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We construct parametric Liouville surfaces corresponding to parametric soliton solutions of the Liouville equation and Darboux-transformed counterparts. We also use a modified variation of parameters method together with the elliptic functions method to obtain the traveling wave solutions to Liouville equation, and express the centroaffine invariant in terms of the Hamiltonian.

Keywords: Liouville equation, Liouville soliton surface, centro-affine invariant, Darboux transformation, Lax pair.

I. INTRODUCTION

The Liouville nonlinear equation first occurred as a result obtained by Liouville when he carefully studied the surfaces of constant curvature, cf. one of his notes, in the fifth edition of the famous textbook by Monge [1] curated by Liouville. He was led to the following partial differential equation

$$(\log \lambda)_{uv} \pm \frac{\lambda}{2a^2} = 0 \ , \tag{1}$$

where a is a real arbitrary constant. By letting $\alpha = \mp \frac{1}{2a^2}$, and denoting $\Lambda = \log \lambda$, this equation may be written in the equivalent form

$$E(\Lambda) \equiv \Lambda_{uv} - \alpha e^{\Lambda} = 0 \ . \tag{2}$$

In his notes, Liouville presented a very simple method to the complete integral of (1) through some geometric considerations related to the properties of the sphere [1, 2]. His solution, involving two arbitrary functions $\Phi(u)$ and $\Psi(v)$ was written as

$$\lambda(u,v) = \frac{2}{\mp\alpha}\frac{\Phi_u\Psi_v \ e^{\Phi(u)+\Psi(v)}}{[1 \pm e^{\Phi(u)+\Psi(v)}]^2}. \tag{3}$$

One can easily see that (3) can be also written as

$$\begin{array}{l} \lambda(u,v) = \mp\frac{1}{2\alpha}\frac{\Phi_u\Psi_v}{\cosh^2\left(\frac{\Phi(u)+\Psi(v)}{2}\right)} \quad \text{or} \\ \lambda(u,v) = \pm\frac{1}{2\alpha}\frac{\Phi_u\Psi_v}{\sinh^2\left(\frac{\Phi(u)+\Psi(v)}{2}\right)} \ , \end{array} \tag{4}$$

corresponding to bound and singular at the origin solutions, respectively. An even simpler form of (3) can be obtained by using the substitutions $\log\phi = \Phi$, and $\log\psi = \Psi$,

$$\lambda(u,v) = \frac{2}{\mp\alpha}\frac{\phi_u\psi_v}{(1 \pm \phi\psi)^2} \ . \tag{5}$$

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In this paper, we focus on the soliton surfaces generated by Darboux/auto-Bäcklund transformations of the Liouville sech^2 soliton obtained when one employs linear functions $\Phi(u)$ and $\Psi(v)$. Although these transformations are known for more than a century in mathematics, and the concept of soliton surface has been introduced in mathematical physics in the 1980s [3, 4], we could not find any discussion of the Liouville soliton surfaces in the recent literature, e.g., [5–7]. This gap in the literature served us as a motivation for writing this paper. Another motivation is that soliton surfaces are geometrical analogs of gauge theories of (super)strings, spin systems, and chiral integrable models of elementary particles in which the interaction is not generated by considering interaction Lagrangians, but has pure geometric origin related to the curvature of the soliton surface and other geometric and topological concepts [3, 4, 8, 9].

II. AN AFFINE SURFACE AND THE BÄCKLUND TRANSFORMATION

It was noted by the Romanian mathematician Gheorghe Țițeica, a.k.a. Georges Tzitzéica [10] who studied under Darboux and Goursat, that in some asymptotic coordinates u, v, surfaces Σ parametrized by $s^1(u,v),\ s^2(u,v),\ s^3(u,v)$ have the property that the Gaussian curvature $\mathcal{K}$ is proportional to the forth power of the distance D from the origin O to the tangent plane to the surface Σ at some point P of coordinates (s^1, s^2, s^3). Thus, Tzitzéica introduced the centroaffine constant invariant $\mathcal{I}$ defined by

$$\mathcal{I} = \frac{\mathcal{K}}{D^4}\ . \tag{6}$$

The idea to construct surfaces in this way was also used by Tzitzéica for the equation rediscovered by Dodd and Bullough (the Dodd-Bullough equation), and further he introduced a mapping (auto-Bäcklund transformation) between surfaces in his papers [11, 12], and book published in French [10].

