4-21-2023

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Qingzhao Kong
Jimei University

Wanting Wang
Jimei University

Dongxiao Zhang
Jimei University

Wenbin Zhang
Michigan Technological University, wenbinzh@mtu.edu

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Received: 18 January 2023 | Revised: 17 February 2023 | Accepted: 8 March 2023

CAAI Transactions on Intelligence Technology

DOI: 10.1049/cit2.12222

ORIGINAL RESEARCH

Two kinds of average approximation accuracy

Qingzhao Kong[1,2] | Wanting Wang[1] | Dongxiao Zhang[1] | Wenbin Zhang[3]

[1]Department of Science, Jimei University, Xiamen, China

[2]Digital Fujian Big Data Modeling and Intelligent Computing Institute, Jimei University, Xiamen, China

[3]Department of Computer Science, Michigan Technological University, Houghton, Michigan, USA

Correspondence

Qingzhao Kong.
Email: kongqingzhao@163.com

Funding information

National Natural Science Foundation of China, Grant/Award Number: 61976254; Natural Science Foundation of Fujian Province, Grant/Award Numbers: 2020J01707, 2020J01710

Abstract

Rough set theory places great importance on approximation accuracy, which is used to gauge how well a rough set model describes a target concept. However, traditional approximation accuracy has limitations since it varies with changes in the target concept and cannot evaluate the overall descriptive ability of a rough set model. To overcome this, two types of average approximation accuracy that objectively assess a rough set model's ability to approximate all information granules is proposed. The first is the relative average approximation accuracy, which is based on all sets in the universe and has several basic properties. The second is the absolute average approximation accuracy, which is based on undefinable sets and has yielded significant conclusions. We also explore the relationship between these two types of average approximation accuracy. Finally, the average approximation accuracy has practical applications in addressing missing attribute values in incomplete information tables.

KEYWORDS

rough sets, rough set theory

1 | INTRODUCTION

Due to the rapid development of technology and information interaction network, the data that people need to process every day expands like a surging tide. The importance of data has become increasingly prominent. Data has become a very important and indispensable factor of production. It is an urgent problem for people to find the theory and method of mining data quickly and effectively to obtain valuable knowledge. For example, people use statistical methods to analyse and describe the data, find and speculate the knowledge and rules hidden in the data, and then make reasonable decisions [1, 2]. Unlike classical data, sometimes we need to deal with some random and fuzzy data. Therefore, Zadeh created a method to describe fuzzy data by introducing appropriate membership functions, namely fuzzy set theory [3]. At present, fuzzy set theory plays an increasingly important and irreplaceable role in processing and analysing various fuzzy data problems [4–6].

At the same time, in view of the increasing scale of data, people urgently need to find effective methods to deal with massive data. Therefore, according to the characteristics of data labels, scholars granulate the data to obtain many knowledge granules or blocks that cannot be subdivided. Then people describe or express the data based on these blocks and finally obtain useful knowledge or rules. This method of processing data is called granular computing theory. Based on this theory, many effective granular computing models have been established and widely used in many data problems [7–13]. Undoubtedly, rough set is one of the most successful and widely used granular computing models for analysing data [14, 15].

Generally speaking, people always want to get accurate information or rules to make reasonable decisions. For any rough set model, the lower and upper approximation sets need to be constructed to describe or express the knowledge. However, we usually cannot make an accurate classification for the samples in the difference set of these two approximation sets. In other words, because of the existence of boundary region, a set can only be described approximately. The larger the boundary region is, the worse the completeness of the description is. In consideration of measuring the accuracy of the approximate description of a given concept, the concept of approximation accuracy is introduced. In this

way, we can get the approximation accuracy of any set in the universe. We all know such a basic fact: in order to better understand the academic performance of all students in a class, it is not enough to only know the test score of each student. More often, we also need to know the average performance of all students. Obviously, it is more reasonable to use the average score to judge the learning level of all students in the class.

