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1-independent percolation on $\mathbb{Z}^2 \times K_n$

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Abstract

A random graph model on a host graph H is said to be *1-independent* if for every pair of vertex-disjoint subsets A, B of $E(H)$, the state of edges (absent or present) in A is independent of the state of edges in B. For an infinite connected graph H, the *1-independent critical percolation probability* $p_{1,c}(H)$ is the infimum of the $p \in [0, 1]$ such that every 1-independent random graph model on H in which each edge is present with probability at least p almost surely contains an infinite connected component. Balister and Bollobás observed in 2012 that $p_{1,c}(\mathbb{Z}^d)$ tends to a limit in $\left[\frac{1}{2}, 1\right]$ as $d \to \infty$, and they asked for the value of this limit. We make progress on a related problem by showing that

$$\lim_{n\to\infty} p_{1,c}(\mathbb{Z}^2 \times K_n) = 4 - 2\sqrt{3} = 0.5358 \ldots .$$

In fact, we show that the equality above remains true if the sequence of complete graphs K_n is replaced by a sequence of weakly pseudorandom graphs on n vertices with average degree $\omega(\log n)$. We conjecture the answer to Balister and Bollobás's question is also $4 - 2\sqrt{3}$.

KEYWORDS

extremal graph theory, locally dependent random graphs, percolation theory

1 | INTRODUCTION

1.1 | Background

Percolation theory lies at the interface of probability theory, statistical physics and combinatorics. Its object of study is, roughly speaking, the connectivity properties of random subgraphs of infinite connected graphs, and in particular the points at which these undergo drastic transitions such as the emergence of infinite components. Since its inception in Oxford in the late 1950s, percolation theory has become a rich field of study (see e.g., the monographs [9, 15, 22]). One of the cornerstones of the discipline is the Harris–Kesten theorem [17, 18], which states that if each edge of the integer square lattice $\mathbb{Z}^2$ is open independently at random with probability p, then if $p \leq \frac{1}{2}$ almost surely all connected components of open edges are finite, while if $p > \frac{1}{2}$ almost surely there exists an infinite connected component of open edges. Thus $1/2$ is what is known as the *critical probability for independent bond percolation* on $\mathbb{Z}^2$.

In general, given an infinite connected graph H, determining the critical probability for independent bond percolation on H is a hard problem, with the answer known exactly only in a handful of cases. There is thus great interest in methods for rigorously estimating such critical probabilities. One of the most powerful and effective techniques for doing just that was developed by Balister et al. [5], and relies on comparing percolation processes with locally dependent bond percolation on $\mathbb{Z}^2$ (to be more precise: 1-independent bond percolation; see below for a definition). The method of Balister, Bollobás and Walters has proved influential, and has been widely applied to obtain the best rigorous confidence interval estimates for the value of the critical parameter in a wide range of models, see, for example, [1, 3–8, 12, 13, 16, 24].

However, as noted by the authors of [5] and again by Balister and Bollobás [2] in 2012, locally dependent bond percolation is poorly understood. To quote from the latter work, "[given that] 1-independent percolation models have become a key tool in establishing bounds on critical probabilities […], it is perhaps surprising that some of the most basic questions about 1-independent models are open." In particular, there is no known locally dependent analog of the Harris–Kesten theorem, nor even until now much of a sense of what the corresponding 1-independent critical probability ought to be. In this article, we contribute to the broader project initiated by Balister and Bollobás of addressing the gap in our knowledge about 1-independent bond percolation by making some first steps toward a 1-independent Harris–Kesten theorem. To state our results and place them in their proper context, we first need to give some definitions.

Let $H = (V, E)$ be a graph. Given a probability measure μ on subsets of E, a *μ-random graph* $\mathbf{H}_\mu$ is a random spanning subgraph of H whose edge-set is chosen randomly from subsets of E according to the law given by μ. Each probability measure μ on subsets of E thus gives rise to a *random graph model* on the host graph H, and we use the two terms (probability measure μ on subsets of E/random graph model $\mathbf{H}_\mu$ on H) interchangeably. In this article we will be interested in random graph models where the state (present/absent) of edges is dependent only on the states of nearby edges. Recall that the *graph distance* between two subsets $A, B \subseteq E$ is the length of the shortest path in H from an endpoint of an edge in A to an endpoint of an edge in B. So in particular if an edge in A shares a vertex with an edge in B, then the graph distance from A to B is zero, while if A and B are supported on disjoint vertex-sets, then the graph distance from A to B is at least one.

Definition 1.1 (k-independence)**.** A random graph model $\mathbf{H}_\mu$ on a host graph H is k-independent if whenever A, B are disjoint subsets of $E(H)$ such that the graph distance between A and B is at least k, the random variables $E(\mathbf{H}_\mu) \cap A$ and $E(\mathbf{H}_\mu) \cap B$ are mutually independent. If $\mathbf{H}_\mu$ is k-independent, we say that the associated probability measure μ *is a* k-independent measure, or k-ipm, on H.

Let $\mathcal{M}_{k,\geq p}(H)$ denote the collection of all k-independent measures μ on $E(H)$ in which each edge of H is included in $\mathbf{H}_\mu$ with probability at least p. We define $\mathcal{M}_{k,\leq p}(H)$ mutatis mutandis, and let $\mathcal{M}_{1,p}(H)$ denote $\mathcal{M}_{k,\geq p} \cap \mathcal{M}_{k,\leq p}$—in other words $\mathcal{M}_{k,p}$ is the collection of all k-ipm μ on H in which each edge of H is included in $\mathbf{H}_\mu$ with probability exactly p.

Observe that a 0-independent measure μ is what is known as a *Bernoulli* or *product* measure on E: each edge in E is included in $\mathbf{H}_\mu$ at random independently of all the others. We refer to such measures as *independent measures*. The collection $\mathcal{M}_{0,p}(H)$ thus consists of a single measure, the *p-random measure*, in which each edge of H is included in the associated random graph with probability p, independently of all the other edges. When the host graph H is K_n, the complete graph on n vertices, this gives rise to the celebrated *Erdős–Rényi random graph model*, while when $H = \mathbb{Z}^2$ this is exactly the *independent bond percolation* model considered in the Harris–Kesten theorem.

In this article, we will focus instead on $\mathcal{M}_{1,\geq p}(H)$ and $\mathcal{M}_{1,p}(H)$, whose probability measures allow for some local dependence between the edges. A simple and well-studied example of a model from $\mathcal{M}_{1,p}(H)$ is given by *site percolation*: build a random spanning subgraph $\mathbf{H}_\theta^{\text{site}}$ of H by assigning each vertex $v \in V(H)$ a state S_v independently at random, with $S_v = 1$ with probability θ and $S_v = 0$ otherwise, and including an edge $uv \in E(H)$ in $\mathbf{H}_\theta^{\text{site}}$ if and only if $S_u = S_v = 1$. Each edge in this random graph is open with probability $p = \theta^2$, and the model is clearly 1-independent since "randomness resides in the vertices," and so what happens inside two disjoint vertex sets is independent. More generally, any *state-based model* obtained by first assigning independent random states S_v to vertices $v \in V(H)$ and then adding an edge uv according to some deterministic or probabilistic rule depending only on the ordered pair (S_u, S_v) will give rise to a 1-ipm on H. State-based models are a generalization of the probabilistic notion of a *two-block factor*, see [20] for details.

Given a 1-ipm μ on an infinite connected graph H, we say that μ *percolates* if $\mathbf{H}_\mu$ almost surely (i.e., with probability 1) contains an infinite connected component.[1]

Definition 1.2. Given an infinite connected graph H, we define the 1-independent critical percolation probability for H to be

$$p_{1,c}(H) := \inf \left\{ p \geq 0 : \ \forall \mu \in \mathcal{M}_{1,\geq p}(H), \ \mu \ \text{percolates} \right\}.$$

Remark 1.3. Given $\mu \in \mathcal{M}_{1,\geq p}(H)$ we can obtain a random graph $\mathbf{H}_\nu$ from $\mathbf{H}_\mu$ by deleting each edge uv of $\mathbf{H}_\mu$ independently at random with probability $1-p/\left(\mathbb{P}\left[uv \in E(\mathbf{H}_\mu)\right]\right)$. Clearly $\mathbf{H}_\mu$ stochastically dominates (i.e., is a supergraph of) $\mathbf{H}_\nu$ and $\nu \in \mathcal{M}_{1,p}(H)$. Thus the definition of $p_{1,c}(H)$ above is unchanged if we replace $\mathcal{M}_{1,\geq p}(H)$ by $\mathcal{M}_{1,p}(H)$.

Remark 1.4. The probability $p_{1,c}(H)$ is in fact one of five natural critical probabilities for 1-independent percolation one could consider, all of which are distinct in general—see [10, Section 11.3, Corollary 50 and Question 53].

Balister et al. [5] devised a highly effective method for giving rigorous confidence interval results for critical parameters in percolation theory via comparison with 1-independent models on the

[1] Note the existence of an infinite connected component is a tail event, in the sense that one cannot create or destroy an infinite connected component by changing the state of finitely many edges, so that by a 1-independent version of Kolmogorov's zero–one law, $\mathbf{H}_\mu$ contains an infinite connected component with probability 0 or 1 (see the discussion below Theorem 1 in [9, Chapter 2]).

square integer lattice $\mathbb{Z}^2$. Their method relies on estimating the probability of certain *finite, bounded* events (usually via Monte Carlo methods, whence the confidence intervals) and on bounds on the 1-independent critical probability $p_{1,c}(\mathbb{Z}^2)$. Work of Liggett et al. [20] on stochastic domination of independent models by 1-independent models implied $p_{1,c}(\mathbb{Z}^2) < 1$. Balister et al. [5, Theorem 2] obtained the effective upper bound $p_{1,c}(\mathbb{Z}^2) < 0.8639$ via a renormalization argument; this upper bound has not been improved since, and the authors of [5] noted "it would be of interest to give significantly better bounds for $p_{1,c}(\mathbb{Z}^2)$; unfortunately, we cannot even hazard a guess as to [its] value." The question of determining $p_{1,c}(\mathbb{Z}^2)$ was raised again by Balister and Bollobás [2, Question 2], who noted the difficulty of the problem:

Problem 1.5 (1-independent Harris–Kesten problem)**.** Determine $p_{1,c}(\mathbb{Z}^2)$.

Balister and Bollobás [2] observed that a simple modification of site percolation due to to Newman shows that $p_{1,c}(\mathbb{Z}^2) \geq (\theta_s)^2 + (1-\theta_s)^2$, where $\theta_s = \theta_s(\mathbb{Z}^2)$ is the critical probability for site percolation in $\mathbb{Z}^2$. Since it is known that $\theta_s \in [0.556, 0.679492]$ (see [26, 27]), this shows that $p_{1,c}(\mathbb{Z}^2) \geq 0.5062$. Non-rigorous simulation-based estimates $\theta_s \approx 0.597246$ [28] improve this to a non-rigorous lower bound of 0.5172. Recently, Day, Hancock and the first author gave significant improvements on these lower bounds. In [10, Theorem 7], they constructed measures based on an idea from the first author's PhD thesis [14, Theorem 62] showing that for any $d \in \mathbb{N}$, $p_{1,c}(\mathbb{Z}^d) \geq 4 - 2\sqrt{3} = 0.5358\ \ldots$. They in fact showed $p_{1,c}(H) \geq 4 - 2\sqrt{3}$ for any host graph H satisfying what they call the *finite 2-percolation property* (see Section 3 for a formal definition), a family which includes the graphs $\mathbb{Z}^2 \times K_n$ for any $n \in \mathbb{N}$. (Recall that the Cartesian product $H \times K_n$ of a graph H with K_n is the graph whose vertices are the pairs $(v, i) \in V(H) \times \{1, 2, \ldots\ n\}$ and in which two distinct vertices (v, i) and (v', i') are joined by an edge if either $v = v'$ or vv' is an edge of H and $i = i'$; see Section 1.4 for an illustration and a more general definition of the Cartesian product of two graphs.) Further, the same authors gave a different construction [10, Theorem 8] showing that

$$p_{1,c}(\mathbb{Z}^2) \geq (\theta_s)^2 + \frac{1-\theta_s}{2}, \tag{1.1}$$

where $\theta_s = \theta_s(\mathbb{Z}^2)$ is the critical probability for site percolation in $\mathbb{Z}^2$. Using the aforementioned simulation-based estimates for θ_s, this gives a non-rigorous lower bound of 0.5549 on $p_{1,c}(\mathbb{Z}^2)$. All these lower bounds remain far apart from the upper bound of 0.8639 from [5], and, as noted in [5], part of the difficulty of Problem 1.5 has been the absence of a clear candidate conjecture to aim for.