These surfaces therefore admit the affine representation given by the coordinates (s^1, s^2, s^3) which are three linearly independent integrals of the linear system

$$\begin{aligned} s_{uu} &= a s_u + b s_v\ , \\ s_{uv} &= \alpha\lambda s\ , \\ s_{vv} &= a' s_u + b' s_v\ . \end{aligned} \tag{7}$$

For the specific values of $a = \frac{\lambda_u}{\lambda}$, $b = a' = 0$, and $b' = \frac{\lambda_v}{\lambda}$, <u>any</u> solution $\lambda(u,v)$ of the Liouville equation represents the compatibility solution of the system

$$\begin{aligned} s_{uu} &= \tfrac{\lambda_u}{\lambda} s_u \\ s_{uv} &= \alpha\lambda s, \\ s_{vv} &= \tfrac{\lambda_v}{\lambda} s_v. \end{aligned} \tag{8}$$

The first Lax pair for (2), known by Tzitzéica, is given by

$$\begin{aligned} S_u &= LS \\ S_v &= AS \end{aligned}\ , \tag{9}$$

where $S(u,v)$ is a vector valued wave function given by $S^T = (s_v, s_u, s)$, and L, A are the third order matrices

$$L = \begin{pmatrix} 0 & 0 & \alpha e^{\Lambda} \\ 0 & \Lambda_u & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} \Lambda_v & 0 & 0 \\ 0 & 0 & \alpha e^{\Lambda} \\ 1 & 0 & 0 \end{pmatrix}. \tag{10}$$

This pair satisfies the compatibility condition $S_{uv} = S_{vu}$ which leads to

$$L_v - A_u + [L, A] = \begin{pmatrix} -E(\Lambda) & 0 & 0 \\ 0 & E(\Lambda) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

We are interested in generating solitonic surfaces corresponding to (1) and first Lax pair (10) for which we use a linear form of the general functions given by

$$\begin{aligned} \Phi(u) &= 2\sqrt{\alpha}\ ku\ , \\ \Psi(v) &= -2\sqrt{\alpha}\ \omega v. \end{aligned} \tag{12}$$

Since we deal with solitons, we also use henceforth the right- and left-moving traveling coordinates, $\xi = ku - \omega v$ and $\zeta = ku + \omega v$, respectively, and also known as light-cone coordinates in the context of linear wave equations. Thus, for $\alpha > 0$, the soliton solution of (1) in the variable ξ has the form,

$$\lambda(u,v) = 2k\omega\ \mathrm{sech}^2\left(\sqrt{\alpha}\xi\right)\ . \tag{13}$$

This will be the seed solution used to generate Σ. To find the surface corresponding to these solutions, we need to solve the linear system (8). By two quadratures the third equation leads to

$$s(u,v) = -c_1(u)\frac{\tanh\left(\sqrt{\alpha}\xi\right)}{\sqrt{\alpha}\omega} + c_2(u) = \left[-\frac{c_1(u)}{\sqrt{\alpha}\omega}\sinh\left(\sqrt{\alpha}\xi\right) + c_2(u)\cosh\left(\sqrt{\alpha}\xi\right)\right]\mathrm{sech}\left(\sqrt{\alpha}\xi\right)\ , \tag{14}$$

while the second equation is satisfied for

$$c_2(u) = \frac{c_1(u)'}{2k\alpha\omega}\ . \tag{15}$$

To find $c_1(u)$ we substitute the last two in the first equation of (8), which leads to the third order linear differential equation

$$c_1(u)''' - 4\alpha k^2 c_1(u)' = 0\ . \tag{16}$$

Thus, the surface Σ is determined by the algebraic curves of the cubic

$$m^3 - 4\alpha k^2 m = 0 \tag{17}$$

with roots $m_1 = 0$, $m_{2,3} = \pm 2\sqrt{\alpha}k$ which lead to the three solutions

$$\begin{aligned} c_1^1(u) &= \sqrt{\alpha}\omega\ , \\ c_1^2(u) &= \sqrt{\alpha}\omega\sinh(2\sqrt{\alpha}k\ u)\ , \\ c_1^3(u) &= \sqrt{\alpha}\omega\cosh(2\sqrt{\alpha}k\ u)\ . \end{aligned} \tag{18}$$

Using these functions in (15) and (14), the three linearly independent solutions of system (8)

$$\begin{aligned} s^1(u,v) &= \sinh\left(\sqrt{\alpha}\xi\right)\mathrm{sech}\left(\sqrt{\alpha}\xi\right)\ , \\ s^2(u,v) &= \cosh\left(\sqrt{\alpha}\zeta\right)\ \mathrm{sech}\left(\sqrt{\alpha}\xi\right)\ , \\ s^3(u,v) &= \sinh\left(\sqrt{\alpha}\zeta\right)\ \mathrm{sech}\left(\sqrt{\alpha}\xi\right)\ . \end{aligned} \tag{19}$$

The Darboux-transformed Liouville solutions can be successively generated using the invariant transformations [13], [14]

$$\tilde{\lambda}^i = -\lambda + \frac{2}{\alpha}(\log s^i)_u(\log s^i)_v\ , \tag{20}$$

where λ is the seed solution given by (13), and s^i with $i = 1, 2, 3$, is any solution of (19). This type of transformation known as Bäcklund transformation represents a relation between two sets of independent solutions λ and $\tilde{\lambda}$ of the same PDE.