From the above analysis, it is obvious that we cannot just be satisfied with knowing the approximate accuracy of each set in the universe. In order to get a more comprehensive understanding of the approximation accuracies of all sets in the universe, we also need to investigate the average value of the approximation accuracies of all sets. Therefore, the main contribution of this paper is to propose the concept of average approximation accuracy. With the help of this concept, we can understand the ability of any rough set model to approximate all sets in the universe.

The contents of the rest of this study are briefly described as follows. In Section 2, some important concepts related to rough set are listed. In Section 3, based on all sets in the universe, the relative average approximation accuracy is introduced, and many meaningful conclusions are obtained. In Section 4, the absolute average approximation accuracy is proposed according to all undefinable sets in an information table. Then, many important properties of absolute average approximation accuracy are discussed in detail. In Section 5, the relationship between these two kinds of average approximation accuracy is deeply studied. In Section 6, taking the absolute average approximation accuracy as an example, we investigate the application of the average approximation accuracy in filling the missing data in the incomplete information tables. Section 7 briefly summarises the main contents of this paper and makes prospects for future research work.

2 | PRELIMINARIES

Usually, an information table can be denoted by [16]:

$$I = (OB, AT, \{V_a | a \in AT\}, \{f_a | a \in AT\}) \tag{1}$$

where OB, AT, V_a and f_a are a non-empty finite set, a non-empty attribute set, the domain of attribute a and an information function respectively.

For $\forall A \subseteq AT$ and $\forall x \in OB$, then

$$xE_A y \Leftrightarrow \forall a \in A(f_a(x) = f_a(y))$$

$$[x]_A = \{y \in OB | xE_A y\}$$

$$OB/E_A = \{[x]_A | x \in OB\}$$

are the equivalence relation E_A, the equivalence class containing x and the partition of OB respectively [15].

The rough set theory created by Pawlak has powerful data processing capabilities [17–20]. The Pawlak model is introduced as follows [14].

Definition 2.1 In an information table shown in Equation (1), $A \subseteq AT$ is a subset of attributes, for $X \subseteq OB$, the lower and upper approximation sets of X are denoted by:

$$\underline{apr}_A(X) = \{x \in OB \mid [x]_A \subseteq X\},$$

$$\overline{apr}_A(X) = \{x \in OB \mid [x]_A \cap X \neq \varnothing\}.$$

For $\forall X \subseteq OB$, the inclusion relation $\underline{apr}_A(X) \subseteq X \subseteq \overline{apr}_A(X)$ holds. And, X is approximately described by two sets $\overline{apr}_A(X)$ and $\underline{apr}_A(X)$. Usually, $\left(\underline{apr}_A(X), \overline{apr}_A(X)\right)$ is called the rough set of X.

In order to characterise the accuracy of knowledge description using a rough set model, Pawlak proposes the approximate accuracy and the approximate roughness [14, 15].

Definition 2.2 In an information table shown in Equation (1), $A \subseteq AT$ is a subset of attributes, for each $X \subseteq OB$,

$$\alpha_A(X) = \frac{|\underline{apr}_A(X)|}{|\overline{apr}_A(X)|},$$

$$\beta_A(X) = 1 - \alpha_A(X)$$

are called the approximation accuracy and approximation roughness of X respectively.

Definition 2.3 In an information table shown in Equation (1), $A \subseteq AT$ is a subset of attributes, for each $\wp = \{P_1, P_2, \ldots, P_t\} \subseteq OB/E_A$, we call $\cup\wp = \cup_{i=1}^{t} P_i$ a definable set. We use $\mathcal{DS}_A$ to denote all the definable sets, that is, $\mathcal{DS}_A = \{\cup\wp | \wp \subseteq OB/E_A\}$. In addition, for each $Q \in \left(2^{OB} - \mathcal{DS}_A\right)$, we call Q an undefinable set, and all the undefinable sets are denoted by $\mathcal{UDS}_A$, that is, $\mathcal{UDS}_A = 2^{OB} - \mathcal{DS}_A$.

