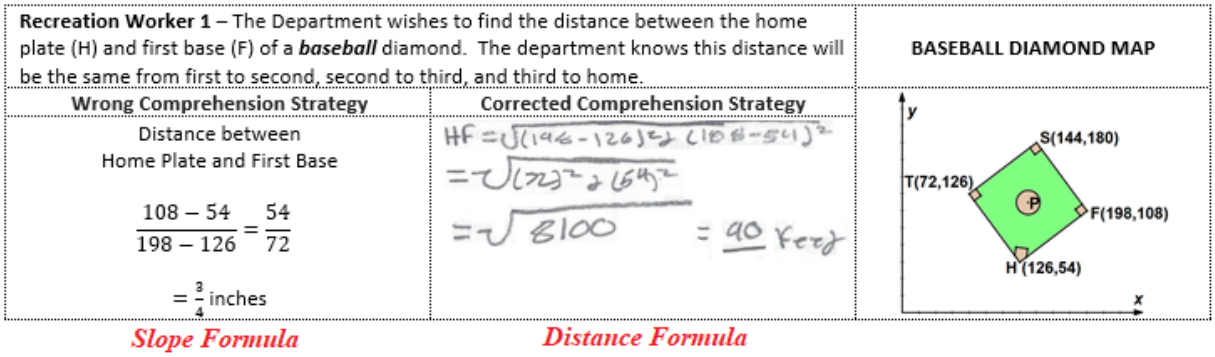
Icis elaborated on why Shiela, the GPS, did not choose the more comfortable, familiar route,

> Yeah it [Shiela, the GPS] kind of was like oh well this one's faster. Let's go this way. Like um, whenever like you pull up instructions on your phone, the map and it like it gives you all these alternate routes. Like oh well this one's like five minutes shorter or this is five minutes longer. Like you can pick one of those but it was saying that this one was like two to three hours like shorter. And we wanted to get there. We want to get there get there, get there, and unpack the car. So like it was already 12 hours in the car. So we kind of wanted to get there as fast as possible. Probably should have stopped, … taking our time to look back, to kind of look at all the other options.

In a wrong strategy problem (see Figure 42) on the literacy document, Icis agreed to use the distance formula instead of the slope formula.

Figure 42

In this Wrong Strategy Problem, Icis Dnew the Distance Formula had to be Used Instead of the Slope Formula

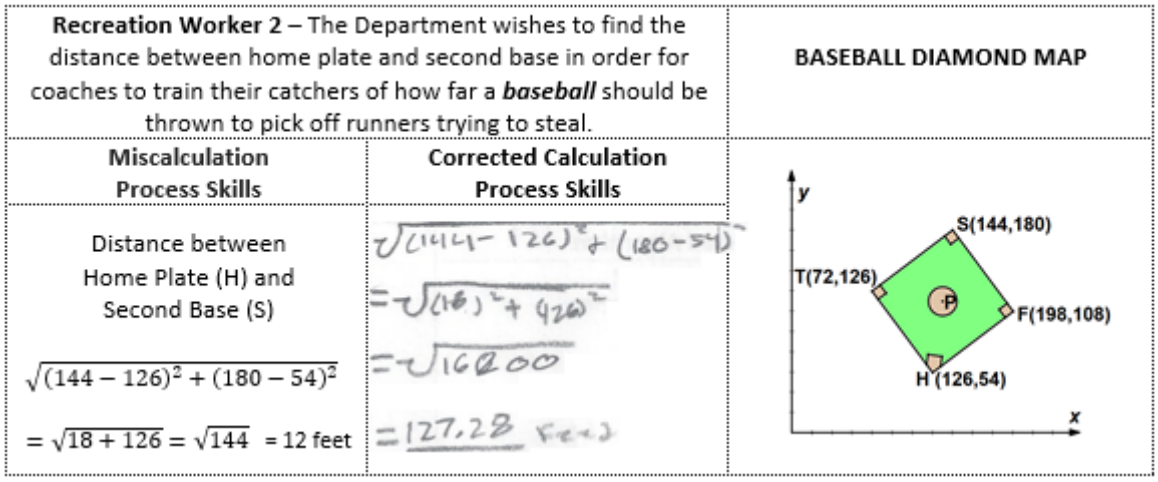


When Icis mentioned searching for alternate routes and looking for other options in her narrative about the GPS, there was intrigue to know of other options she knew to use for other problems in the literacy document:

> Like right here [see Figure 44] all they messed up was is instead of putting, they kind of did what I thought like … what I did with the parentheses. They took away … the squared part and they're like oh this isn't going to make a big deal. But it ended up like messing up their calculation. So they thought okay I'm gonna use this strategy and this is what's going to work for me but then it actually ended up hurting them more than it helped them.

Figure 44

In this Miscalculation Problem, Icis Saw the Values that Needed to be Squared for a Correct Answer



I questioned this error related to her GPS:

> In a way it's like okay you [Sheila, the GPS] ended up hurting me more than you helped me. Dang it.

When asking for a metaphor that represented her experience, Icis was hesitant with thinking of a response. When mentioning about her GPS narrative:

> Kind of like directions. Like um to like try and choose the right path. Like go, like trust your instincts, choose the path that you think was meant for you and not the other one. I guess I'm not that good with metaphors.

Her words sparked an image of a parachute in my mind. I asked if what she was mentioning was like being on a parachute?

> Oh yeah. Like, yeah, when you're jumping off an airplane, you have a parachute. Oh yeah, yeah, direction. You gotta choose your direction, which way your feet is going to go? [She humorously moved her hands in a horizontal fashion.]

I asked if it was safe to say that an experience with distance, midpoint, and slope is similar to navigating on a parachute:

> Yeah I have to choose where I'm going to go. Am I going to like here where it's five and six or am I going to go here where it's four and eight?

When she mentioned five and six along with four and eight, she referred to change in horizontal and vertical values of the distance and slope formulas.

Week 2 - What Wrong Strategy Story did Icis Relate to Perimeter and Area?

> Icis recalled a narrative of rearranging furniture in her room:
>
> Oh I can remember when I was trying to, my room has a very kind of it's a weird shape. So what we have to do is when I was trying to fill it up it kind of looks smaller. And I have put some stuff in the wrong area and it ended up looking really cluttered. But then when I fixed it and I moved things back around it looked a lot more open and had more room.

When Icis mentioned "weird shape," I questioned whether the shape was considered an extension our lesson in class beyond rectangles:

> Yeah, like um it's kind of my room size is kind of like a hexagon in a way. It has weird kind of curves and all that. And like when you're trying to work within this basic, when you're looking at a rectangle, from above you can kind of see like okay well this is this and this and this. When you're actually like working in it, it's a lot harder to position things around.

When Icis considered the type of error:

> I probably did the wrong strategy of moving things around. I probably could have done it differently which I did eventually move things around. Like I got rid of the big desk I had and I got a smaller one. And it was a lot easier after like I kind of corrected my mistake. Yeah I got … rid of some things that I moved things around.

When asking what could have done differently:

> I mean, you know, I have like, I probably could have if I wanted to keep it in there. I could have moved other things around. I've been seeing like I could have moved my bed to a different angle. And then put everything kind of together. I could have in a way, like I like to draw things out. So I did I drew like a map kind of looking above it. So I guess I probably could have taken it and moved it. Like moved it to the other side of my room and kind of like move things around. Like when um, like I guess like when you're playing chess and you're moving everything, sometimes you have to go back on like a move because you're like okay that's not a good area to have it.

Probing if the drawing of the map is an overview:

> Yeah like … when you're looking at something like area or a perimeter, then you don't look at, you're not looking at it as if like you're in it. You're looking at it as if you're looking above it.

I provided a rough summary of how the perimeter formula is a sum and the area formula is product. I asked if the perimeter and area formulas change or stay the same when applying her drawing of the map:

> I mean [the perimeter and area formulas] kind of adapt sometimes. It just matters on like what you're drawing and what you're doing with it.

I referred to some of the words that Icis mentioned, which included "adapt," "moving other things around," and "playing chess." I questioned whether these actions were going on with her literacy on perimeter and area:

> Oh yeah. Like you have to look at, you like okay, well what if I put this here and then you have to like see what that outcome would be. Or okay, well, what if I put it here? I mean like, it's kind of I don't want to say it's really a chance game. Because it's not really a chance. Because you can always go back and fix it. Now, not every mistake you can always go back and fix it. But there are some like okay I can fix this. Like okay this is on paper, it's not finalized. I can go back and fix this. Or I'm typing a report. I can go back and fix my misspelling or my punctuation or my capitalization. You can fix that

> type of stuff. But there are just some things that it's really harder to go back and fix. But with the literacy it was really easy. You kind of look at it. You found out what was wrong with it. You looked at the formulas you had and then you kind of position things back to where it needed to be.

I asked if she felt being more observant:

> Yeah I feel like observing things is kind of key. When it comes to anything, math, literacy, history, anything especially in math and history, I've learned. Because with history, a lot of people are like well and math it's like oh it's memorization. I'm not very good at memorizing things. I suck at it even. I could sit there and read it and read it and read it and read and read it. And I still will not be able to memorize it. It's just something with that brain, with this brain up here.

Icis described a strategy to overcome:

> So what I've learned is if you just look at your surroundings and you observe what's going around you. You 'll be able to put two and two together. Like you, I hate saying this, but you could be the dumbest person in the room. But if you just stop and you observe, you can know everything that has happened in that room. And a lot of people are like, no, no, you can't. Well you kind of can if you're sitting in a room with four other people. The thing is like a lot of people use it.

She provided a scenario:

> If you put a girl in a room with like five guys, … at my standpoint if she would stop and just observe what's going on, you [she] can really, if you stop, and just look at, somebody looking at them, sometimes you can understand what they're going through, sometimes you can't. Sometimes, you actually have to dig a little deeper. And I feel like people nowadays, they don't really do that.

She gave her perspective on assumptions:

> They [people in general] kind of look and they assume things, which assuming is never a good choice, ever. And like I personally have assumed some things before. And especially with math, you can't really assume things. Because you can assume one thing, it's like a hypothesis. Your hypothesis could be wrong, it could be right, it could be dead on. But maybe that's not the way you're supposed to do it. And to a lot of people, I'm like, look you can.

Week 3 - What Wrong Strategy Story did Icis Relate to Counting Points and Segments?

> Icis shared a narrative about counting coins:
>
> I remember um at one point I was counting some coins. And I got down to the end and I noticed that I had miscounted some of them. And I had to go back and I had to fix my

> problem. What I had done was I had decided to count by threes. And I noticed after a while I ended up just counting normally instead of counting by the threes.

When inquiring about type of error:

> Probably the wrong strategy because I should have probably just gone with counting normally instead of trying to count by threes and get it over with quickly.

I asked what counting normally meant to her:

> Just like one, two, three, four, just counting with the normal strategy that most people in like kindergarten would use. My brain kind of works a little slower when it comes down to trying to double everything or triple it all. Yes, just counting by the ones.

When asking whether she is able to maintain or lose focus when counting by ones, "I probably say I was able to maintain it [focus]. It's usually a lot easier when going into it." I questioning what made her to decide to count by threes:

> I have a kind of it's kind of like an obsession by doing everything by threes. So after a while it's kind of just it's kind of started to get normal just I have everything in threes in my room. I do everything by threes when I organize anything. So I was like well let's try counting by threes and that did not work.

I asked whether there are a lot of students that she knows that count by a different number:

> I probably really say that a lot of people count by ones because it's kind of what we're taught like this the base of what we're taught. But I think some people do have like that kind of like ability to be able to count by like twos or fives or sixes or stuff like that.

I asked if two, three, and five were the popular numbers to count by:

> Probably yeah because usually twos, like everybody knows their twos, their threes, and their fives. And then when they go up to like sixes, eights, nines, tens, it starts to get a little more difficult to keep up.

I sought to confirm if counting by ones is what she would have done differently:

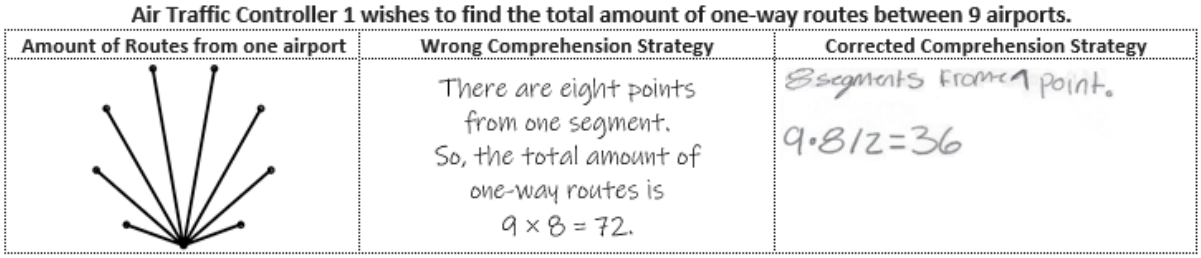
> Yeah I probably would have gone back and after like counting I noticed I had missed, had done it wrong. So I had to go back and I just separated everything. Then I count them all by ones and then yeah. I just I need to know how many coins I had in all.

I questioned if the counting by one practice was applied in her literacy:

> Oh yeah like with um with this one [see Figure 45], I noticed that like after reading and going over it and looking at how much we had and all. Like at the very beginning, it says … we have eight points. And I went back and I was like okay you do. And then it's from one point I was like okay. But when you look at how she [Air Traffic Controller 1 as a female] set it up, you notice that she didn't divide by two. So I was like okay, so you have to go back and fix her mistake.

Figure 45

Icis Figured out that Air Traffic Controller 1 Missed the Double Count by Needing to Divide by 2



When demonstrating to Icis how a segment remains one and the same regardless of how the segment travels from one point to another, "Yeah we'd have to divide by two instead of double counting it over it." She agreed that the behavior of one segment is the behavior of all segments, "Yeah, like one behavior kind of affects everything else." I questioned whether this behavior was present in her real life experiences:

> Just one mistake can kind of affects everything else. Like, for example, if I know, when I was younger, I wanted a lollipop. So I took the lollipop and put in my pocket and we left the store. And then I noticed that later on like when I got caught, I got in trouble. And then I thought about I was like okay well if I didn't, if I hadn't done that, and I had asked, maybe I wouldn't be in the same situation I am now.

I questioned if Icis felt the same mentality was going on with the wrong strategy problem:

> Oh yeah. Right here [see Figure 44] she, when she multiplies it, you have the 72. But that's not technically the right answer. So when she had to divide it by two, she would, if she were to look at this, she'd be like, okay that's what I did wrong.

Ironically, we humorously agreed that the lollipop represents a point and a line at the same time.

When discussing the behavior of one is the behavior of all, I questioned whether the behavior of one mistake causes more mistakes:

> Yeah probably because like once you make one mistake, it's hard to kind of go back and fix it sometimes. So then after a while, it kind of turns into something that you do over and over and over and over again.

When seeing a mistake, I inquired whether Icis knew of a need to stop (she completed my sentence):

> Yeah kind of stop and like read like freshen or like refresh my mind. And be like okay, what did I do? What's wrong? And then I could want to find out. You kind of go back … in the situation. Okay, this is what I did wrong. Like when you do like a paper like this [see Figure 44], … if she [Air Traffic Controller 1 as a female] stops and just looks at it and notice okay this is the wrong answer. Let me go back and rethink how I did this. And then she would notice okay I have to divide by two to get the 36.

Week 4 - What Wrong Strategy Story did Icis Relate to Sequences of Transformations?

> Icis shared a narrative about navigating a boat:
>
> I would say when my parents used to own a boat, we would go out and my dad was trying to teach me how to sail in a way. And there was a time where I had kind of like, I went to turn the bow. I had to turn it to Eastport and I went the wrong way. I turned it to North instead of East.

She classified the error:

> Probably the wrong strategy because I wasn't thinking about it. And I like, when he said it, I was like oh yeah. And then I was like, okay, if I do this, I should be able to do this. And I messed it up in my head in a way.

Since Icis mentioned "turn," I stated that her navigation of the boat definitely involved a rotation, "Yeah, oh yeah. Gosh, we almost crashed that day too." When questioning whether the navigation involved translating and reflecting too or just rotations:

> I would probably just say the rotation because … I don't think you're really reflecting anything and I don't think you're translating anything there.

I asked if it was just like a sequence of rotations, “Yeah, it’s just kind of like a constant thing you have to kind of do.” I asked what she would have done differently with the turns:

> With the boat, I probably should have like stopped and thought about it. Like okay, if I do this, what are my outcomes gonna be? Instead I was kind of in a rush trying to get it over with and because I was too busy watching the seagulls. Couldn’t help it, cute birds [as she humored].

I probed for imagination when she stops and thinks about a situation:

> Yeah, like you can kind of stop and if you think. I don’t know if anybody else does it but it’s kind of like a movie. But like if you stop, you can think, okay, if I do this, this is what it might look like. But if I do this, this is what it might look like. You think of all the possible outcomes to it.

I questioned the presence of multiple outcomes in her mind:

> Yeah, I mean it’s like a lot of people kind of use like the domino effect. As they think, okay, well if I do this, then … it’s going to be like all a straight line. Well, I mean you can take a lot of paths. So at one point that even though it’s a domino effect, it [the paths] has to stop. Because maybe there’s like three paths you can take. So then you have to be like okay, you’re that domino. Okay which one am I gonna knock over? I can’t choose all three and if you do you’re very talented [as she humored].

I summarized her response as following a pattern of paths until crossroads occur where she had to make a decision:

> Yeah. Okay am I gonna go down this this way or am I gonna go this way? How do I want… to see this outcome? So probably stop and take your time before you turn a boat [as she humored].

Week 5 - What Wrong Strategy Story did Icis Relate to Rigid Motions?

> Icis recounted the baking of a cake with her mom:
>
> Probably like cooking. My mom, we like to bake a lot, especially me and my mom, like we love like making cakes. There’s a recipe I just recently got out of a book that we ordered for Christmas for me. And I’m not a big chocolate fan. So we thought we could change out the chocolate for vanilla. And then my mom was like oh let’s add strawberry. And we ended up making it, it wasn’t like the recipe was not supposed to change at all. And like we got there and then it was just kaput. And like my mom, we sat there, we kind of talked about it. And I was like ooof. You know we could have done this instead of this. We had it like, I looked at her, I was like, we could have strategized about it.

I questioned for more clarification of her strategy:

> Like whatever like you bake. Like whatever baking, like how much flavoring you put in, how much like vanilla chocolate chips you put in, the vanilla chips.

She agreed of deviating from the recipe. When asking to describe the deviation:

> Well, … the recipe normally calls for two cups of chocolate chips, a … spoonful of vanilla and then I think it's like … some type of … brownie mix in a way. And you're supposed to put all that in. And we ended up trying, we ended up doing … one cup and one half cup of vanilla chips. And my mom, instead of using the vanilla extract like we were supposed to, she ended up having like some strawberry stuff. And that's what we used. And then the more and more we added, the more and more we kind of drifted away from what we were supposed to be doing. And then by the ending … it was a good cake but um a lot, I mean a lot, and it was like really sweet and everything. So I kept telling my mom's like you know we could probably strategize just a little more, but a bit more efficient.

I inquired about the types of rigid motions that Icis noticed with the cake recipe:

> I probably say it was more like translations and reflections.… Okay when you're reflecting something, like you're taking it, you're putting it over here [as she moved her hands with her fingertips on a table from left to right as a flip]. When you're using a recipe book, you're taking this recipe and you're putting it into like actual things. You're putting into a physical form translation. Um translation was when ah [she hesitated, trying to think of the meaning of translations] when you're moving things. So when it comes down to that it's like when you're moving like the chocolate chips or something from one area to another oh

I asked Icis what could have been done differently to bake the cake properly:

> Probably could have followed the book [the recipe] a little more, only modified certain parts, could have tried to stay on track instead of like going off.

I clarified by reiterating the concept of not deviating too much:

> Yeah don't deviate too much. You can deviate that little bit but that's it. Like stick to the recipe. Stick to the strategy.

Regarding literacy, when Icis discussed following the recipe, I asked if her vocabulary played the role as a substitute for a recipe or if she had her own "recipe" for finding the correct rigid motions for erroneous problems:

> I could have paid attention a lot more. And probably taking my time, observed what I was doing. When it comes to vocabulary, I'm not very good at it. Because vocabulary kind of like it's like you have to memorize it. I'm not very good at memorization. I suck at it, honestly. So when it comes to vocabulary, like apparently I'm really good at using my vocabulary. I have very extended vocabulary for a high schooler. But when it comes down to like memorizing something or taking like vocabulary tests or anything on a test that involves having to remember the vocabulary, that's something like if we're offered a cheat sheet, I usually have to write it down. Because you can use the vocabulary and it can be useful in certain like ways. Like, like oh let me look back at my vocabulary to see how this would describe this. Like if you're having to compare or match or you're using it and you have the formula, but you can't remember how to use the formula. So you look back at the vocabulary to see what you have done. Or how it would look. So I mean yes in some takes, vocabulary can be extremely useful. It's just for some students, it's not, maybe not for them.

I inquired whether looking back at the vocabulary is similar to looking back at the recipe when cooking:

> Oh yeah like my mom's one that usually follows everything by the book. So I'm the one who likes to deviate from everything. I like to experiment. That's just kind of that's just how my brain kind of works. So when it comes down to like deviating that's kind of like I guess my strong suit. But when it's like following everything by the book, it's my mom. Because a lot of cooking, as my mom explains it, it's like when you're cooking, you don't really experiment. As she sees it, it's something that you have to follow by the book. I see it as something that I kind of like you can experiment with. You can throw maybe a pepper in here or um you know an onion and a piece of bread or like I like to like mix things up and see how the outcome would come out. Because you never know what's actually going to taste good. So like, whenever, even when I'm doing literacy, it's like okay well what if I did this? well what if I threw this in here? oh that don't work. Okay let me take that out. It's just like when you're cooking you can't take anything out. You can only learn from it and it comes sometimes to the same thing when you're doing literacy. Because I prefer, like I like to write in pen, and there are certain things that you can't erase with a pen. So you kind of like scribble it out and you do it again.

I questioned whether the vocabulary takes away the excitement and adventure of doing literacy:

> [Icis laughed.] Yeah personally I like the adventure. I like the adrenaline rush. So my idea of … a good time is skydiving. Like I have gone skydiving more than like one time so far. I've had to be more than once soon. So my idea of like when I do work, I try to like not to be disrespectful to teachers but I usually have like a show going or I'm listening to music or a podcast and it's mostly just so I can focus. Because like … when I'm sitting beside somebody and I finally get to talk to that person and then like a teacher moves them and it's like oh. When I have somebody here that I may not like or I just don't like talking to or maybe we don't share the same interest. So now I'm like okay I have to find

> something else to keep me on track…. So I have to like, like in a lot of my classes, I'll watch Netflix and I'll have NCIS going or I have like a different show on Netflix going and I'll have it going. And I can do my work and just listen to them. I just have to have that background noise. It's just when it comes to an in-school environment, it's like sometimes I can live with listening to what's going on around me but sometimes I don't want to hear that, which is understandable so.

I asked if skydiving is that metaphor for her experience with rigid motions:

> Yeah it's [rigid motions] kind of like skydiving. You have to like kind of, it's like once you do it, you can't really back out. It's like it's at all one end way choice. You start with it, you're starting to learn. Like you go through your basics when you're learning about it. You go through like the jumping off the wall, the jumping into the mat, all that all. Like how can you swim if you jump into the water? How, what you need to do if you're off course. Like you start with the learning, the basic stuff. And then when you finally get into it, it's like oh I have committed to this. Here we go.

I inquired if it is almost like creating her own vocabulary:

> Yeah, it's like when it comes to vocabulary I kind of I have to like in a way find my own connections to certain things. Like okay, this sounds like chocolate so chocolates got it. It's just kind of you have to like make your own connections to certain things. And I know with certain kids it's like okay … I have to connect what I'm learning in here with something that I'm doing on the outside that I'm really interested in so I can focus at the same time.

Icis's Stories of Miscomputations

Icis reflected on stories that involved irregularities. She perceived rechecking practices as a major factor for avoiding mishaps with calculations. Once more, humor was used to help cope with the errors.

Week 1 - What Miscomputation Story did Icis Relate to Distance, Midpoint, or Slope?

After requesting a confirmation of the definition of miscomputation, Icis recounted a documentary about marine biologists analyzing the travels of a shark:

> I don't know if you've watched documentaries but there's a documentary on a shark called Big Blue. Um they [the marine biologists of the documentary] actually like, they calculated a number of like how many days or like weeks she [the shark Big Blue] would be down in the depths of the ocean while she was giving birth.

Icis, a fan of the documentary, included herself as part of the team of marine biologists:

> And we completely lost her and we didn't understand what was going on. They actually found out that they had miscounted the numbers. Like I think their estimate was like five to like six weeks when it actually was supposed to be like 20 weeks. And um like they had uh they had messed up. With it I'd say that's probably like slope because she's having to go down.

I questioned whether going down was like slope to her:

> Yeah because like she's going down into the depth. Because if you start here and then you have to go down. And she also has to like ascend up whenever she comes up. Because they can't spend too long into the deep.

I inquired if the shark's travel was related to time:

> Uh yeah it was like um whenever they [the marine biologists of the documentary] do it because they have … a tracker on her fin. And they monitor how long like she's in the deep and then how long she's not like how deep she goes and everything. And they had estimated that she would be down into like the deepest part, uh I think it was like the Mariana Trench, for like five to six weeks. And like her depth would be a certain amount of numbers. And they had messed up with their counting. Like um, like their marine biologists had done their math wrong somewhere.