Using successively these three solutions, the new potentials $\tilde{\lambda}$ are

$$\begin{aligned} \tilde{\lambda}^1(u,v) &= -2k\omega\ \mathrm{csch}^2\left(\sqrt{\alpha}\xi\right)\ , \\ \tilde{\lambda}^2(u,v) &= -2k\omega\ \mathrm{sech}^2\left(\sqrt{\alpha}\zeta\right)\ , \\ \tilde{\lambda}^3(u,v) &= 2k\omega\ \mathrm{csch}^2\left(\sqrt{\alpha}\zeta\right)\ . \end{aligned} \tag{21}$$

These are also solutions to the Liouville equation (1), and can be infinitely many times generated using (20).

The three solutions $s^i(u,v)$ allow us to generate a parametric surface $\Sigma(s^1, s^2, s^3)$ in $\mathbb{R}^3$ known as the Liouville soliton surface. To classify the surface, notice the quadratic relation between these solutions defining the one-sheeted hyperboloid

$$\Sigma(u,v) \equiv [s^1(u,v)]^2 + [s^2(u,v)]^2 - [s^3(u,v)]^2 = \frac{4k^2}{\alpha}\ , \tag{22}$$

for which

$$s^3 = h(s^1, s^2) = \sqrt{[s^1(u,v)]^2 + [s^2(u,v)]^2 - \frac{4k^2}{\alpha}} \ . \tag{23}$$

To calculate the centroaffine invariant for this surface, we use the curvature

$$\mathcal{K} = \frac{|h_{s^1 s^1} h_{s^2 s^2} - (h_{s^1 s^2})^2|}{[1 + (h_{s^1})^2 + (h_{s^2})^2]^2} \ , \tag{24}$$

and distance D

$$D = \frac{|s^1 h_{s^1} + s^2 h_{s^2} - h(s^1, s^2)|}{[1 + (h_{s^1})^2 + (h_{s^2})^2]^{1/2}} \ . \tag{25}$$

For Σ, we have then the invariant is

$$\mathcal{I} = \frac{\alpha^3}{(2k)^6} \ . \tag{26}$$

This indeed shows that the Gaussian curvature at the point P on Σ is proportional to the fourth power of the distance from the origin to the tangent plane to Σ at P. In Fig. 1, we plot the Liouville surface Σ for the seed sech square soliton solution for some fixed values of the set k, ω, and α.

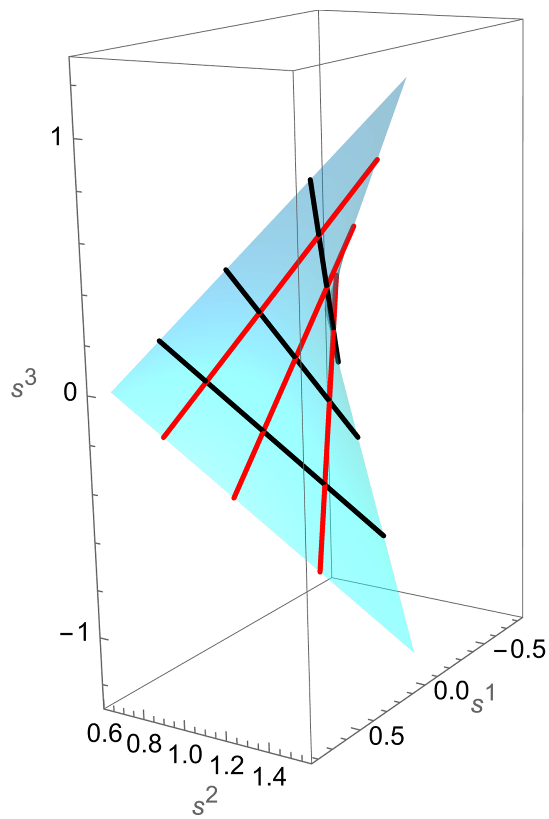

FIG. 1: One octant of the Liouville soliton surface Σ (hyperboloid of one sheet) for the solitary wave solution (13) with $k = 1, \omega = 1, \alpha = 1/4$. The centroaffine invariant is $\mathcal{I} = 1/4^6$. The red curves are found using $v = -1/2, 0, 1/2$, while the black curves correspond to $u = -1/2, 0, 1/2$.

One can generalize these type of surfaces by choosing any particular functions of interest, such as general power-functions that depend on some parameter p. One example, given by $\Phi(u) = 2\sqrt{\alpha}\ ku^p$ and $\Psi(v) = -2\sqrt{\alpha}\ \omega v^p$ leads to the parametric soliton solution

$$\lambda(u, v; p) = 2p^2\, k\omega\, (uv)^{p-1}\, \mathrm{sech}^2\left(\sqrt{\alpha}\xi^p\right) \ , \tag{27}$$

where $\xi^p = ku^p - \omega v^p$. The soliton solutions corresponding to $p = 1, 2, 3$ are presented in Fig. 2.