According to Definition 2.3, each set in 2^{OB} is either a definable set or an undefined set. And the definable sets and undefinable sets can be further illustrated as follows:

(1) The definable sets are closed under set complement, intersection and union [21]. And the undefinable sets are only closed under set complement.
(2) If $|OB| = n$, $|OB/E_A| = s$, it is not difficult to verify that there are 2^s definable sets and $2^n - 2^s$ undefinable sets in 2^{OB} [22].
(3) For each $X \in \mathcal{DS}_A$, the equation $\alpha_A(X) = 1$ holds and X is also called a descriptive set. For each $Y \in \mathcal{UDS}_A$, we have $0 \leq \alpha_A(Y) < 1$.

3 | THE RELATIVE AVERAGE APPROXIMATION ACCURACY

For any $X \subseteq OB$, we know that the traditional approximation accuracy reflects the ability of the definable sets related to X to approximate X. However, we note the fact that only part of the definable sets are involved in the calculation of approximation accuracy $\alpha_A(X)$. The traditional approximation accuracy does not reflect the ability of all definable sets to express knowledge. $\alpha_A(X)$ changes as X changes. Thus, it can be seen that $\alpha_A(X)$ is only a local concept. Hence, a new measure needs to be proposed to truly describe the accuracy of knowledge representation by all definable sets from the global perspective. Here, the relative average approximation accuracy and roughness are first developed as follows.

Definition 3.1 In an information table shown in Equation (1), $A \subseteq AT$ is a subset of attributes, then

$$\alpha_A\left(2^{OB}\right) = \frac{\sum_{X \in 2^{OB}} \alpha_A(X)}{|2^{OB}|},$$

$$\beta_A\left(2^{OB}\right) = 1 - \alpha_A\left(2^{OB}\right)$$

are the relative average approximation accuracy and the relative average approximation roughness respectively.

Obviously, $\alpha_A(2^{OB})$ is the average value of approximation accuracies of all sets in a universe, that is, $\alpha_A(2^{OB})$ characterises the ability of all definable sets to describe information granules in the information table. It will only change with the change of the rough set model or the information table.

Example 3.1 In an information table shown in Equation (1), where $OB = \{x_1, x_2, x_3\}$ and $OB/E_A = \{\{x_1\}, \{x_2, x_3\}\}$, $X_1 = \{x_2\}$, $X_2 = \{x_3\}$, $X_3 = \{x_1, x_2\}$ and $X_4 = \{x_1, x_3\}$ are all the undefinable sets. And $X_5 = \varnothing$, $X_6 = \{x_1\}$, $X_7 = \{x_2, x_3\}$ and $X_8 = OB$ are all the definable sets.

Based on Definition 2.2,

$$\alpha_A(X_1) = \alpha_A(X_2) = 0,\ \alpha_A(X_3) = \alpha_A(X_4) = \frac{1}{3};$$

$$\alpha_A(X_5) = \alpha_A(X_6) = \alpha_A(X_7) = \alpha_A(X_8) = 1.$$

So, by Definition 3.1,

$$\alpha_A\left(2^{OB}\right) = (\alpha_A(X_1) + \alpha_A(X_2) + \cdots + \alpha_A(X_8))/8 = \frac{7}{12},$$

$$\beta_A\left(2^{OB}\right) = 1 - \alpha_A\left(2^{OB}\right) = 1 - \frac{7}{12} = \frac{5}{12}.$$

From Definition 3.1, we can define the arithmetic mean of the approximation accuracies of all sets in the universe as the relative average approximation accuracy.