In view of the difficulty of Problem 1.5, there has been interest in increasing our understanding of 1-independent models on other host graphs than $\mathbb{Z}^2$. Balister and Bollobás noted $p_{1,c}(\mathbb{Z}^d)$ is non-increasing in d and must therefore converge to a limit as $d \to \infty$. They showed this limit is at least $1/2$ and posed the following problem [2, Question 2]:

Problem 1.6 (Balister and Bollobás problem)**.** Determine $\lim_{d\to\infty} p_{1,c}(\mathbb{Z}^d)$.

By the construction of Day, Falgas-Ravry and Hancock mentioned above, this limit is in fact at least $4 - 2\sqrt{3}$; the only known upper bound is again the 0.8639 upper bound on $p_{1,c}(\mathbb{Z}^2)$ from [5].

Balister and Bollobás have further studied 1-independent models on infinite trees, obtaining in this setting 1-independent analogs of classical results of Lyons [21] for independent bond percolation. Day, Hancock and the first author for their part gave a number of results on the connectivity of 1-independent random graphs on paths and complete graphs, and on the almost sure emergence of arbitrarily long paths in 1-independent models. More precisely, they introduced the *Long Paths critical probability* $p_{1,LP}(H)$ of H, given by

$$p_{1,LP}(H) := \inf \left\{ p \in [0,1] : \ \forall \mu \in \mathcal{M}_{1,p}, \forall \ell \in \mathbb{N}, \ \mathbb{P}\left[\mathbf{H}_\mu \ \text{contains a path of length} \ \ell\right] > 0 \right\},$$

and showed $p_{1,LP}(\mathbb{Z}) = 3/4$, $p_{1,LP}(\mathbb{Z} \times K_2) = 2/3$. Since the sequence $p_{1,LP}(\mathbb{Z} \times K_n)$ is non-increasing in n, it tends to a limit in $[0,1]$ as $n \to \infty$. Day, Hancock and the first author showed in [10, Theorem 12(v)] that this limit lies in the interval $[4-2\sqrt{3}, 5/9]$ and asked [10, Problem 54]:

Problem 1.7 (Day, Falgas-Ravry and Hancock)**.** Determine $\lim_{n\to\infty} p_{1,LP}(\mathbb{Z} \times K_n)$.

1.2 | Contributions of this article

Our main result in this article is determining the limit of the 1-independent critical probability for percolation in $\mathbb{Z}^2 \times K_n$ as $n \to \infty$:

Theorem 1.8. *The following hold*:

(i) if $p > 4 - 2\sqrt{3}$ is fixed, then there exists $N \in \mathbb{N}$ such that $p_{1,c}\left(\mathbb{Z}^2 \times K_N\right) \leq p$;
(ii) for every $n \in \mathbb{N}$, $p_{1,c}\left(\mathbb{Z}^2 \times K_n\right) \geq 4 - 2\sqrt{3}$.

In particular, we have $\lim_{n\to\infty} p_{1,c}(\mathbb{Z}^2 \times K_n) = 4 - 2\sqrt{3} = 0.5358 \ldots$.

As a corollary to the key result in our proof of Theorem 1.8, we also obtain a solution to the problem of Day, Falgas-Ravry and Hancock on long paths in 1-independent percolation, Problem 1.7 above:

Theorem 1.9. $\lim_{n\to\infty} p_{1,LP}\left(\mathbb{Z} \times K_n\right) = 4 - 2\sqrt{3}$.

In fact, we are able to show the conclusions of Theorems 1.8 and 1.9 still hold if we replace the complete graph K_n by a suitable *pseudorandom graph*. Recall that the study of pseudorandom graphs originates in the ground-breaking work of Thomason [25]. In this article we shall use the following notion of weak pseudorandomness (see Condition (3) in the survey of Krivelevich and Sudakov [19]):

Definition 1.10. Let $q = q(n)$ be a sequence in $[0,1]$. A sequence $(G_n)_{n\in\mathbb{N}}$ of n-vertex graphs is weakly q-pseudorandom if

$$\max \left\{ \left| e(G_n[U]) - q\frac{|U|^2}{2} \right| : \ U \subseteq V(G_n) \right\} = o(qn^2).$$

Note that if $(G_n)_{n\in\mathbb{N}}$ is a sequence of weakly q-pseudorandom graphs, then for any $U_1, U_2 \subseteq V(G_n)$ with $U_1 \cap U_2 = \emptyset$, we have

$$e(G_n[U_1, U_2]) = q\,|U_1|\,|U_2| + o(qn^2).$$

Theorem 1.11. *Let $q = q(n)$ satisfy $nq(n) \gg \log n$. Then for any sequence $(G_n)_{n\in\mathbb{N}}$ of n-vertex graphs which is weakly q-pseudorandom, we have* $\lim_{n\to\infty} p_{1,c}(\mathbb{Z}^2 \times G_n) = 4 - 2\sqrt{3}$.

Theorem 1.12. *Let $q = q(n)$ satisfy $nq(n) \gg \log n$. Then for any sequence $(G_n)_{n\in\mathbb{N}}$ of n-vertex graphs which is weakly q-pseudorandom, we have* $\lim_{n\to\infty} p_{1,LP}(\mathbb{Z} \times G_n) = 4 - 2\sqrt{3}$.

We conjecture that the conclusion of Theorem 1.8 still holds if we replace the complete graph K_n by an n-dimensional hypercube.

Conjecture 1.13. $\lim_{n\to\infty} p_{1,c}(\mathbb{Z}^2 \times Q_n) = 4 - 2\sqrt{3}$.

Observe that, since $\mathbb{Z}^2 \times Q_n$ is a subgraph of $\mathbb{Z}^{n+2}$ and $p_{1,c}(\mathbb{Z}^{n+2}) \geq 4 - 2\sqrt{3}$ [10, Theorem 7], Conjecture 1.13 implies that the answer to the problem of Balister and Bollobás (Problem 1.6 above) is $4 - 2\sqrt{3}$. In fact, we make the following bolder conjecture:

Conjecture 1.14 (1-independent percolation in high dimension)**.** *There exists $d \geq 3$ such that*

$$p_{1,c}(\mathbb{Z}^d) = 4 - 2\sqrt{3}.$$

Finally we prove some modest results on component evolution in 1-independent models on K_n and on pseudorandom graphs. The main point of these results is that "the two-state measure minimizes the size of the largest component," a heuristic which in turn guides our Conjecture 1.13. Here by the *two-state measure*, we mean the following variant of site percolation, due to Newman (see [23]):

Definition 1.15 (Two-state measure)**.** Let H be a graph, and let $p \in \left[\frac{1}{2}, 1\right]$. The two-state measure $\mu_{2s,p} \in \mathcal{M}_{1,p}(H)$ is constructed as follows: assign to each vertex $v \in V(H)$ a state S_v independently and uniformly at random, with $S_v = 1$ with probability $\theta = \theta(p) = (1 + \sqrt{2p-1})/2$ and $S_v = 0$ otherwise. Then let $\mathbf{H}_{\mu_{2s,p}}$ be the random subgraph of H obtained by including an edge if and only if its endpoints are in the same state.

Day, Hancock and the first author showed in [10, Theorem 16] that $\mu_{2s,p}$ minimizes the probability of connected subgraphs over all 1-ipm $\mu \in \mathcal{M}_{1,p}(K_{2n})$. We show below that it also minimizes the probability of having a component of size greater than n. Explicitly, given a set of edges $F \subseteq E(H)$ in a graph H, we let $C_i(F)$ denote the ith largest connected component in the associated subgraph $(V(H), F)$ of H. Then:

Proposition 1.16. *Set $p_{2n} = \frac{1}{2}\left(1 - \tan^2\left(\frac{\pi}{4n}\right)\right)$ and $H = K_{2n}$. Then for all $p \in [p_{2n}, 1]$,*

$$\min\left\{\mathbb{P}\left[|C_1(\mathbf{H}_\mu)| > n\right] \;:\; \mu \in \mathcal{M}_{1,\geq p}(K_{2n})\right\} = 1 - \binom{2n}{n}\left(\frac{1-p}{2}\right)^n.$$

Further, we show that the two-state measure also asymptotically minimizes the likely size of a largest component in 1-independent models on pseudorandom graphs:

Theorem 1.17. *Let $r \in \mathbb{N}$, and let $p \in \left(\frac{1}{r+1}, \frac{1}{r}\right]$ be fixed. Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of weakly q-pseudorandom graphs on n vertices with $q = q(n) \gg \log(n)/n$. Then the following hold for $H = H_n$:*

(i) For every $\mu \in \mathcal{M}_{1,p}(H)$, with probability $1 - o(1)$ we have $|C_1(\mathbf{H}_\mu)| \geq (1 - o(1))\frac{1+\sqrt{\frac{(r+1)p-1}{r}}}{r+1}n$.
(ii) There exists $\mu \in \mathcal{M}_{1,p}(H)$ such that with probability $1 - o(1)$ the random graph $\mathbf{H}_\mu$ satisfies $|C_1(\mathbf{H}_\mu)| \leq (1 + o(1))\frac{1+\sqrt{\frac{(r+1)p-1}{r}}}{r+1}n$.

This leads us to the natural conjecture that the two-state measure asymptotically minimizes the size of a largest component in 1-independent models on the hypercube Q_n:

Conjecture 1.18. *Let $p \in (\frac{1}{2}, 1]$ be fixed, and let $H = Q_n$. Then for all $\mu \in \mathcal{M}_{1,\geq p}(Q_n)$, with probability $1 - o(1)$ we have $|C_1(\mathbf{H}_\mu)| \geq \left(\frac{1+\sqrt{2p-1}}{2} - o(1)\right) 2^n$.*

We suspect that a proof of this conjecture combined with the ideas in the present paper would yield a proof of Conjecture 1.13.

Overall, our results would lead us to speculate that the true value of $p_{1,c}(\mathbb{Z}^2)$ is probably a lot closer to the lower bound of 0.5549 from (1.1) than to the upper bound of 0.8639 obtained from renormalization arguments in [5]. However a rigorous proof of improved upper bounds on $p_{1,c}(\mathbb{Z}^2)$ remains elusive for the time being.

1.3 | Organization of the article

The key step in the proof of our main results, Theorem 2.1, is proved in Section 2; it establishes that $p = 4 - 2\sqrt{3}$ is the threshold for ensuring there is a high probability in any 1-independent model of finding a path between the largest components in two disjoint copies of K_n joined by a matching. The argument in a sense captures "what makes the $4 - 2\sqrt{3}$ measure of [10, 14] tick." We then use Theorem 2.1 in Section 3 to prove Theorems 1.8–1.12. Our component evolution results, Proposition 1.16 and Theorem 1.17 are proved in Section 4.

1.4 | Notation

Given $n \in \mathbb{N}$ we write $[n]$ for the discrete interval $\{1, 2, \ldots, n\}$. We write $S^{(2)}$ for the collection of all unordered pairs from a set S. We use standard graph-theoretic notation throughout the article. Given a graph H, we use $V = V(H)$ and $E = E(H)$ to refer to its vertex-set and edge-set respectively, and write $e(H)$ for the size of $E(H)$. Given $X \subseteq V$, we write $H[X]$ for the subgraph of H induced by X, that is, the graph $(X, E(H) \cap X^{(2)})$. For disjoint subsets X, Y of V we also write $H[X, Y]$ for the bipartite subgraph of H induced by $X \sqcup Y$, that is the graph $(X \cup Y, \{xy \in E(H) : x \in X, y \in Y\})$. We denote by K_n the complete graph on n vertices, $K_n = ([n], [n]^{(2)})$.