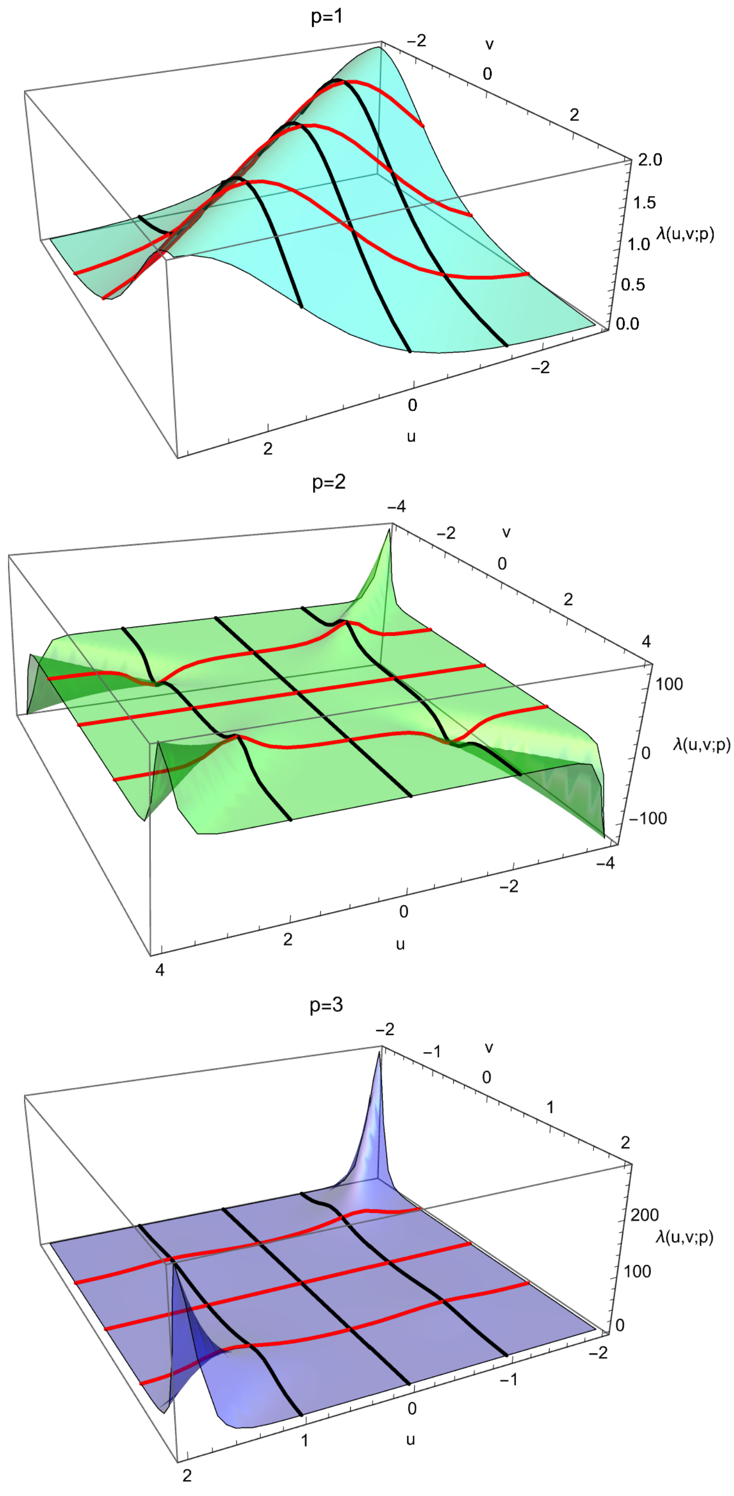

FIG. 2: The parametric solution from (27) with $k = 1,\ \omega = 1,\ \alpha = 1/4$, for $p = 1, 2, 3$ from top to bottom. The traces $u = const.$ (black color), and $v = const.$ (red color) are also displayed.

III. THE MODIFIED VARIATION OF PARAMETERS METHOD

Next, we present an equivalent method to obtain (13) based on a traveling wave *ansatz*, and by using a modified variation of parameters method which was first introduced by Kečkić in 1946 [15].