After prompting for how the shark was supposed to be down in the ocean, "Yeah, she was supposed to be down for like 20 weeks, 20 or 30." She agreed to be thinking of time as slope. However, I meant to ask like a rate of distance per time. I questioned how the error could have been avoided:

> Um I feel like when they [the marine biologists of the documentary] calculated it. Because my family, we like to go and like watch when she comes up or see if we can go see her. So one of the things I feel like when they calculated it, they probably could have like recheck their math, like gone over it, double check things, make like, have another, like have a second opinion look at it too.

Icis elaborated on rechecking and double checking:

> Yeah that would have like probably solved that. Because like originally, she [Big Blue the shark] was supposed, like I said, she was supposed to be down for five to six weeks is what they estimated. She ended up being down longer and a lot of people went to go see her like emerge. So as like a lot of people saw was like a waste of time and a lot of people got really upset about it. I feel like if the like the … marine biologist would have gone over what they were like doing with their math. Like would have rechecked what they

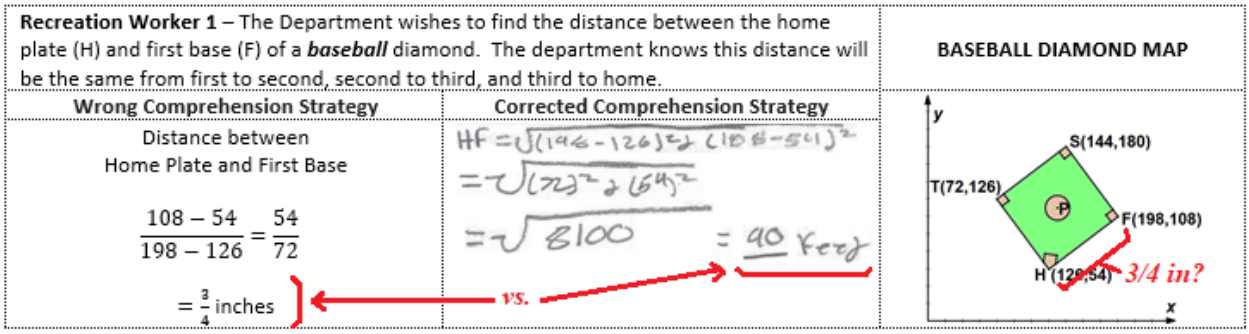
were looking. Like double check it. Like hey will you make sure my math is right. It probably could have been avoided.

When she mentioned rechecking and double checking, I asked her to talk about her sense making with one particular problem where it made sense to use feet instead of inches:

> Um … when it comes to like the three fourths and then like you finally solve it like that's really off [see Figure 46]. So it was kind of like once like if you look at their math, like they messed up. They used the wrong formula. They ended up doing the wrong strategy when they, uh, were calculating it. So when you go back and use the correct strategy, find out what strategy you need to use and you use it, you end up … getting 90 feet instead of … three-fourths inches. And then if you come here like it looks like a graph. If you look at it and you can like see , where three would be and it's like way up here and you needed to be like over here.

Figure 46

Icis Knew ¾ Inches was way off Because the Wrong Formula was Used



She agreed that it was similar to her narrative of Big Blue:

> Yeah like how like okay you're supposed to like they thought okay you're supposed you're gonna be here but you actually ended up being here.

Week 2 - What Miscomputation Story did Icis Relate to Perimeter and Area?

After requesting for clarification of the definition of a miscomputation, Icis reminisced a narrative of a game that used to play as a child:

> I know as a kid we used to like walk around and as a kid like you want to walk like if there are squares or something on the ground you want to walk in the square and not out.

> I remember this one time everything was going really good. I was playing with my sister and then I kind of took the wrong step and I didn't notice it. And then I got like off in a way offbeat. Or like when you're learning music and you have to stay on beat at one point I was a little too far behind and I had to redo it.

As a child, I played a similar game. Icis agreed to step inside the squares to avoid the line. I questioned if she thought of imagining being zapped in the line when stepping on the line:

> Yeah or something like that yeah. It was kind of like a game like I remember. Um my sister would always feel like if you step on a crack, you break your … mother's back and if you step on a line you break your father's spine. So it was always like oooooh what was gonna happen.

I asked if stepping in the square was a good way to think about area:

> Yeah, yeah. Like if you think about it, like it could be like okay well like tic-tac-toe. You have to stay within this grade. You can't like do it on the lines. You can't do it outside of it, it's inside.

Icis agreed that the miscomputation involved the wrong steps that she was taking during the game. I reminded her that she did not want to step on the lines, "It's like Hopscotch. You don't want to step outside of it." I asked what she could have done to avoid stepping on the lines:

> I mean like, I could probably watch where I was walking. Like kind of taking in, for example. Like there are times where you can be walking and the shape will change. Because it's like okay this area is older than this one. So they haven't gotten to it. So I probably could have been like watching where I was going.

I questioned being more observant, "Yeah kind of observe like um taking your surroundings." I inquired if there would always be an error no matter how observant you are:

> Yeah, I mean like as a human because every everybody makes a mistakes. It's kind of inevitable like it's gonna happen. Even if you can, if you try, and make sure it doesn't happen. It's gonna happen and it could happen right under your nose and you'll never notice it. And I feel like people forget that because it's like okay, nobody's perfect but everybody wants to be.

Icis elaborated by transitioning her response to a conversation with her mom:

> And that's like with um, I was talking to my mom about it. Yesterday I made the comment that circles don't really exist and people like she was like well they kind of do. I was like no. It was like if you look into it, even if you draw a circle, if you zoom in, it's

> made of little squares. That's like circles don't exist. I was like it's not really. It's kind of not there. Like when we look at it from a distance, it's a circle. But when you get closer and closer like you can look at it, that perimeter [circumference] is not a circle it's squares [she was thinking of pixels on a computer screen]. And then the inside isn't circles, they're squares or maybe they're triangles or something. But to me like circles don't exist.

She continued with her elaboration:

> And people, like they're always, I think people, when it comes down to making a mistake or like doing something wrong, they're so hard on stuff. Like oh I could, I should have got that right. I should have got that right. But I feel like people forget the mistakes kind of like, they do they help us grow. Even how even though it's cliche it really does it helps us grow. Because you, nobody in the world could tell me they have never made a mistake. At least one time in your life, you had to have made a mistake to get the outcome that you're in now. Like again, it's a domino effect. When you, you're gonna do this and something down the line is going to happen or you can do this in something down or maybe in two days, this is gonna happen. It kind of just matters. Like I know people who have said they have had dreams and then they happen in real life. A lot of the time, yeah, some people are like oh it's a parallel universe. I'm sitting here, I'm thinking okay, it could be that or it could be the fact that you're thinking about it 24/7 so after a while you see yourself. If you observe yourself, you kind of see yourself doing that which would affect this happening. Like you know what I mean, like you're gonna do this and then this is going to give you the outcome you want. But I feel like people, no matter what, anybody's gonna make a mistake.

I shifted the interview to inquire a relationship between Icis's overall view of mistakes as a growing experience with error analysis as literacy:

> Oh yeah where they've made a mistake and you have to correct it. Yeah I think that's all, it helps us because when you're looking at it you're like okay, I could have made that mistake. Well now I know what it looks like and I know how to correct it at the same time. So it's kind of like you're putting us, it's like a side by side, you're looking at, you're like okay this is the wrong way but this is the right way. So now, if I make this mistake and I see like it looks like this or I have a similar problem, I'm like okay, I know what this is now. I know how to fix it.

Metaphorically, she described her experience with perimeter and area:

> Like hopscotch and chess. When it comes down to like perimeter area, when playing chess, you have the, you can have like the black and white spaces. And let's say you have to stay in the white spaces. It's kind of like, okay, if I go into this area, it's not gonna work. Like the Hopscotch you have … to remember, you have to stay, you can't go outside of like the box, you can't go outside the perimeter, you have to stay inside of the area.

Icis concurred that area is the inside and perimeter is the outside of a shape. I questioned whether perimeter can be thought of a boundary:

> Yeah like a boundary, like in a video game, like okay this is … your square and only your square you can't go outside.

Week 3 - What Miscomputation Story did Icis Relate to Counting Points and Segments?

After providing a review to Icis of the definition of a miscomputation error type, I slipped up by relating it to a faulty algorithm. In either case, Icis gave her version of a narrative that she interpreted to be a miscomputation:

> … I think it's like maybe last week I was doing uh me and my little sister were doing some chores. And we have a system where we kind of split everything up. And we usually do it equally. But, by the end, like we noticed that I had done more, she had done more. And we had to go back and we had to like fix how we kind of strategized everything. So I think that would be kind of like a miscomputation.

Interest entered my mind of whether Icis thought of the chores as points or lines:

> I'd probably say the chores were more the point. Because like that's where we started off. Like we had to separate everything we had. We had the chores down. And then, it kind of like, because … once by the end, we noticed that it had kind of gone wrong. We had to go back and fix it.

I prompted with Icis having more points:

> Yeah we had, like I had more points than she did or she had more points than I did. I thought we had to split it. We tried … to make sure everything's equal because my sister has like an obsession with thinking if she does more it's bad.

I asked what she would do for a correction:

> … I think I would have, … we went back and I looked at them [all the chores]. And I was like … okay … well if I do this Wednesdays and you do this Tuesdays, we were able to kind of go back and separate … which ones we had to do. Yeah we had to balance it out.

I asked her to think of a metaphor that describes her experience with counting points and segments:

> I mean like when you look at a bookshelf, you can look at all the corners and those are your points. And then the shelves part of your lines. Or if like you're looking at a brick, the corners are the points and then the part, like the parts of the shape are your line segments.

I reiterated with Icis concerning her ability to see points and segments on these objects:

> Yeah like I can see. It's like I've been told I'm kind of a visual learner. So, I'm really good at puzzles or mazes. So I can look at a maze and I can tell you how to get from one side to the other side without running into a dead end or something. It's kind of hard to explain. It's like when you look … at a brick, usually bricks are going to be a rectangle. So, if you look at them and all four corners, those are your points and then … the lines connecting them are your lines. And then you can look at it and you're like okay, well this point could go to this point, this point could go here. Like there are different ways of setting it up.

She agreed that counting points and line segments are mazes that she loves. She also agreed that mazes are effective for helping students, in general, to visualize in Geometry:

> Oh yeah because I know, okay, as a student, I know everybody learns differently. And when you're trying to help somebody else explain it, you have to … take yourself and kind of look at how they're looking through. Like the ... saying that's like you have to put yourself in their shoes. At times, you have to like you have to look at it like as if they're looking at you. Like okay, I see this as a maze. Maybe they see this … as words.

Icis talked about a comparison of visualization between her and a friend:

> Like I know one of my friends, he struggles.… You can give him the formula and he can look at it. But if you don't explain it, he's not gonna know what to do with it. You give me a formula and I can look at it. And most of the time, if it's already been taught, I can tell you exactly how to use it. Him? You have to explain it a little more. So when I'm looking at it, I can see. I can look at this maze and I can … literally just use my eyes. I'm like okay this goes here and it all connects. And then he kind of looks at he has to like stop for a moment. He has … to go through all the paths that leads to a dead end. Then he'll find that one right path that works.

Week 4 - What Miscomputation Story did Icis Relate to Sequences of Transformations?

After receiving a reminder of what a miscomputation was, Icis talked about a light bulb:

> We used to have like an older house where we would always have to go and kind of buy light bulbs all the time for the specific … light. It was the downstairs bathroom light. You would like flip the light switch and it would … always turn off into a short circuit after like, I would say maybe like a few days. So we always had like … enough light bulbs to

> kind of last us at least two months. And we … usually had a stock of at least 20. And I was like, oh well, if we use this one, it's a stronger light bulb. We won't need it. And we only bought 15. And we had to go out and spend even more money because these the ones that were supposed to be stronger light bulbs didn't work. They ended up actually being worse for us than better.

She reiterated about the circuit:

> It was like a short circuit.… Because if you're going to the bathroom you'd flip on the light and it would … flicker. [She posed humor by imagining the light bulb as a character.] And then it would just say, no not today maybe tomorrow.

Since Icis mentioned "flip," I questioned whether the light bulb just involved reflecting or all three transformations:

> It kind of reflects because it's like if I do this, then I'm gonna get this outcome. So it's kind of like reflecting your life. So okay, if I move this point here, it's going to reflect this shape. Or it's gonna reflect … on my future and my past or something like that. Like if you're relating into like real life situations. So yeah I would probably say it reflects.

I asked what could be done differently about the light bulb:

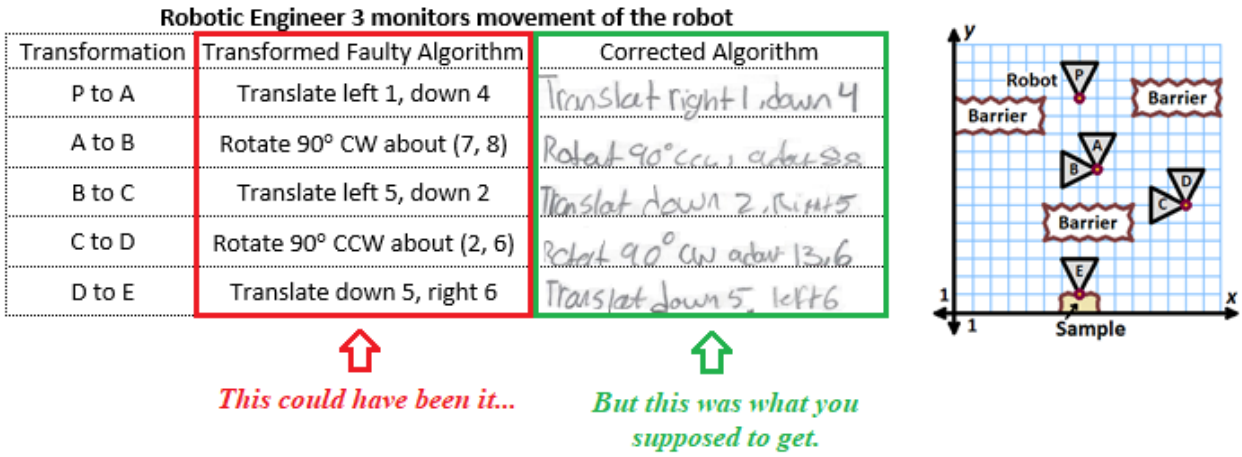
> Honestly, I probably could have read more about this before changing. Because we changed, we had a brand that was working good. And in my head, I was like, okay, if we get this, we can save money. But then we ended up just spending more money. I should have thought. I should have stopped and researched what was gonna happen. Like I know a lot of people are always like, well we can't freeze time. No you can't but you can always stop and slow down and think about … what this might have to do with later on in life.

I inquired if Icis had to stop and do research on her literacy problems before responding:

> You had to like look at it [see Figure 47]. You had to stop and you had to kind of look at, what you, what they were kind of supposed to, like what it kind, what it looks like when they were going over it. So this is their answer. And then you have to go okay, well where do they get this? Where did they get this information? So you go back to look at where they got their information from and then you kind of like go through and … you kind of go through and you correct it. Like okay, this could have been it but this was what you were supposed to get.

Figure 47

Icis Stopped, Looked at the Errors, Questioned the Origin of the Errors, and Then Corrected Them



I probed for other possibilities:

> I mean like some of these [see Figure 46] that they did wrong if you … change some of how this looked, they could have been right, they could have actually ended up being right.

Week 5 - What Miscomputation Story did Icis Relate to Rigid Motions?

> Icis unpacked a narrative about her mom withdrawing money from a bank:
>
> We go to the bank a lot. So, whenever are we doing anything, my mom likes to like take out a certain amount of money. I think everybody does, like you take out a certain amount every time because you can only take a certain amount anyway. But my mom will only take like a certain amount for like the month or the week. And there was one time I guess she over, like she just didn't take as much as she was supposed to and or took too much. Because she got a call from the bank they're like "Hi, we think somebody might have been taking money from your account." And my mom was like, "Well no I was just at the bank." It was a whole ordeal. It's kind of like she miscounted how much she needed. And so like she kind of assumed it. Like oh …, for instance, I'm going to need 200 for this and you actually needed like 104 or something.

I questioned about the types of rigid motions that Icis noticed with the money withdrawal:

> I mean, I see like when it pops up on the screen it's like okay 200 being like extracted. And when I see it, it's kind of like … reflecting what you're taking. So it's like okay one

> dollar, two dollar, three dollar, like it counts. It's trying to make sure like feel like people, we kind of miscount. So you would expect a robot not to miscount. So it also counts kind of like on the screen. So you can actually see how much you're taking out.

Icis confirmed that the screen providing information of the withdrawal acts like a reflection.

When asking Icis how to prevent the miscount:

> I mean we probably could have double checked to make sure we put the right amount in. Made sure … like the computer also counted correctly. Maybe it wasn't on us. Maybe the computer had like an error in the system or something. Like we could have… actually been I guess precautious.

Icis agreed being precautious was paying attention to fine details.

Icis' Stories of Faulty Algorithms

Icis shared stories that involved a need for more focus and concentration on her part. She perceived slowing down, taking her time, taking a step backwards, and reading information more in detail as potential remedies for better outcomes. She considered multiple pathways for solving an erroneous problem as ways to help her grow with understanding how to overcome challenges. Icis once again used humor to help her cope with the blunders.

Week 1 - What Faulty Algorithm Story did Icis Relate to Distance, Midpoint, or Slope?

> Although a game ended not to her liking, Icis reminisced:
>
> Um for midpoint I would say like ax throwing. Like when you have to throw the ax, you have to get it right in the middle to get a certain amount of points or right on the outside to get a certain amount of points. We were doing it one time with the family. And I guess I kind of used like … like it was going good and then it kind of nicked at the end. Like I lost, I didn't have enough points to win.

I asked if the ax throwing was like a faulty algorithm where the game was going smooth, "They [the game] were going great and then I missed a throw." I questioned if ax throwing is like darts too in a way:

> Yeah like it's almost the exact same, almost the same amount of points I think. They're just kind of doubled or tripled.

I inquired what exactly was going smooth in the game:

> Like I was getting like a decent amount of points. I kept hitting in the middle, which is a good, like a really good amount of points. But then I had somebody catching up and I guess I got really nervous. And I went to throw and I wasn't paying attention. Somebody's talking to me. I threw it while I looked away. And I kind of I missed my target completely.

I asked if she was influenced by a distractor, "Yeah I had a little distracted there." I asked what could be done differently to avoid the distractor?

> Yeah I think thinking about it. I probably could have been like, okay, like told this person, hey hold on one second. And then thrown it. Like I was … kind of all up in my feelings because somebody was catching up. So I was like, oh no you can't win. I'm very competitive. And when it got like I got really distracted, I think I should have just like calmed down and just gone with it, just gone with the flow.

I inquired, "So just talk to the person first?"

> Yeah or like been like, hey give me a second, let me throw this, and then I'll be like right with you.

I asked if there was someone in class that distracted her from making sense of answers on her literacy:

> I was, yeah, I was talking to one of my, um, one of my friends. I was helping her out and then I kind of oh it's like um I struggle when it comes down to focusing on something sometimes. So I was helping her and then I totally forgot about this. And I was like oh well now I have to do this. So I um yeah I was a little distracted.

I reminded her of her strategy of telling her friend about the need to focus:

> Oh yeah I probably could have been like hey hold on just one second, I will help you. It's just I gotta finish this really quick. I probably should have been like hey you could give me a second. Let me finish this one and I'll be like right with you. I will help you in just one minute.

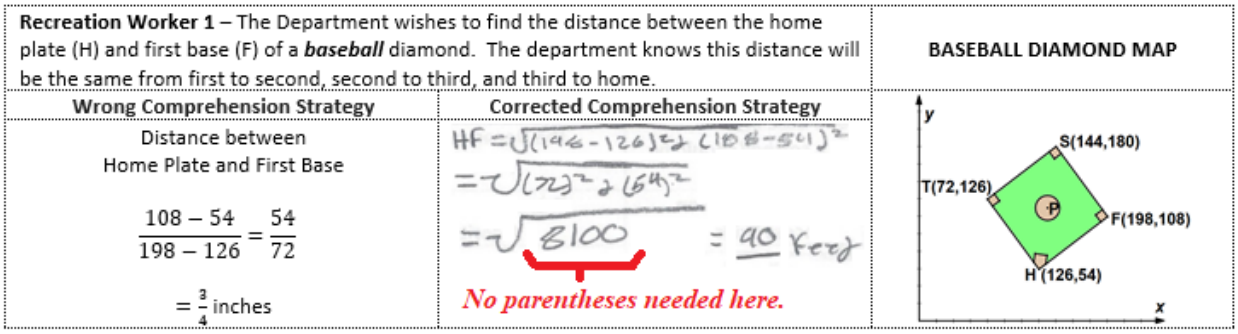
I was curious to know whether the square root and square exponents in the distance formula were distractors for Icis to make sense of the literacy problems. Although this on faulty algorithms, a

wrong strategy problem was used as the focus of interest. When questioning whether she gets lost with the abstract appearance of the distance formula:

> Um, with the parentheses, there are a few too many. So it's like hard to keep up. I would like, there are sometimes where like the parentheses are unneeded. So it's like okay well this doesn't really need to be here. It's just separating this and this. So, I mean, like that could be a distractor. Like if there are too many. Like I'm gonna be honest, like sometimes looking after you do the math and then you go back to looking, you're like wow I did that [see Figure 48]? That looks so like difficult and it's usually because they're like a lot of parentheses. There's a lot of square roots or something like that. So I mean, I mean I don't personally think it's a distractor because I look at it. I'm like okay I understand what I did here and I like to remove parentheses where they aren't needed. So like whatever like you put a square root on and they're like two parentheses right. Here, I probably wouldn't have put them there. Well I put like, I put them there because we were going over it. It was one of the ones that we were going over but like later on I didn't put them there because to me I was like okay well they're not really necessary right now.

Figure 48

Icis's Reaction of Surprise After Correcting a Wrong Strategy Problem She Performed Using the Distance Formula. She Removed Parentheses Where They Weren't Needed



I mentioned how she likes to pick and choose:

> Yeah like I look at it. I'm like okay, well, this isn't gonna like affect my problem later. So we don't need these. Let's take them away right now. I'm like put them in this little bag.

I elaborated with her picking and choosing so that it makes sense to her:

> Yeah, I kind of modify it just enough where it's not going to mess up the problem but it's still gonna make sense to me.

Week 2 - What Faulty Algorithm Story did Icis Relate to Perimeter and Area?

> I mean whatever my mom, well my stepdad like whatever, we're building … a desk for my room. I also like putting computers together and PCs [personal computers]. I remember like everything was going really good when we were building this desk. And then all of a sudden at the very end, nothing was right. Like everything was offline. Like you've had this part and one of the … legs were like all the way in here. It was in the inside instead of being on the outside. Apparently I had miscalculated how I was doing things.

I inquired whether she felt the miscalculation was part of the faulty algorithm in a way:

> Yeah it was like a little of both. Because it was like by the end it was going good and everything was fitting where it was supposed to go. We just had the wrong pieces in the wrong area. And it's like really hard because with perimeter and area, when you're working with it, area you're usually working on the inside and the perimeter you're working on the out. But when you're building something, you have to kind of take in the fact, okay this is the inside and out. So you have to be like okay I have to work with both sides.

Icis agreed that she was working with perimeter and area at the same time. She also concurred that the desk was wobbling and a little off track. I asked what could have been done differently:

> I mean I probably could have slowed down, taking my time with it, read the directions more clearly, read over the directions, reload, like relooked over what I was doing. Maybe taking a step back and looked at it before I put it all together.

Icis elaborated by providing an overall status of why she perceived situations like a wobbling desk occurs:

> Because a lot of people, like I know some people personally, who they'll do something really fast and don't instead of taking a step back and looking at it. You're like okay I'm done and then I'll walk away. And a lot of people are like oh well if you stop and you look and you critique and you grow, you're like, you're um, it's like you're not over exaggerating but like, what is it called, it's kind of, it's not like a clean freak, but like when you're looking and looking and looking. Uh it was on the tip of my tongue and now it's gone. But it's like looking at it and then they're like oh well you're, it's like you're over exaggerating. No, I'm being persistent. I'm looking at it at a different standpoint. I'm putting myself in other people's shoes to look at this type of problem in a different way.

Icis commented on how people in general perceives an investment of time:

> And people don't really do that anymore. They kind of, I feel like people just were like okay let's just get this done and over with. And like, I get it, like some people are like

> okay well I need to do this and this. I don't really have time for this. Let me get this out of the way even if it's wrong, it's done. And a lot of people they like, they don't take their time anymore because time is, yeah, it's a valuable thing.

She reminisced watching a movie with her sister Christina that involved a perception of time:

> … I was talking to my sister last night, I made a comment because we were watching a movie.… My sister was cooking something and she made a comment she would go sit down [to just watch the movie for a minute. Her sister was worried about being away from cooking and not paying attention to cooking]. I was like, "Christina, one minute isn't that long." She goes, "Yeah but it feels forever.…" She [Christina] goes, "Well do you know why it feels like that?" "Yeah it's because it's your brain."

Icis implied that the way her brain thinks about time has an effect on the way one feels about completing a literacy problem with quality. She elaborated:

> … I've noticed it like if I'm sitting in class and I don't feel good or I don't really want to be here, it feels like it's taking forever to go by. But if I really like that class and maybe I could be in there for an hour and it's gonna be like 30 seconds. And I feel like people, when they … have to do work, if it's not enjoyable to the kind of their standards, it feels like it's so long.