The *Cartesian product* of two graphs G_1 and G_2 is the graph $G_1 \times G_2$ with $V(G_1 \times G_2) = \{(v_1, v_2) : v_1 \in V(G_1), v_2 \in V(G_2)\}$ and $E(G_1 \times G_2)$ consisting of all pairs $\{(u_1, u_2), (v_1, v_2)\}$ with either $u_1 = v_1 \in V(G_1)$ and $u_2 v_2 \in E(G_2)$ or $u_1 v_1 \in E(G_1)$ and $u_2 = v_2 \in V(G_2)$. In particular if $G_1 = K_2$, that is, a single edge, then $G_1 \times G_2$ is the *bunkbed graph* of G_2 consisting of two disjoint copies of G_2, the *left copy* $\{1\} \times G_2$ and the *right copy* $\{2\} \times G_2$, together with a perfect matching joining each vertex $(1, v)$ in the left copy to its image $(2, v)$ in the right copy. See Figure 1 for an example.

Finally we use the standard Landau notation for asymptotic behavior: given functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f = O(g)$ if $|f(n)| \leq C|g(n)|$ for some $C > 0$ and all n sufficiently large, and $f = o(g)$ if $\lim_{n\to\infty} |f(n)/g(n)| = 0$. We use $f = \Omega(g)$ and $f = \omega(g)$ to denote $g = O(f)$ and $g = o(f)$, respectively. We also sometimes use $f \ll g$ and $f \gg g$ as a shorthand for $f = o(g)$ and $f = \omega(g)$, respectively. Given

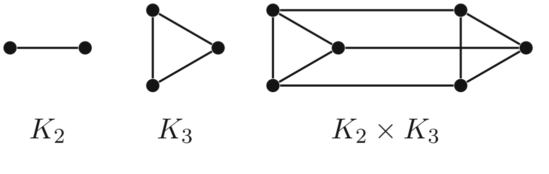

FIGURE 1 The Cartesian product $K_2 \times K_3$

a sequence of events $(E_n)_{n\in\mathbb{N}}$ in some probability space, we say that E_n occurs *with high probability (whp)* if $\mathbb{P}[E_n] = 1 - o(1)$.

2 | WHEN LEFT MEETS RIGHT: JOINING THE LARGEST COMPONENTS ON EITHER SIDE OF $K_2 \times G_n$

Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of weakly q-pseudorandom n-vertex graphs where $qn \gg \log n$. Consider the Cartesian product $H = K_2 \times G_n$. Given $\mu \in \mathcal{M}_{1,p}(H)$, let “Left meets Right” denote the event that the μ-random graph $\mathbf{H}_\mu$ contains a connected component containing both strictly more than half of the vertices in $\{1\} \times [n]$ and strictly more than half of the vertices in $\{2\} \times [n]$. Our main result in this section is showing that the event “Left meets Right” undergoes a sharp transition at $p = 4 - 2\sqrt{3}$, in the sense that for $p \leq 4 - 2\sqrt{3}$ it is possible to construct 1-independent measures $\mu \in \mathcal{M}_{1,p}(H)$ such that whp the event “Left meets Right” does not occur, while for $p > 4 - 2\sqrt{3}$ it occurs whp regardless of the choice of μ.

Theorem 2.1.

(i) Let $p > 4 - 2\sqrt{3}$ be fixed. Then for every $\mu \in \mathcal{M}_{1,p}(H)$,

$$\mathbb{P}\left[\text{Left meets Right}\right] = 1 - o(1).$$

(ii) Let $\frac{1}{2} < p \leq 4 - 2\sqrt{3}$ be fixed. Then there exists $\mu \in \mathcal{M}_{1,\geq p}(H)$ such that

$$\mathbb{P}\left[\text{Left meets Right}\right] = o(1).$$

For $p \in \left(\frac{1}{2}, 1\right]$, let $\theta = \theta(p)$ be given by

$$\theta(p) := \frac{1 + \sqrt{2p - 1}}{2}.$$

The quantity θ will play an important role in the proof of both parts of Theorem 2.1. Observe that $\theta \in [p, 1]$ and satisfies

$$\theta^2 + (1 - \theta)^2 = p \qquad \text{and} \qquad 2\theta(1 - \theta) = 1 - p.$$

Using the latter of these relations, we see that for $p \in [0, 1]$,

$$\begin{aligned} \theta\sqrt{p} \leq 1 - p = 2\theta(1 - \theta) \qquad &\Leftrightarrow \; p \leq 4(1 - \theta)^2 = 2p - 2\sqrt{2p - 1} \Leftrightarrow \; 8p - 4 \leq p^2 \\ &\Leftrightarrow \; p \leq 4 - 2\sqrt{3}. \end{aligned} \tag{2.1}$$

Our proofs will also make extensive use of the following Chernoff bound: given a binomial random variable $X \sim \text{Binom}(N, p)$ and $\varepsilon \in (0, 1)$, we have

$$\mathbb{P}\left[|X - Np| \geq \varepsilon Np\right] \leq 2e^{-\frac{\varepsilon^2 Np}{3}}. \tag{2.2}$$

2.1 | Lower bound construction: Proof of Theorem 2.1(ii)

For each $1/2 < p \leq 4 - 2\sqrt{3}$, we construct a state-based measure $\mu_F \in \mathcal{M}_{\geq p}(K_2 \times G_n)$, based on the ideas behind constructions in [10, 14]. Assume without loss of generality that $V(G_n) = [n]$. We randomly assign to each vertex $(i, v) \in [2] \times [n]$ a state $S_v \in \{0, 1, \star\}$, independently of all the other vertices, with

(a) $S_{(1,v)} = 1$ with probability θ and $S_{(1,v)} = 0$ otherwise;
(b) $S_{(2,v)} = 0$ with probability $\sqrt{p}$ and $S_{(2,v)} = \star$ otherwise.

We then include edges of $H = K_2 \times G_n$ in our random subgraph $\mathbf{H}_{\mu_F}$ according to the following rules:

(i) an edge $\{(1, u), (1, v)\}$ is included if $S_{(1,u)} = S_{(1,v)}$;
(ii) an edge $\{(2, u), (2, v)\}$ is included if $S_{(2,u)} = S_{(2,v)} = 0$;
(iii) an edge $\{(1, v), (2, v)\}$ is included if $S_{(2,v)} = \star$ or if $S_{(1,v)} = S_{(2,v)} = 0$.

See Figure 2 for an illustration of the construction. Since μ_F is state-based, it is clearly a 1-ipm. Our state distributions (a)–(b) imply that every edge in the left copy of G_n is open (included in our random graph) with probability $\theta^2 + (1 - \theta)^2 = p$ (by the edge-rule (i) above), and that every edge in the right copy of G_n is open with probability $(\sqrt{p})^2 = p$ (by the edge-rule (ii) above). On the other hand, (by the edge-rule (iii) above) an edge $\{(1, v), (2, v)\}$ from the left copy to the right copy is closed if and only if $S_{(1,v)} = 1$ and $S_{(2,v)} = 0$, which by (2.1) occurs with probability $\theta\sqrt{p} \leq 1 - p$ provided $p \leq 4 - 2\sqrt{3}$. Thus $\mu_F \in \mathcal{M}_{1,\geq p}(K_2 \times G_n)$ as claimed.

All that remains to show is that for this measure the event "Left meets Right" occurs with probability $o(1)$ in the random graph $\mathbf{H}_{\mu_F}$. Observe that the construction of μ_F ensures there is no path in $\mathbf{H}_{\mu_F}$ from the vertices in $\{1\} \times [n]$ in state 1 to the vertices in $\{2\} \times [n]$ in state 0. Indeed the only edges of $\mathbf{H}_{\mu_F}$ in which the endpoints are in different states are those edges containing a vertex $(2, v)$ in state $S_{(2,v)} = \star$. Since by construction vertices in state $\star$ have degree exactly one in $\mathbf{H}_{\mu_F}$, it follows that there is no component of $\mathbf{H}_{\mu_F}$ containing both vertices in state 1 and vertices in state 0.

Since the expected number of vertices in $\{1\} \times [n]$ in state 1 is $\theta n > pn$ and the expected number of vertices in $\{2\} \times [n]$ in state 0 is $\sqrt{p}n > pn$, and since states are assigned independently, it follows

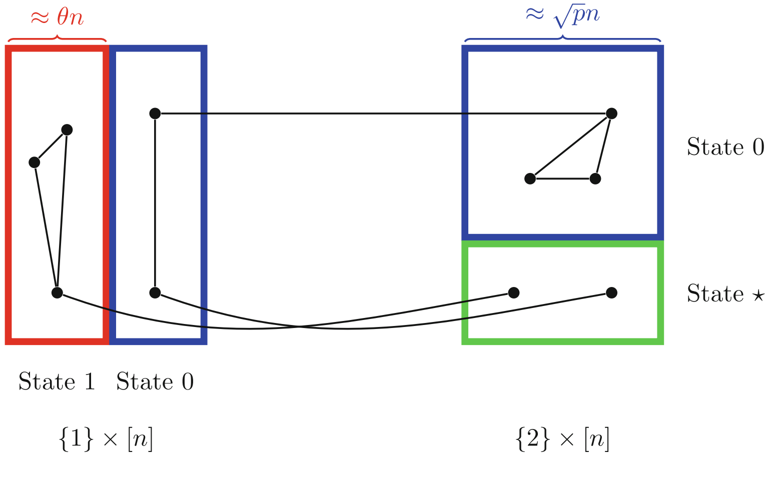

FIGURE 2 The lower bound construction

from (2.2) that for all fixed p with $1/2 < p \leq 4 - 2\sqrt{3}$, with probability $1 - o(1)$ there is no connected component in $\mathbf{H}_{\mu_F}$ containing at least half of the vertices of both $\{1\} \times [n]$ and $\{2\} \times [n]$. Thus "Left meets Right" occurs with probability $o(1)$ for $\mathbf{H}_{\mu_F}$, as claimed. □

2.2 | Upper bound: Proof of Theorem 2.1(i)

Suppose $p > 4 - 2\sqrt{3}$ is fixed. We shall show that for n sufficiently large this implies that for any $\mu \in \mathcal{M}_{1,p}(H)$, whp "Left meets Right" occurs. Our strategy for doing this is as follows: first of all we show in Lemma 2.5 that, for each $i \in [2]$, in any fixed tripartition $\sqcup_{j=1}^{3} V_j$ of $\{i\} \times [n]$, whp each of the parts V_j contains roughly the expected number of edges of $\mathbf{H}_\mu$, that is, $(p + o(1))\, e(H[V_j])$. This immediately implies that whp there is a component C_L of $\mathbf{H}_\mu$ containing strictly more than half of the vertices of $\{1\} \times [n]$, and another component C_R containing at least half of the vertices of $\{2\} \times [n]$.

If these two components C_L and C_R are not the same, then we color vertices of $[2] \times [n]$ green if they lie in a small component of $\mathbf{H}_\mu[\{i\} \times [n]]$ for some $i \in [2]$, and otherwise red if they are part of C_L and blue if not (so in particular vertices in C_R are colored blue). This gives rise to a partition of $[n]$ into 9 sets $V_{c,c'}$, corresponding to the possible ordered color pairs assigned to the vertex pairs $((1, v), (2, v))$, $v \in [n]$. Since whp at least $(p - o(1))n$ of the n edges from $\{1\} \times [n]$ to $\{2\} \times [n]$ are present in $\mathbf{H}_\mu$, we can combine the probabilistic information from Lemma 2.5 to show that whp the relative sizes of the $V_{c,c'}$ almost satisfy a certain system $\mathcal{S} = \mathcal{S}(p)$ of inequalities (2.7)–(2.10) (or more precisely that we can extract from the $|V_{c,c'}|/n$ a solution to $\mathcal{S}(p_\star)$ for some $p_\star$ a little smaller than p). For $p > 4 - 2\sqrt{3}$ and n sufficiently large, we are able to show this leads to a contradiction (Lemma 2.6). Having outlined our proof strategy, we now fill in the details. We shall use the following path-decomposition theorem due to Dean and Kouider.