Effecting the logarithmic derivative, (1) becomes

$$\lambda\lambda_{uv} - \lambda_u\lambda_v = \alpha\lambda^3 \ , \tag{28}$$

which can be turned into the ordinary differential equation

$$\lambda_\xi{}^2 - \lambda\lambda_{\xi\xi} = \frac{\alpha}{k\omega}\lambda^3 \ , \tag{29}$$

using the traveling wave variable $\xi = ku - \omega v$. This equation can be written as a system of first order equations

$$\begin{aligned} \lambda_\xi &= \eta \equiv M(\lambda,\eta) \ , \\ \eta_\xi &= \tfrac{1}{\lambda}\left(\eta^2 - \tfrac{\alpha}{k\omega}\lambda^3\right) \equiv N(\lambda,\eta) \ . \end{aligned} \tag{30}$$

The system (30) is integrable since there is a first integral $\mathcal{H}(\lambda,\eta) \equiv const$ of (29) such that

$$\begin{aligned} M(\lambda,\eta) &= \mu(\lambda,\eta)\tfrac{\partial\mathcal{H}}{\partial\eta} \ , \\ N(\lambda,\eta) &= -\mu(\lambda,\eta)\tfrac{\partial\mathcal{H}}{\partial\lambda} \ . \end{aligned} \tag{31}$$

For the special case of unitary integrating factor $\mu(\lambda,\eta) \equiv 1$, then (30) is also Hamiltonian with first integral given by

$$\mathcal{H}(\lambda,\eta) = \left(\frac{\eta}{\lambda}\right)^2 + \frac{2\alpha}{k\omega}\lambda \equiv const \ , \tag{32}$$

and curves shown in Fig. 3. However, our system (31) holds true when the integrating factor is $\mu(\lambda,\eta) = \lambda^2/2$ for

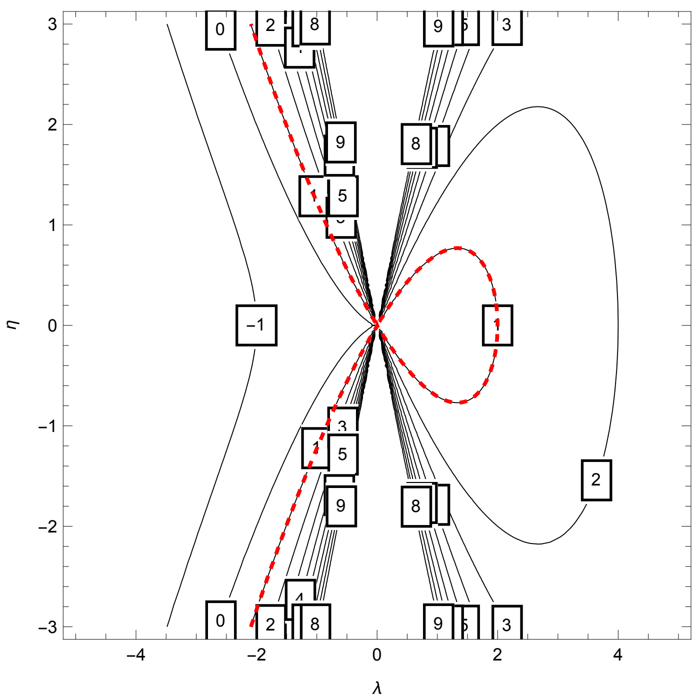

FIG. 3: Hamiltonian curves for $k = 1$, $\omega = 1$, and $\alpha = 1/4$. The dotted red curve corresponds to $\mathcal{H} = 1$ and solution (44).

which (29) yields to the degenerate elliptic equation

$$\lambda_\xi{}^2 = \mathcal{H}\lambda^2 - \frac{2\alpha}{k\omega}\lambda^3 \ . \tag{33}$$

When $a \to \infty$ then $\alpha \to 0$, and the associated equation of (29) becomes

$$\lambda_\xi{}^2 - \lambda\lambda_{\xi\xi} = 0 \ . \tag{34}$$

By one quadrature we obtain

$$\lambda_\xi = K\lambda \ , \tag{35}$$

while the second quadrature yields to the exponential solution

$$\lambda(\xi) = K_0 e^{K\xi} \ , \tag{36}$$

with K and K_0 arbitrary constants.

If we now suppose that K is not a constant, i.e., $K = K(\lambda)$, by substituting (36) into (29) we obtain

$$\lambda_\xi{}^2 - \lambda\lambda_{\xi\xi} = -KK_\lambda \lambda^3 \ , \tag{37}$$

which can be matched with (29) if $K(\lambda)$ is such that,

$$-KK_\lambda = \frac{\alpha}{k\omega} \ . \tag{38}$$

The latter equation leads to

$$K(\lambda) = \sqrt{a_2 - \frac{2\alpha}{k\omega}\lambda} \ , \tag{39}$$

where a_2 is an arbitrary constant. Letting $a_3 = -\frac{2\alpha}{k\omega}$ then (39) becomes

$$\lambda_\xi = \lambda\sqrt{a_2 + a_3\lambda} \ , \tag{40}$$

which is exactly (33) for $a_2 = \mathcal{H}$. This equation belongs to the general class of elliptic equations of the form (see also section 4 in [7])