Icis continued by talking about an outcome of her perception of time:

> And I wish people would just kind of take their time. Because I think, like I've noticed when I'm working and I don't get distracted to you, like I get distracted really easy, but if I'm like zoned, like zoomed in on something, I finish it so fast. And I'm like well now what do I do? Gosh just kind of sit here.… Then it's like okay. But it's like time, it feels sometimes, it just feels like time is like working against you.

I questioned whether the problem with literacy is that students progress through it too quickly:

> I'm not gonna lie, I'm not one for literacy. I don't, I have, I know it's a down the line it's gonna become a problem, but I don't like reading things. To me, like, it's not enjoyable. Like if I read like the first two sentences and I don't find an interest in it, then I can read the whole thing and I can't tell you what it's about. And I know that like, down the line, that's going to be a problem. Because you know sometimes you're gonna have to read what you don't want to but you're gonna have to understand it. And I'm working on it but I feel like a lot of people when it comes to literacy they feel like okay we have to do this, you have to do it.

Icis made an analogy to home chores:

> And I've noticed even at home when you're doing something, like let's say you're just doing the laundry, if somebody comes in and tells you need to do it while you're doing it,

> you don't want to do it anymore. Because it feels like you're kind of being told to do it. And I know like some people are like oh you just have a problem with authority. Well, everybody kind of has a problem with authority. It's because everybody likes to feel in control of themselves which I hate saying it.

Icis transitioned to describing emotions:

> You're not all the time. Like you're in control of your actions. But sometimes whether it's I'm bored or you're just you're maybe you're angry, scared, anything, sometimes your emotions will take over those actions. And you can like afterwards you can sit there and be like dang I shouldn't have done that. Why did I do that? and you won't be able to like answer it.

I questioned if these same feelings were going on during the literacy that we have in class:

> Most recently, not quite. I quite like literacy but it has happened before when it comes down to literacy to other classes. It's just boring, it's just sometimes it's just so boring. I'm sitting there I don't want to do this.

Week 3 - What Faulty Algorithm Story did Icis Relate to Counting Points and Segments?

Icis limned a time of baking cookies with her mom:

> I know when I was younger I was doing something with my mom. And then right in the middle, we noticed that, like we were cooking. I think we were cooking cookies and then in the middle, like we noticed, that the batter was too thick. So it was it was kind of a problem because when you bake and everything, it's a little harder to do that when everything's a little too thick.

Icis concurred that the cookies represented points and the thick batter represented a line. She

elucidated:

> Yeah because the point is kind of like where you kind of like start it and after a while when you're going through, you notice okay well now it's kind of like a line, yeah.

Icis confirmed that the faulty algorithm was the thickness of the batter when making the cookies.

I asked what she could have done differently:

> I probably, if I knew that it was going to turn out like that, probably wouldn't have added as much flour as we did. uh could have gone back and been like okay well let's not add this much. Let's add half of this instead of like two cups.

I inquired if an estimation error occurred as a form of miscomputation:

> We, my mom and me, I like doing things by the book. But at the same time, when you cook I kind of like to explore a little more. So it was an error in kind of like how much we estimated where we were going to have in all.

She agreed that the cooking was like a combination of faulty algorithm and miscomputation. She confirmed that she enjoyed exploring estimations:

> Oh yeah like um we baked yesterday and I made a cake and then it ended up being way too big and we all were gonna eat.

Icis elicited confidence of being able to detect and correct a wrong strategy problem. Curiosity entered my mind about how Icis felt of a transformed faulty algorithm, particularly where a right answer occurred at the end. When questioning if Icis was able to detect a correct answer at the end but a faulty algorithm in the middle, she answered:

> Yeah, like we're like you kind of miscalculate what you did or you have one wrong … number but you got the right answer somehow. Yeah. You I think it was when I was doing one of these, we had our literacy. I had … the right answer but I had put my numbers in the wrong area. But I somehow got the right answer. So I had to go back and kind of redo how I put my problem together.

I probed whether a lot of students in general come up with a right answer but wrong process:

> Well it's like you did this right but you did, you set it up wrong, yeah. I probably say a lot of students would go through it and notice okay, this is the right answer but maybe I went about it wrong.

I interjected by saying it's a phenomenon of how this happens. Icis elaborated:

> There have been times where I have gotten the right answer. And then I look back at my math I'm like I have no idea how I got that. I was like hold on. Like um one of my friends and me, we were … doing something. And I noticed halfway through that yes this is the right answer. But like I went back to explain how I did it. And I was like I have no idea how I got that. And it's just like you have to just go back. And you have to like refresh your mind and kind of think how maybe outside the box how you got that answer.

I probed if there was some kind of mysterious instinct occurring in the minds of students:

> It could be something that just kind of clicks at one area. And it clicks for just that one by one split second and you got it. But when you go back and you're like I don't, what?

> There have been times when I'm doing math and it's just like … two plus two is four. There have been times where I've gotten that but it wasn't even two plus two and I'm like, what? Hold on. That, that's not even, excuse me? Like you have to kind of stop. I think it could be after a while a student's mind is running at a certain pace. But, like I know when I'm reading, my eyes move faster, my brain and my eyes move faster than my mouth can talk. So, at times when like you're doing something like that, you could be you could have the right answer the whole time. And you can know the right answer. But when you go back and you're trying to like read it or you're trying to like explain it to another person, you're looking at your paper as a reference or as a guide. You're like I don't know how I got that but I know this is the right answer.

Week 4 - What Faulty Algorithm Story did Icis Relate to Sequences of Transformations?

Icis reminisced about an occasion where there was a kink in a water hose:

> We have a hose. We want to like turn it on. And we couldn't find out why it wasn't coming out of … the hose. There was a kink … in a line. So we're trying to find out. And it was all going good. And then my sister was like, oh well let's unscrew the hose and find out what's … wrong. And because that hose is still there like it would like go everywhere if the water was still running. And the water was still running. And she unscrewed it and the water went everywhere. So it was kind of it was a false faulty algorithm [as she humored].

I asked whether this faulty algorithm involved turning inward:

> Yeah and kind of flipping. Because like with the kink in the hose, we had to like undo it. And there are times like because how are … hoses, you can fold it. It's one of the foldable ones that you can kind of fold up and then put into its little place. And you have to like open, there's like a whole box you gotta open. And then there's a switch you have to flick so you can like take the hose out properly. It's … a whole process.

Icis concurred that the process involved all three transformations of translating, reflecting, and rotating. When asking what could have been done differently, she responded:

> With the hose, … I probably should have gone through and looked at all the other possibilities before just jumping to conclusions. Because when we jump to conclusions, … your mind tends to wander. And then you think of all the bad outcomes instead of maybe this is a good outcome. You don't really think of all the possibilities.

I probed for some of those possibilities:

> Probably could have turned the water off, inspected what was going on, could have like searched up the brand of the hose, and found out if this was normal… There's a lot that you could do. It's … like I could have shoulda woulda. But you kind of just have to stop

> and be like okay, well this is this is what I did choose. So now I'm stuck with it. Now, I'm stuck with wet clothes and it's cold outside so yay [as she humored].

I questioned whether there would be a kink in the hose no matter what she did. Icis perceived the error from an initial negative circumstance to a more positive one:

> Yeah I mean after a while like it was really bad. But then like of course I got really mad at my sister for undoing the hose. I'm like, great, now I'm all wet. But it's like a hot summer day. At first that water when it hits you it's cold. But then it's like, oh it's water, let's play in the water. So it was a bad outcome that kind of turned into a better day.

When asking if the practice of looking at possibilities was present in her literacy:

> Oh yeah.… It's like a lot of people are like you gotta stop and smell the roses…. Like let's say, let's stop and look at how they did it wrong… Now think of all the possibilities of how you could do this wrong or how you could do it right. If you stop and you kind of take your time, eventually you're gonna get there. Like you're … gonna be able to do it correctly.

Icis referred to a miscomputation problem when applying possibilities, particularly with choices between rotations of clockwise and counterclockwise:

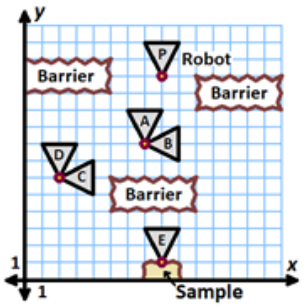
> Yeah, like with one of them, like okay, let's say I took this one [see Figure 49]. And instead … of going clockwise I could have gone counterclockwise. It's just that would have been a bigger turn than this one.… I hate saying this, but like a lot of people take the easy way out. So they're gonna take the easier path. But you could think, okay well if I did this, this … could have been different than it is now. So when going through, you're kind of like you're taking what they did wrong and you're fixing it.

Figure 49

Icis Writing Clockwise From A To B By Considering That Possibility Over Counterclockwise

Robotic Engineer 2 monitors movement of the robot

Transformation	Miscalculation Process Skills	Corrected Calculation Skills
P to A	Translate right 1, down 5	Translate right 1, down 4
A to B	Rotate 90° CW about (8,7)	Rotate 90° CW about (7,8)
B to C	Translate left 6, up 1	Translate left 5 down 2
C to D	Rotate 90° CCW about (2,8)	Rotate 90° about (2,6)
D to E	Translate right 1, down 7	Translat Right 6, down 6



I questioned if her advice would be not to take the easy way out:

> Yeah personally I don't believe in the easy way out. I feel like even if you're really stressed, when life starts to get really hard and it feels like you should just give up, … that's when you start to push…. Because it's like when life throws you that curveball … at that point it's like okay this is the test … I could see it as a curveball and other people could see it as an opportunity. You just you never know.

As a metaphor, Icis extended on her mention of the curveball with a book she read about choices:

> When … I say something like when life throws you that curveball, there used to be a book where like the girl would continue and it was a path in the woods. And every turn she took out like was an outcome and how she was in real life. So she was constantly in her head it was like, you had it was like two characters. But it was one same character you had her conscience and then you had her physical like body. And everything that was going on the outside, like on the outside world. So you were mostly in the book in her conscience. So she would talk about how okay well, if I got on this path, it seems dark and scary but this one's full of light. And a lot of the time instead of her choosing the like really bright path, she would choose the dark path. But it always ended up being a good outcome for her outside. So like I feel like when you get to that that strike that it's like, it's just like at a stop sign or a four-way stop.

Week 5 - What faulty algorithm story did Icis relate to rigid motions?

> Icis shared a story about partaking in a kart racing video came with her family:
>
> I can probably say with the wrong strategy we, me and my family, likes to play Mario Kart [a kart racing video game] a lot. And I was in the lead. And then what I tried to do, I tried to throw like a blue shell [a power-up item of Mario Kart that stops a first place racer on impact]. And it was not a good idea because I was in the lead. So kind of messed it up for myself.

After initially classifying her narrative as a wrong strategy, I questioned whether her narrative was a faulty algorithm where the racing was going smoothly for her with a sudden error, "That might be yeah. Things were going smooth, so that's might be a faulty algorithm." I asked what happened to cause Icis to lose her lead:

> So when you're in the lead, there is something called like a blue shell. And if you throw it, it's gonna go at the person who's in front, so in first place. And it's gonna hit them. It's very … rare to get the game. I was like oh this is my chance. And I threw it. And right as I threw it, I was like oh I'm in first place. That was not a good decision. So I ended up losing. And then we had a big laugh about it later.

When inquiring if the throwing error happened at the end, she responded:

> Yeah it was like, right at the end, I was like a good maybe like a few feet away from the finish line. And I threw it [the blue shell]. I was like yay and then no [as she emphasized the yay and the no with humor].

I inquired about the types of rigid motions that Icis experienced with Mario Kart:

> Yeah I mean, well with rotation, you kind of rotate around the whole track a lot. And then there are times where … there's something like you can use that'll kind of like teleport you in a way. It's like teleportation, like you go like to the beginning, but you're not in first. Like you go to like third if you're in fifth or something. So it's kind of like you know I'm reflecting myself over. I reflected over the track [as she humored].

When asking Icis what could have been done differently to secure her win with Mario Kart: "Not throw that shell [as she laughed]. I would have said no. Probably thought before I acted." I questioned for an alternative:

> You have okay, so in the corner of the thing you have two options. So when you try like it looks like a dominoes thing. When you hit it, you get this and then you have like. I had two things I could have used. I could hit the button to switch over from the blue shell to like the mark the to the mushroom. The mushroom helps you speed up. Could have won than throwing the blue shell and get me after I had already won. [She confirmed.] If I used the mushroom before the blue shell, then I probably would have won.

Chapter 5: Discussion and Implications

This ending chapter describes participants' responses about their stories by summarizing the results and comparing results to current research. This discussion addresses three research questions with tables that portray the themes from participants' stories. For validity and inclusion of all participants, each emerging theme was observed in at least one quotation from each participant. The themes relate to at least one of the standards of mathematical practice (see Figure 50). Theoretical and practical implications are presented along with suggestions to follow-up on the narratives with extended research. This chapter concludes with a summary.

Figure 50

The Standards of Mathematical Practice

1.	Make sense of problems and persevere in solving them.	2.	Reason abstractly and quantitatively.
3.	Construct viable arguments and critique the reasoning of others.	4.	Model with mathematics.
5.	Use appropriate tools strategically.	6.	Attend to precision.
7.	Look for and make use of structure.	8.	Look for and express regularity in repeated reasoning.

Research Question 1 Wrong Strategy Narrative Themes

Summary of Wrong Strategy Narrative Findings

The first research question is, "What stories can students tell about using a wrong strategy in a Geometry error analysis problem?" All participants shared what they thought most related to an error analysis problem involving a wrong strategy. Each participant told wrong strategy stories that involved unique contexts. Athena shared stories of being lost. Lyssa talked about narratives that involved analyzing choices. Xana depicted stories about accidents with tasks or activities. The stories of Icis also focused on decision-making but emphasized the

importance of being a catalyst of change and deviating from norms. Despite the wide range of context, five themes emerged that represented characteristics of the narratives being shared about wrong strategies with subthemes as support for details (see Table 2).

Table 2

Number of Quotations by Themes and Subthemes from Stories Told about Wrong Strategies

Wrong Strategy Theme	Number of Quotations				
	Athena	Lyssa	Xana	Icis	Totals
1.1 - Participants critiqued and reasoned:					
1.1a – Self Reflection	9	23	10	16	58
1.1b – Context	10	5	12	14	41
1.1c – Objects, Others, Literacy	1	6	2	8	17
1.2 - Participants looked at…					
1.2a - Environment	6	2	1	6	15
1.2b - Information	4	5	1	3	13
1.3 - Participants modeled:					
1.3a - Relationship	1	2	4	7	14
1.3b - Content	3	3	1	1	8
1.4 - Participants persevered and showed precision with:					
1.4a - Comprehension	3	12	1	1	17
1.4b - Finding solutions	1	2	2	4	9
1.4c - Organizing	1	4	1	1	7
1.5 - Participants used visual tools:	4	12	3	9	28
Totals	44	80	39	70	233

Theme 1.1. Critiquing and Reasoning

All participants simultaneously critique and reason about several aspects that primarily focus on their performance and their demeanor for managing a wrong strategy in their story and literacy. Assessment exists about the choice of strategy and use of knowledge, emphasizing

making sense. Their narratives also evaluate the presence of objects, other persons, and literacy error analysis problems.

Subtheme 1.1a unfolds that all participants valued analyzing and explaining their rendition and persona of committing a wrong strategy. Each participant felt that assessing their wrong strategy was essential to rediscover and remind who they are and what they can do. Athena stated her liking of reading and listening to herself:

> I think I try to understand in my head before writing things down…. Personally, I'm a person who reads directions and I do them by a fine line. That's how I do things … I listen to what my gut tells me.

Lyssa shared her ways of breaking things down:

> You gotta look at stuff first before you do it…. I break [things] down into a lot of things because I do process by process.

Xana declared her preference of experiencing reality over theory:

> Because you know like something can be … fine in theory. But when you test it and you experience it, it could go completely different from what you thought could happen.

Icis mentioned her need to connect learning inside and outside the classroom:

> I have to connect what I'm learning … with something that I'm doing on the outside that I'm really really interested in … so I can focus at the same time.

Subtheme 1.1b discloses that participants thought deeply about knowledge and chosen strategies in their stories. The thoughts allowed the connection of a Geometry topic to a real-life situation. All participants felt that when a Geometry topic was part of their context, there was an opportunity to describe and relate the topic to real life. Athena talked about her perception of perimeter and area based on the structure of a building. Athena showed a progression to inquire about the acquisition of values for perimeter and area:

> I feel area is going to be larger than perimeter because perimeter is really just the outline of a place ... am I using perimeter for area [or vice-versa]?

Lyssa mentioned her relationship to slope to a golf ball rolling down the hill:

> I feel like slope kind of applies with the hill thing because it's like they're [golf balls] falling down on the sides.

Similar to Athena, Xana discussed her vision of perimeter to kitchen space:

> I think I'm thinking about perimeter. I think it's there was a lot of, there was a lot of space around the everything in the kitchen.

Icis mentioned translations and reflections as applications of a recipe book:

> I probably say it [her recipe book] was more like translations and reflections.… Okay when you're reflecting something, like you're taking it, you're putting it over here [as she moved her hands with her fingertips on a table from left to right as a flip]. When you're using a recipe book, you're taking this recipe and you're putting it into like actual things. You're putting into a physical form translation.

Subtheme 1.1c reveals that discussing attributes about objects, other people, and literacy problems allows participants to discover a subconscious characteristic about their persona. Participants feel they have a platform to express a phenomenon that they should have observed or did observe. Athena mentioned what she should have noticed about the construction of a model pyramid:

> I think I could compare the puzzle pieces and kind of see like how they would look in the pyramid. Because, like by the end of it, the pyramid is always going to look the same. There is always going to be a pyramid. There's not going to be anything funky. And there was something funky about it. And I guess I didn't notice it and so it caused it to fall.

Lyssa evaluated how the presentation of literacy error analysis problems in this study is perceived as a synchronicity of Geometry and literacy rather than two separate entities:

> I think it [the Geometry and math as literacy together] is good because it's … math and we're doing all that stuff right now. So, … I feel like it ties in just fine because I don't want to be too much. Because if it's too much, then it's gonna like, it's two different subjects mixing together and then that's too much.

Xana appraised how assumptions about objects leads to the importance of proper planning:

> There was one time in elementary school where we had this project where we had to build a pyramid out of toothpicks and marshmallows. Me and my team decided we would

> try to determine a ratio of how many toothpicks we should use to marshmallows. And we thought that the amount of toothpicks would be better to use. Like a greater amount of toothpicks would be better to use than marshmallows because they provided structure. But we didn't draw a blueprint or anything. So it ended up failing what my teacher called the earthquake test. Because there were more toothpicks than there were marshmallows. And they were not evenly stacked on top of each other.

Icis expanded how assumptions are subjective:

> They [people in general] kind of look and they assume things, which assuming is never a good choice, ever. And like I personally have assumed some things before. And especially with math, you can't really assume things. Because you can assume one thing, it's like a hypothesis. Your hypothesis could be wrong, it could be right, it could be dead on.

Theme 1.2. Looking

All participants felt that analyzing their environment led to more favorable outcomes to counter the wrong strategies that transpired in their narratives. Participants looked for information that led to a solution to a problem. Objects served minor roles in aiding the search for clues to deal with circumstances in their narratives.

Subtheme 1.2a reveals that participants observed their environment to manage a dilemma. Participants felt that familiarizing themselves with their surroundings eased tension and led them on a path of problem-solving that made sense. Athena shared what she could have done differently to avoid disorientation in a store:

> I think I it would have not gone lost if I would have immediately knew to like look around where I was. And then also eventually look at the signs but also to examine my surroundings and see where I was.

Lyssa depicted what she needs to do to get her golf ball on the green from a pit:

> So I have to kind of like look around and see how far up I have to go to get it out of the hole [pit before the green].

Xana shared why she chose a path to ride her bike:

> I think um because when I was going down that path, I assumed it would be easier. Because it looked easier just because it didn't have all the complicated like steps and like the roots and stuff and it wasn't as crowded.

Icis mentioned her need to connect learning inside and outside the classroom:

> I have to connect what I'm learning … with something that I'm doing on the outside that I'm really really interested in … so I can focus at the same time.

Subtheme 1.2b uncovers that participants searched for information to enhance their comprehension of solving a problem. The information is specific, improving the likelihood of correction. The information also reminds participants of what they consider to be their specialties. Athena confirmed:

> I look at every single, little detail. I examine every single little thing so I understand it [the siutation at hand].

Lyssa declared her quest for familiarity:

> So I was just trying to make this [a literacy problem] look like what I knew.… I think about everything else we've done.

With drawing, Xana implied how correspondence between entities spurred the finding of locations as information:

> When there's like a better reference image that matches a canvas better, I know exactly where to put things and then more.

With literacy, Icis talked about her inspection of correcting an error analysis problem:

> You kind of look at it. You found out what was wrong with it. You looked at the formulas you had and then you kind of position things back to where it needed to be.

Theme 1.3. Modeling

Participants perceived the modeling of a relationship between their decision-making and some entity as an inner confirmation for choosing an appropriate pathway. To grasp the content

through a description of tangible objects, participants applied the modeling of a relationship between content and an object better. Both modeling themes imply the need for participants to take risks to discover outcomes of truth and fallacy based on their decisions and comprehension of content.

Subtheme 1.3a imparts that participants relied on modeling a relationship between their intuition and an object to convince themselves that their pathway is sensible. A hunch spurs the relationship that an approach to responding to a dilemma is legitimate based on the comfort the object provides the participant. Whether the pathway is right or wrong is not a factor. The primary component is the relationship between what the participant models with choices and a trustworthy object. Athena teamed up her intuition with a part of her body, "I have to go with my gut … and so that's what I went with." Lyssa compared comprehension and relevance on her literacy error analysis problems in this study:

> It's [a literacy problem is] a good balance to where it's like it gives you the base of what's going on and then it tells you like it gives you real life situation so it makes more sense.

Xana contrasted theory with experience:

> Because you know like something can be, like from from viewing it, something can be like fine in theory. But when you test it and you experience it, it could go completely different from what you thought could happen. Like … a road could be clear or you could um but you could like go through it and it could like take you way too fast or way too far or way too slow or not far enough.

Icis paired pathways taken in life with random trials:

> Oh yeah … when you're jumping off an airplane, you have a parachute.… You gotta choose your direction, which way your feet is going to go.

Subtheme 1.3b demonstrates that participants depended on modeling a Geometry concept to an object to represent and describe the concept's meaning. The model is on a situation that the participant experienced in real life and aids with comprehending and interpreting the concept.

Athena modeled her understanding of perimeter and area to the outside and inside of a house, respectively:

> I feel like perimeter like outside it can seem smaller. Like say you walk into a house like on the outside it can seem smaller than it actually is. But once you walk inside it's bigger than you think. I feel area is going to be larger than perimeter because perimeter is really just the outline of the place. Like if you're walking around a house and it's so many square feet around, once you walk inside you have to take it apart. Every single bit of the area is like inside of that house and it's a lot bigger than the outside.

Lyssa modeled slope based on her experiences with the concept, "When I think of slope, I think of stairs because that's all I've seen in algebra." Xana shared a few ways to model translations:

> I think … getting from my home to school is kind of like translating. I think a lot of things to me could be compared to translating. I think any time anyone moves, it's kind of like translating somewhere like going from like my desk to the front of the classroom to get like paper or something. Or going on the bus and then going home. I think things like that they're very similar.

Icis thought of rigid motions as one of her favorite hobbies:

> Yeah it's [rigid motions] kind of like skydiving. You have to like kind of, it's like once you do it, you can't really back out. It's like it's at all one end way choice. You start with it, you're starting to learn. Like you go through your basics when you're learning about it. You go through like the jumping off the wall, the jumping into the mat, all that all.

Theme 1.4. Persevering and Showing Precision

When quotations began with the word "and," participants tended to show perseverance and attendance to precision. Comprehension of why a situation was happening was the motivating factor for their determination and rectification. Finding solutions served as an overarching goal. Participants implied a need for help with organizing the tasks while enduring and attempting solutions for their situation.

Subtheme 1.4a reveals that all participants tended to understand a circumstance through persistence and exploration of correction. A particular word was said by each participant that demonstrated their continuous ambition for comprehension. Each word was a subconscious

motivator, encouraging the participant to continue seeking comprehension. Athena used “kept” to deal with her understanding of conflict:

> I was like just looking at it [the same answer for perimeter and area on a test]. And I kept just staring at it. And I kept like looking at both of them.

Lyssa said “reread” to understand a literacy problem:

> I have to reread things a couple of times to like actually get it stuck in my brain. So like, if I don’t reread it a couple times, then I might not completely understand what’s going on.

Xana used “different” to comprehend the experimentation of multiple ways, “And so we tried seven different ways and we still didn’t get it right.” Icis said “back” as a review of her thoughts to better understand a situation, “Let me go back and rethink how I did this.”

Subtheme 1.4b divulges that all participants have in hindsight, that a solution exists for their situation. There is a reliance on the feeling of each participant that the appearance of the outcome of their circumstance needs to look correct or better. Athena was devoted to finding a solution with finding her friend:

> All right, I step out for a minute and I come back in and she is nowhere to be seen in the gym so I run down the bleachers. I’m on the dance floor. I’m looking around and when it comes to like different points, I’m really thinking of like the four different corners of the room because there’s the two corners that are really just used for seating, the one that’s used for dancing, and there’s the photo booth. And so I felt like I was kind of going from point to point just looking for her.

Lyssa was persistent for a match with her records as a basketball manager and the scoreboard:

> So it all goes into the book and then at the end of the like at the end of the game you have to add everything up together and there’s been like more times than I can count that I’ve messed up and not added all right. And then I look at the scoreboard and I’m like uh-oh where did I mess up? Or I have to go back and recount everything.