Theorem 2.2 (Dean and Kouider [11]). *Let G be an n-vertex graph. Then there exists a set $\mathcal{P}$ of edge-disjoint paths in G such that $|\mathcal{P}| \leq \frac{2n}{3}$ and $\bigcup_{P \in \mathcal{P}} E(P) = E(G)$.*

Recall that a *matching* in a graph is a set of vertex-disjoint edges.

Corollary 2.3. *Let $\varepsilon > 0$ and let G be an n-vertex graph with $e(G) \geq 2n/\varepsilon$. Then there exists a set $\mathcal{M}$ of edge-disjoint matchings in G such that*

(M1) $|\mathcal{M}| \leq 2n$,
(M2) $\left|E(G) \setminus \bigcup_{M \in \mathcal{M}} M\right| \leq 2\varepsilon e(G)$, *and*
(M3) $|M| \geq \frac{\varepsilon e(G)}{2n}$ *for every* $M \in \mathcal{M}$.

Proof. By Theorem 2.2, there exists a set $\mathcal{P}$ of edge-disjoint paths in G such that $|\mathcal{P}| \leq \frac{2n}{3}$ and $E(G) = \bigcup_{P \in \mathcal{P}} E(P)$. Let $\mathcal{P}_{\text{short}} = \{P \in \mathcal{P} : e(P) \leq 2\varepsilon \frac{e(G)}{n}\}$. Let $\mathcal{M}$ be the set of matchings obtained by decomposing each path in $\mathcal{P} \setminus \mathcal{P}_{\text{short}}$ into two matchings. We have $|\mathcal{M}| \leq 2|\mathcal{P}| \leq 2n$. Moreover, each $M \in \mathcal{M}$ satisfies $|M| \geq \left\lfloor \frac{\varepsilon e(G)}{n} \right\rfloor \geq \frac{\varepsilon e(G)}{2n}$. Finally, $\left|E(G) \setminus \bigcup_{M \in \mathcal{M}} E(M)\right| \leq \frac{2n}{3} \cdot 2\varepsilon \frac{e(G)}{n} \leq 2\varepsilon e(G)$. ■

Matchings are useful in a 1-independent context since the states of their edges (present or absent) are independent. We can thus combine Corollary 2.3 with a Chernoff bound to show the number of edges in a 1-independent model is concentrated around its mean.

Lemma 2.4. *Let $\varepsilon > 0$ and $p \in (0,1]$. Let G be an n-vertex graph with $e(G) \geq 2n/\varepsilon$ and let $\mu \in \mathcal{M}_{1,p}(G)$. Then*

$$\mathbb{P}\left[e(\mathbf{G}_\mu) \leq (1-3\varepsilon)pe(G)\right] \leq 4n\exp\left(-\frac{\varepsilon^3 pe(G)}{6n}\right).$$

Proof. We apply Corollary 2.3 to obtain a set $\mathcal{M}$ of edge-disjoint matchings in G such that properties (M1) to (M3) hold. For every $M \in \mathcal{M}$, we have $|M| \geq \frac{\varepsilon e(G)}{2n}$. Thus by (2.2) and 1-independence,

$$\mathbb{P}\left[e(\mathbf{G}_\mu \cap M) \leq (1-\varepsilon)p\,|M|\right] \leq 2\exp\left(-\frac{\varepsilon^3 pe(G)}{6n}\right).$$

By a union bound, we have

$$\begin{aligned}\mathbb{P}\left[e(\mathbf{G}_\mu \cap M) \geq (1-\varepsilon)p\,|M| \quad \text{for all } \; M \in \mathcal{M}\right] &\geq 1 - 2|M|\exp\left(-\frac{\varepsilon^3 pe(G)}{6n}\right)\\ &\geq 1 - 4n\exp\left(-\frac{\varepsilon^3 pe(G)}{6n}\right).\end{aligned}$$

Thus with probability at least $1 - 4n\exp\left(-\frac{\varepsilon^3 pe(G)}{6n}\right)$ we have

$$e(\mathbf{G}_\mu) \geq \sum_{M\in\mathcal{M}} (1-\varepsilon)p\,|M| \geq (1-\varepsilon)p(1-2\varepsilon)e(G) \geq (1-3\varepsilon)pe(G).$$

This completes the proof. ∎

Lemma 2.5. *Let $p \in \left(\frac{1}{2}, 1\right]$, and let $\varepsilon = \varepsilon(p) > 0$ be fixed and sufficiently small. Let G be an n-vertex graph satisfying*

$$\left|e(G[U]) - q\frac{|U|^2}{2}\right| \leq \frac{\varepsilon^2}{4}qn^2 \tag{2.3}$$

for all $U \subseteq V(G)$, where $q(n) \gg \frac{\log n}{n}$. Consider a fixed tripartition $V(G) = V_1 \sqcup V_2 \sqcup V_3$. Then for every $\mu \in \mathcal{M}_{1,p}(G)$, the following hold whp:

(P1) $e(\mathbf{G}_\mu[V_i]) \geq pq\frac{|V_i|^2}{2} - \varepsilon qn^2$ *for every* $i \in [3]$.
(P2) $e(\mathbf{G}_\mu[V_i, V_j]) \geq pq\,|V_i|\,|V_j| - \varepsilon qn^2$ *for all* $1 \leq i < j \leq 3$.
(P3) For every $i \in [3]$ with $|V_i| \geq \varepsilon^{1/4}n$, $\mathbf{G}_\mu[V_i]$ contains a unique largest connected component C_i of order at least $(\theta - \varepsilon^{1/4})\,|V_i|$.
(P4) For all $1 \leq i < j \leq 3$ with $|V_i|, |V_j| \geq \varepsilon^{1/4}n$, there exists a path from C_i to C_j in $\mathbf{G}_\mu[V_i, V_j]$.
(P5) There is a unique largest connected component C in $\mathbf{G}_\mu$ such that $|C| \geq (\theta - 3\varepsilon^{1/4})n$ and for each $i \in [3]$ with $|V_i| \geq \varepsilon^{1/4}n$, $C_i \subseteq C$.

Proof. We first show that (P1) holds whp. Fix $i \in [3]$. If $|V_i| \leq \sqrt{\varepsilon}n$, then (P1) trivially holds. Hence we assume that $|V_i| \geq \sqrt{\varepsilon}n$. By our pseudorandomness assumption (2.3) on G we have $e(G[V_i]) \geq q\frac{|V_i|^2}{2} - \frac{\varepsilon}{2}qn^2$ (which for n sufficiently large is greater than $\frac{2n}{\varepsilon}$ so that we can apply Lemma 2.4). Thus we have

$$\mathbb{P}\left[e(\mathbf{G}_\mu[V_i]) \le pq\frac{|V_i|^2}{2} - \varepsilon qn^2\right] \le \mathbb{P}\left[e(\mathbf{G}_\mu[V_i]) \le pe(G[V_i]) - \frac{\varepsilon}{2}qn^2\right]$$
$$\le \mathbb{P}\left[e(\mathbf{G}_\mu[V_i]) \le \left(1-\frac{\varepsilon}{3}\right)pe(G[V_i])\right]$$
$$\le 4n\exp\left(-\Omega\left(\frac{e(G[V_i])}{n}\right)\right) = 4n\exp(-\Omega(qn)) = o(1),$$

where the inequality in the third line follows from Lemma 2.4. So (P1) holds whp.

Next we show that (P2) holds whp. Fix $1 \le i < j \le 3$. If $|V_i| \le \varepsilon n$ or $\left|V_j\right| \le \varepsilon n$, then (P2) trivially holds. Hence we may assume that $|V_i|, \left|V_j\right| \ge \varepsilon n$. By (2.3) applied three times (to V_i, V_j and $V_i \cup V_j$), we have $e(G[V_i, V_j]) \ge q\,|V_i|\left|V_j\right| - 3\frac{\varepsilon^2}{4}qn^2$. In particular, $e(G[V_i, V_j]) \ge \frac{\varepsilon^2}{4}qn^2$, which for n sufficiently large is greater than $\frac{2n}{\varepsilon}$. We now apply Lemma 2.4 to show that (P2) holds whp. We have

$$\mathbb{P}\left[e(\mathbf{G}_\mu[V_i, V_j]) \le pq\,|V_i|\left|V_j\right| - \varepsilon qn^2\right] \le \mathbb{P}\left[e(\mathbf{G}_\mu[V_i, V_j]) \le pe(G[V_i, V_j]) - \frac{\varepsilon}{2}qn^2\right]$$
$$\le \mathbb{P}\left[e(\mathbf{G}_\mu[V_i, V_j]) \le \left(1-\frac{\varepsilon}{3}\right)pe(G[V_i, V_j])\right]$$
$$\le 4n\exp\left(-\Omega\left(\frac{e(G[V_i, V_j])}{n}\right)\right) = 4n\exp(-\Omega(qn)) = o(1).$$

So (P2) holds whp.

Now we show that (P1) implies (P3). Assume that (P1) holds. Fix $i \in [3]$ and assume that $|V_i| \ge \varepsilon^{1/4}n$. Let $C \subseteq V_i$ be a largest connected component in $\mathbf{G}_\mu[V_i]$ and suppose for a contradiction that $|C| \le (\theta - \varepsilon^{1/4})\,|V_i|$.

If $|C| \le \frac{|V_i|}{2}$, then there is a partition of V_i into at most 4 sets, each of size at most $\frac{|V_i|}{2}$, such that every connected component of $\mathbf{G}_\mu[V_i]$ is entirely contained in one of the sets of the partition. Indeed, such a partition can be obtained by starting with a partition of V_i into the connected components of $\mathbf{G}_\mu[V_i]$ and then as long as the partition contains two parts of size at most $\frac{|V_i|}{4}$ choosing two such parts arbitrarily and merging them into a single part. Since for any quadruple (x_1, x_2, x_3, x_4) with $\frac{1}{2} \ge x_i \ge 0$ and $\sum_i x_i = 1$ we have $\sum_i (x_i)^2 \le \frac{1}{2}$, it follows from (P1) and (2.3) that

$$pq\frac{|V_i|^2}{2} - \varepsilon qn^2 \le e(\mathbf{G}_\mu[V_i]) \le q\frac{|V_i|^2}{4} + \varepsilon^2 qn^2.$$

Rearranging terms, this gives

$$\left(p - \frac{1}{2}\right)q\frac{\varepsilon^{1/2}n^2}{2} \le \left(p - \frac{1}{2}\right)q\frac{|V_i|^2}{2} \le q(\varepsilon + \varepsilon^2)n^2,$$

which is a contradiction for ε chosen sufficiently small. Thus we may assume $|C| \ge \frac{|V_i|}{2}$. Now by (P1) and (2.3) again, we have

$$pq\frac{|V_i|^2}{2} - \varepsilon qn^2 \le e(\mathbf{G}_\mu[V_i]) \le e(\mathbf{G}_\mu[C]) + e(\mathbf{G}_\mu[V_i \setminus C]) \le q\frac{|C|^2}{2} + q\frac{(|V_i| - |C|)^2}{2} + \frac{\varepsilon^2}{2}qn^2.$$

Dividing by $q\frac{|V_i|^2}{2}$ and using $|V_i| \ge \varepsilon^{1/4}n$, we deduce that

$$p - 3\sqrt{\varepsilon} \le \left(\frac{|C|}{|V_i|}\right)^2 + \left(1 - \frac{|C|}{|V_i|}\right)^2. \tag{2.4}$$

Since $x \mapsto x^2 + (1-x)^2$ is an increasing function in the interval $\left[\frac{1}{2}, 1\right]$, $\frac{1}{2}|V_i| \le |C| \le (\theta - \varepsilon^{1/4})|V_i|$, and $\theta^2 + (1-\theta)^2 = p$, we have

$$\left(\frac{|C|}{|V_i|}\right)^2 + \left(1 - \frac{|C|}{|V_i|}\right)^2 \le (\theta - \varepsilon^{1/4})^2 + (1 - \theta + \varepsilon^{1/4})^2$$
$$= \theta^2 + (1-\theta)^2 - 2\varepsilon^{1/4}(2\theta - 1) + 2\sqrt{\varepsilon} \le p - 4\sqrt{\varepsilon},$$

contradicting (2.4). Hence $|C| \ge (\theta - \varepsilon^{1/4})|V_i|$. Note that since $\theta - \varepsilon^{1/4} > 1/2$ (for $\varepsilon = \varepsilon(p)$ chosen sufficiently small), C is the unique largest component in $\mathbf{G}_\mu[V_i]$. So (P3) holds whp.