$$\lambda_\xi{}^2 = \sum_{i=0}^{3} a_i \lambda(\xi)^i \equiv Q_3(\lambda) \ , \tag{41}$$

which admit in general both cnoidal and solitary waves solutions. In our case, since $a_0 = a_1 = 0$, then $\lambda = 0$ is a double root of $Q_3(\lambda)$, and (41) admits only the solitary wave solutions

$$\lambda(\xi) = -\frac{a_2}{a_3}\mathrm{sech}^2\left[\frac{1}{2}\sqrt{a_2}(\xi - \xi_0)\right], \quad a_2 > 0, \tag{42}$$

with ξ_0 an arbitrary constant. For $a_2 = 4\alpha$, and by taking $\xi_0 = 0$ with $a_3 = -\frac{2\alpha}{k\omega}$ this solution corresponds to (13) with integral of motion given by $\mathcal{H} = 1$, and

$$\lambda_\xi{}^2 = \lambda^2 - \frac{\lambda^3}{2} \ . \tag{43}$$

In particular, for $\xi_0 = 0$, we obtain the soliton

$$\lambda(\xi) = 2k\omega \ \mathrm{sech}^2\left(\frac{\xi}{2}\right) \ . \tag{44}$$

To end up this section, we notice that according to the above discussion the centroaffine invariant can be written directly in terms of the Hamiltonian constant of the soliton solution as

$$\mathcal{I} = \left(\frac{\mathcal{H}}{k^2}\right)^3 \ , \tag{45}$$

since for such solutions the Hamiltonian is a disguised form of the parameter entering the Liouville equation (2).

IV. A ONE-PARAMETER LAX PAIR

The second parametric third order matrix Lax pair for (2), known by Mikhailov [16], is given by

$$M = \begin{pmatrix} L_{11}\Lambda_u & 0 & L_{13}\ e^{\Lambda/2} \\ 0 & L_{22}\Lambda_u & 0 \\ 0 & L_{32}e^{\Lambda/2} & 0 \end{pmatrix}, \quad N = \begin{pmatrix} A_{11}\Lambda_v & 0 & 0 \\ 0 & A_{22}\Lambda_v & A_{23}\ e^{\Lambda/2} \\ A_{31}\ e^{\Lambda/2} & 0 & 0 \end{pmatrix} \ . \tag{46}$$

These matrices also satisfy

$$\begin{array}{l} R_u = MR \ , \\ R_v = NR \ , \end{array} \tag{47}$$

for some general vector valued wave function $R^T(u,v) = (F(u,v), G(u,v), H(u,v))$. This time, the compatibility condition leads to

$$\begin{pmatrix} A_{31}L_{13}\lambda - \frac{(A_{11}-L_{11})(\lambda_u\lambda_v-\lambda\lambda_{uv})}{\lambda^2} & 0 & \frac{(1-2A_{11})L_{13}\lambda_v}{2\sqrt{\lambda}} \\ 0 & -A_{23}L_{32}\lambda + \frac{(A_{22}-L_{22})(\lambda_u\lambda_v-\lambda\lambda_{uv})}{\lambda^2} & \frac{A_{23}(L_{22}-1)}{2\sqrt{\lambda}}\lambda_u \\ \frac{-(1+2L_{11})A_{31}}{2\sqrt{\lambda}}\lambda_u & \frac{(1+2A_{22})L_{32}}{2\sqrt{\lambda}}\lambda_v & (A_{23}L_{23} - A_{31}L_{13})\lambda \end{pmatrix} \equiv 0 \ . \tag{48}$$

Identifying the terms in the matrix, we require

$$\begin{array}{l} A_{11} = \frac{1}{2} \ , \\ L_{11} = -\frac{1}{2} \ , \\ L_{22} = \frac{1}{2} \ , \\ A_{22} = -\frac{1}{2} \ , \\ A_{23}L_{32} = A_{31}L_{13} \ . \end{array} \tag{49}$$

Choosing the parameters $L_{13} = \beta$ and $A_{31} = \frac{\alpha}{\beta}$, then $A_{23} = \beta$, and $B_{32} = \frac{\alpha}{\beta}$, and substituted into (46) leads to the one-parameter Lax pair

$$M = \begin{pmatrix} -\frac{\lambda_u}{2\lambda} & 0 & \beta\sqrt{\lambda} \\ 0 & \frac{\lambda_u}{2\lambda} & 0 \\ 0 & \frac{\alpha}{\beta}\sqrt{\lambda} & 0 \end{pmatrix} , \quad N = \begin{pmatrix} \frac{\lambda_v}{2\lambda} & 0 & 0 \\ 0 & -\frac{\lambda_v}{2\lambda} & \beta\sqrt{\lambda} \\ \frac{\alpha}{\beta}\sqrt{\lambda} & 0 & 0 \end{pmatrix} \ . \tag{50}$$