Although an outcome did not work out in her favor, Xana was adamant about a finding a solution for driving appropriately in her driveway:

> There was a time where my dad was trying to teach me how to drive. And he let me use his truck. And we were in … my parents driveway. And he was trying to teach me how to back out of a driveway. And it wasn't the best space to do it because there was a lot of stuff surrounding the car. And there were other cars in the driveway as well. So when I tried to back out of the driveway, I didn't know how to properly turn the wheel. So I ended up turning like a little too far and I backed into a trash can the first time. And the second time when I tried to move back into the driveway, I kind of pushed the car into the other cars in the driveway.

Icis talked about her persistence for finding a solution to organizing her room:

> And I have put some stuff in the wrong area and it ended up looking really cluttered. But then when I fixed it and I moved things back around it looked a lot more open and had more room.

Subtheme 1.4c shows that participants need assistance from the organization when progressing through procedures for understanding and finding a solution to their circumstances. There is a description of multiple tasks occurring at the same time and at random that portrays overwhelmingness. Each participant indirectly hinted for intervention and guidance for managing the tasks. Athena implied a persistent need to organize pieces of a puzzle:

> When I was gonna be like fifth grade I got a gift for Christmas. And it was this little puzzle game and like you had to basically take all the pieces out and it would give you a little diagram. And then it would have like some pieces placed but then you had to fill in the rest of the pieces. And so I was on the hard levels which is basically where they're not flat anymore but you build them into a pyramid. And so I um was stacking them and I was like shifting things around to basically like doing transformations. Like trying to figure out what goes where and how to position each one. And I ended up placing one wrong puzzle piece and the entire thing fell.

Lyssa described her perseverance for keeping track of a multitude of tasks as a basketball manager:

> So … I manage boys basketball right. So when you have … every shot that they make, I have to count down. And I have to write down and they have their own like a little section where I write it down. But there's when there's a lot of people on the court, you can't like really see what's going on and you kind of have to just look at their feet. So there's … a three point so everything outside of that one line is all three points and then everything inside and then there's the free throw line which is only one point.

Xana discussed a need to organize as an "even distribution" of marshmallows and toothpicks to build a model pyramid:

> We didn't consider the amount of the mass between the toothpicks and the marshmallows. We didn't try to evenly distribute them.

Icis mentioned cooking and responding to literacy error analysis problems as a random set of experiments as procedures to organize:

> I see it [cooking] as something that I kind of like you can experiment with. You can throw maybe a pepper in here or um you know an onion and a piece of bread or like. I like to like mix things up and see how the outcome would come out. Because you never know what's actually going to taste good. So like, whenever, even when I'm doing literacy, it's like okay well what if I did this? well what if I threw this in here? oh that don't work. Okay let me take that out. It's just like when you're cooking you can't take anything out.

Theme 1.5. Using Tools

Participants used secondary reinforcer tools like signs, highlighters, manipulatives, and technology apps to help visualize sense-making to solve a problem and retrieve back to normalcy and comfort in their lives. The tools provided information and clues that helped all participants devise a solution. The tools served as a reliever for stress that emanates when participants encountered the wrong strategy committed. The tools helped participants gather together, knowing a pathway exists for resolving their dilemma. The tools also come in different tangible or doable forms corresponding to the participant's liking. The one commonality between all the tools is visualization. Athena stated how she first feels when lost in a store. She shared what she should have done with palpable tools to get herself out of this situation should it happen again:

> I didn't know which direction to go … I remember really getting lost.… So I think I would have immediately looked up at the signs and read what they said and then I would have went from there.

Lyssa talked about a feeling of confusion when too much information was presented to her either in a situation or literacy problem. She mentioned an achievable tool to overcome the immensity:

> If everything's all messy, then … it doesn't make sense to me at all.… It won't like click in my brain.… I specifically highlight every different section.… I underline stuff to make sure I know what's going on.

Xana expressed disappointment when a model construction falters and implies how an obtainable tool could have been used to avoid the collapse:

> We thought that … a greater amount of toothpicks would be better to use than marshmallows. But we didn't draw a blueprint or anything. So it ended up failing what my teacher called the earthquake test.

Icis conveyed a feeling of surprise to a reliance on technology and talked about how to modify the technology as a visual tool for other outcomes:

> Yeah it [Sheila the GPS] kind of was like oh well this [route is] faster.… Whenever like you pull up instructions on your phone, the map … gives you all these alternate routes. Like, oh well, this one's like five minutes shorter or this is five minutes longer. Like you can pick one of those but it was saying that this one was like two to three hours like shorter.

Summary of Wrong Strategy Narratives to Current Research

When managing a wrong strategy, this study's findings relate to how prior mathematicians and participants critiqued and reasoned about their persona, knowledge, and stimuli for realistic situations. As Lakatos (1976) and Borasi (1996) implied, the mathematicians de Jonquières and Möbius demonstrated their talents, challenged norms, and revealed sense-making for correcting wrong strategies based on their experiential situations. Similarly, each participant in this study critiqued and reasoned by challenging their mindsets to go beyond what they believed was true to what they need to believe for reality, specifically in their narratives (Polkinghorne, 1988; Clandinin & Connelly, 2000). Similarly, participants showcased their attributes for analyzing and explaining, cogitated corrections beyond the norm, and revealed

sense-making attributes about themselves through their stories (Polkinghorne; Clandinin & Connelly). Loibl and Leuders (2019) supported the findings with participants in this study encountering a wrong strategy as a non-example and contemplating a strategy that works better as an example. This finding is consistent with Riessman (2008), who suggested the revelation of students' identities through narrative comprehension. Moreover, this study parallels Riessman by connecting their persona of critiquing and reasoning a real-life narrative to solving a wrong strategy, error analysis Geometry problem.

This study demonstrates how participants parallel mathematicians with understanding a situation by looking for observations and information. Usiskin (2008) implied how modern mathematicians rely on dynamic geometry software for their observations and elicited information for comprehension. Participants in this study acquire comprehension on a different channel: what pictures and words participants see in their surroundings as information (Risastuti et al., 2017). Consistent with Mertova and Webster (2020), each participant's story stresses looking at the environment and information to figure out how to untangle themselves from a dilemma. This study supports Große and Renkl (2007) with the presence of conflict and reflection from each participant to evaluate why a chosen pathway does not work through observations and how another strategy potentially works better.

Through prior experiences, this study shows how participants model content context in their narratives and error analysis problems for conviction and sense-making. This finding is consistent with how mathematicians model their prior knowledge to explain mathematics in various contexts (Draper, 2002; Ratnaningsih & Hidayat, 2020). In contrast to the studies of Sfard (2007) and Hillman (2014), this study shows how participants describe representations of content in their narratives through a unique language and prior knowledge developed from their

persona and experiences rather than interaction with family members and peers. The study shows models in the dominant form of similes, infused with the word “like.” The similes contrast the works of Lakoff and Johnson (1980), who supported metaphors for comprehension. However, the emphasis on the relationship in this study between each participant’s thoughts and an object as a model supports the purpose of metaphors used for understanding content (Lakoff & Johnson).

Parallel to Clandinin and Connelly (2000), the findings show a story in each participant’s life that involves perseverance and precision in knowing and finding an existing solution to solve a problem. Overall, Coles (1989) supported the findings with a Geometry teacher, as a researcher, understanding the organization of a general process with how students persevere and attend to precision with correcting wrong strategies. More specifically, Bateson (1994) supported the findings with a Geometry teacher acquiring how students uniquely organize their thoughts. Moreover, the study demonstrates how teachers can gain insight into providing spontaneous, natural interventions to maintain or improve the organization of a participant’s thoughts (Schön, 1983; Bateson). Geertz (1995) supported the findings with teachers discovering the subconscious ways students blend comprehension and personal interpretation of facts.

Concerning the use of tools, Clandinin and Connelly (2000) supported the findings with each participant having a story that applies visual clues and information for devising solutions as endeavors to solve a problem. Polkinghorne (1988) also upheld the findings with participants using tools to undergo temporal connections from a state of disarray back to normalcy when responding to a wrong strategy. As championed by Mertova and Webster (2020), the findings reveal how tools for comprehension accentuate a pathway of sense-making as participants convey visually in their narratives. As Lakoff and Johnson (1980) suggested, the tools applied by

participants in this study are representative or symbolic of abstract states of mind: mindsets of making sense and returning to typical comfort zones. In addition, prior research supports how the tools used by participants as visual reinforcers serve as valuable information for Geometry teachers to devise meaningful interventions to improve literacy via error analysis (Schön, 1983; Coles, 1989; Bateson, 1994).

Research Question 2 Miscomputation Narrative Themes

Summary of Miscomputation Narrative Findings

The second research question is, "What stories can students tell about using a miscalculation within a Geometry error analysis problem?" Participants unpacked what they pondered most associated with an error analysis problem involving a miscalculation. Participants shared miscomputation stories that involved distinctive situations. Athena shared stories that entailed an extracurricular activity or event that either she engaged in or observed from someone close to her. Lyssa recounted stories that included an experiment or task. Xana illustrated stories that considered alternatives with decision-making over the performance of one instinctive action. Icis's stories focused on anomalies. Notwithstanding the contrasts of circumstances, five themes emerged that epitomize attributes of the miscomputation narratives with subthemes as a foundation (see Table 3).

Table 3

Number of Quotations by Themes and Subthemes from Stories Told About Miscomputations

Miscomputation Strategy Theme	Number of Quotations				
	Athena	Lyssa	Xana	Icis	Totals
2.1 - Participants critiqued and reasoned about:					
2.1a – Corrections	4	2	5	7	18
2.1b – Explanations	4	2	5	6	17
2.1c - Generalizations	4	5	2	3	14
2.1d – Self Performance	4	3	3	2	12
2.2 - Participants looked at…					
2.2a - Details	2	4	1	11	18
2.2b - Perspectives	2	2	1	8	13
2.3 - Participants modeled context to content	3	2	4	10	19
2.4 - Participants persevered and showed precision with:					
2.4a - Collaboration	6	14	7	6	33
2.4b - Task Focus	3	4	10	5	22
2.4c - Outcome Mindset	1	1	9	2	13
2.5 - Participants used retention tools:	5	8	6	3	22
Totals	38	47	53	63	201

Theme 2.1. Critiquing and Reasoning

Participants analyzed and cogitated about the miscomputations that occurred in their narratives. Explanations are extensive about the context of the narratives where miscomputations emanate. There are generalizations that participants make that involve why the miscomputations occur. There are also discussions of self-based on the deterrence and management of miscomputations.

Subtheme 2.1a finds that participants preferred to analyze and explain the correction of a miscomputation for a chance at a better outcome. Participants felt that assessing the process and

providing suggestions were essential before the computation occurs. Athena talked about how to avoid miscomputations of formulas like distance, midpoint, and slope by decomposition:

> I feel it [a formula] is something you have to break down because especially with a lot of long formulas that have a lot of steps to them. Immediately looking at them, you can immediately get a sense of like over overwhelmed. Like just because there's so many steps. But like once you break it down and focus on every single little thing, it becomes so much easier.

Lyssa shared a prevention of miscomputations by studying the comparison of what was being asked about a topic like rigid motions versus what she was writing about the topic:

> So like, if I had looked at the whole thing first and then like been like oh yeah this literally doesn't make sense. Then … the sentence would have made sense instead of me messing up the entire thing [so fully look at the sentence first] and trying to like make it make sense in my brain.

Xana discussed the avoidance of miscomputation by considering a sense of estimation:

> I jumped off when it was swinging forward, which propelled me past the center. So I think maybe I should have waited till I swung back and then jumped off into the center. Because if I was farther back, I could propel myself more accurately into the center rather than waiting for me to pass it and then jump off.

Icis mentioned collaborative reviews of information to hinder miscomputation:

> So one of the things I feel like when they [marine biologists] calculated it [the time a shark would remain at a cetain depth of the ocean], they [marine biologists] probably could have like recheck their math, like gone over it, double check things, … have a second opinion… I feel like if the… marine biologist would have gone over what they were like doing with their math. Like would have rechecked what they were looking. Like double check it. Like hey will you make sure my math is right. It probably could have been avoided.

Subtheme 2.1b suggests that participants favored explaining the details of the pathway of a miscomputation in ways they understand it. Participants felt that an explanation was an historical account of how the miscomputation originates and led to its occurrence. Athena explained how an overwhelming amount of plays in volleyball led to an errant pass on her part during a game:

> So I played volleyball for about one and a half years. And so in volleyball there's a lot of, it's not as simple other sports. There are there's different moves, you have settings, you have spiking, you have passing … like once you touch the ball you can hit it again.… So one time, I was … back middle and there was the ball heading straight for me.… It should have been an easy set but instead I passed it. And it ended up causing the ball to not go as high as it needed to. And so it hit the ground which caused us to lose the point.

Lyssa expounded about how a miscomputation of a gun's starting location led to an aberrant outcome of a simulation activity in her law class:

> We had a gun and there was two chairs.… We started at what we thought was zero but it wasn't that. So we messed that up. So like that was like the distance of the outside which messed our whole thing up. Because we watched a video that said we sort of started at 10. And then just like, so like, if we started at 10, like the whole thing was 65. Then minus 10 from 65 would never get the whole thing.

Xana talked about specifics of doing more with the tools of a video game to rotate a building:

> I couldn't properly figure out how to how to rotate the building exactly. I didn't … use the right materials to try to rotate the building.… Well without the the levers and the pistols, it's difficult to complete the task. Like actually follow through with it. Like you don't necessarily need it but it helps out a lot more. And it, you know, it's kind of, it kind of pushes you to what you need to do.

Icis explained the details of a miscomputation of time of a shark's depth in the ocean:

> Whenever they [the marine biologists of the documentary] … have … a tracker on her [a shark's] fin. And they monitor how long she's in the deep.… And they had estimated that she would be down … for like five to six weeks…. And they had messed up with their counting. Like … the marine biologists had done their math wrong somewhere.

Subtheme 2.1c reveals that participants generalized why miscomputations occur based on their experiences. The generalizations helped participants cope with the occurrence of a miscomputation. The generalizations showed that miscomputations were bound to happen due to contrasting ways of managing prior knowledge constructed from experiences. For Athena, memorization had a role in causing and avoiding miscomputations:

> I feel like [memorization to avoid miscomputation] is different with certain things. Like if my mom asks me to … remember [a code].… I'll remember it for that time but then once it's once it's already finished …, I'll basically just kind of like throw it out of my brain I

> guess. But when it comes to stuff like math and reading and school and stuff that I might need for my future, I try and remember most of it as much as I can.

Lyssa talked about how a closeness between perimeter and area caused her to miscompute area for perimeter and vice-versa:

> Since area and perimeter are literally like related to each other, that's when it flip flops.… Yeah because they're [perimeter and area] like, they're so, they're in like the same section of my mind, so they're super similar to me.

For Xana, counting intervals for perimeter and counting squares for area was preferred over tools like a tape measure and applying patterns due to feasibility on her part:

> I think it's easier for me to do [measuring] by counting [over a tape measure] because I don't confuse myself by like looking at all the different lines and numbers and stuff. It was a little hard [to see the doubling pattern in the literacy]. I had to like think about it because I didn't think it at first. I didn't really notice it. But when I thought about it more, I definitely noticed it later.

Icis viewed miscomputations as inevitable but turns to a proactive approach:

> Yeah, I mean like as a human because … everybody makes mistakes. It's kind of inevitable, like it's gonna happen. Even if you can, if you try, and make sure it doesn't happen.… It could happen right under your nose and you'll never notice it. And I feel like people forget that because it's like okay, nobody's perfect but everybody wants to be…. As a student, I know everybody learns differently. And when you're trying to help somebody else explain it, you have to … take yourself and kind of look at how they're looking through. Like the ... saying that's like you have to put yourself in their shoes. At times, you have to like you have to look at it like as if they're looking at you. Like okay, I see this as a maze. Maybe they see this … as words.

Subtheme 2.1d implies that participants assess their self-performances to determine why a miscomputation occurs and minimize the chances of the miscomputation occurring again. The self-assessments help participants realize that the miscomputation can occur again. The self-assessments provide participants with a foundational knowledge of managing the potential reoccurrence of the miscomputation. For Athena, line placement with face painting required attention to care:

> Oh I was face painting … the one thing about it was that I did the one … design on one kid and then I messed up the next time with the other kid. I was like, I did it [the line placement] right the first time, then I messed up the next time.… I feel like because I did it the first time I got maybe a little bit careless. Because I did it the first time and it was fine but then the second time around, I messed up. So it was like oh that's that's not right.

For Lyssa, success with literacy error analysis assignments required understanding all information, including information that appears trivial:

> If I think too fast, … it seems so easy in my brain that I already know what's happening so I just take it as it is. And I don't really like take it back and take every single step as important. I'm just like oh okay well I already know what's going on. So it makes sense but then I mess up on little things.

For Xana, conducting transformations properly involved using wax paper over gandering or glancing at where images would be from pre-images:

> I think the wax paper really helps me look at it [a transformation].… I'm kind of a visual learner. And seeing … what is actually happening and how it's turning and rotating is helpful for me. Because it's kind of hard for me to like just eyeball it [a transformation] I guess.

Like Xana, Icis felt that her visual talents allowed her to understand any topic of Geometry that related to a phenomenal way she looks at puzzles and mazes:

> Yeah like I can see. It's like I've been told I'm kind of a visual learner. So, I'm really really good at puzzles or mazes. So I can look at a maze and I can tell you how to get from one side to the other side without running into a dead end or something. It's kind of hard to explain.

Theme 2.2. Looking

Observing for details and considering perspectives of a situation allows a better chance of preventing miscomputations. The inspection provides conciseness and focuses on an overall summary of what transpires between relevant information in a narrative. The inspection aims to blur out distractors. The perspectives serve as alternatives for finding the best approach for a succinct and sensible response.

Subtheme 2.2a shows that all participants focused on a relationship between themselves and relevant information. Athena talked about better finding a sequence of transformations between two shapes when focusing only on those rather than all shapes:

> It [inspecting just the pair of shapes] really does help because it's just like you're focused on that one thing [relationship] and you can get the answer easier than having to look at the whole image [entire set of shapes].

Lyssa emphasized a relationship between details of the process used in her law simulation activity. She craved a bigger picture of understanding a process by focusing on what the procedures do together rather than individual, separated parts:

> If we had took more time and like processed what was happening and then done everything instead of just going like step by step. Like plan the steps out ahead of time instead of going, like, okay, we do this and then oh no what do we do next? Oh, we do this next.… But if we had like written it down or like even just in our minds, we're like okay so we're gonna do this and then do this.

Xana shared about better applying the distance, slope, and midpoint formulas when she focused only on a relationship between corresponding coordinates:

> When you look at … the image with the coordinates and everything, it helps you better to like know what you're going to place and where [referring to how to write coordinates within a formula].

To correctly count how many geometric parts exist in an object, Icis stressed the relationship of relevant, geometric parts that compose the object:

> It's like when you look … at a brick, usually bricks are going to be a rectangle. So, if you look at them and all four corners, those are your points and then … the lines connecting them are your lines.

Subtheme 2.2b unfolds that participants contemplate looking at perspectives to either prevent or substitute a miscomputation or misestimation. Athena avoided a double count of segments by perceiving them as the same path that can travel in different directions:

> I feel like I look at it [a segment] both ways [from one point to another]. It depends on how … the story that it goes along with it or like what you immediately think of that goes

> with it. I think of it as the same way because it's a really the same path because either way going in or out it's still the same like basically it's the same direction you're going opposite or forward.

Lyssa discovered how reading a sentence in its entirety was beneficial before completing an assignment that depended on the interpretation of that sentence:

> I think the thing I messed up on was I just didn't fully look at like the sentence. I was writing first and I didn't look at the pronoun beforehand.… So like, if I had looked at the whole thing first and then like been like oh yeah this literally doesn't make sense. Then … the sentence would have made sense instead of me messing up the entire thing [so fully look at the sentence first] and trying to like make it make sense in my brain.

Xana prevented a misinterpretation of transformation sequences by realizing how wax paper served her better for seeing rotations:

> I think the wax paper really helps me look at it. Seeing what is actually happening and how it's turning and rotating is helpful for me.

Icis considered the perspective of number sense, particularly with how distance between bases of baseball diamond should be 90 feet rather than ¾ inches:

> So it was kind of like once like if you look at their math, like they messed up. They used the wrong formula. They ended up doing the wrong strategy when they, uh, were calculating it. So when you go back and use the correct strategy, find out what strategy you need to use and you use it, you end up … getting 90 feet instead of … three-fourths inches.

Theme 2.3. Modeling Context to Content

All participants resembled the context of their narratives to the content learned in Geometry. The resemblance connects their reality of experiences to a visualization of a topic in Geometry. The quotations from each participant portrays how the similitude livens a topic of Geometry, particularly as some dynamic phenomenon that is moving and changing. Athena invigorated an image of points and lines moving in her soccer game:

> And the one thing about sports is that everyone has their own area.… I kind of see all the players as points and whenever you're running with soccer it's kind of like a line because

> you're getting from here to here because you're trying to get across the field to go to the other goal.

Lyssa implied the way she moves her eyes to visualize congruencies in a pair of triangles models the same way she moved her eyes to visualize relevant information in a sentence:

> I think the thing I messed up on was I just didn't fully look at … the sentence [of her Spanish assignment]. I was writing first and I didn't look at the pronoun beforehand. Kind of like if you don't look at the [congruence of] angles [of a pair of triangles] enough, then it [the triangle congruence criteria] looks like it could be side angle side when it's really HL [hypotenuse-leg].

Xana modeled herself and her cousin as two points moving like ants on various paths as line segments:

> I could compare [us her and her cousin] to … ants in a colony. Like … we were the ants … location was the colony. Like … ants use a specific trail and follow each other to get to one point to another, that's how we were trying to get from one point to another… I think it [a segment] can [move] in both ways. I think it could be like back and forth, or up and down… traveling from one place to another.

Icis vitalized an outline of perimeter and an inside of area as motion on board games:

> If you think about it, it [perimeter and area] could be like … tic-tac-toe. You have to stay within this grade. You can't like do it on the lines. You can't do it outside of it, it's inside. It's like Hopscotch. You don't want to step outside of it.

Theme 2.4. Persevering and Showing Precision

Participants persevere and show precision through collaborating, focusing on the task, and developing a mindset for the outcome. Collaboration is an exchange of working with someone planned, mainly with a family member, friends, or classmates. Concentration on the task involves continuous attempts of completion with positive intentions. The attitudes associated with correcting outcomes with miscomputations depend upon the importance related to the participant's experiences.

Subtheme 2.4a reveals that participants felt obliged to work with someone to overcome the presence of a miscomputation during a situation. Participants underwent accountability with their input for resolution and negotiation for compromise. Participants agreed upon a level of wrongness for the miscomputation that leads to a decision on how to correct the miscomputation based on that level. A considerable amount of time occurred, with each participant having a unique point of view about the purpose of collaboration. Athena talked about the eventual collaboration with her brother about a garden needing to be more significant:

> We were building the base of the garden and it turns out we didn't make it big enough. We had bought too many plants to fit in there and so everything was on top of each other and things were overgrowing other things. And so it was just this huge mess. And so eventually we have decided … we're going to expand it to make it thicker because well too many plants and then we also want to add more plants.

Lyssa implied a redoing of collaboration with her team about the start of a measurement:

> And what we did is, you know how on the tape measure it starts at zero but like it has the things that sometimes cover it? So yesterday we started at what we thought was zero but it wasn't that. So we messed that up. So like that was like the distance of the outside which messed our whole thing up. Because we watched a video that said we sort of started at 10 … if we started at 10, like the whole thing was 65. Then minus 10 from 65 would never get the whole thing.… So we had to go over and redo all of that yesterday.

Xana implied that collaboration is continuous, particularly when guesses result in miscomputations:

> So we used the wrong amounts of flour and … sugar. So they [the lemon squares] turned out watery and they weren't … fully cooked. And the proportions were wrong and everything. And so every time that we tried to cook it and we'd retry, we just guessed on the amount that we would need for… each ingredient. So it would always end up being wrong or undercooked or things like that.

Icis insinuated that collaboration is *efficient* and *equitable* when managing miscomputations:

> Me and my little sister were doing some chores. And we have a system where we kind of split everything up. And we usually do it equally. But, by the end, like we noticed that I had done more, she had done more.… And I was like … okay … well if I do this Wednesdays and you do this Tuesdays, we were able to kind of go back and separate … which ones we had to do. Yeah we had to balance it out.

Subtheme 2.4b imparts that participants progressed through a task with trials of tenacity and ambition for precision. Participants thought of these trials as a self-competition and challenge for success. Participants were inquisitive to know if a favorable task completion was attainable, mainly when miscomputations occur. Athena displayed determination with her neighborhood familiarization and face-painting line placement:

> I have like just now like been like trying to learn the roots of like the neighborhood and how it's mapped out. It's like you're trying to keep the straight line straight.

Lyssa was persistent with positioning the gun of her law simulation activity in the correct place:

> So the gun was kind of like it was the middle … and you had to try and triangulate it so you had to like go from the corner.

Xana was assiduous with her landing from a swing as her midpoint and a search for a gas station as her sequence of transformations:

> I tried to land in the center but I ended up going way farther out. I tried to guess how far the nearest gas station would be by using what I knew and how to get to the places around the ones I knew.

Icis was sedulous in her quest to redo steps in a square:

> As a kid … you want to walk… on the ground… in the square… I was playing with my sister and then I kind of took the wrong step and I didn't notice it. And then I got like off … I was a little too far behind and I had to redo it.