Next we show that (P2) and (P3) together imply (P4). Assume that (P2) and (P3) hold. Fix $1 \le i < j \le 3$ and assume that $|V_i|, |V_j| \ge \varepsilon^{1/4}n$. Suppose for a contradiction that there is no path in $\mathbf{G}_\mu[V_i, V_j]$ from C_i to C_j. Let $A_i \subseteq V_i$ and $A_j \subseteq V_j$ be the sets of vertices which cannot be reached by a path in $\mathbf{G}_\mu[V_i, V_j]$ from C_j and C_i, respectively. Since there is no path from C_i to C_j, we must have $C_i \subseteq A_i$ and $C_j \subseteq A_j$. By (P2), by the definition of A_i and A_j, and by (2.3) (applied in A_i, A_j, $V_i \setminus A_i$, $V_j \setminus A_j$, $A_i \cup (V_j \setminus A_j)$ and $A_j \cup (V_i \setminus A_i)$), we have

$$pq|V_i||V_j| - \varepsilon qn^2 \le e(\mathbf{G}_\mu[V_i, V_j]) \le e(\mathbf{G}_\mu[A_i, V_j \setminus A_j]) + e(\mathbf{G}_\mu[V_i \setminus A_i, A_j])$$
$$\le q|A_i|(|V_j| - |A_j|) + q|A_j|(|V_i| - |A_i|) + \frac{3\varepsilon^2}{2}qn^2. \tag{2.5}$$

Let $x_i = \frac{|A_i|}{|V_i|}$ and $x_j = \frac{|A_j|}{|V_j|}$. By (P3), $x_i \ge \frac{|C_i|}{|V_i|} \ge \theta - \varepsilon^{1/4} \ge \frac{1}{2}$ and similarly $x_j \ge \frac{1}{2}$. From (2.5) we get by dividing by $q|V_i||V_j|$ and using $|V_i|, |V_j| \ge \varepsilon^{1/4}n$, that

$$p - 2\sqrt{\varepsilon} \le x_i(1 - x_j) + x_j(1 - x_i) = x_i + x_j - 2x_ix_j \le \frac{1}{2}, \tag{2.6}$$

where the last inequality follows since $(x, y) \mapsto x + y - 2xy$ is non-increasing in both x and y for $x, y \ge \frac{1}{2}$. Note that (2.6) gives a contradiction for ε sufficiently small since $p > \frac{1}{2}$. So (P4) holds whp.

Finally, we observe that (P5) follows directly from (P3) and (P4). Indeed let $k \in [3]$ denote the number of $i \in [3]$ for which $|V_i| \ge \varepsilon^{1/4}n$ (note we can guarantee $k \ge 1$ provided $\varepsilon < 3^{-4}$). Then (P3) and (P4) together imply there is a unique connected component C in $\mathbf{G}_\mu$ of size at least $(\theta - \varepsilon^{1/4})(1 - (3-k)\varepsilon^{1/4})n > (\theta - 3\varepsilon^{1/4})n$ and containing C_i for each $i \in [3]$ with $|V_i| \ge \varepsilon^{1/4}n$. ■

Let $\mathcal{S}(p)$ denote the collection of 3×3 matrices A with non-negative entries $A_{ij} \ge 0$, $i,j \in [3]$, satisfying the following inequalities:

$$A_{11} + A_{22} + p \le \sum_{i,j} A_{ij} \le 1, \tag{2.7}$$

$$A_{1j} \ge \frac{1}{2}\sum_i A_{ij} \quad \forall j \in [3] \qquad \text{and} \qquad A_{i1} \ge \frac{1}{2}\sum_j A_{ij} \quad \forall i \in [3], \tag{2.8}$$

$$(A_{1j})^2 + (A_{2j})^2 \ge p\left(\sum_i A_{ij}\right)^2 \quad \forall j \in [3], \tag{2.9}$$

$$(A_{i1})^2 + (A_{i2})^2 \ge p\left(\sum_j A_{ij}\right)^2 \quad \forall i \in [3]. \tag{2.10}$$

The key step in our proof of Theorem 2.1 will be, assuming that "Left meets Right" does not occur whp, to use Lemma 2.5 to exhibit a partition of $[n]$ into 9 parts whose relative sizes can be used to find a solution to $\mathcal{S}(p_\star)$, for some $p_\star$ satisfying $4-2\sqrt{3} < p_\star < p$. We will then be able to use the following lemma to derive a contradiction.

Lemma 2.6. *For $4-2\sqrt{3} < p \le 1$, $\mathcal{S}(p) = \emptyset$.*

Proof. Suppose not and let $A \in \mathcal{S}(p)$. Note that the bound for $\sum_{i,j} A_{ij}$ in (2.7) implies

$$A_{11} + A_{22} \le 1 - p. \tag{2.11}$$

By transpose-symmetry of $\mathcal{S}(p)$ and (2.7), we may assume without loss of generality that

$$w := A_{21} + A_{31} + A_{32} + A_{33} \ge \frac{p}{2}. \tag{2.12}$$

Note that if $\sum_j A_{3j} > \frac{A_{31}}{\theta}$, then, since $x \mapsto x^2 + (1-x)^2$ is an increasing function of x in the interval $\left[\frac{1}{2}, 1\right]$ and since $A_{31} \ge \frac{1}{2}\sum_j A_{33j}$ by (2.8),

$$\left(\frac{A_{31}}{\sum_j A_{3j}}\right)^2 + \left(\frac{A_{32}}{\sum_j A_{3j}}\right)^2 \le \left(\frac{A_{31}}{\sum_j A_{3j}}\right)^2 + \left(1 - \frac{A_{31}}{\sum_j A_{3j}}\right)^2 < \theta^2 + (1-\theta)^2 = p,$$

contradicting (2.10). Hence

$$\sum_j A_{3j} \le \frac{A_{31}}{\theta}. \tag{2.13}$$

By an analogous argument, we have $\sum_i A_{i1} \le \frac{A_{11}}{\theta}$ and thus

$$A_{21} \le A_{21} + A_{31} \le \frac{1-\theta}{\theta} A_{11}. \tag{2.14}$$

Now, by (2.13) we have $w \le A_{21} + \frac{A_{31}}{\theta}$. By (2.9), we have that

$$A_{31} \le \frac{\sqrt{(A_{11})^2 + (A_{21})^2}}{\sqrt{p}} - A_{11} - A_{21}.$$

Substituting this expression into our upper bound on w, we get

$$w \le -\frac{(1-\theta)A_{21}}{\theta} - \frac{A_{11}}{\theta} + \frac{\sqrt{(A_{11})^2 + (A_{21})^2}}{\theta\sqrt{p}}.$$

For A_{11} fixed, the continuous function $f_{A_{11}}(y) = -\frac{(1-\theta)y}{\theta} - \frac{A_{11}}{\theta} + \frac{\sqrt{(A_{11})^2+y^2}}{\theta\sqrt{p}}$ is convex in $(0, +\infty)$ as its derivative $f'_{A_{11}}(y) = -\frac{(1-\theta)}{\theta} + \frac{1}{\theta\sqrt{p}\sqrt{(A_{11}/y)^2+1}}$ is increasing in y in that interval. By (2.14), $0 \le A_{21} \le \frac{1-\theta}{\theta}A_{11}$, which together with the convexity of $f_{A_{11}}$ gives:

$$
\begin{aligned}
w &\le \max\left\{ f_{A_{11}}(0),\, f_{A_{11}}\left(\frac{1-\theta}{\theta}A_{11}\right)\right\} \\
&\le \max\left\{ -\frac{A_{11}}{\theta} + \frac{A_{11}}{\theta\sqrt{p}},\, -\left(\frac{1-\theta}{\theta}\right)^2 A_{11} - \frac{A_{11}}{\theta} + A_{11}\frac{\sqrt{1+\left(\frac{1-\theta}{\theta}\right)^2}}{\theta\sqrt{p}} \right\} \\
&\le \max\left\{ \frac{A_{11}}{\theta}\left(\frac{1}{\sqrt{p}}-1\right),\, \frac{A_{11}}{\theta}(1-\theta)\right\} \\
&\le \max\left\{ \frac{1-p}{\theta}\left(\frac{1}{\sqrt{p}}-1\right),\, \frac{1-p}{\theta}(1-\theta)\right\},
\end{aligned}
$$

where the last inequality follows from the upper bound (2.11) on A_{11}. We now claim that this contradicts (2.12), that is, that

$$
\max\left\{ \frac{1-p}{\theta}\left(\frac{1}{\sqrt{p}}-1\right), \frac{1-p}{\theta}(1-\theta)\right\} < \frac{p}{2}.
$$

Note that $p \mapsto \frac{1-p}{\theta}\left(\frac{1}{\sqrt{p}}-1\right) - \frac{p}{2}$ and $p \mapsto \frac{1-p}{\theta}(1-\theta) - \frac{p}{2}$ are both strictly decreasing functions (as θ is increasing in p). Hence to prove the claim above, it suffices to show that for $p = 4-2\sqrt{3}$, we have $\frac{1-p}{\theta}\left(\frac{1}{\sqrt{p}}-1\right) \le \frac{p}{2}$ and $\frac{1-p}{\theta}(1-\theta) \le \frac{p}{2}$. Let $p = 4-2\sqrt{3}$. Note that $(\sqrt{3}-1)^2 = 4-2\sqrt{3}$ and $(2-\sqrt{3})^2 = 7-4\sqrt{3}$. Hence $\sqrt{p} = \sqrt{3}-1$, $\sqrt{2p-1} = 2-\sqrt{3}$, and $\theta = (3-\sqrt{3})/2$. Now it is easy to check that

$$
\frac{1}{\sqrt{p}} - 1 = 1-\theta = \frac{\theta}{(1-p)}\frac{p}{2} = \frac{\sqrt{3}-1}{2},
$$

which completes the proof. ■

We are now ready to complete the proof of Theorem 2.1(i).

Proof. Let $p > 4-2\sqrt{3}$ be fixed. Let $\varepsilon = \varepsilon(p) > 0$ be fixed and chosen sufficiently small. Let $p_\star = \frac{1}{2}\left(4-2\sqrt{3}+p\right)$. Finally, let n be sufficiently large so that for $G = G_n$ the pseudorandomness assumption (2.3) holds, and let $\mu \in \mathcal{M}_{1,p}(H)$, where $H = K_2 \times G_n$.

For $i \in [2]$, let $\mathbf{G}^i_\mu = \mathbf{H}_\mu[\{i\}\times[n]]$. For $i,j \in [2]$ with $i \neq j$, let $\mathcal{E}_{ij}$ be the event that for any partition $(\{i\}\times V_1) \sqcup (\{i\}\times V_2) \sqcup (\{i\}\times V_3)$ of $\{i\}\times[n]$ such that $\{i\}\times V_1$ and $\{i\}\times V_2$ are each a union of components of order at least $\varepsilon^{1/4}n$ in $\mathbf{G}^i_\mu$, we have that $\mathbf{G}^j_\mu$ satisfies (P1) to (P5) of Lemma 2.5 with $\{j\}\times V_1$, $\{j\}\times V_2$, $\{j\}\times V_3$ playing the roles of V_1, V_2, V_3. Given $\mathbf{G}^i_\mu$ and ε fixed, the number of such partitions is at most $3^{\varepsilon^{-1/4}} = O(1)$. Hence Lemma 2.5 implies that $\mathcal{E}_{ij}$ holds whp.