Now we will need to determine the eigenfunction $R(u,v)$ of the commutating differential operators which satisfy (47). The first equation of the system leads to

$$\begin{pmatrix} F_u + \frac{\lambda_u}{2\lambda}F - \beta\sqrt{\lambda}H \\ G_u - \frac{\lambda_u}{2\lambda}G \\ H_u - \frac{\alpha}{\beta}\sqrt{\lambda}G \end{pmatrix} = 0 \ , \tag{51}$$

while the second one gives

$$\begin{pmatrix} F_v - \frac{\lambda_v}{2\lambda}F \\ G_v + \frac{\lambda_v}{2\lambda}G - \beta\sqrt{\lambda}H \\ H_v - \frac{\alpha}{\beta}\sqrt{\lambda}F \end{pmatrix} = 0 \ . \tag{52}$$

To solve these systems, we notice that we can integrate the second equation of the first set (51) to obtain

$$G(u,v) = c_1(v)\mathrm{sech}\left(\sqrt{\alpha}\xi\right) \ . \tag{53}$$

Using this result into the third equation and integrating again we can find

$$H(u,v) = \left[c_1(v)\frac{\sqrt{2\alpha\omega}}{\beta\sqrt{k}}\sinh\left(\sqrt{\alpha}\xi\right) + c_2(v)\cosh\left(\sqrt{\alpha}\xi\right)\right]\mathrm{sech}\left(\sqrt{\alpha}\xi\right) \ . \tag{54}$$

Since we have $H(u,v)$ up to the two integration constants, we integrate the first equation to obtain

$$F(u,v) = \left\{\frac{\left[c_1(v)\sqrt{\alpha}\omega\sinh\left(\sqrt{\alpha}\xi\right) + c_2(v)\sqrt{k\omega/2}\beta\cosh\left(\sqrt{\alpha}\xi\right)\right]^2}{\alpha k\omega c_1(v)} + c_3(v)\cosh^2\left(\sqrt{\alpha}\xi\right)\right\}\mathrm{sech}\left(\sqrt{\alpha}\xi\right) \ . \tag{55}$$

The arbitrary constants c_i will be determined from the second system (52) by eliminating all the functions F, H and their partial derivatives in v to obtain only a third order equation on G, which after some cumbersome algebra reads

$$G_{vvv} - 3\sqrt{\alpha}\omega\tanh\left(\sqrt{\alpha}\xi\right)G_{vv} - \alpha\omega^2 G_v + 3\alpha^{3/2}\omega^3\tanh\left(\sqrt{\alpha}\xi\right)G = 0 \ . \tag{56}$$

Once we obtained this equation, we substitute $G(u,v)$ together with its partial derivatives, G_v, G_{vv}, and G_{vvv} into (56), to hopefully obtain a manageable equation in c_1. Surprisingly, but not unexpected, the new ODE in c_1 is

$$c_1(v)''' - 4\alpha\omega^2 c_1(v)' = 0 \ , \tag{57}$$

with the same equation as (16) but with k replaced by ω, and u by v. Using

$$\begin{aligned} c_1^1(v) &= 1 \ , \\ c_1^2(v) &= \sinh(2\sqrt{\alpha}\omega v) \ , \\ c_1^3(v) &= \cosh(2\sqrt{\alpha}\omega v) \ , \end{aligned} \tag{58}$$

in (53), we obtain the solutions corresponding to $G(u,v)$ that are

$$\begin{aligned} g^1(u,v) &= \operatorname{sech}\left(\sqrt{\alpha}\xi\right) \ , \\ g^2(u,v) &= \sinh(2\sqrt{\alpha}\omega v) \operatorname{sech}\left(\sqrt{\alpha}\xi\right) \ , \\ g^3(u,v) &= \cosh(2\sqrt{\alpha}\omega v) \operatorname{sech}\left(\sqrt{\alpha}\xi\right) \ . \end{aligned} \tag{59}$$

The three solutions corresponding to $H(u,v)$ found from the second equation of (52) are

$$\begin{aligned} h^1(u,v) &= \sinh\left(\sqrt{\alpha}\xi\right) \operatorname{sech}\left(\sqrt{\alpha}\xi\right) \ , \\ h^2(u,v) &= \cosh\left(\sqrt{\alpha}\zeta\right) \ \operatorname{sech}\left(\sqrt{\alpha}\xi\right) \ , \\ h^3(u,v) &= \sinh\left(\sqrt{\alpha}\zeta\right) \ \operatorname{sech}\left(\sqrt{\alpha}\xi\right) \ . \end{aligned} \tag{60}$$