Subtheme 2.4c demonstrates that participants strategized an outcome for preventing miscomputations based on a level of relevance. The level depended upon the mentality of the participant for dealing with the miscomputation. Athena memorized because she felt avoiding miscomputations had an impact on her future:

> When it comes to stuff like math and reading and school and stuff that I might need for my future, I try and remember most of it as much as I can. Like I can still basically tell you the plot of what I read for lit last semester. I feel like I sort of used memorization for error analysis. Because well along with just looking at it, you kind of have to also like see the errors, remember what went wrong with them until you can like look at it and remember and go like fix it as you're going along.

There was a sense of urgency for Lyssa to have correct computations to avoid an overall errant outcome that followed a domino effect of getting everything wrong because of one miscomputation:

> You have to like, every process has to be right. Because, like if I mess up, … then the rest of them are going to be wrong. Everything following suit is gonna be wrong. Like if I mess up the first one, everything after that is going to be wrong. So you have to like make it [sensible], make sure.

Xana believed in devising sensible connections with her procedures to reaching an outcome without miscomputations:

> Like just like how like ants use a specific trail and follow each other to get to one point to another, that's how we were trying to get from one point to another.

Icis upheld a carefree, predestined expectation for dealing with an outcome:

> It's kind of inevitable like it's gonna happen. Even if you can, if you try, and make sure it doesn't happen. It's gonna happen and it could happen right under your nose and you'll never notice it.

Theme 2.5. Using Tools of Retention

Participants value the maintenance of information to deal with miscomputations using a tool for retention. These tools vary amongst participants and serve as habitual responses learned and developed over time and experience. Athena talked about memory for recalling information she needs:

> I feel like in a lot of subjects memorization is a huge thing. It's something that is a big skill you need like to learn and to further yourself.

Lyssa emphasized *writing* for retaining information, "I just have to write down and make sure I know everything that's happening." Xana applied *tactile tools* for acquiring information that she knows to exist:

> I think the wax paper really helps me look at it [transformations between two shapes]. There are tools like pistols and … switches and stuff [from the video game Minecraft].

> And there's also something called redstone. So, if you needed to push something or move something in a certain direction, you could use pistols to like shove it one way. And you could put another object in front of it so whatever was in front of the pistol would move whenever you flipped a lever. I'd have to measure everything and try to calculate what could go where. I think either like a tape measure or whatever measure.

Icis needed comparative tools to preserve knowledge, mainly with a miscomputation versus a corrected computation:

> Well now I know what it looks like and I know how to correct it at the same time. So it's kind of like you're putting us, it's like a side by side, you're looking at, you're like okay this is the wrong way but this is the right way. So now, if I make this mistake and I see like it looks like this or I have a similar problem, I'm like okay, I know what this is now. I know how to fix it. Like would have rechecked what they [marine biologists] were looking. Like double check it [their calculations]. Like hey will you make sure my math is right.

Summary of Miscomputation Narratives to Current Research

When correcting a miscomputation, this study's findings associate with how prior mathematicians and participants critiqued and reasoned about their corrections, explanations, generalizations, and self-performance. Like mathematics predecessors, each participant in this study critiqued and reasoned by transgressing from perceptions to reality by assessing the miscomputation (Lakatos, 1976; Polkinghorne, 1988; Borasi, 1996; Clandinin & Connelly, 2000). The findings support prior research of how the perceptions, formed by prior knowledge within the narratives of participants, reshape to actualities when deriving a correction to a miscomputation (Große & Renkl, 2007; Durkin & Rittle-Johnson, 2012; Loibl & Rummel, 2014; Loibl & Leuders, 2019; Barbieri and Booth, 2020). Both de Jonquières and Möbius explained a correction, demonstrated their comprehension, shared their experience, and showed why a miscomputation occurs based on a non-example and example approach (Borasi; Kawasaki, 2010; Loibl & Leuders, 2019).

Similarly, this study shows how participants unpack a correction, exhibit personal understanding, reveal their experience, and unfold why a miscomputation transpires through their narratives (Polkinghorne, 1988; Clandinin & Connelly, 2000). The findings show how the miscomputation of each participant's story portrays the non-example and how an anticipated correction serves as an example (Loibl & Leuders, 2019). This finding is consistent with Riessman (2008), who suggested how participants discover a persona of coping with the miscomputation occurring again.

This study exhibits how participants align with mathematicians by looking for relationships between information and looking at their perspectives for dealing with a miscomputation. Adhering to Mertova and Webster (2020), each participant's story parallels the conjectures of mathematicians by mentioning a relationship and considering perspectives that stakeholders prefer to see initially followed by ones that correspond with actuality that can cause debate amongst stakeholders. This finding is consistent with Geertz (1995), who suggested how the blend of process with miscomputation offers incentive as to why the miscomputation occurs based on what stakeholders want to see by instinct versus what should be seen by reality. Usiskin (2008) implied that modern mathematicians think more efficiently by focusing on relationships with quadrilaterals and replacing prior knowledge with their perspectives acquired from a dynamic geometry environment. Similarly, participants in this study think more productively by looking at a functional, goal-oriented relationship between the steps of a process and looking for a replacement for a miscomputation to improve the relationship (Risastuti et al., 2017). This study upholds Große and Renkl (2007) with conflict and reflection on how participants look for comparisons and contrasts with phenomena in their narratives.

This study shows how participants model content context for meaningful livelihood in their narratives and error analysis problems. The modeling for bringing the content alive supports Geertz (1995), where participants relate an appealing attribute of a phenomenon to Geometry content by instinct. The findings mirror Clandinin and Connelly (2000), with participants in this study either modeling content based on their narrative or on prior experiences. In contrast to Lakoff and Johnson (1980), most models represent similes where participants model content indirectly to some phenomenon triggered by the word "like."

The findings of this study demonstrate how participants felt obligation for collaboration, focused on trials of precision, and chose what they felt is relevant. Similar to Borasi (1987), a "degree of wrongness" depicts in the narratives, where participants and others agreed about an errant outcome resulting from a miscomputation with the context of the narrative but not necessarily other contexts. The findings support Polkinghorne (1988), where occurrences at different junctures of a task are compared to pinpoint when a miscomputation occurs. In turn, the findings uphold Reissman (2008) as participants in this study uncover their attributes for rendering collaboration, precision, and relevance levels for managing miscomputations.

The findings support Schön (1983), who implied how the use of retention tools by participants in this study explores ways of thinking that are valuable to the Geometry teacher for designing lessons that either reinforce or go beyond the tools. The findings of this study show how participants merit retention as habitual tools to recall prior knowledge for managing miscomputations. Consistent with Loibl and Leuders (2019), participants were aware of their retention tools and applied them by comparing an incorrect process containing a miscalculation to a correct process. The findings support prior research of participants relating the correction of a miscalculated process in their narrative to one in a Geometry error analysis problem (Coles,

1989; Bateson, 1994). The findings uphold Ingram et al. (2015), with participants desiring to apply a retention tool beneficially in situations containing a miscomputation.

Research Question 3 Faulty Algorithm Narrative Themes

Summary of Faulty Algorithm Narrative Findings

The third research question is, "What stories can students tell about using a faulty algorithm in a Geometry error analysis problem?" Participants informed narratives they thought most related to an error analysis problem encompassing a faulty algorithm. Participants shared faulty algorithm stories that involved noteworthy circumstances. Athena recounted stories that entailed alternatives or replacements. Lyssa shared narratives about tasks that needed better measurements for better outcomes. Xana described stories of family activities that transpired either inside or outside the home. Icis shared stories that required more focus and concentration on her part. Three themes emerged that embodied faulty algorithm narratives with subthemes as descriptors (see Table 4).

Table 4

Number of Quotations by Themes and Subthemes from Stories Told About Faulty Algorithms

Faulty Algorithm Theme	Number of Quotations				
	Athena	Lyssa	Xana	Icis	Totals
3.1 - Participants critiqued and reasoned about:					
3.1a - Alternatives	4	5	11	7	27
3.1b - Familiarity	8	6	4	9	27
3.1c - Limitations	2	5	3	5	15
3.1d - Visualization	3	7	2	3	15
3.2 - Participants persevered and showed precision with:					
3.2a - Reflection	5	16	3	16	40
3.2b - Obligation	9	12	1	12	34
3.2c - Measures	6	13	7	3	29
3.3 - Participants looked at …					
3.3a - Specifics	10	6	5	9	30
3.3b - General	6	6	4	5	21
Totals	53	76	40	69	238

Theme 3.1. Critiquing and Reasoning

All participants considered alternative solutions to situations in their narratives as support for acquiring better outcomes. Participants evaluated and thought about the faulty algorithms within their narratives. There existed familiarity with prior experiences that depicted the faultiness of either the narrative or Geometry content related to the narrative. Participants referenced constraints and visual observations that considered either the presence or prevention of faulty algorithms.

Subtheme 3.1a shows that participants preferred to think of alternatives ahead of time after a faulty algorithm occurred to deter hasty, spontaneous decisions. Participants felt the alternatives needed consideration for an amount of time before the task began. Athena talked

about precautionary measures to avoid faultiness with the baking of a cake and helping her friend make a proper quiz with an answer key:

> I feel like I probably would have asked one of my parents to go to the store and buy eggs. Yes but we didn't … we weren't able to do that because we were making cakes like very late at the night. I feel like I probably should have like warned her because like [the teacher] didn't really like specifically tell us. Like I had to ask her myself to make an answer, if I need to make an answer key. She [the teacher] didn't like announce it. So I was like I probably should have just told her before like she had started that. But I forgot to.

Lyssa referred to dividing a task into sections to avoid faultiness:

> I section it out into different … steps. So … first you read.… So like if I were to do this again, I would read this first and then look at this. And then so I would go through count all of these and then look at this. And be like okay where did they mess up here and then fix what they messed up on and then write it all down. So like writing it all down is like the final part. You have to like break it into like little parts so you can roll in one place to another place or like if you're going multiple places at once.

Xana preferred to take into account precautionary actions and limitations before conductance of a task after committing faulty algorithms in her narratives:

> I think we should have budgeted for everything that we had, you know, like the food. I think we definitely should have tried to evenly distribute food and things like that. I think we could have tried to … use, like turn it a different direction that wasn't like up or down. I think we we could have fed it [a vase] through the door if we tried it vertically or maybe if we'd … wrap something around it, it wouldn't have broken.

Icis emphasized an investment of time to explore all alternatives to gear her mindset towards a positive outcome:

> I probably should have gone through and looked at all the other possibilities before just jumping to conclusions. Because when we jump to conclusions, … your mind tends to wander. And then you think of all the bad outcomes instead of maybe this is a good outcome. You don't really think of all the possibilities.

Subtheme 3.1b exposes that familiarity based on analogies and feelings aids the participant in understanding how to cope with a faulty algorithm. The analogies are comparisons of context that are similar to modeling as a theme. However, the analogies are implicit or

incidental, meaning no direct evidence suggests that all comparisons serve as models for the narrative context. The analogies uphold preferred states of mind to manage the faulty algorithm. Athena referred to points as steps in a process and talked about adjustments:

> I feel like the eggs could be points, kind of because I feel like the different ingredients could be points. Because it's kind of like a step. I see it as you add like you go from here like you have milk and batter but then you add eggs to the mix which makes it a different point in the way of baking. I feel like the adjustment makes me comfortable because deep down in my head I know there's something wrong by seeing the mistake. You can tell it's there's no really other way to go around it because it's a wrong answer. I feel like if you're doing a math problem, your main goal is to get the answer. So I guess it kind of makes me comfortable to know I have the right answer.

Lyssa thought of her map as a sequence of locations and mentioned memorization as a form to familiarize herself with what was happening:

> But you have to make sure it's like, since like my parents don't like let me use a map, if it's like places I know where I'm going, you have to make sure you have this like the sequence of places that you're going to memorize. So it's like you have to go out and you have to make a right and then you can, like for that, you can just go straight for a little bit and then you make it right in there.

Xana offered a sense of coping by familiarizing herself with an expected, mercurial path of struggle. She compared the use of the distance, midpoint, and slope formulas to a pathway filled with expectations of uncertainty and change:

> It [the use of formulas] kind of can be … compared to like driving on a really bumpy road or something like that or maybe going on a roller coaster. Because you know you start off smooth but then you end up going like all different directions and you never really know what you're gonna get into because it just always changes.

Icis related the inside and outside of a desk to area and perimeter, respectively, and familiarized herself with working with both simultaneously to manage a faulty algorithm:

> Because it was like by the end it was going good and everything was fitting where it was supposed to go. We just had the wrong pieces in the wrong area. And it's like really hard because with perimeter and area, when you're working with it, area you're usually working on the inside and the perimeter you're working on the out. But when you're building something, you have to kind of take in the fact, okay this is the inside and out. So you have to be like okay I have to work with both sides.

Subtheme 3.1c manifests that participants need awareness of constraints before conducting a task to avoid faulty algorithms. Initially, these limitations are unnoticeable. The limitations appear when faultiness occurs. Participants implied the importance of predicting these limitations before a task. Athena mentioned needing to know the amount of area first before placing and organizing furniture in a room, "But in reality you won't know how much space it takes up until you have like the actual area." Lyssa talked about reading her map first before driving to avoid the reality of deviating away from her destination and into unknown territory:

> Like this was Main Road and this is the neighborhood, the neighborhood was not even on the map. Shouldn't have even been there [at this neighborhood]. So it was kind of like um it was like if I had read it [the map] first and like thought more common sense like. And was like okay so this doesn't make sense…. Yeah basically I don't want to go off the grid.

Xana discussed having a budget first before expenditures in a situation occurs:

> And for a while we thought it was okay because you know we had a pretty steady income. And both my grandparents were working jobs and my dad. So we thought we had enough money to feed everyone which at the time we were living in this big house where like all of our family lived in.

Icis mentioned a relationship between limitations of time and interest with a topic of literacy:

> I've noticed it like if I'm sitting in class and I don't feel good or I don't really want to be here, it feels like it's taking forever to go by. But if I really really like that class and maybe I could be in there for an hour and it's gonna be like 30 seconds.… Because I think, like I've noticed when I'm working and I don't get distracted… but if I'm like … zoomed in on something, I finish it so fast. And I'm like well now what do I do? Gosh just kind of sit here.… Then it's like okay. But it's like time, it feels sometimes, it just feels like time is like working against you.

Subtheme 3.1d indicates that participants needed visual clues to deter faulty algorithms. These clues serve as support to be on a pathway that prevents faultiness or acts as reasons why a faulty algorithm occurs. Participants either strengthened their prior knowledge or discovered hindrances that need management. Athena explained how she differentiated between perimeter and area through visualization:

> I feel like I see like the little gaps in between my room, like the little, like spaces where there's like open space.… I think I have a real distinction because with perimeter you can basically see like the outline of things but you really with area it's just the space it takes up.

Lyssa shared how changing the representation of tick marks to boxed marks allowed her to make sense of counting correctly to avoid a faulty algorithm:

> I think since it [interval tick marks] was unboxed, it didn't make sense.… Because these were just lines, it confused me. I'm used to seeing boxes so like it changed it and it made sense.

When counting, Xana demonstrated *trust and self-assurance* with visualization over pattern recognition and application:

> I think counting for me is easier [than applying a pattern]. I think I like to see it visually and be able to try to group it together on my own.

Icis talked about how she needed to be *aware of distractors* in her visualization to deter faulty algorithms:

> Like I was getting like a decent amount of points. I kept hitting in the middle, which is a good, like a really good amount of points. But then I had somebody catching up and I guess I got really nervous. And I went to throw and I wasn't paying attention. Somebody's talking to me. I threw it while I looked away. And I kind of I missed my target completely…. Yeah I had a little distractor there. With the parentheses, there are a few too many. So it's like hard to keep up. I would like, there are sometimes where like the parentheses are unneeded. So it's like okay well this doesn't really need to be here. It's just separating this and this. So, I mean, like that could be a distractor.

Theme 3.2. Persevering and Showing Precision

Participants showed persistence with overcoming faultiness and continued endeavors to refine a situation. The quotations showed the presence of reflection as a means for participants to understand, manage, or fix what was transpiring by taking a step back to get a step forward. There was a feeling of obligation by participants to be responsible and accountable for the

management and correction of faultiness. The measurements within the quotations inclined to support favorable outcomes of faulty algorithms.

Subtheme 3.2a indicates that participants often reflected on responding to faulty algorithms during the task. Athena felt that her response to faultiness occurred because of a need for a transparent exchange between comprehension and instruction:

> And she [Athena's friend] was basically done and then she messed up with her … quiz and an answer key. She messed up by not making an answer key. And so I had to remind her to make one. And so then she had to restart another quiz. Because while she had done the answer key basically like she'd filled in the answer so she shouldn't have. But she just made another quiz … [the teacher] didn't really like specifically tell us … [the teacher] didn't like announce it.

Lyssa thought an investment of time to go back, question where the faultiness occurs in a process and review for sense-making, helped alleviate the effects of a faulty algorithm:

> So you have to like take a step back. Then you have to kind of back up. And you're like, okay, where did I mess up at?... So it's kind of like you have to like back track and make sure it makes sense. That's like my whole thing, making sure things make sense. Because things don't make sense when it's not gonna work

Xana needed to observe specific details to help think of a process to put together to overcome faultiness:

> I knew I had to like fix something about it but I didn't quite know what so I had to try to like um piece together what went wrong. I think at first I believe it because when I don't notice little details when I look at things like that … I kind of … thought about it a bit more.

Icis also posed about going back to review a situation even when she had a right answer. She felt it important to find out how she got an answer for credibility:

> I had … the right answer but I had put my numbers in the wrong area. But I somehow got the right answer. So I had to go back and kind of redo how I put my problem together … There have been times where I have gotten the right answer. And then I look back at my math I'm like I have no idea how I got that. And it's just like you have to just go back. And you have to like refresh your mind and kind of think how maybe outside the box how you got that answer.

Subtheme 3.2b suggests that participants felt a duty to handle and resolve a faulty algorithm. Athena talked about feeling comfortable with accomplishing a goal. She shared what needed completion to be part of a marching band team that relied on correct movements without faultiness:

> I feel like if you're doing a math problem, your main goal is to get the answer. So I guess it kind of makes me comfortable to know I have the right answer. At least movement wise, you have to shift your body to go like if they said go left two, you have to shift your body like this [she demonstrated a rotation of her body] but keep your head straight and then keep going that way.

Lyssa discussed focusing on information for certainty. She shared the certainty of performing her task as a basketball manager by going back to check her work multiple times to correct faultiness:

> Yeah you have to like focus in on the subject [information] because … you have to like make sure that's exactly what you want. Yeah because then when I like, if I add it all up and it's like say, we had 65 and I only have 63 now. And I'm like uh-oh and I have to go back and triple check.

Xana describes struggle to rectify faulty algorithms that involved care for a family member and construction of a bed mattress:

> We had this family member … not financially stable. And so we invited him to come stay with us… We were spending a lot of money focusing on him.… So we ended up having to try to all work jobs to find a home for him. And no matter what we did, it was always like there was not enough money. We needed … a bed frame for my … mattress…. But we didn't have the money … to buy another one. So we tried to make one ourselves.… But…the legs were too short … too thin so they couldn't hold up… the bed. And it would end up breaking apart and … falling.… So the size of the of the bed frame … wasn't proportionate.

Icis talked about exceeding expectations when correcting a faulty algorithm by thinking in different ways and by having bilateral perspectives:

> And you have to like refresh your mind and kind of think how maybe outside the box how you got that answer. But when you're building something, you have to kind of take in the fact, okay this is the inside [area] and out [perimeter]. So you have to be like okay I have to work with both sides.

Subtheme 3.2c demonstrates that participants valued measurements to avoid or prevent faulty algorithms. Measurements are either on the mind of participants during a task *or* thoughts after a task. Athena talked about the measurements she performed for the marching band and the measurements she should have thought about after organizing her room:

> So it's basically, every single section is divided into how many feet there are like a yard line to a yard line is eight steps in marching bandwise. And so I think we kind of underestimated like how the room was planned out because I wanted to put all these things in there but I couldn't.

Lyssa explained how she organized measurements during a task and how she had to re-evaluate prior knowledge that made sense in theory but not necessarily experimentation:

> So figuring out when I messed up and then going through and doing them by sections. So like sectioning out first quarter and then doing that and sectioning out second quarter and doing that. And then seeing like which one it's not right, like which one isn't right. Because if I recount them by sections, then I add them all up at the end. Then I might be like oh okay so this is the one that I messed up in before. So like taking it and dividing it into little like separate parts... Yeah basically I don't want to go off the grid. Like I tried to use two things that literally didn't even go with a third of a cup. Because a fourth and a half don't even like there. A third is in between them but you can't mix them together … and I found that out at the hard way.

Xana considered categories of measurements in the course of a situation and how she needed to apply preciseness of measurements:

> I think like … whenever you're buying clothes and stuff, you always have to consider, you know, like … your body size, like your weight. And like … sometimes you have to measure like around your waist and stuff, your pants, things like that. Or whenever you're like buying gloves to like, you have to make sure, you have like the right size hands for the gloves because if they're too small, it's not gonna go on easily. So I should have I should have measured everything right. Like I should have figured out how much exactly in inches that I was going to be cutting off. Because I wasn't actually like trying to take off a certain amount I was just cutting.

Icis related the estimation of measurements to a Geometry concept to refine the measurement in her mind and implied how estimates needed to be better after a faulty algorithm:

> Like when you have to throw the ax, you have to get it right in the middle to get a certain amount of points or right on the outside to get a certain amount of points. So it was an error in kind of like how much we estimated where we were going to have in all.

Theme 3.3. Looking

Within quotations, participants categorized their observations based on what they perceive as specific and general information. Looking at detailed information leads to discovering what participants can accomplish by applying a subconscious attribute that leads to emotion. Inspecting general information leads to a faulty algorithm in the midst or end of a situation.

Subtheme 3.3a finds that participants uncovered the application of a latent talent about themselves that led to a feeling of wonderment. Athena talked about how a particular way of observing details drove her to effective performance in the marching band:

> I feel like I could have looked at my peripherals more because with marching band, you rely a lot on peripheral vision because the entire time your head is facing straight towards the stadium. Like your head cannot move. And so um I feel like I could have looked at my peripherals more instead of just like looking dead ahead.… We have this thing called, um, focalizing. And basically, it's where you look at your peripheral vision and make sure you're lined up with the two people beside you. It's like and then even with when we make curves, we still have to look at people beside us because if you're too close to this person too or too far from that person, it throws off the entire thing.

Lyssa noticed how her investment of time with guessing and calculating persuaded her to endeavor a more worthwhile strategy even if it takes more time:

> I think if I had found a different way to make a third. Like maybe like looked up how many tablespoons are in a third of a cup, even though that would have taken forever. Instead of just trying to guess and just keep guessing if I had like taken the back and been like well I know that this these can make a perfect amount of a third of a cup instead of like kind of guesstimating in between [a half and a fourth].

Xana deduced that an inspection of details allowed her to reach a significant learning moment:

> I think at first I believe it because when I don't notice little details when I look at things like that like counterclockwise. But when I … recognize the way that it was turning, I

> was, I kind of … thought about it a bit more. I was like, oh yeah the counterclockwise is this way [she moved her hand and arm in a counterclockwise direction] and then clockwise is this way [she moved her hand and arm in a clockwise direction]. And I had to … think about a clock for a minute.

Icis implied how witnessing details leads to a feeling of reverence about herself:

> Like I'm gonna be honest, like sometimes looking after you do the math and then you go back to looking, you're like wow I did that? That looks so like difficult and it's usually because they're like a lot of parentheses. There's a lot of square roots or something like that. So I mean, I mean I don't personally think it's a distractor because I look at it.

Subtheme 3.3b suggests that participants precipitated faultiness by looking at information in general. In some situations, there was a feeling of concern when initially viewing information. At the same time, there was hope that a task can still be accomplished despite the concern. In other situations, there was a feeling of satisfaction of everything transpiring well. Then, there was a sudden concern when the faultiness occurs. Athena experienced discomfort at first glance at her desk but minimally achieves the task of placing the desk in her room even though it was not at the level of achievement that she preferred:

> Like when I originally had like got my desk for Christmas, I was very nervous because it looked bigger than it actually was. Because I didn't see it next to anything else in my room so I was like that's gonna be too big. I was like stressing but once I moved it in there, I was like looking and I actually do have some space left. And it's just like you have to like look at the area of it first because the perimeter I guess can kind of make it things seem bigger in reality.

Lyssa went through inceptive optimism leading to a *sudden turn of perturbation*:

> At first it seemed great. Because we were like oh my goodness this couch looks great. It'll look great in the living room. And then it gets there we're like hey this is way too big to fit in the living room. And we like spun it around and everything. So like, it just like, the ending of it messed up the whole beginning.

Xana felt contentment in the beginning stages of her task but *discovers disappointment* of the results from her need of experience at the time:

> It was going relatively well because my hair from the back part is I need to pull it down to cut it. I could just cut it from where I was looking. And it looked okay. Because I had

> my hair pulled out to the front. But when I pulled it to the back, everything looked choppy. Because obviously when I push it to the front, the hair is positioned in different pieces and stuff.

Icis progressed with comfort with leading in a game but a distractor she deemed as irrelevant ended up leaving her in surprise:

> Like I was getting like a decent amount of points. I kept hitting in the middle, which is a good, like a really good amount of points. But then I had somebody catching up and I guess I got really nervous. And I went to throw and I wasn't paying attention. Somebody's talking to me. I threw it while I looked away. And I kind of I missed my target completely.