Further, by 1-independence and (2.2), whp there are at least $(p-\varepsilon)n$ edges in the matching $\mathbf{H}_\mu[\{1\}\times[n], \{2\}\times[n]]$. Let $\mathcal{E}_{\text{good}}$ be the event that $\mathcal{E}_{12}$ and $\mathcal{E}_{21}$ both occur and that in addition $e(\mathbf{H}_\mu[\{1\}\times[n], \{2\}\times[n]]) \ge (p-\varepsilon)n$. Then $\mathcal{E}_{\text{good}}$ holds whp. We claim that if $\mathcal{E}_{\text{good}}$ holds, then so does "Left meets Right" (which implies the statement of the theorem).

Suppose for a contradiction that $\mathcal{E}_{\text{good}}$ holds but "Left meets Right" does not. For $i \in [2]$, let C^i be the unique largest connected component in $\mathbf{G}^i_\mu$ (this exist by (P5)). Let $U_1 \sqcup U_2 \sqcup U_3 = [n]$ and $W_1 \sqcup W_2 \sqcup W_3 = [n]$ be such that the following hold.

(a) $\{1\} \times U_1$ is the union of C^1 and all connected components in $\mathbf{G}^1_\mu$ of order at least $\varepsilon^{1/4}n$ that can be reached from C^1 by a path in $\mathbf{H}_\mu$.
(b) $\{1\} \times U_2$ is the union of all other connected components in $\mathbf{G}^1_\mu$ of order at least $\varepsilon^{1/4}n$.
(c) $\{1\} \times U_3$ is the union of all connected components of order less than $\varepsilon^{1/4}n$ in $\mathbf{G}^1_\mu$.
(d) $\{2\} \times W_1$ is the union of all connected components in $\mathbf{G}^2_\mu$ of order at least $\varepsilon^{1/4}n$ that cannot be reached from C^1 by a path in $\mathbf{H}_\mu$.
(e) $\{2\}\times W_2$ is the union of all connected components in $\mathbf{G}^2_\mu$ of order at least $\varepsilon^{1/4}n$ that can be reached from C^1 by a path in $\mathbf{H}_\mu$.
(f) $\{2\} \times W_3$ is the union of all connected components in $\mathbf{G}^2_\mu$ of order less than $\varepsilon^{1/4}n$.

We can think of these partitions as giving us a 3-coloring of the vertices in $V(H)$: a vertex in $\{i\}\times V_n$ is colored red if it belongs to a large component in $\mathbf{G}^i_\mu$ and can be reached from C^1 in $\mathbf{H}_\mu$, blue if it belongs to a large component in $\mathbf{G}^i_\mu$ and cannot be reached by C^1 in $\mathbf{H}_\mu$, and green if it belongs to a small component in $\mathbf{G}^i_\mu$. The key properties of this coloring are that the large components C^1 and C^2 in $\mathbf{G}^1_\mu$ and $\mathbf{G}^2_\mu$ are colored red and blue respectively, that there are no edges from red vertices to blue vertices, and that the green vertices span few edges in $\mathbf{G}^i_\mu$, $i \in [2]$. Our 3-coloring of $V(H)$ gives rise to a partition of $[n]$ into 9 sets in a natural way, by considering the possible color pairs for $((1, v), (2, v))$, $v \in [n]$. This partition is illustrated in Figure 3.

We now investigate the relative sizes of this 9-partition. For $i,j \in [3]$, let $V_{ij} = U_i \cap W_j$. Since there is no path from C^1 to C^2 in $\mathbf{H}_\mu$, there are no edges present in the bipartite graphs $\mathbf{H}_\mu[\{1\}\times V_{11}, \{2\}\times V_{11}]$ and $\mathbf{H}_\mu[\{1\}\times V_{22}, \{2\}\times V_{22}]$. Since $\mathcal{E}_{\text{good}}$ holds, there are at least $(p-\varepsilon)n$ edges in $\mathbf{H}_\mu[\{1\}\times[n], \{2\}\times[n]]$ in total, which implies

$$|V_{11}| + |V_{22}| \leq (1 - p + \varepsilon)n. \tag{2.15}$$

Moreover, $\sum_{i,j} |V_{ij}| = n$. Hence

$$\sum_{i,j} |V_{ij}| - |V_{11}| - |V_{22}| \geq (p - \varepsilon)n. \tag{2.16}$$

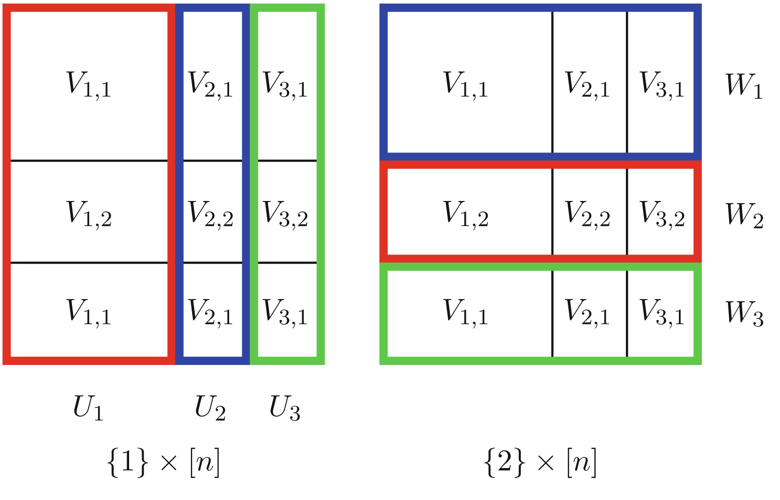

FIGURE 3 The partition of $V(H)$

For $j \in [3]$, if $|W_j| \geq \varepsilon^{1/4}n$, we have by (P3) to (P5) that there is a unique largest connected component C_j^1 in $\mathbf{G}_\mu^1[\{1\} \times W_j]$, and that this component satisfies $C_j^1 \subseteq C^1$ and $|C_j^1| \geq (\theta - \varepsilon^{1/4})|W_j|$, which for $\varepsilon = \varepsilon(p)$ chosen sufficiently small is greater than $\frac{1}{2}|W_j|$. Translating this in terms of our 9-partition, we have that for all $j \in [3]$ such that $\sum_i V_{ij} \geq \varepsilon^{1/4}n$

$$\left|V_{1j}\right| \geq \frac{1}{2}\sum_i \left|V_{ij}\right| \tag{2.17}$$

holds. By a symmetric argument, for every $i \in [3]$ such that $\sum_j V_{ij} \geq \varepsilon^{1/4}n$ we have

$$|V_{i1}| \geq \frac{1}{2}\sum_j \left|V_{ij}\right|. \tag{2.18}$$

Let $j \in [3]$. Note that $\mathbf{G}_\mu^1[U_3]$ contains only connected components of size at most $\varepsilon^{1/4}n$. These components can be covered by at most $\frac{2}{\varepsilon^{1/4}}$ sets, each of order at least $\frac{\varepsilon^{1/4}n}{2}$ and at most $\varepsilon^{1/4}n$. By (2.3) (which holds by our choice of n), each of these sets contains at most $q\frac{\varepsilon^{1/2}n^2}{2} + \frac{\varepsilon^2}{4}qn^2 < q\varepsilon^{1/2}n^2$ edges. Hence we have $e(\mathbf{G}_\mu^1[U_3]) \leq 2\varepsilon^{1/4}qn^2$. Since $V_{3j} \subseteq U_3$, we have $e(\mathbf{G}_\mu^1[V_{3j}]) \leq 2\varepsilon^{1/4}qn^2$. By (P1) and the pseudorandomness assumption (2.3), we have

$$\begin{aligned}
pq\frac{\left|W_j\right|^2}{2} - \varepsilon qn^2 &\leq e(\mathbf{G}_\mu^1[\{1\} \times W_j]) \\
&= e(\mathbf{G}_\mu^1[\{1\} \times V_{1j}]) + e(\mathbf{G}_\mu^1[\{1\} \times V_{2j}]) + e(\mathbf{G}_\mu^1[\{1\} \times V_{3j}]) \\
&\leq q\frac{\left|V_{1j}\right|^2}{2} + q\frac{\left|V_{2j}\right|^2}{2} + 2\varepsilon^{1/4}qn^2 + \frac{\varepsilon^2}{2}qn^2 < q\frac{\left|V_{1j}\right|^2}{2} + q\frac{\left|V_{2j}\right|^2}{2} + 3\varepsilon^{1/4}qn^2.
\end{aligned}$$

Hence, for every $j \in [3]$ and ε chosen sufficiently small,

$$\left|V_{1j}\right|^2 + \left|V_{2j}\right|^2 \geq p\left(\sum_i \left|V_{ij}\right|\right)^2 - 7\varepsilon^{1/4}n^2. \tag{2.19}$$

Similarly, for every $i \in [3]$,

$$|V_{i1}|^2 + |V_{i2}|^2 \geq p\left(\sum_j \left|V_{ij}\right|\right)^2 - 7\varepsilon^{1/4}n^2. \tag{2.20}$$

Let A be the 3×3 matrix with entries

$$A_{ij} = \begin{cases} \frac{\left|V_{ij}\right|}{n}, & \text{if } \left|V_{ij}\right| \geq \varepsilon^{1/9}n, \\ 0, & \text{otherwise.} \end{cases}$$

We claim that, provided $\varepsilon = \varepsilon(p)$ was chosen sufficiently small, $A \in \mathcal{S}(p_\star)$. Indeed, A clearly has nonnegative entries summing up to at most 1, thus the second inequality of (2.7) is satisfied, while the first inequality (with $p_\star$ instead of p) follows from (2.16) and an appropriately small choice of ε (more specifically, we need $p_\star \leq p - \varepsilon - 7\varepsilon^{1/9}$). Indeed,

$$\sum_{i,j} A_{ij} - A_{11} - A_{22} \geq \sum_{i,j} \frac{|V_{ij}|}{n} - \frac{|V_{11}|}{n} - \frac{|V_{22}|}{n} - 7\varepsilon^{1/9} \geq p - \varepsilon - 7\varepsilon^{1/9} \geq p_\star,$$

where the penultimate inequality uses (2.16).

Next, consider $j \in [3]$. If $\sum_i |V_i| \geq \varepsilon^{1/4}n$, then by (2.17) we have $A_{1j} \geq \frac{1}{2}\sum_i A_{ij}$ (regardless of whether some of the V_{ij}, $i \in [3]$ have size less than $\varepsilon^{1/9}n$). Other the other hand if $\sum_i |V_i| < \varepsilon^{1/4}n$, then $A_{1j} = A_{2j} = A_{3j} = 0$. In either case, $A_{1j} \geq \frac{1}{2}\sum_i A_{ij}$ holds. By a symmetric argument we obtain that $A_{i1} \geq \frac{1}{2}\sum_j A_{ij}$ holds for every $i \in [3]$. Thus (2.8) is satisfied by A.

Finally, pick $j \in [3]$. If $|V_{i2}| \geq \varepsilon^{1/9}n$, then by (2.8) which we have just established and the definition of A_{i1}, we have $|V_{i1}| \geq \varepsilon^{1/9}n$ also. In this case (2.19) and an appropriately small choice of ε ensure that $(A_{1j})^2 + (A_{2j})^2 \geq p_\star\left(\sum_i A_{ij}\right)^2$. On the other hand, suppose $|V_{i2}| < \varepsilon^{1/9}n$. If $|V_{i1}| < \varepsilon^{1/9}n$, then by (2.8) the inequality $(A_{1j})^2 + (A_{2j})^2 \geq p_\star\left(\sum_i A_{ij}\right)^2$ holds trivially, since the right hand-side is zero. So suppose that $|V_{i1}| \geq \varepsilon^{1/9}n > |V_{i2}|$. Then (2.19), and $p > 1/2$ imply that

$$|V_{i1}|^2 > |V_{i1}|^2 - |V_{i2}|\,(2p\,|V_{i1}| - (1-p)\,|V_{i2}|) \geq p(|V_{i1}| + |V_{i3}|)^2 - 7\varepsilon^{1/4}n^2.$$

Together with an appropriately small choice of ε, this ensures $(A_{1j})^2 + (A_{2j})^2 \geq p_\star\left(\sum_i A_{ij}\right)^2$ again. Thus in every case (2.9) is satisfied by A (with $p_\star$ instead of p). A symmetric argument shows A satisfies (2.10) for $p_\star$ as well.