Finally, since we have H, we can use the third equation of system (52) to find the three corresponding $F(u,v)$, which are

$$\begin{aligned} f^1(u,v) &= \operatorname{sech}\left(\sqrt{\alpha}\xi\right) \ , \\ f^2(u,v) &= \sinh(2\sqrt{\alpha}ku) \operatorname{sech}\left(\sqrt{\alpha}\xi\right) \ , \\ f^3(u,v) &= \cosh(2\sqrt{\alpha}ku) \operatorname{sech}\left(\sqrt{\alpha}\xi\right) . \end{aligned} \tag{61}$$

Notice that these three functions satisfy the first equation of system (52), as they should. Combining all these functions (59)- (61) we can finally construct the two Darboux-transformed Liouville surfaces $\Sigma_2(f^2,g^2,h^2)$, and $\Sigma_3(f^3,g^3,h^3)$, which are presented in Fig. 4.

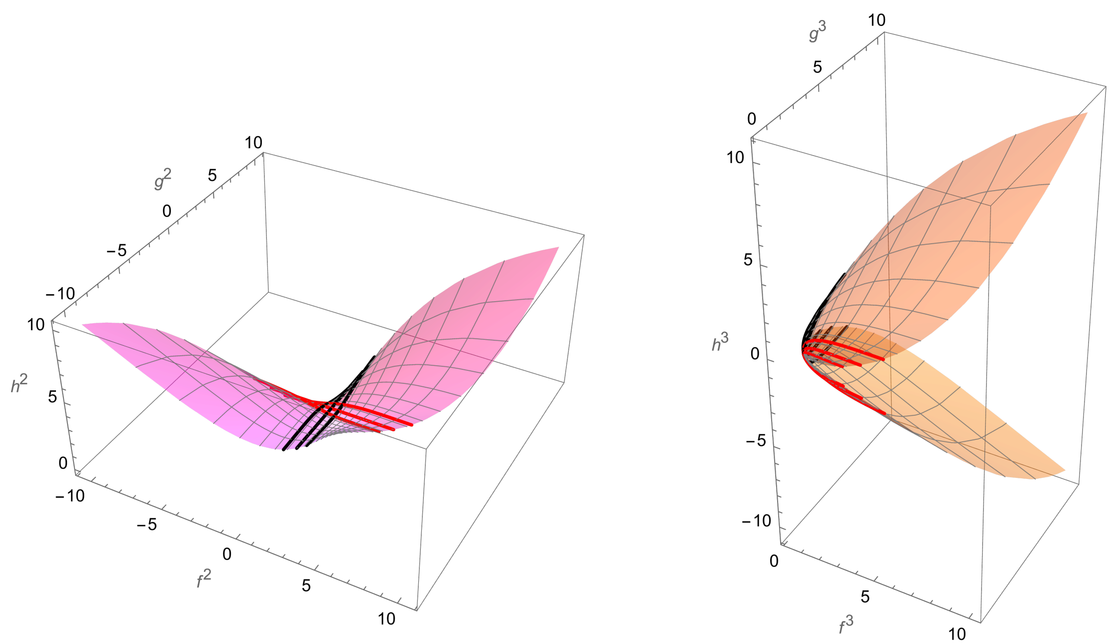

FIG. 4: The Liouville soliton surfaces $\Sigma_{2,3}$ corresponding to the Darboux-transformed Liouville solutions (20) for the same values $k=1, \omega=1, \alpha=1/4$.

V. SUMMARY AND FUTURE PROSPECTS

In this paper, we have shown explicitly how soliton surfaces can be constructed based on the soliton solution of the Liouville equation in the travelling variable and its Darboux partner solutions. Similar surfaces can be constructed for other equations in the same class, such as the Tzitzéica equation (also known as Dodd-Bullough-Tzitéica equation), the Dodd-Bullough-Mikhailov (DBM) equation, and the Tzitzéica-Dodd-Bullough-Mikhailov(TDBM) equation, which written as (29), but with the coupling constant(s) in the power terms set to unity, have the form

$$\begin{aligned} {\lambda_\xi}^2 - \lambda\lambda_{\xi\xi} &= -\tfrac{1-\lambda^3}{k^2-\omega^2}\ , \\ {\lambda_\xi}^2 - \lambda\lambda_{\xi\xi} &= -\tfrac{1+\lambda^3}{k\omega}\ , \\ {\lambda_\xi}^2 - \lambda\lambda_{\xi\xi} &= -\tfrac{(1+\lambda)\lambda^3}{k\omega}\ , \end{aligned} \tag{62}$$

respectively. This set of equations have similar sech^2 soliton solutions [17], but the more complicated nonlinearities can generate cnoidal solutions. Consequently, periodic 'cnoidal' surfaces can be constructed. In fact, for any cubic or quartic polynomial in the right hand side of nonlinear equations of this type one can construct surfaces associated to their solutions following the method described here for the monomic Liouville case. We plan to study these surfaces in a future publication.

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