Summary of Faulty Algorithm Narratives to Current Research

When managing a faulty algorithm, this study's findings show how mathematicians and participants critique and reason about proactive alternatives, familiarity, awareness of limitations, and visual clues for realistic situations. As Scriba and Schreiber (2015) implied, mathematicians generally take a proactive approach before accepting the widely accepted Euclid's parallel postulate, amongst others, as accurate by perceiving the familiar alternative of our world as three-dimensional. Similarly, each participant in this study critiqued and reasoned in their narratives a familiar, proactive alternative that took them beyond what is viewed as true to what is true (Polkinghorne, 1988; Clandinin & Connelly, 2000). However, one contrast between earlier mathematicians and participants involves awareness and visualization. Scriba and Schreiber (2015) suggested that mathematicians extend upon notions through visualization, whereas this study shows that participants considered limitations of situations for more favorable outcomes. Overall, the findings are consistent with Riessman (2008), with participants connecting to their identity for critiquing and reasoning with familiar prior knowledge based on analogies and feelings that help understand how to endure a faulty algorithm. The findings parallel Lakoff and Johnson (1980), where participants checked on awareness of the possible

existence of faultiness through constraints via familiarity of prior experiences. The awareness of faultiness adheres to Polkinghorne (1988), where participants in this study critiqued and reasoned through various junctures of a situation to either correct or prevent faulty algorithms. In addition, the findings support Große and Renkl (2007) with participants learning more in depth and stimulating their algorithmic reasoning via enterprising alternatives, relatable contexts, realization of constraints, and ocular traces of information.

This study shows how participants frequently reflected with faulty algorithms, possessed a sense of morality for corrected responses, and relied on measurements for management and deterrence. Consistent with Mertova and Webster (2020), the reflections support participants in making sense of the situation in their narratives. The findings uphold Clandinin and Connelly (2000), where participants displayed a virtuous need to correct faulty algorithms by paying attention to the detail of their measurements for problem-solving. Similarly, these findings advocate Rushton (2018), with participants demonstrating overall perseverance and focus on precision for reflecting on sense-making and thinking of ways to correct/prevent faulty algorithms. Riessman (2008) implied that participants define another part of their identity by having the integrity to transform or encode a sequence of operations within their narrative that intuitively needs their intervention for accountability. This finding dealing with obligation coincides with Durkin and Rittle-Johnson (2012) with participants yearning to correct their faulty algorithms thereby simultaneously reshaping and expanding upon prior knowledge.

The study's findings manifest that participants looked for information in their surroundings specifically to enter a realm of bewilderment to correct faultiness or look generally to subconsciously cause faultiness. This finding supports Riessman (2008) with participants having identified with various emotions when they observed specifically versus

generally. Prior research supported how participants looked for information to evaluate for the existence of a faulty algorithm and how they indirectly promoted the existence or correct it (Coles, 1989; Bateson, 1994). This dual way of looking supported Geertz (1995), who implied how participants acquired insight into what needs inspection for an algorithm of operations.

Overall Summary of Key Findings as Connected to Previous Research

Ultimately, the findings venture to answer three research questions apropos the narratives of students about error types. Five themes emerge and relate to the standards of mathematical practice, as supported by Rushin (2018). Three out the five themes share with all research questions: critiquing/reasoning, persevering/attending to precision, and looking. The other two themes share with the research questions about wrong strategies and miscomputations: modeling and using tools. There are variations with subthemes from one error type to another. A summary of prior research corresponding with key findings illustrates in Table 5.

Table 5

Summary Comparisons of Prior Research to Key Findings by Error Type

Wrong Strategy Narratives	
Prior Research	Key Findings
A transition exists in narratives to go beyond perceptions of truth to the reality of truths (Polkinghorne, 1988; Clandinin & Connelly, 2000).	Participants critique and reason about their persona, knowledge, and stimuli for realistic situations.
Comprehension occurs from environmental observations and information (Risastuti et al., 2017).	Participants look for observations and information based on the pictures and words they see in their surroundings.
Information modeling occurs through a unique language and prior knowledge that develop from discourse with peers (Sfard, 2007; Hillman, 2014).	In contrast, participants model content in their narratives from their unique language and prior knowledge of their persona and experiences.
There is a narrative that describes a process of some phenomenon (Schön, 1983; Coles, 1989; Bateson, 1994).	Narratives show an organization of a general process with how students persevere and attend to precision with correcting wrong strategies.
There is a narrative of a phenomenon that shows progressions through temporal connections of context (Polkinghorne, 1988) and abstract states of mind (Lakoff & Johnson, 1980).	Students apply tools for comprehension to undergo temporal connections from a state of disarray back to normalcy when responding to a wrong strategy with the tools being representative of experiencing abstract states of mind: mindsets of making sense and returning to typical comfort zones.
Miscomputation Narratives	
Prior Research	Key Findings
A transition exists in narratives to go beyond perceptions of truth to the reality of truths (Polkinghorne, 1988; Clandinin & Connelly, 2000).	Participants critique and reason about their corrections, explanations, generalizations, and self-performance.
A phenomenon offers incentives to see what exists in reality over expectations of what to see (Geertz, 1995; Risastuti et al., 2017).	Participants look for functional, goal-oriented relationships between information and look at their perspectives for dealing with a miscomputation to think more productively between the steps of a process and to find a replacement for a miscomputation to improve the relationship.

There exists an appealing attribute of a phenomenon to content by instinct (Geertz, 1995).	Participants model content context for meaningful livelihood.
There exists a "degree of wrongness" about an errant outcome with one context but not necessarily all contexts (Borasi,1987). There exist comparisons of occurrences at different junctures of a task to pinpoint where a phenomenon occurs (Polkinghorne, 1988).	Participants feel obligation for collaboration, focus on trials of precision, and choose what they feel is relevant to determine where a miscomputation occurs and how to correct it.
There are beneficial applications of tools to overcome conflict in a situation (Ingram et al., 2015; Loibl & Leuders, 2019).	Participants merit retention as habitual tools to recall prior knowledge for managing miscomputations.

Faulty Algorithm Narratives

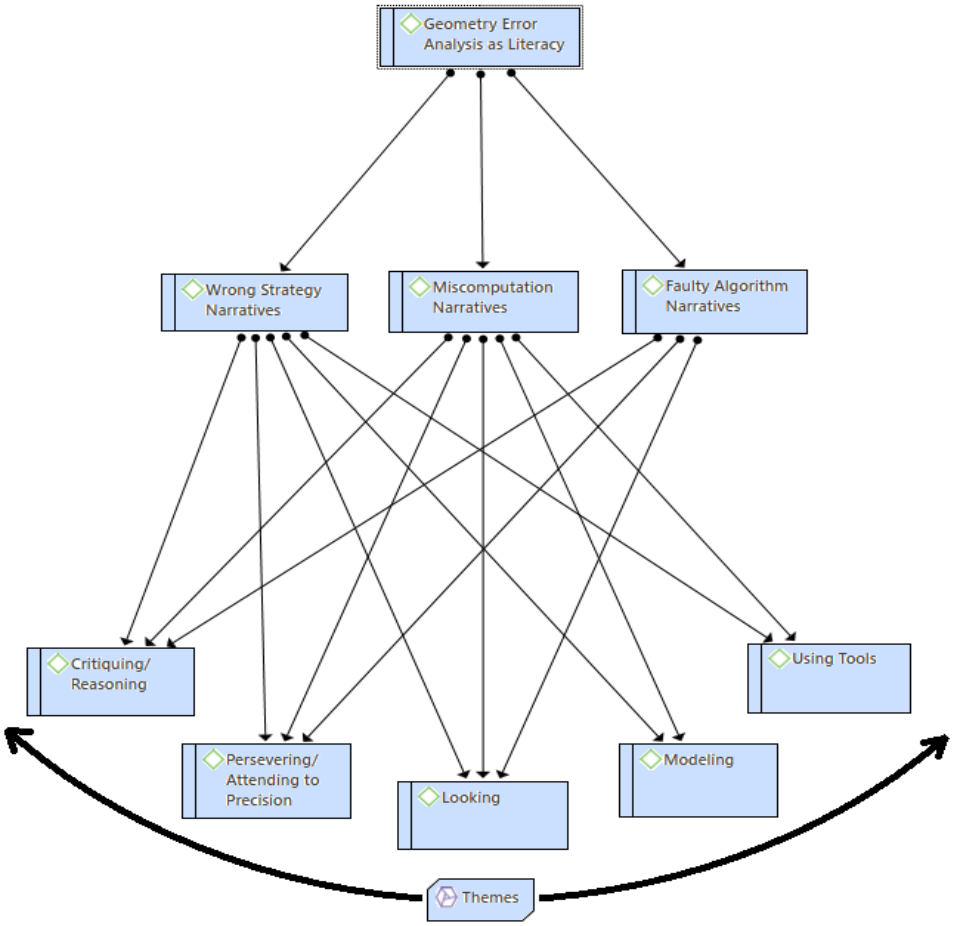
Prior Research	Key Findings
A transition exists in narratives to go beyond perceptions of truth to the reality of truths (Polkinghorne, 1988; Clandinin & Connelly, 2000).	Participants critique and reason proactive alternatives, familiarity, awareness of limitations, and visual clues for realistic situations.
Participants define their identity through their narratives (Riessman, 2008) and yearn to correct faulty algorithms through reshaping and expanding upon prior knowledge (Durkin & Rittle-Johnson, 2012).	Participants frequently reflect with faulty algorithms, possess a sense of morality for corrected responses, and rely on measurements for management and deterrence.
There exist observations for analysis to evaluate the existence and response to a phenomenon (Coles, 1989; Bateson, 1994).	Participants look for information in their surroundings specifically to enter a realm of bewilderment to correct faultiness or look for information generally to subconsciously and potentially cause faultiness.

Implications

Through narratives involving Geometry error analysis as literacy, the findings of this study bespeak a relationship between participants undergoing standards of mathematical practice (SMPs) (see Figure 51) and teachers acquiring insight into a sensible progression through the standards of mathematical teaching practices (MTPs) (see Figure 52). Theoretical implications revolve around the SMPs and include applying prior knowledge, a vision of MTPs in action, and a vision of learning models. Practical implications include a host of experiences for students that the participants of this study exhibited.

Figure 51

An Overall, General Network of Themes Emerging from this Study that Relate to the Standards of Mathematical Practice for Students

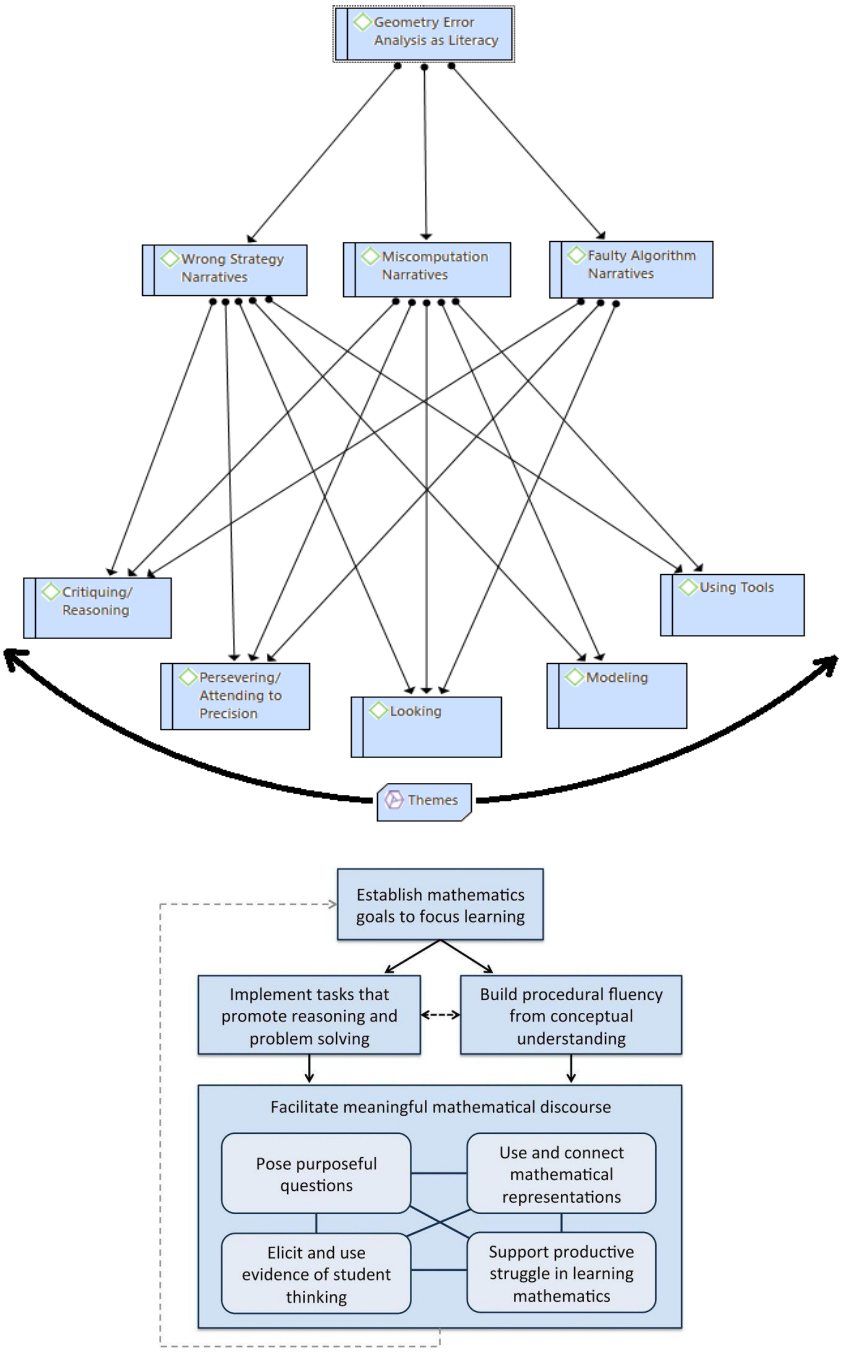


The standards of mathematical practice.

1. Make sense of problems and persevere in solving them.	2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.	4. Model with mathematics.
5. Use appropriate tools strategically.	6. Attend to precision.
7. Look for and make use of structure.	8. Look for and express regularity in repeated reasoning.

Figure 52

This Study's Network of Themes as a Model for Mathematical Teaching Practices for Teachers



Note: This figure was produced by Boston et al. (2017, pg. 215) showing the fluidity of relationships between implementation of the standards of mathematical teaching practices.

Theoretical Implications

This study explores relationships between Geometry error analysis problems as literacy and students' narratives. The application of prior knowledge and the prevalence of SMPs is evident through the voices and actions of participants, as supported by prior research (Kawasaki, 2010; Rushin, 2018; Durkin & Rittle-Johnson, 2012; McLaren et al., 2012). At the same time, a Geometry teacher acquires an exemplar of how error analysis as literacy supports the formation of mathematics goals to focus on learning (Boston et al., 2017). Although metaphors only insinuate in this study, the findings support Lakoff and Johnson (1980) with participants progressing through a model of learning with error analysis as literacy that is relatable and meaningful to their experiences.

Throughout this research, participants elicit behaviors through SMPs that are analogous to prior mathematicians, where beliefs thought to be accurate as perceptions transform into beliefs that are true as realities (Kline, 1980; Polkinghorne, 1988; Borasi, 1996; Clandinin & Connelly, 2000). As Rushin (2018) implied, the transformation to the truth as reality results from this study's findings that show how students naturally progress through SMPs by applying and reshaping prior knowledge. This study supports Kawasaki (2010), where participants detect and respond to errors by reshaping their prior knowledge based on reality. The findings uphold Durkin and Rittle-Johnson (2012) by participants demonstrating exuberance for expanding their prior knowledge from narratives to correct errors. The findings also advocate McLaren et al. (2012) with participants building more comprehension of Geometry content over time by applying prior knowledge in their narratives.

Simultaneously, a teacher instinctively undergoes a pathway that progresses through the MTPs (Boston et al., 2017). Große and Renkl (2007) supported this study with how participants engage with real-life, erroneous situations that serve as tasks to apply reasoning and critique with

prior knowledge. Simultaneously, the transformation to truths as realities upholds Tsovaltzi et al. (2010), where participants demonstrate procedural fluency from reshaping prior knowledge for comprehension of concepts via SMPs. Like past mathematicians, this study's participants persevere and attend to precision for sense-making by inquiry and assessing their self-performances when managing and correcting errors (Tsovaltzi et al.; Borasi, 1996; Große & Renkl).

Consistent with Trninic et al. (2018), each participant demonstrates a "dialogic" discourse that is primarily between themselves and the information acquired by looking at their surroundings for spontaneous, open-ended ideas to confirm truths and develop responses to errors. Concerning narratives with wrong strategies and miscomputations, the findings support Sfard (2007) and Hillman (2014) with "visual mediators" that constitute how participants in this study use mathematical representations as models and various tools for sense-making (Sfard; Hillman). Pomalato et al. (2020) uphold the findings with participants in this study exhibiting productive struggle with error analysis in their narratives and Geometry problems by applying information from their environment and reading passages to model more specific truth in mathematics learning and life in general.

Despite a direct reference to metaphors, this study underpins Lakoff and Johnson (1980) as a model of learning conducive to relating Geometry content in a meaningful context to the narrative experiences of students. As Collins et al. (1991) suggested, this study demonstrates a "cognitive apprenticeship" where participants visualize and express their thinking to improve errant situations by applying evidence in their surroundings and using their inherent talents to reshape prior knowledge. The findings of this study uphold Fast and Hankes (2010) by participants perceiving Geometry error analysis problems as "anchors" to relating the

management of error types to personal, real-life narrative situations. The findings of this study suggest how students progress through “cognitive apprenticeship” and “anchoring” to potentially transfer their management of errors in their narratives and Geometry problems to various fields of future employment (Collins et al.; Fast & Hankes).

This research reinforces Fast and Hankes (2010) concerning how students acquire an assiduous, optimistic mentality for applying the ways they overcome errors in their narratives in a congruent manner for overcoming errors in Geometry problems and life in general. As Trninic et al. (2018) implied, participants in this study showed how a building comprehension of abstract Geometry concepts kindles into a relationship between content and application. The knowledge participants acquired from their narratives transforms from embedded subconsciousness to uprooted visibility, as Collins et al. (1991) supported. Overall, the findings of this study build upon prior research on how a union of cognitive apprenticeship and anchoring provides students with the autonomy to reshape prior knowledge acquired from their narratives into a leadership role of correcting erroneous solutions (Collins et al.; Fast & Hankes; İlhan et al., 2021).

Practical Implications

When connecting with narrative experiences that involve errant situations, this study shows as a model for literacy of how students attain discernment by managing and correcting the situation, which led to improved opportunities for correctly solving Geometry problems. Based on the findings of this study, there are nine practical implications for the implementation of Geometry error analysis problems as literacy for both students and teachers.

First, students discovered and corrected different errors that included a wrong strategy, miscomputation, and faulty algorithm. Students often reflected on how to improve erroneous situations. They considered the correction of errors as an obligation. Students valued the

precision of measurements to avoid or deter errors from occurring. Professors and teachers gained insight of how to establish a Geometry error analysis literacy assignment that involved reasoning and problem solving. The design started with a search of an employment field that related to a Geometry concept. The design continued with choosing error analysis problems for students to perform in a supervisory role for novice employees.

Second, students analyzed and talked about their identity. They rediscovered who they are and what they can do. They uncovered hidden attributes of their identity and strove to improve upon them. Discovering a latent talent gave students a feeling of pride and respect for themselves. Hope existed that a task can be successfully completed despite arising concerns. The encounter provided opportunities for professors and teachers to have students witness the effects of the positive outcomes of progressing through productive struggle and the beneficial effects of accomplishing mathematical goals.

Third, students tended to understand each part of a situation through persistence and exploration of correction. A particular word was said by each participant that demonstrated their continuous goal for comprehension. The word was a subconscious motivator encouraging the participant to continue seeking comprehension. Students went through a task with trials of tenacity and ambition for precision. Professors and teachers acquired insight of the word that stimulated students to think in a productive, positive manner and to enhance their perseverance (Rushton, 2018).

Fourth, students connected a Geometry topic to a real-life situation in their narratives. This connection served as a model to represent and describe the meaning of a Geometry concept. Students focused on a relationship between themselves and relevant information. For students, the topic of Geometry was enlivened, particularly as some dynamic phenomenon that

was moving and changing. Professors and teachers acquired a method of applying and relating mathematical representations using the narratives of students. Carefully listening to the stories that students live through had a profound impact for helping students realize the relevance their lived experiences have to Geometry concepts.

Fifth, students self-assessed their thinking before acting on a task. The assessment served as a historical account of the origin of a task's process leading up to an outcome. The history served as foundational knowledge for managing potential errors. This self-assessment helped to minimize the occurrence of errors, particularly miscomputations. The assessment helped to consider all alternatives before doing a task, particularly a faulty algorithm. The assessment was a way for professors and teachers to expand upon evidence of student thinking into a realm of purposeful inquiry. For example, teachers can challenge students to think how a problem can be rewritten for an errant solution to be correct.

Sixth, students discovered the difference between observing their environment in general and observing their environment in detail for information. Students sought awareness of limitations, particularly with faulty algorithms, and valued the importance of predicting limitations. Familiarity with their environment led to analogies and feelings that helped them understand how to cope with errors, specifically faulty algorithms. Familiarity eased tension and led to sense-making. Students looked at specific information to help increase comprehension of a situation. Students looked at different perspectives from their observations to prevent or substitute errors, particularly miscomputations. Students relied on a hunch that a solution exists for a situation based on how the appearance of the result needs to look correct or better. The difference of noticing generally versus specifically allowed professors and teachers to assist students with productive struggle. Teachers and professors acquired a way to teach students how

to observe phenomena in two perspectives. For example, teachers can have students view an optical illusion with two embedded pictures; students can be challenged to see both pictures, particularly those who see just one.

Seventh, students used various secondary reinforcer tools like signs, highlighters, manipulatives, and technology apps to help visualize sense-making to solve problems. These tools helped students retrieve a state of normalcy in their lives and gather their thoughts together to solve a problem. These tools differed among students and served as habitual responses learned and developed over time and experience. This research served as another way for professors and teachers to help students cope with productive struggle. Professors and teachers have a way for students to practice overcoming the presence of vast amounts of information and to focus on relevant information by using secondary reinforcer tools.

Eighth, students felt obligated to collaborate with anyone to overcome the presence of a miscomputation in a process. Students experienced accountability with their input for resolution and negotiation for compromise. Students agreed upon a level of wrongness with their relatives for the miscomputation that led to a decision on how to correct the miscomputation based on that level. A considerable amount of time occurred, with each student having a unique perspective on the purpose of collaboration. The narratives involving collaboration with relatives helps professors and teachers understand the cohesion that is occurring. That cohesion can be used to devise ways for students to collaborate with their peers in the classroom. An example is the presentation of a problem with multiple solutions. The collaboration arrives full circle when students reach common ground of perceiving each other's perspectives.

Ninth, students felt the need for teachers to intervene with organizing a process that implements literacy with their learning of Geometry. This research serves as a way for professors

and teachers to pose purposeful questions for students to refine their organization skills for thinking. In turn, the inquiry allows students to build procedural fluency by reshaping knowledge of what they consider to be absolute truth to open-ended knowledge of curiosities as realities (Polkinghorne, 1988; Clandinin & Connelly, 2000). This narrative study serves as a bridge for professors and teachers to assist students organize a path of literacy through error analysis that connects a process of their authentic experiences to the learning of Geometry.

Recommendations for Further Research

Based on the theoretical and practical implications of this study, the following are recommendations for future research:

1. Duplicate this study with twice the number of students with participants from another class where the researcher is not their teacher. Doubling the participants is helpful for attaining more variability and improved relationships among responses.
2. Conduct a study that differentiates critiquing from reasoning and persevering from attending to precision within interviews. In this study, it is challenging to determine differences with responses amongst the SMPs.
3. Conduct a study with focus groups rather than individual, semi-structured interviews to acquire the interaction of discourse and collaboration between participants. Using focus groups brings to light more the nature of student exchanges with each other.
4. Conduct a study that clearly defines tools as habitual responses. Perhaps a different word other than tool should be used to represent responses like reading signs, highlighting, drawing blueprints, and working with technology.
5. Conduct a study that differentiates modeling as direct versus indirect relationships between content and objects.

6. Study the works of Lakoff and Johnson (1980) to acquire indirect, genuine uses of metaphors rather than inquiring for examples of metaphors. In this study, metaphors are more like similes.
7. Consider replacing the words “error,” “wrong,” “miscalculation,” and “faulty” to “focus,” “comprehension,” “process,” and “transformational/encoding,” respectively. The substitutions convey more of a positive vibe to participants of building phenomena that is constructive.

Limitations

This study had a few limitations. First, there needed to be at least one male participant. The inclusion of a male participant would have validated the study more and follow my school’s initiative of having a blend of heterogenous students by gender. It was difficult for me to determine why a boy was not interested in being a part of this study.

Second, participants needed to be independent from the researcher’s classroom. The inclusion of participants from another classroom was an impossible feat due to scheduling and my responsibilities for being in the classroom. It was challenging to figure out a way that participants were not the students in my classroom.

Third, there were interruptions and distractions during the interviews. These distractions took the form of a third party knocking of my door and announcements being given on the intercom. There was a couple of times where my camera lost power and shut down during the second of two interviews in a row.

Conclusion

In this narrative inquiry, four high school students shared narratives that relate to a wrong strategy, miscomputation, and faulty algorithm. The students initially responded to Geometry error analysis problems, exposing them to each error type. After, the students described the management of each error type in stories that are meaningful to them. The findings of this study suggest that students handled the errors in their narratives and Geometry error analysis problems similarly to prior mathematicians. Each student operated on practices that match the context and are similar to mathematical standards of practice.