Thus $A \in \mathcal{S}(p_\star)$ as claimed. However, since $p_\star > 4 - 2\sqrt{3}$, Lemma 2.6 implies that $\mathcal{S}(p_\star) = \emptyset$, a contradiction. Thus the event $\mathcal{E}_{\text{good}}$, which holds whp, does imply the event "Left meets Right," proving the theorem. ∎

3 | PROOF OF THEOREMS 1.8, 1.9, 1.11, AND 1.12

Our main theorems are all proved via a renormalization argument combined with Theorem 2.1. Given two graphs G and H, we may view the Cartesian product $H \times G$ as a kind of "augmented" version of H, and use any 1-independent random graph $(\mathbf{H}\times\mathbf{G})_\mu$ on $H \times G$ to construct a new 1-independent random graph $\mathbf{H}_\nu$ on H as follows: given an edge $uv \in E(H)$, we let uv be present in $\mathbf{H}_\nu$ if in the restriction of $(\mathbf{H}\times\mathbf{G})_\mu$ to $\{u,v\} \times V(G)$ there is a connected component containing strictly more than half of the vertices in each of $\{u\} \times V(G)$ and $\{v\} \times V(G)$.

That $\mathbf{H}_\nu$ is a 1-independent random graph follows immediately from the fact that $(\mathbf{H}\times\mathbf{G})_\mu$ was 1-independent: the states of edges inside vertex-disjoint edge-sets in $\mathbf{H}_\nu$ are determined by the states of edges inside vertex-disjoint edge sets in $(\mathbf{H}\times\mathbf{G})_\mu$. Further, any path in $\mathbf{H}_\nu$ can be "lifted" up to a path in $(\mathbf{H}\times\mathbf{G})_\mu$ of equal or greater length: if uv, vw are present in $\mathbf{H}_\nu$, then there exist connected subgraphs C_{uv} and C_{vw} in $(\mathbf{H}\times\mathbf{G})_\mu$ with $C_{uv} \subseteq \{u,v\}\times V(G)$, $C_{vw} \subseteq \{v,w\}\times V(G)$, $C_{uv} \cap (\{u\} \times V(G))$ and $C_{vw} \cap (\{w\} \times V(G))$ both non-empty, and $C_{uv}, C_{v,w}$ both containing strictly more than half of the vertices in $\{v\} \times V(G)$ (and hence having non-empty intersection).

Now the likelihood of an edge uv being present in $\mathbf{H}_\nu$ is exactly the probability of the event corresponding to "Left meets Right" occurring in the restriction of $(\mathbf{H}\times\mathbf{G})_\mu$ to the vertex-set $\{u,v\}\times V(G)$ (which induces a copy of $K_2 \times G$ in $H \times G$). Thus for $p > 4 - 2\sqrt{3}$ and a suitable choice of G, we can use Theorem 2.1(i) to ensure that each edge in the 1-independent random graph $\mathbf{H}_\nu$ is present with probability $1 - o(1)$. With such a high edge probability, we can then establish the almost sure existence of infinite components or long paths in $\mathbf{H}_\nu$ in a straightforward way—either by using results in the literature, or by a direct argument.

On the other hand if $p \leq 4 - 2\sqrt{3}$, we can use ideas from the lower bound construction in the proof of Theorem 2.1(ii), which date back to [10, 14], in order to construct a 1-independent random subgraph $\mathbf{G}$ of $H \times K_n$ that fails to percolate (or, if $H = \mathbb{Z}$, that only contain paths of length $O(n)$). For the convenience of the reader, we sketch below how this works in the special case $H = \mathbb{Z}^2$.

Take $p = 4 - 2\sqrt{3}$, and set $\theta = (1 + \sqrt{2p-1})/2$. Independently assign to each vertex $(x, y, z) \in \mathbb{Z}^2 \times V(K_n)$ a random state $S_{x,y,z} \in \{0, 1, \star\}$ as follows:

- if $\|(x, y)\|_\infty \cong 0$ *mod* 6, set $S_{x,y,z} = 1$ with probability 1;
- if $\|(x, y)\|_\infty \cong 1$ *mod* 6, set $S_{x,y,z} = 1$ with probability θ, and 0 otherwise;
- if $\|(x, y)\|_\infty \cong 2$ *mod* 6, set $S_{x,y,z} = 0$ with probability $\sqrt{p}$, and $\star$ otherwise;
- if $\|(x, y)\|_\infty \cong 3$ *mod* 6, set $S_{x,y,z} = 0$ with probability 1;
- if $\|(x, y)\|_\infty \cong 4$ *mod* 6, set $S_{x,y,z} = 0$ with probability θ, and 1 otherwise;
- if $\|(x, y)\|_\infty \cong 5$ *mod* 6, set $S_{x,y,z} = 1$ with probability $\sqrt{p}$, and $\star$ otherwise.

We now use these random states to build a 1-independent random graph $\mathbf{G}$ as follows. Given an edge$\{(x_1, y_1, z_1), (x_2, y_2, z_2)\}$ of $H \times K_n$, include it in $\mathbf{G}$ if one of the following holds:

- $S_{x_1,y_1,z_1} = S_{x_2,y_2,z_2} \neq \star$.
- $\|(x_1, y_1)\|_\infty < \|(x_2, y_2)\|_\infty$ and $S_{x_2,y_2,z_2} = \star$.

Then the choice of probabilities for our random states ensure each edge is open with probability at least $p = 4 - 2\sqrt{3}$, and our edge rules further imply that every connected component C in $\mathbf{G}$ meets at most four consecutive cylinders $C_r := \{(x, y, z) : \|(x, y)\|_\infty = r\}$, $r \in \mathbb{Z}_{\geq 0}$ since, as is easily checked, a connected component in $\mathbf{G}$ cannot both contain a vertex assigned state 0 and a vertex assigned state 1—we leave this as an exercise to the reader, and refer them to [10, Corollary 24] for a proof of this fact in a more general setting. In particular, we have that $\mathbf{G}$ does not percolate.

Having thus outlined our proof ideas, we now fill in the details. First we formalize our renormalization argument with the following lemma.

Lemma 3.1 (Renormalization lemma). *Let H be a graph. Let $q = q(n)$ satisfy $nq(n) \gg \log n$, and let $(G_n)_{n\in\mathbb{N}}$ be a sequence of n-vertex graphs which is weakly q-pseudorandom. Then for every $\varepsilon > 0$ and every $p > 4 - 2\sqrt{3}$ fixed, there exists n_0 such that for all $n \geq n_0$, $G = G_n$ and $\mu \in \mathcal{M}_{1,\geq p}(H \times G)$ there exists $\nu \in \mathcal{M}_{1,\geq 1-\varepsilon}(H)$ and a coupling between $\mathbf{H}_\nu$ and $(\mathbf{H} \times \mathbf{G})_\mu$ such that there exists a path from u to v in $\mathbf{H}_\nu$ only if there exists a path from $\{u\} \times V(G)$ to $\{v\} \times V(G)$ in $(\mathbf{H} \times \mathbf{G})_\mu$.*

Proof. Let $p > 4 - 2\sqrt{3}$ and $\varepsilon > 0$ be fixed. By Theorem 2.1(i), there exists $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ and all $\mu \in \mathcal{M}_{1,\geq p}(K_2 \times G_n)$, the μ-probability of the event "Left meets Right" is at least $1 - \varepsilon$. For $n \geq n_0$, $G = G_n$ and $\mu \in \mathcal{M}_{1,\geq p}(H \times G)$, define a random graph model $\mathbf{H}_\nu$ from $(\mathbf{H} \times \mathbf{G})_\mu$ as follows: for each edge $uv \in E(H)$, we add uv to $\mathbf{H}_\nu$ if and only if there is a connected component in $(\mathbf{H} \times \mathbf{G})_\mu\,[\{u, v\} \times V(G_n)]$ containing strictly more than half of the vertices in $\{u\} \times V(G_n)$ and strictly more than half of the vertices $\{v\} \times V(G_n)$. The model $\mathbf{H}_\nu$ is clearly 1-independent, has edge-probability at least $1 - \varepsilon$, and has the property that any path in $\mathbf{H}_\nu$ can be lifted up to a path in $(\mathbf{H} \times \mathbf{G})_\mu$. This proves the lemma. ∎

Recall that 2-neighbor bootstrap percolation on a graph G is a discrete-time process defined as follows. At time $t = 0$, an initial set of infected vertices $A = A_0$ is given. At every time $t \geq 1$, every vertex of G which has at least 2 neighbors in A_{t-1} becomes infected and is added to A_{t-1} to form A_t. We denote by $\overline{A}$ the set of all vertices of G which are eventually infected, $\overline{A} = \bigcup_{t\geq 0} A_t$. Following Day et al. [10], we say that a graph G has the *finite 2-percolation property* if for every finite set of initially infected vertices A, the set of eventually infected vertices $\overline{A}$ is finite. The content of [10, Corollary 24] is, informally, that the construction based on random-states we outlined above "works on all host graphs that have the finite 2-percolation property."

Proof of Theorem 1.11. Let $H = \mathbb{Z}^2$. Pick $\varepsilon > 0$ such that $1 - \varepsilon > 0.8639$. Then by Lemma 3.1, for any $p > 4 - 2\sqrt{3}$, n sufficiently large and $G = G_n$, we can couple a random graph $(\mathbf{H} \times \mathbf{G})_\mu$, $\mu \in \mathcal{M}_{1,\geq p}(H)$ with a random graph $\mathbf{H}_\nu$, $\mu \in \mathcal{M}_{1,\geq 1-\varepsilon}(H)$ such that if $\mathbf{H}_\nu$ percolates then so does $(\mathbf{H} \times \mathbf{G})_\mu$. Since $p_{1,c}(H) < 0.86339$, as proved in [5, Theorem 2], it follows that $p_{1,c}(H \times G) \leq p$. Since $p > 4 - 2\sqrt{3}$ was arbitrary, we have the claimed upper bound $\lim_{n\to\infty} p_{1,c}(H \times G_n) \leq 4 - 2\sqrt{3}$. The lower bound $\lim_{n\to\infty} p_{1,c}(H \times G_n) \geq 4 - 2\sqrt{3}$ follows from [10, Corollary 24] and the fact that $\mathbb{Z}^2 \times G_n$ is easily seen to have the finite 2-percolation property. Indeed, for any finite set of vertices A in $\mathbb{Z}^2 \times G_n$, there is some finite N such that $A \subseteq [N]^2 \times V(G_n)$. Now every vertex outside $[N]^2 \times V(G_n)$ has at most one neighbor in $[N]^2 \times V(G_n)$, and thus can never be infected by a 2-neighbor bootstrap percolation process started from A. ■

Remark 3.2. The proof above in fact works in a more general setting than $\mathbb{Z}^2$: suppose H has the finite 2-percolation property and satisfies $p_{1,c}(H) < 1$. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of weakly q-pseudorandom n-vertex graphs with $nq(n) \gg \log n$. Then $H \times G_n$ also has the finite 2-percolation property, and the proof above shows

$$\lim_{n\to\infty} p_{1,c}(H \times G_n) = 4 - 2\sqrt{3}.$$

Examples of graphs with the finite 2-percolation property include many of the standard lattices studied in percolation theory, such as the honeycomb (hexagonal) lattice, the dice (rhombille) lattice or the tetrakis ("Union Jack") lattice.

Proof of Theorem 1.8. Since K_n is 1-pseudorandom, Theorem 1.8 is immediate from Theorem 1.11. ■

Proof of Theorem 1.12. Let $H = \mathbb{Z}^2$. Pick $\varepsilon > 0$ such that $1 - \varepsilon > 3/4$. Then by Lemma 3.1, for any $p > 4 - 2\sqrt{3}$, n sufficiently large and $G = G_n$, we can couple a random graph $(\mathbf{H} \times \mathbf{G})_\mu$, $\mu \in \mathcal{M}_{1,\geq p}(H)$ with a random graph $\mathbf{H}_\nu$, $\mu \in \mathcal{M}_{1,\geq 1-\varepsilon}(H)$ such that if $\mathbf{H}_\nu$ contains a path of length ℓ then so does $(\mathbf{H} \times \mathbf{G})_\mu$. Since $p_{1,LP}(H) = \frac{3}{4}$, as proved in [10, Theorem 11(i)][2] it follows that $p_{1,LP}(H \times G) \leq p$. Since $p > 4 - 2\sqrt{3}$ was arbitrary, we have the claimed upper bound $\lim_{n\to\infty} p_{1,LP}(H \times G_n) \leq 4 - 2\sqrt{3}$. The lower bound $\lim_{n\to\infty} p_{1,c}(H \times G_n) \geq 4 - 2\sqrt{3}$ was proved in [10, Theorem 12(v)] (with the same construction as we outlined at the beginning of this section, adapted mutatis mutandis to the setting $H = \mathbb{Z}$). ■

Proof of Theorem 1.9. Since K_n is 1-pseudorandom, Theorem 1.9 is immediate from Theorem 1.12. ■

4 | COMPONENT EVOLUTION IN 1-INDEPENDENT MODELS

Recall that the independence number $\alpha(G)$ of a graph G is the size of a largest independent (edge-free) subset of $V(G)$, and that a *perfect matching* in a graph G is a matching whose edges together cover all

[2]For the proof of this theorem, all we need is $p_{1,LP}(H) < 1$, and thus the weaker bound $p_{1,LP}(H) \leq 1 - 1/3e$ (which follows directly from an application of the Lovász local lemma) would suffice for our purposes here.

the vertices in $V(G)$. Moreover, a graph G is a *complete multipartite graph* if there exists a partition of $V(G)$ such that two vertices in $V(G)$ are joined by an edge in G if and only if they are contained in different parts of the partition. Finally, the *complement* G^c of a graph G is the graph on $V(G)$ whose edges are the non-edges of G, $G^c := (V(G), V(G)^{(2)} \setminus E(G))$.

Lemma 4.1. *If G is a complete multipartite graph on $2n$ vertices with independence number $\alpha(G) \leq n$, then G contains at least $n!$ perfect matchings.*

Proof. Let G be a complete multipartite graph on $2n$ vertices with the minimum number of perfect matchings subject to $\alpha(G) \leq n$. Let $V_1, V_2, \ldots, V_r$ denote the parts of G with $|V_1| \geq |V_2| \geq \ldots \geq |V_r|$. If $|V_{r-1}| + |V_r| \leq n$, then the graph G' obtained from G by deleting all edges in $G[V_{r-1}, V_r]$ satisfies $\alpha(G') \leq n$ and has at most as many perfect matchings as G. We may therefore assume that $|V_{r-1}| + |V_r| \geq n$, and thus in particular that $r \leq 3$. Consider a perfect matching M in G and let i be the number of edges in $E(G[V_1, V_2]) \cap M$. Clearly $|E(G[V_1, V_3]) \cap M| = |V_1| - i$ and $|E(G[V_2, V_3]) \cap M| = |V_2| - i = |V_3| - (|V_1| - i)$. From this we deduce that $i = \frac{1}{2}(|V_1| + |V_2| - |V_3|) = n - |V_3|$. Hence the number $\mathrm{PM}(G)$ of perfect matchings in G is:

$$\mathrm{PM}(G) = \binom{|V_1|}{i}\binom{|V_2|}{i}\binom{|V_3|}{|V_1| - i} i!(|V_2| - i)!(|V_1| - i)! = \frac{|V_1|!\,|V_2|!\,|V_3|!}{(n - |V_1|)!(n - |V_2|)!(n - |V_3|)!}.$$

(Here $\binom{|V_1|}{i}\binom{|V_2|}{i} i!$ counts the number of different ways of selecting i-sets of vertices from each of V_1 and V_2 and joining them by a perfect matching, while $\binom{|V_3|}{|V_1|-i}(|V_2| - i)!(|V_1| - i)!$ counts the number of ways of joining the vertices of V_3 by a perfect matching to the remaining vertices of $V_1 \cup V_2$.) If $|V_3| > 0$, then let G' be the complete tripartite graph with parts of size $|V_1|, |V_2| + 1, |V_3| - 1$. Note that $\alpha(G') \leq n$. By the formula above, we have

$$\frac{\mathrm{PM}(G)}{\mathrm{PM}(G')} = \frac{|V_3|\,(n - |V_3| + 1)}{(|V_2| + 1)(n - |V_2|)} \geq 1,$$

since $|V_3|\,(n - |V_3| + 1) - (|V_2| + 1)(n - |V_2|) = (|V_2| - |V_3| + 1)(|V_2| + |V_3| - n) \geq 0$ (as $|V_2| \geq |V_3|$ and $|V_2| + |V_3| \geq n$). It follows that $\mathrm{PM}(G) \geq \mathrm{PM}(K_{n,n}) = n!$ as claimed. ∎

Proof of Proposition 1.16. Let $H = K_{2n}$. For all $p \in [\frac{1}{2}, 1]$, we may construct the two-state measure $\mu_{2s,p} \in \mathcal{M}_{1,p}(H)$ which satisfies:

$$\mathbb{P}\left[|C_1(\mathbf{H}_{\mu_{2s},p})| \leq n\right] = \mathbb{P}\left[|C_1(\mathbf{H}_{\mu_{2s},p})| = n\right] = \binom{2n}{n}\theta^n(1-\theta)^n = \binom{2n}{n}\left(\frac{1-p}{2}\right)^n,$$

proving the upper bound in that range. For $p_{2n} \leq p \leq \frac{1}{2}$, we note that $\theta = \theta(p)$ is no longer a real number. However, as shown in [10, Section 7.1], we may take a "complex limit" of the 2-state measure $\mu_{2s,p}$, and the conclusion above still holds.

For the lower bound, let $C_1, C_2, \ldots, C_r$ be the connected components of a μ-random subgraph $\mathbf{H}_\mu$ of K_{2n}. Let $\mathbf{G}$ denote the complete multipartite graph associated with the partition $\sqcup_i C_i$ of $V(K_{2n}) = [2n]$. Observe that $\mathbf{G}$ is a subgraph of the complement $\mathbf{H}_\mu^c$ of $\mathbf{H}_\mu$. If $|C_i| \leq n$ for all i, then $\alpha(\mathbf{G}) \leq n$, whence by Lemma 4.1 $\mathbf{G}$ contains at least $n!$ perfect matchings. In particular, $\mathbf{H}_\mu^c$ must contain at least $n!$ perfect matchings. By Markov's inequality, we thus have

$$\begin{aligned}\mathbb{P}\left[|C_1(\mathbf{H}_\mu)| \leq n\right] &\leq \mathbb{P}\left[\mathbf{H}_\mu^c \text{ contains} \geq n! \text{ perfect matchings}\right]\\ &\leq \frac{1}{n!}\mathbb{E}\left[\#\{\text{perfect matchings in } \mathbf{H}_\mu^c\}\right]\\ &= \frac{1}{n!}\left(\frac{1}{n!}\prod_{i=0}^{n-1}\binom{2n-2i}{2}\right)(1-p)^n = \binom{2n}{n}\left(\frac{1-p}{2}\right)^n.\end{aligned}$$

(Here $\left(\frac{1}{n!}\prod_{i=0}^{n-1}\binom{2n-2i}{2}\right)$ counts the number of perfect matchings in K_{2n} by selecting n vertex-disjoint edges sequentially one after the other, and dividing through by $n!$.) The lower bound follows. ∎

Proof of Theorem 1.17. Let $p \in \left(\frac{1}{r+1}, \frac{1}{r}\right]$ be fixed. Fix $\varepsilon = \varepsilon(p) > 0$ sufficiently small. For n large enough, we have by the pseudorandomness assumption on H_n that for every $U \subseteq V(H_n)$, $e(H_n[U]) \leq q\frac{|U|^2}{2} + \varepsilon^2 pqn^2$. It then follows from Lemma 2.4 that whp

$$e(\mathbf{H}_\mu) \geq pq\frac{n^2}{2}(1-4\varepsilon^2), \tag{4.1}$$

which is strictly greater than $\frac{qn^2}{2(r+1)}$ for $\varepsilon = \varepsilon(p)$ chosen sufficiently small. Assume (4.1). We show this implies the claimed lower bound on the size of a largest component.

If $|C_1(\mathbf{H}_\mu)| \leq \frac{n}{r+1} - \varepsilon n$, then for ε sufficiently small there is a partition of $V(H)$ into at most $2(r+1)+1$ sets, each of which has size at most $\frac{n}{r+1} - \varepsilon n$, such that every connected component of $\mathbf{H}_\mu$ is wholly contained in one of the sets of the partition. Indeed, such a partition can be obtained by starting with a partition of $V(H)$ into the connected components of $\mathbf{H}_\mu$, and then as long as the partition contains two parts of size at most $\frac{1}{2}\left(\frac{n}{r+1} - \varepsilon n\right)$, choosing two such parts arbitrarily and merging them into a single part. Since for any $(2r+3)$-tuple $(x_1, \ldots, x_{2r+3})$ with $\frac{1}{r+1} - \varepsilon \geq x_i \geq 0$ and $\sum_i x_i = 1$ we have $\sum_i (x_i)^2 \leq (r+1)\left(\frac{1}{r+1} - \varepsilon\right)^2 + ((r+1)\varepsilon)^2$, we have by our pseudorandomness assumption that

$$e(\mathbf{H}_\mu) \leq \frac{q(r+1)}{2}\left(\frac{1}{r+1} - \varepsilon\right)^2 n^2 + \frac{q}{2}((r+1)\varepsilon)^2 n^2 + (2r+3)\varepsilon^2 pqn^2 < \frac{qn^2}{2(r+1)}$$

for ε sufficiently small, contradicting (4.1). Thus we may assume that $|C_1(\mathbf{H}_\mu)| > \frac{n}{r+1} - \varepsilon n$.

If $|C_1(\mathbf{H}_\mu)| \geq \frac{n}{r}$, then we have nothing to show. Finally if $\frac{n}{r+1} - \varepsilon n \leq |C_1(\mathbf{H}_\mu)| < \frac{n}{r}$, then $\mathbf{H}_\mu$ contains at least $r+1$ components. Let αn denote the size of a largest component, where $\frac{1}{r+1} - \varepsilon < \alpha < \frac{1}{r}$. Then

$$\left(r\alpha^2 + (1-r\alpha)^2\right) q\frac{n^2}{2} + (r+2)\varepsilon^2 pqn^2 \geq e(\mathbf{H}_\mu) \geq pq\frac{n^2}{2}(1-4\varepsilon^2).$$

Dividing through by $qn^2/2$, rearranging terms and using the fact ε is chosen sufficiently small, we get

$$r\alpha^2 + (1-r\alpha)^2 \geq p - \varepsilon.$$

Solving for α, we get that

$$\alpha \geq \frac{1 + \sqrt{\frac{(r+1)(p-\varepsilon)-1}{r}}}{r+1},$$

giving part (i).

For part (ii), consider the $r+1$-state measure in which each vertex is assigned state $r+1$ with probability $\frac{1-\sqrt{r((r+1)p-1)}}{r+1}$ and a uniform random state from the set $\{1, 2, \dots, r\}$ otherwise, and in which an edge is open if and only if its vertices are in the same state. This is easily seen to be a 1-ipm with the requisite properties. ∎

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