The students valued reflections on improving errant situations and considered enhancements a moral obligation. They reinvented their identity by knowing their persona and applying their abilities to give them optimism about themselves and improve situations. Students have a predilection for understanding each part of a situation to attain a favorable outcome through perseverance and attendance to precision. Students have a latent proclivity to enliven the concepts learned in Geometry class through real situations in their narratives, particularly when finding corrections to errors. After errors occur in their narratives, students strove to self-assess alternatives to build historical accounts of foundational knowledge as a reference for preventing or deterring errors before a task. Students learned the difference between looking specifically and generally at their surroundings to gather information for countering errors in their narratives and Geometry problems. Observing detailed information allowed students to increase their understanding of the occurrences of a situation and how to return to a state of normalcy when errors occur. Students applied tools in various habitual responses developed over time and experience. Students preferred collaborating with teachers and peers to reach a consensus for solving problems in their narratives and Geometry class. The display of the organization of

processes by students was valuable for teachers to provide interventions using the MTPs naturally.

From a teacher's perspective, a vision occurs of one possible way of implementing literacy using error analysis of Geometry problems and student narratives. This study provides insight into how a dual framework of cognitive apprenticeship and anchoring operates for students to develop meaning and relevance to learning. Through error analysis as literacy, students manage and counter the "degree of wrongness" (Borasi, 1987, p. 5) for situations.

References

Barbieri, C. & Booth, J. (2020). Mistakes on display: Incorrect examples refine equation solving and algebraic feature knowledge. *Applied Cognitive Psychology*, *3*(4), 862–878.

Bateson, M. (1994). *Peripheral visions: Learning along the way*. HarperCollins.

Borasi, R. (1987). Exploring mathematics through the analysis of errors. *For the Learning of Mathematics*, *7*(3), 2–8.

Borasi, R. (1996). *Reconceiving mathematics instruction: A focus on errors*. Greenwood Publishing Group.

Boston, M., Dillon, F.L., Smith, M.S., & Miller, S. (2017). *Taking action: Implementing effective mathematics teaching practices in grades 9-12*. National Council of Teachers of Mathematics.

Clandinin, D. J. & Connelly, F. M. (2000). *Narrative inquiry: Experience and story in qualitative research*. John Wiley & Sons, Inc.

Coles, R. (1989). *The call of stories: Teaching and the moral imagination*. Houghton Mifflin.

Collins, A., Brown, J., & Holum, A. (1991). Cognitive apprenticeship: Making thinking visible. *American Educator*, *15*(3), 1–18.

Creswell, J. W. & Poth, C. N. (2018). *Qualitative inquiry and research design: Choosing among five approaches* (4th ed.). Sage.

Draper, R. J. (2002). School mathematics reform, constructivism, and literacy: A case for literacy instruction in the reform-oriented math classroom. *Journal of Adolescent & Adult Literacy*, *45*(6), 520–529.

Durkin, K., & Rittle-Johnson, B. (2012). The effectiveness of using incorrect examples to support learning about decimal magnitude. *Learning and Instruction*, *22*(3), 206–214.

Enderson, M., Klerlein, J., & Johnson, J. (2010). Developing mathematics content knowledge along new paths: Shifting from student to teacher. *New England Mathematics Journal, 42*(1), 31-40.

Erickson, A., & Herbst, P. (2018). Will teachers create opportunities for discussion when teaching proof in a Geometry classroom? *International Journal of Science & Mathematics Education, 16*(1), 167–181.

Fast, G., & Hankes, J. (2010). Intentional integration of mathematics content instruction with constructivist pedagogy in elementary mathematics education. *School Science & Mathematics*, *110*(7), 20–30.

Geertz, C. (1995). *After the fact: Two countries, four decades, one anthropologist*. Harvard University Press.

Glesne, C. (2016). *Becoming qualitative researchers: An introduction* (5th ed.). Pearson Education.

Große, C., & Renkl, A. (2007). Finding and fixing errors in worked examples: Can this foster learning outcomes? *Learning and Instruction*, *17*(6), 612–663.

Heath, T. (1956). *The thirteen books of the elements, vol. 1: Books 1-2* (2nd ed.). Dover.

Hillman, A. M. (2014). A literature review on disciplinary literacy: How do secondary teachers apprentice students into mathematical literacy? *Journal of Adolescent & Adult Literacy*, *57*(5), 397–406.

İlhan, A., Gemcioğlu, M., & Poçan, S. (2021). Investigation of the relationship between mathematics achievement, geometry attitudes, and beliefs towards Geometry. *Mersin University Journal of the Faculty of Education, 17*(1), 77–91.

Ingram, J., Pitt, A., & Baldry, F. (2015). Handling errors as they arise in whole-class interactions. *Research in Mathematics Education*, *17*(3), 183–197.

Kawasaki, M. (2010). Learning to solve mathematics problems: The impact of incorrect solutions in fifth-grade peers' presentations. *Japanese Journal of Developmental Psychology*, *21*(1), 12–22.

Keefe, E., & Copeland, S. (2011). What is literacy? The power of definition. *Research & Practice for Persons with Severe Disabilities*, *36*(3-4), 92–99.

Kontrová, L., Biba, V., & Šusteková, D. (2022). Teaching and exploring mathematics through the analysis of student's errors in solving mathematical problems. *European Journal of Contemporary Education*, *11*(1), 89–98.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press.

Loibl, K., & Leuders, T. (2019). How to make failure productive: Fostering learning from errors through elaboration prompts. *Learning and Instruction*, *62*(1), 1–10.

Loibl, K., & Rummel, N. (2014). Knowing what you don't know makes failure productive. *Learning and Instruction*, *3*(1), 74–85.

MetaMetrics Inc. (n.d.). *Lexile grade level charts*. https://hub.lexile.com/Lexile-grade-level-charts.

McLaren, B. M., Adams, D., Durkin, K., Goguadze, G., Mayer, R.E., Rittle-Johnson, B., Sosnovsky, S., Isotani, S., & Van Velsen, M. (2012, Sept. 18). To err is human, to explain and correct is divine: A study of interactive erroneous examples with middle school math students. In A. Ravenscroft, S. Lindstaedt, C. Delgado Kloos, & D.

Hernándex-Leo (Eds.), *Proceedings of ECTEL 2012: Seventh European Conference on Technology Enhanced Learning, LNCS 7563* (pp. 222–24), Saarbrücken, Germany.

Mertova, P., & Webster, L. (2019). *Using narrative inquiry as a research method: An introduction to critical event narrative analysis in research, teaching, and professional practice* (2nd ed.). Routledge.

Pinnegar, S., & Daynes, J. G. (2007). Locating narrative inquiry historically: Thematics in the turn to narrative. In D. J. Clandinin (Ed.), *Handbook of narrative inquiry: Mapping a methodology* (p. 3). Sage.

Polkinghorne, J. (1988). *Narrative knowing and the human sciences*. State University of New York Press.

Pomalato, S., Ili, L., Ningsi, B., Fadhilaturrahmi, F., Hasibuan, A., & Primayana, K. (2020). Student error analysis in solving mathematical problems. *Universal Journal of Educational Research*, *8*(11), 5183–5187.

Prayitno, L., Purwanto, P., & Subanji, S. (2020). Exploring student's representation process in solving ill-structured problems Geometry. *Participatory Educational Research*, *7*(2), 183–202.

Rapp, J., Marvin, K., Nystedt, A., Swanson, G., Paananen, L., & Tabatt, J. (2011). Response repetition as an error-correction procedure for acquisition of math facts and math computation. *Behavioral Interventions*, *27*(1), 16–32.

Ratnaningsih, N., & Hidayat, E. (2020). Error analysis and its causal factors in solving mathematical literacy problems in terms of habits of mind. *Journal of Physics: Conference Series*, 1764(1), 1–6.

Ravitch, S. & Riggan, M. (2017). *Reason and rigor: How conceptual frameworks guide research* (2nd ed.). Sage.

Riastuti, N., Mardiyana, M., & Pramudya, S. (2017). Students' errors in geometry viewed from spatial intelligence. *Journal of Physics: Conference Series*, *895*(1), 1–6.

Riessman C. (2008). *Narrative methods for the human sciences*. Sage.

Rushton, S. (2018). Teaching and learning mathematics through error analysis. *Fields Mathematics Education Journal*, 3(4), 1–12.

Saldaña, J. (2021). *The coding manual for qualitative researchers* (4th ed.). Sage.

Scriba, C., & Schreiber, P. (2015). *5000 years of geometry: Mathematics in history and culture*. Springer.

Schön, D. (1983). *The reflective practitioner: How professionals think in action*. Basic Books.

Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *The Journal of the Learning Sciences*, *16*(4), 565–613.

Shanahan, T., & Shanahan, C. (2008). Teaching disciplinary literacy to adolescents: Rethinking content area literacy. *Harvard Educational Review*, *78*(1), 40-59.

Simons, R., & Gold, B. (2011). *Proof and other dilemmas: Mathematics and philosophy*. Mathematical Association of America.

Stake, R. (2010). *Qualitative research–Studying how things work*. The Guilford Press.

Trninic, D., Wagner, R., & Kapur, M. (2018). Rethinking failure in mathematics education: A historical appeal. *Thinking Skills and Creativity*, *30*(1), 76–89.

Tsovaltzi, D., Melis, E., McLaren, B. M., Meyer, A. K., Dietrich, M., & Goguadze, G. (2010). Learning from erroneous examples: When and how do students benefit from them? In M.

Wolpers, P. A. Kirschner, M. Scheffel, S. Lindstaedt, & V. Dimitrova (Eds.), *Proceedings of the 5th European Conference on Technology Enhanced Learning, Sustaining TEL: From Innovation to Learning and Practice (EC-TEL 2010), LNCS 6383*, September/October, Barcelona, Spain. (pp. 357–373). Springer-Verlag Berlin Heidelberg.

Usiskin, Z. (2008). *The classification of quadrilaterals: A study in definition*. Information Age Publishing Inc.

Zeybek, Z. (2016). Productive struggle in a geometry class. *International Journal of Research in Education and Science,* 2(2), 396–415.

Appendix A

Participant Assent Form

PARTICIPANT ASSENT FORM

Title of Research Study: High School Student Narratives of Error Analysis: A Qualitative Investigation into Literacy for Geometry Learning.
Principal Investigator: Mr. Dean Petti, dpetti@paulding.k12.ga.us, Ph: (770)687-5823

Purpose of Study
You are being asked to participate in a research study conducted under the guidance of faculty from Kennesaw State University. This project has been reviewed and approved by the KSU Institutional Review Board under tracking number IRB-FY23-165. Before you decide to participate in this study, it is important that you understand why the research is being done and what it will involve. Please read the following information carefully. Please ask the researcher if there is anything that needs to be clear or if you need more information.
The purpose of this study is to discover, analyze and interpret how student narratives play a role in the presence of errors in student responses to Geometry problems. The goals are to improve an understanding that student narratives can have on the existence of errors within responses to Geometry problems and to design lessons that can better match this understanding. This research serves as a means of mathematical literacy, where students make sense of math in various individual and related contexts.

Study Procedures
You will write corrected versions of given erroneous responses to error analysis problems relating to literacy passages. Documents, observations, and interviews occur where you will share stories. The stories will relate to three common errors: a selected wrong strategy, a miscalculation, or a faulty algorithm. Observations and interviews will video record verbal and nonverbal responses for discovery, analysis, and interpretation of the roles that stories have for the existence of these errors.
Time Required
The study will take place during class while studying various Geometry concepts sometime during the Spring 2023 semester from January 11 to May 21. Interviews will occur either in or outside class and last about 10 to 15 minutes after a literacy passage.
Risks
No risks can be thought of other than you learning in a different way using error analysis, literacy, and the sharing of narratives. I will encourage you to express individuality and self-thought in a sincere process of how narratives affect the existence of errors subconsciously.
Benefits
You will have the opportunity to discover how their narratives can subconsciously affect their performance of writing responses to Geometry problems. You can enhance your learning of Geometry concepts with literacy by connecting narratives to error analysis. You can help teachers gain insight on how and why student narratives support comprehension, communication, and learning of Geometry.

Confidentiality
Your responses to any reports obtained from this study will be anonymous and not identified by your name. A code name will be used for you for all research notes and documents. The results of your participation will be confidential and remain secure. Deidentified or coded data will be in a private folder for study validation and further research. Subsequent uses of records and data will be subject to standard data use policies, which protect the confidentiality of individuals and institutions. Participant data will be kept confidential except in cases where the researcher is legally obligated to report specific incidents.
Voluntary Participation: You will be a volunteer and free to withdraw from responses at any time without consequences. If you decide to take part in this study, you will be asked to sign this assent form. After you sign the assent form, you are still free to withdraw at any time and without giving a reason. Withdrawing from this study will not have any consequences. If you withdraw from the study before data collection is completed, your data will not be part of the study and will be shredded.
Right to Ask Questions: Research at Kennesaw State University that involves human participants is carried out under the oversight of an Institutional Review Board. Address questions or problems regarding these activities to the Institutional Review Board, Kennesaw State University, 585 Cobb Avenue, KH3417, Kennesaw, GA 30144-5591, (470) 578-7721.

Assent to Participate
I have read and I understand the provided information and have had the opportunity to ask questions. I understand that my participation in this research project is voluntary and that I am free to withdraw at any time, without giving a reason and without penalty. I understand that I will be given a copy of this assent form. I voluntarily agree to take part in this study.

Participant Name (printed): .....

Participant Signature: .....

Parent/Guardian (printed): .....

Parent/Guardian Signature: ..... Date: ../../23

I verify that the above informed assent procedure has been followed.

Investigator Signature: ..... Date: ../../23

Please sign both copies of this form, keep one, and return the other to the investigator.

Appendix B

Parent/Guardian Consent Form

PARENTAL CONSENT FORM

Title of Research Study: High School Student Narratives of Error Analysis: A Qualitative Investigation into Literacy for Geometry Learning.

Principal Investigator: Mr. Dean Petti, dpetti@paulding.k12.ga.us, Ph: (770)687-5823

Purpose of Study
Your child is being asked to participate in a research study conducted under the guidance of faculty from Kennesaw State University. This project has been reviewed and approved by the KSU Institutional Review Board under tracking number IRB-FY23-165. Before you decide for your child to participate in this study, it is important that you understand why the research is being done and what it will involve. Please read the following information carefully. Please ask the researcher if there is anything that needs to be clear or if you need more information.

The purpose of this study is to discover, analyze and interpret how student narratives play a role in the presence of errors in student responses to Geometry problems. The goals are to improve an understanding that student narratives can have on the existence of errors within responses to Geometry problems and to design lessons that can better match this understanding. This research serves as a means of mathematical literacy, where students make sense of math in various individual and related contexts.

Study Procedures
Your child will write corrected versions of given erroneous responses to error analysis problems relating to literacy passages. Documents, observations, and interviews occur where your child will share stories. The stories will relate to three common errors: a selected wrong strategy, a miscalculation, or a faulty algorithm. Observations and interviews will video record verbal and nonverbal responses for discovery, analysis, and interpretation of the roles that stories have for the existence of these errors.

Time Required
The study will take place during class while studying various Geometry concepts sometime during the Spring 2023 semester from January 11 to May 21. Interviews will occur either in or outside class and last about 10 to 15 minutes after a literacy passage.

Risks
No risks can be thought of other than your child learning in a different way using error analysis, literacy, and the sharing of narratives. I will encourage your child to express individuality and self-thought in a sincere process of how narratives affect the existence of errors subconsciously.

Benefits
Your child will have the opportunity to discover how their narratives can subconsciously affect their performance of writing responses to Geometry problems. Your child can enhance their learning of Geometry concepts with literacy by connecting narratives to error analysis. Your child can help teachers gain insight on how and why student narratives support comprehension, communication, and learning of Geometry.

Confidentiality
Your child's responses to any reports obtained from this study will be anonymous and not identified by name. A code name will be used for your child for all research notes and documents. The results of your child's participation will be confidential and remain secure. Deidentified or coded data will be in a private folder for study validation and further research. Subsequent uses of records and data will be subject to standard data use policies, which protect the confidentiality of individuals and institutions. Participant data will be kept confidential except in cases where the researcher is legally obligated to report specific incidents.

Voluntary Participation: Your child will be a volunteer and free to withdraw from responses at any time without consequences. If you decide for your child to take part in this study, you will be asked to sign this consent form. After you sign the consent form, your child is still free to withdraw at any time and without giving a reason. Withdrawing from this study will not have any consequences. If your child withdraws from the study before data collection is completed, your child's data will not be part of the study and will be shredded.

Right to Ask Questions: Research at Kennesaw State University that involves human participants is carried out under the oversight of an Institutional Review Board. Address questions or problems regarding these activities to the Institutional Review Board, Kennesaw State University, 585 Cobb Avenue, KH3417, Kennesaw, GA 30144-5591, (470) 578-7721.

Consent to Participate
I have read and I understand the provided information and have had the opportunity to ask questions. I understand that my child's participation in this research project is voluntary and that I am free to withdraw at any time, without giving a reason and without penalty. I understand that I will be given a copy of this consent form. I voluntarily agree to have my child take part in this study.

Student Name (printed): .....

Parent/Guardian (printed): .....

Parent/Guardian Signature: ..... Date: ../../23

I verify that the above informed consent procedure has been followed.

Investigator Signature: ..... Date: ../../23

Please sign both copies of this form, keep one, and return the other to the investigator.

Appendix C

Reading passage and error analysis problems of a leadership role of a recreations department for applications of the distance, midpoint, and slope formulas.

THE DIAMOND OF DREAMS

Abner Doubleday (1819-1893)

George Hancock (1846-1956)

Jackie Robinson (1919-1972)

According to the National Baseball Hall of Fame, both baseball and softball are American family pastimes and have active roles in shaping and unifying our nation. Civil War hero Abner Doubleday is credited with creating baseball in 1839, while George Hancock is known for inventing softball in 1887. In the Civil War, both sports were played as diversions to promote commonality between the North and South. In World War 2, both sports continued to rally and support the 500 Major Leaguers serving the war and the founding of the first woman's baseball league, documented by the movie *A League of Their Own*. In 1997, the number 42 was retired for every Major League baseball team to commemorate baseball hero Jackie Robinson for overcoming the hardship he experienced as the first African American player to play baseball and for being a civil rights leader for all sports players. In 2001, baseball and softball continued despite terrorist attacks against the United States to promote strength and unity for all Americans.

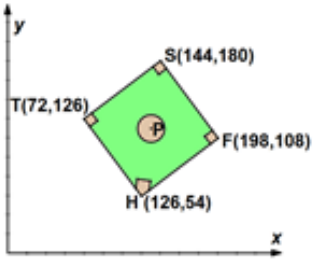
The main part of the baseball and softball fields is in the shape of a square. Known as the *diamond*, three square bases and a pentagonal home plate are connected at right angles with a circular pitching mound in the center. In Geometry, a square is a four-sided shape called a quadrilateral with two pairs of opposite parallel sides, consecutive perpendicular sides, all congruent sides, and all congruent right interior angles. A local recreation department has created a map of a ***baseball*** and ***softball*** diamond. However, there are mistakes in the responses of eight novice employees to solving problems using the distance, slope, and midpoint formulas. Suppose you are the supervisor of this department. How should you write corrections?

WRONG STRATEGY

Recreation Worker 1 – The Department wishes to find the distance between the home plate (H) and first base (F) of a ***baseball*** diamond. The department knows this distance will be the same from first to second, second to third, and third to home.

Wrong Comprehension Strategy	Corrected Comprehension Strategy
Distance between Home Plate and First Base $\frac{108-54}{198-126} = \frac{54}{72} = \frac{3}{4}$ inches	

BASEBALL DIAMOND MAP

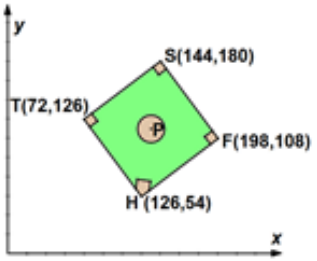


MISCOMPUTATION

Recreation Worker 2 – The Department wishes to find the distance between home plate and second base in order for coaches to train their catchers of how far a ***baseball*** should be thrown to pick off runners trying to steal.

Miscalculation Process Skills	Corrected Calculation Process Skills
Distance between Home Plate (H) and Second Base (S) $\sqrt{(144-126)^2 + (180-54)^2}$ $= \sqrt{18+126} = \sqrt{144}$ = 12 feet	

BASEBALL DIAMOND MAP

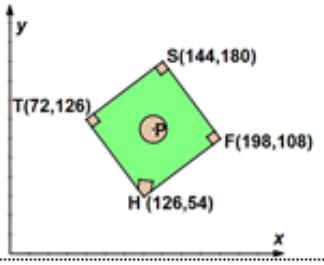


FAULTY ALGORITHM - TRANSFORMATIONAL

Recreation Worker 3 – The Department wishes to find the coordinates of the pitcher's mound for the ***baseball*** diamond.

Transformed Faulty Algorithm	Corrected Algorithm
Midpoint of Home Plate (H) and Second Base (S) $\left(\frac{126+54}{2}, \frac{144+180}{2}\right)$ $= \left(\frac{180}{2}, \frac{324}{2}\right) = (90,162)$	

BASEBALL DIAMOND MAP

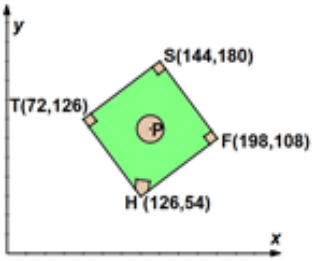


FAULTY ALGORITHM - ENCODING

Recreation Worker 4 – The Department wishes to know slopes between consecutive bases of a ***baseball*** diamond to make sure the angle between bases are right angles.

Encoded Faulty Algorithm	Corrected Algorithm
Slope between Home (H) and First (F) $\frac{108-54}{198-126} = \frac{54}{72} = \frac{2}{3}$ Slope between First (F) and Second (S) $\frac{108-180}{198-144} = \frac{-72}{54} = -\frac{3}{2}$	

BASEBALL DIAMOND MAP

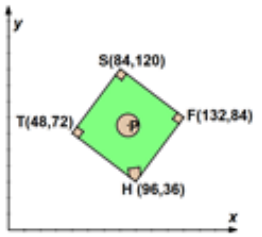


WRONG STRATEGY

Recreation Worker 5 – The Department wishes to find the distance between home plate and first base of a ***softball*** diamond. The department knows this distance will be the same from first to second, second to third, and third to home.

Wrong Comprehension Strategy	Corrected Comprehension Strategy
Distance between Home Plate (H) and First Base (F) $\left(\frac{96+132}{2}, \frac{36+84}{2}\right)$ $= \left(\frac{228}{2}, \frac{120}{2}\right) = (114, 60) = 174$ feet	

SOFTBALL DIAMOND MAP

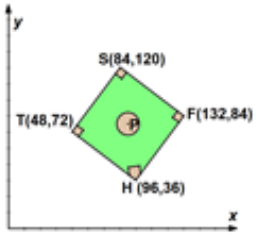


MISCOMPUTATION

Recreation Worker 6 – The Department wishes to find the distance between home plate and second base in order for coaches to train their catchers of how far a ***softball*** should be thrown to pick off runners trying to steal.

Miscalculation Process Skills	Corrected Calculation Process Skills
Distance between Home Plate (H) and Second Base (S) $\sqrt{(84-96)^2 + (120-36)^2}$ $= 144 + 7056 = 7200$ feet	

SOFTBALL DIAMOND MAP

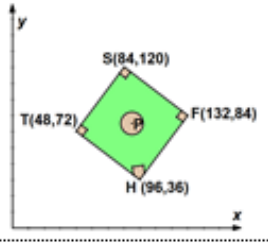


FAULTY ALGORITHM - TRANSFORMATIONAL

Recreation Worker 7 – The Department wishes to find the coordinates of the pitcher's mound for the ***softball*** diamond.

Transformed Faulty Algorithm	Corrected Algorithm
Midpoint of Home Plate (H) and Second Base (S) $\left(\frac{36+120}{2}, \frac{96+84}{2}\right)$ $= \left(\frac{156}{2}, \frac{180}{2}\right) = (78, 90)$	

SOFTBALL DIAMOND MAP

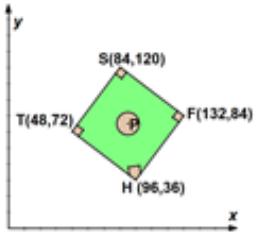


FAULTY ALGORITHM - ENCODING

Recreation Worker 8 – The Department wishes to know slopes between consecutive bases of a ***softball*** diamond to make sure the angle between bases are right angles.

Encoded Faulty Algorithm	Corrected Algorithm
Slope between Home (H) and First (F) $\frac{84-36}{132-96} = \frac{48}{36} = \frac{3}{2}$ Slope between First (F) and Second (S) $\frac{84-120}{132-84} = \frac{-36}{48} = -\frac{2}{3}$	

SOFTBALL DIAMOND MAP

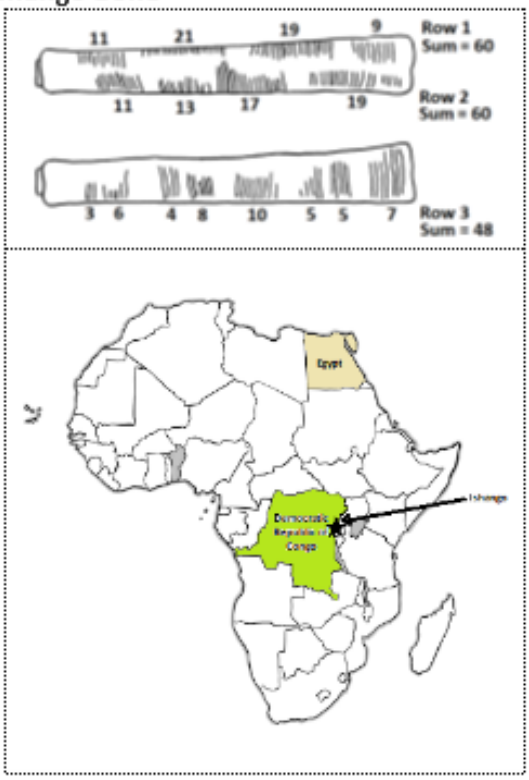


Appendix D

Reading passage and error analysis problems of a leadership role of an archaeologist for applications of perimeter and area.

Archaeology and the Ishango Bone

Archaeology studies human activity through the analysis of artifacts. In 1950, archaeologists discovered a 10 centimeter long bone with sequences of tally marks in a fisherman settlement of Ishango in the Democratic Republic of Congo in central Africa. The bone dates back to approximately 20,000 years old. Many scientists believe that the Ishango bone is the first tool in the world to be ever used as a mathematics ruler of measurement.

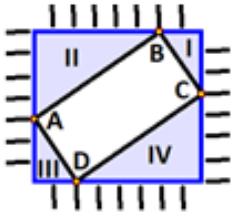


The Ishango bone has three rows, each with a series of tally marks. The sum of the first two rows each equal to 60 with each row containing at least two prime numbers. The third row shows a doubling pattern from 3 to 6 and 4 to 8 and a halving pattern from 10 to 5. There is a theory that the ancient Egyptians used these patterns to find the perimeter and area of rectangles using the Ishango bone.

Suppose you are the lead supervisor of eight archaeologists who have made interpretation errors of calculating perimeters and areas of rectangles using the Ishango bone as a ruler. How can you provide corrected intepretations?

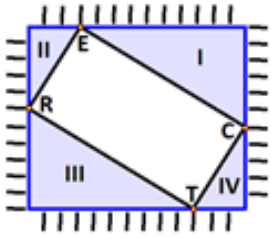
WRONG STRATEGY

Archaeologist 1 wishes to find the PERIMETER of rectangle ABCD

Rectangle	Wrong Comprehension Strategy	Corrected Comprehension Strategy
	Entire Region: 8 x 7 = 56 Region I: 2 x 3 /2 = 3 Region II: 6 x 4 / 2 = 12 Region III: 2 x 3 /2 = 3 Region IV: 6 x 4 / 2 = 12 Perimeter = 56 – 3 – 12 – 3 – 12 = 26 units	

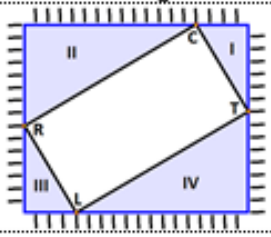
MISCOMPUTATION

Archaeologist 2 wishes to find the PERIMETER of rectangle RECT

Rectangle	Miscalculation Process Skills	Corrected Calculation Process Skills
	$RE = \sqrt{3^2 + 5^2} = \sqrt{6 + 10} = \sqrt{16}$ $EC = \sqrt{10^2 + 6^2} = \sqrt{20 + 12} = \sqrt{32}$ $TC = \sqrt{3^2 + 5^2} = \sqrt{6 + 10} = \sqrt{16}$ $RT = \sqrt{10^2 + 6^2} = \sqrt{20 + 12} = \sqrt{32}$ Perimeter = $2\sqrt{16} + 2\sqrt{32}$ = $8 + 8\sqrt{2} \approx 19.31$ units	

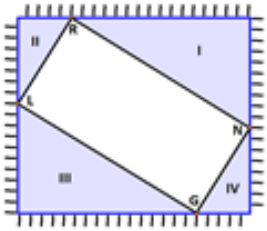
FAULTY ALGORITHM - TRANSFORMATIONAL

Archaeologist 3 wishes to find the PERIMETER of rectangle RCTL

Rectangle	Transformed Faulty Algorithm	Corrected Algorithm
	$RC = \sqrt{13^2 + 7^2} = \sqrt{218}$ $CT = \sqrt{3^2 + 6^2} = \sqrt{45}$ $LT = \sqrt{13^2 + 7^2} = \sqrt{218}$ $RL = \sqrt{3^2 + 6^2} = \sqrt{45}$ Perimeter = $2\sqrt{218} + 2\sqrt{45}$ = $2\sqrt{218} + 6\sqrt{5} \approx 42.95$ units	

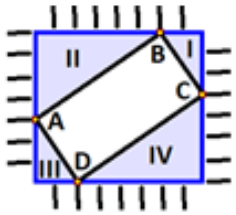
FAULTY ALGORITHM - ENCODING

Archaeologist 4 wishes to find the PERIMETER of rectangle RNGL

Rectangle	Encoded Faulty Algorithm	Corrected Algorithm
	$RN = \sqrt{16^2 + 10^2} = \sqrt{356}$ $GN = \sqrt{5^2 + 8^2} = \sqrt{89}$ $LG = \sqrt{16^2 + 10^2} = \sqrt{356}$ $LR = \sqrt{5^2 + 8^2} = \sqrt{89}$ Perimeter = $\sqrt{356 + 89 + 356 + 89}$ = $\sqrt{890} \approx 29.83$ units	

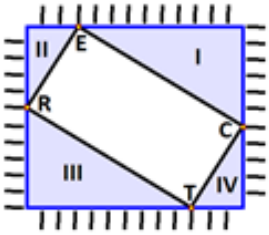
WRONG STRATEGY

Archaeologist 5 wishes to find the AREA of rectangle ABCD

Rectangle	Wrong Comprehension Strategy	Corrected Comprehension Strategy
	$AB = \sqrt{6^2 + 4^2} = \sqrt{52}$ $BC = \sqrt{2^2 + 3^2} = \sqrt{13}$ $DC = \sqrt{6^2 + 4^2} = \sqrt{52}$ $AD = \sqrt{2^2 + 3^2} = \sqrt{13}$ Area = $2\sqrt{52} + 2\sqrt{13}$ = $6\sqrt{13} \approx 21.63$ square units	

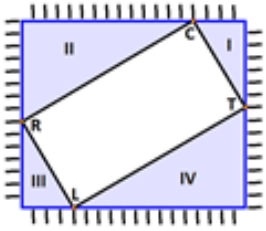
MISCOMPUTATION

Archaeologist 6 wishes to find the AREA of rectangle RECT

Rectangle	Miscalculation Process Skills	Corrected Calculation Process Skills
	Entire Region: 13 x 11 = 143 Region I: 10 x 6 /2 = 60 Region II: 3 x 5 / 2 = 15 Region III: 9 x 5 /2 = 45 Region IV: 2 x 4 / 2 = 8 Area = 143 – 60 – 15 – 45 – 8 = 15 square units	

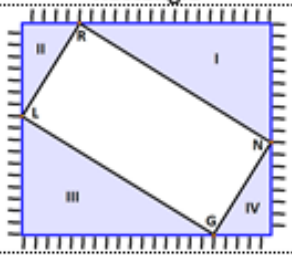
FAULTY ALGORITHM - TRANSFORMATIONAL

Archaeologist 7 wishes to find the AREA of rectangle RCTL

Rectangle	Transformed Faulty Algorithm	Corrected Algorithm
	Entire Region: 18 x 15 = 270 Region I: 4 x 7 = 28 Region II: 14 x 8 = 112 Region III: 4 x 7 = 28 Region IV: 14 x 8 = 112 Area = 270 – 28 – 112 – 28 – 112 = -10 square units	

FAULTY ALGORITHM - ENCODING

Archaeologist 8 wishes to find the AREA of rectangle RNGL

Rectangle	Encoded Faulty Algorithm	Corrected Algorithm
	Entire Region: 21 x 18 = 378 Region I: 16 x 10/2 = 80 Region II: 5 x 8/2 = 20 Region III: 16 x 10/2 = 80 Region IV: 5 x 8/2 = 20 Area = 378 – 80 – 20 = 278 square units	

Appendix E

Reading passage and error analysis problems of a leadership role of an air traffic controller for applications of counting points and segments.

Air Traffic Controllers

Air traffic controllers (ATCs) direct aircraft through a given section of controlled airspace. The primary purpose of ATCs is to prevent collisions, organize the flow of air traffic, and provide information and support for pilots. ATCs monitor the location of aircraft in their assigned airspace by radar to communicate with the pilots by radio. Every ATC needs to determine the total amount of one-way routes that can occur between several airports to set up non-collision times for planes to fly. Geometrically, a one-way route is a distance along a line between two points called segments with a diagonal one-way route being a segment betweenairports not next to each other. The skills of an ATC are to apply the amount of routes originating from one airport as the same as any other airport within a network and to avoid counting the same one-way route twice between any two airports.

Suppose you are the manager of eight novice ATC's trying to find the number of one-way routes between a certain number of airports. Each ATC uses technology to acquire a drawing of the routes. You analyze and notice errors in their problem solving. How can you as their supervisor correct their responses?

WRONG STRATEGY

Air Traffic Controller 1 wishes to find the total amount of one-way routes between 9 airports.

Amount of Routes from one airport	Wrong Comprehension Strategy	Corrected Comprehension Strategy
	There are eight points from one segment. So, the total amount of one-way routes is $9 \times 8 = 72$.	

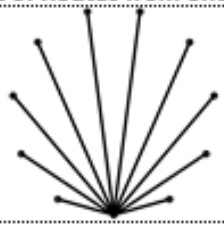
MISCOMPUTATION

Air Traffic Controller 2 wishes to find the total amount of one-way routes between 10 airports.

Amount of Routes from one airport	Miscalculation Process Skills	Corrected Calculation Process Skills
	There are eight segments from one point. So, the total amount of one-way routes is $10 \times 8 / 2 = 40$.	

FAULTY ALGORITHM - TRANSFORMATIONAL

Air Traffic Controller 3 wishes to find the total amount of one-way routes between 11 airports.

Amount of Routes from one airport	Transformed Faulty Algorithm	Corrected Algorithm
	There are nine segments from one point. So, the total amount of one-way routes is $10 \times 9 / 2 = 45$.	

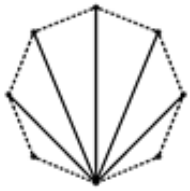
FAULTY ALGORITHM - ENCODING

Air Traffic Controller 4 wishes to find the total amount of one-way routes between 12 airports.

Amount of Routes from one airport	Encoded Faulty Algorithm	Corrected Algorithm
	There are eleven segments from one point. So, the total amount of one-way routes is $12 \times 11 / 2 = 132$.	

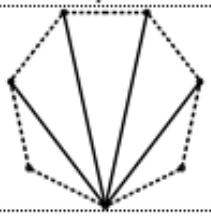
WRONG STRATEGY

Air Traffic Controller 5 wishes to find the total amount of one-way DIAGONAL routes between 8 airports.

Amount of Diagonal Routes from one airport	Wrong Comprehension Strategy	Corrected Comprehension Strategy
	There are seven diagonals from one point. So, the total amount of one-way diagonal routes is 8 × 7 / 2 = 28.	

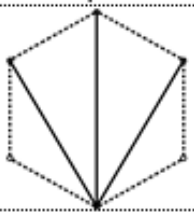
MISCOMPUTATION

Air Traffic Controller 6 wishes to find the total amount of one-way DIAGONAL routes between 7 airports.

Amount of Diagonal Routes from one airport	Miscalculation Process Skills	Corrected Calculation Process Skills
	There are four diagonals from one point. So, the total amount of one-way diagonal routes is 7 × 4 / 2 = 28.	

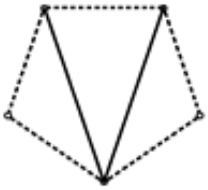
FAULTY ALGORITHM - TRANSFORMATIONAL

Air Traffic Controller 7 wishes to find the total amount of one-way DIAGONAL routes between 6 airports.

Amount of Diagonal Routes from one airport	Transformed Faulty Algorithm	Corrected Algorithm
	There are six diagonals from one point. So, the total amount of one-way diagonal routes is 6 x 3/ 2 = 9.	

FAULTY ALGORITHM - ENCODING

Air Traffic Controller 8 wishes to find the total amount of one-way DIAGONAL routes between 5 airports.

Amount of Diagonal Routes from one airport	Encoded Faulty Algorithm	Corrected Algorithm
	There are two diagonals from one point. So, the total amount of one-way diagonal routes is 5 x 2 = 10.	

Appendix F

Reading passage and error analysis problems of a leadership role of a robotic engineer for sequences of transformations.

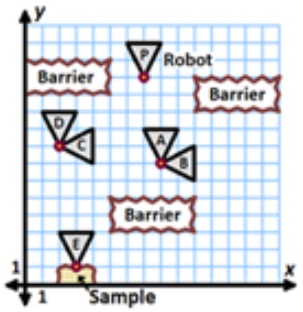
Robotic Engineering

Robotic engineering is a math, computer science, and engineering technical field of work that deals with the design, construction, and operation of robots. ***Opportunity*** was a robotic rover that was active on Mars from 2004 to 2018. It maintained functionality through recharging of its batteries with solar power and hibernated during dust storms to save power. During its venture, Opportunity searched for rock samples that contained clues of past water activity on Mars. Opportunity could translate and rotate as long as it avoided barriers in its vicinity. In order to function, the robot was at least 1 unit away from any barrier. The point of rotation was at the front of the robot. A diagram represented a map of the Martian surface, including the robot, the barriers to avoid, and the rock sample analyzed. Once a rock sample was in its grasp, Opportunity was able to reflect and rotate the surface of the rock. Suppose your job is to correct erroneous instructions provided by eight robotic engineers trying to program Opportunity to move towards a sample and to have Opportunity place samples in different orientations. How can you as their supervisor rewrite corrected solutions?

WRONG STRATEGY

Robotic Engineer 1 monitors movement of the robot

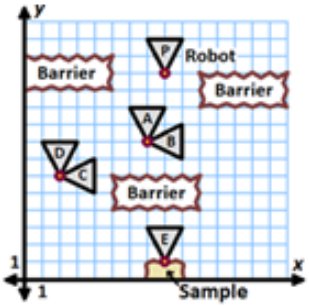
Transformation	Wrong Comprehension	Corrected Comprehension
P to A	Reflect right 1, down 5	
A to B	Translate right 1, down 3	
B to C	Reflect left 6, up 1	
C to D	Translate left 3, up 1	
D to E	Reflect right 1, down 7	



MISCOMPUTATION

Robotic Engineer 2 monitors movement of the robot

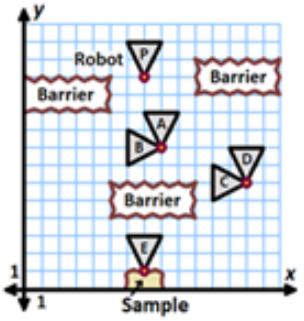
Transformation	Miscalculation Process Skills	Corrected Calculation Skills
P to A	Translate right 1, down 5	
A to B	Rotate 90° CW about (8,7)	
B to C	Translate left 6, up 1	
C to D	Rotate 90° CCW about (2,8)	
D to E	Translate right 1, down 7	



FAULTY ALGORITHM - TRANSFORMATIONAL

Robotic Engineer 3 monitors movement of the robot

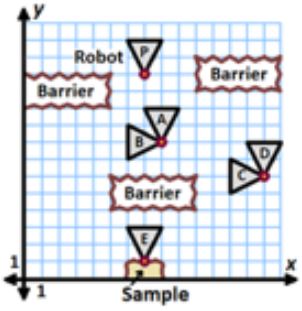
Transformation	Transformed Faulty Algorithm	Corrected Algorithm
P to A	Translate left 1, down 4	
A to B	Rotate 90° CW about (7, 8)	
B to C	Translate left 5, down 2	
C to D	Rotate 90° CCW about (2, 6)	
D to E	Translate down 5, right 6	



FAULTY ALGORITHM - ENCODING

Robotic Engineer 4 monitors movement of the robot

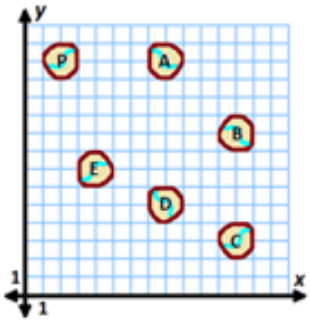
Transformation	Encoded Faulty Algorithm	Corrected Algorithm
P to A	Translate right 1, down 4	
A to B	Rotate 90° CCW about (7, 8)	
B to C	Translate right 5, down 2	
C to D	Rotate 90° CCW about (13, 6)	
D to E	Translate down 5, left 6	



WRONG STRATEGY

Robotic Engineer 5 monitors movement of a sample rock by the robot

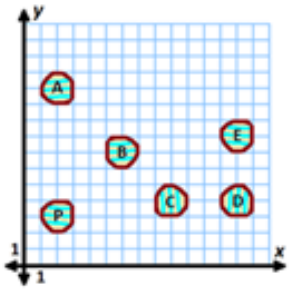
Transformation	Wrong Comprehension	Corrected Comprehension
P to A	Rotate over x = 5	
A to B	Reflect 180° about (10, 11)	
B to C	Rotate over y = 6	
C to D	Reflect 90° CW about (9, 2)	
D to E	Reflect 90° CW about (5, 4)	



MISCOMPUTATION

Robotic Engineer 6 monitors movement of a sample rock by the robot

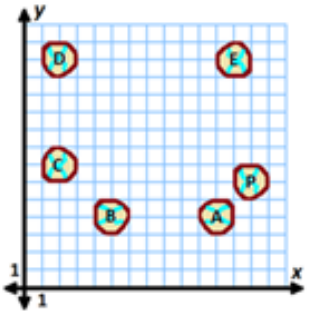
Transformation	Miscalculation Process Skills	Corrected Calculation Skills
P to A	Reflect over y = 6	
A to B	Rotate 90° CW about (4, 9)	
B to C	Rotate 90° CW about (9, 7)	
C to D	Reflect over x = 12	
D to E	Rotate 90° CW about (11, 6)	



FAULTY ALGORITHM - TRANSFORMATIONAL

Robotic Engineer 7 monitors movement of a sample rock by the robot

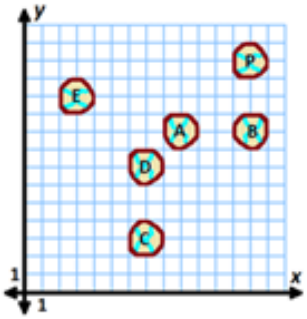
Transformation	Transformed Faulty Algorithm	Corrected Algorithm
P to A	Rotate 90° CCW about (11, 6)	
A to B	Reflect over y = 8	
B to C	Rotate 90° CCW about (2, 4)	
C to D	Reflect over y = 10	
D to E	Rotate 180° about (7, 13)	



FAULTY ALGORITHM - ENCODING

Robotic Engineer 8 monitors movement of a sample rock by the robot

Transformation	Encoded Faulty Algorithm	Corrected Algorithm
P to A	Rotate 90° CW about (9, 13)	
A to B	Reflect over x = 11	
B to C	Rotate 180° about (10, 6)	
C to D	Reflect over x = 5	
D to E	Rotate 90° CW about (3, 7)	



Appendix G

Reading passage of a leadership role of a video game designer for rigid motions.

Robotic Engineering

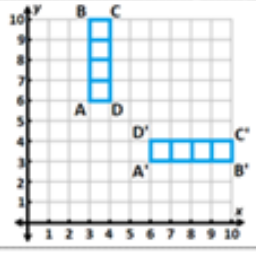
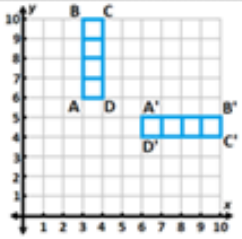

Alexey Pajitnov

Tetris is a tile-matching video game created by Russian software engineer Alexey Pajitnov in 1984. The tiles of Tetris are called tetriminos. A tetromino is a geometric shape composed of four squares that connect orthogonally (meaning at the edges and not corners). You may be familiar with the related domino consisting of two squares. The prefix tetra means four in Greek. The rigid motion of translation moves a tetromino left and right while shifting downward. The rigid motion of rotation is used to turn a tetrimino 90 degrees clockwise or counterclockwise and 180 degrees. Video game designers use these rigid motions to geometrically describe the location and form of the seven tetrominoes, named after letters of the alphabet, located below:

Suppose your job is to manage eight beginning video game designers. You analyze and notice errors from each constructing a blueprint of one of the tetrominoes. How can you as their supervisor write the error made and rewrite the corrected rigid motion in detail?

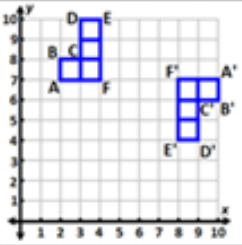
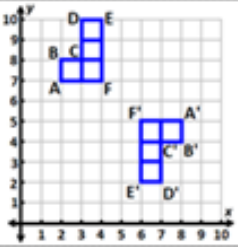
WRONG STRATEGY

Designer 1 – A line instead of a point was used to transform the "I" Tetromino.

Wrong Comprehension Strategy	Corrected Comprehension Strategy
 ABCD was ... to form A'B'C'D'.	 ABCD was ... to form A'B'C'D'.

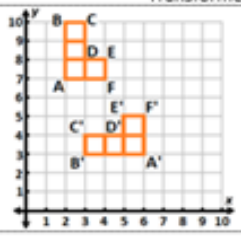
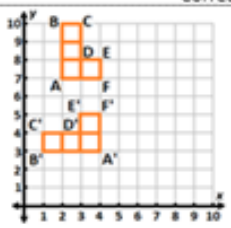
MISCOMPUTATION

Designer 2 – Another point was supposed to be used to transform the "J" Tetromino.

Miscalculation Process Skills	Corrected Calculation Process Skills
 ABCD was ... to form A'B'C'D'.	 ABCD was ... to form A'B'C'D'.

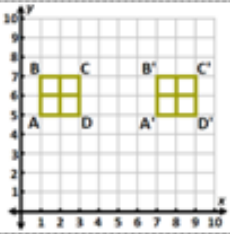
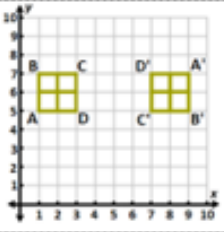
FAULTY ALGORITHM - TRANSFORMATIONAL

Designer 3 – Another point was supposed to be used to transform the "L" Tetromino.

Transformed Faulty Algorithm	Corrected Algorithm
 ABCD was ... to form A'B'C'D'.	 ABCD was ... to form A'B'C'D'.

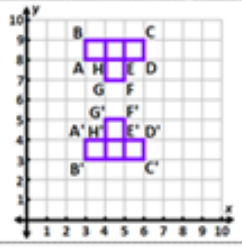
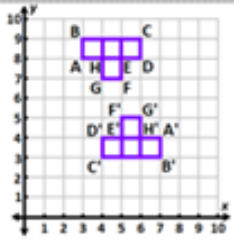
FAULTY ALGORITHM - ENCODING

Designer 4 – A point was supposed to be used to transform the "O" Tetromino.

Encoded Faulty Algorithm	Corrected Algorithm
 ABCD was ... to form A'B'C'D'.	 ABCD was ... to form A'B'C'D'.

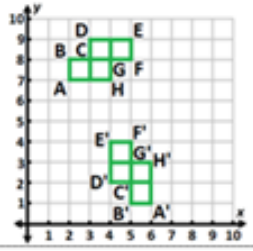
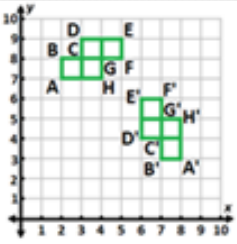
WRONG STRATEGY

Designer 5 – A line instead of a point was used to transform the "T" Tetromino.

Wrong Comprehension Strategy	Corrected Comprehension Strategy
 ABCD was ... to form A'B'C'D'.	 ABCD was ... to form A'B'C'D'.

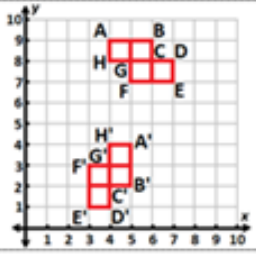
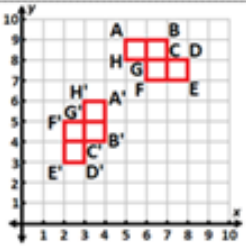
MISCOMPUTATION

Designer 6 – Another point was supposed to be used to transform the "S" Tetromino.

Miscalculation Process Skills	Corrected Calculation Process Skills
 ABCD was ... to form A'B'C'D'.	 ABCD was ... to form A'B'C'D'.

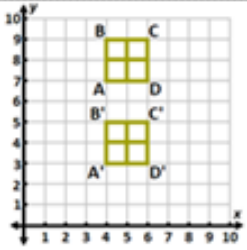
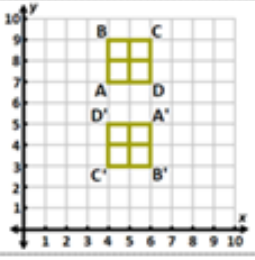
FAULTY ALGORITHM - TRANSFORMATIONAL

Designer 7 – Another point was supposed to be used to transform the "S" Tetromino.

Transformed Faulty Algorithm	Corrected Algorithm
 ABCD was ... to form A'B'C'D'.	 ABCD was ... to form A'B'C'D'.

FAULTY ALGORITHM - ENCODING

Designer 8 – A point was supposed to be used to transform the "O" Tetromino.

Encoded Faulty Algorithm	Corrected Algorithm
 ABCD was ... to form A'B'C'D'.	 ABCD was ... to form A'B'C'D'.