Proposition 3.1 *In an information table shown in Equation (1), $A \subseteq AT$ is a subset of attributes. Then we have the following equation*

$$\beta_A\left(2^{OB}\right) = \frac{\sum_{X \in 2^{OB}} \beta_A(X)}{|2^{OB}|}.$$

Proof. By Definition 3.1, we have

$$\begin{aligned}\beta_A\left(2^{OB}\right) &= 1 - \alpha_A\left(2^{OB}\right)\\ &= 1 - \frac{\sum_{X \in 2^{OB}} \alpha_A(X)}{|2^{OB}|}\\ &= \frac{2^{|OB|} - \sum_{X \in 2^{OB}} \alpha_A(X)}{|2^{OB}|}\\ &= \frac{\sum_{X \in 2^{OB}} (1 - \alpha_A(X))}{|2^{OB}|}\\ &= \frac{\sum_{X \in 2^{OB}} \beta_A(X)}{|2^{OB}|}.\square\end{aligned}$$

Proposition 3.1 indicates that the relative average approximation roughness in Definition 3.1 can be expressed as the arithmetic mean of the approximation roughnesses of all sets in the universe.

Since equation $\beta_A(2^{OB}) = 1 - \alpha_A(2^{OB})$ holds, so we only study the properties of the relative average approximation accuracy, and the conclusions of the relative average approximation roughness will not be mentioned here.

In Definition 3.1, we propose the relative average approximation accuracy. Next, we give a formula to calculate the relative average approximation accuracy. Through the formula and its proof process, we can have a deeper understanding of the relative average approximation accuracy.

Proposition 3.2 *In an information table shown in Equation (1), let $OB/E_A = \{P_1, P_2, \ldots, P_s\}$ $(s \geq 2)$ be the partition of OB induced by the subset of attributes $A \subseteq AT$. Then*

$$\alpha_A\left(2^{OB}\right) = \left(2^s + \sum_{i=1}^{s-1} \sum_{j=1}^{s-i} C_s^i C_{s-i}^j \left(\prod_{k=1}^{j} \left(2^{|P_{vk}|} - 2\right)\right) / |2^{OB}|\right.$$

$$\left.\frac{|P_{u1}| + |P_{u2}| + \cdots + |P_{ui}|}{(|P_{u1}| + |P_{u2}| + \cdots + |P_{ui}|) + (|P_{v1}| + |P_{v2}| + \cdots + |P_{vj}|)}\right)$$

where $\mathcal{P}_u = \{P_{u1}, P_{u2}, \ldots, P_{ui}\}, \mathcal{P}_v = \{P_{v1}, P_{v2}, \ldots, P_{vj}\}$ are any two subsets of OB/E_A and $\mathcal{P}_u \cap \mathcal{P}_v = \varnothing$.

Proof. Suppose that $\mathcal{P}_u = \{P_{u1}, P_{u2}, \ldots, P_{ui}\}, \mathcal{P}_v = \{P_{v1}, P_{v2}, \ldots, P_{vj}\}$ are two subsets of OB/E_A and $\mathcal{P}_u \cap \mathcal{P}_v = \varnothing$. Let $X \subseteq OB$ be an undefinable set, which meets the following conditions:

(1) For each $P \in \mathcal{P}_u$, the set-inclusion relation $P \subseteq X$ holds;
(2) For each $P \in \mathcal{P}_v$, one can find that $P \nsubseteq X$ and $P \cap X \neq \varnothing$.

At this time, we have

$$|\underline{apr}(X)| = |P_{u1}| + |P_{u2}| + \cdots + |P_{ui}|,$$

$$|\overline{apr}(X)| = (|P_{u1}| + |P_{u2}| + \cdots + |P_{ui}|) + \left(|P_{v1}| + |P_{v2}| + \cdots + |P_{vj}|\right).$$

Then the approximate accuracy of X is

$$\alpha_A(X) = \frac{|P_{u1}| + |P_{u2}| + \cdots + |P_{ui}|}{(|P_{u1}| + |P_{u2}| + \cdots + |P_{ui}|) + \left(|P_{v1}| + |P_{v2}| + \cdots + |P_{vj}|\right)}.$$

It is easy to see that the number of the undefinable sets meeting conditions (1) and (2) is

$$\left(2^{|P_{v1}|} - 2\right) \cdot \left(2^{|P_{v2}|} - 2\right) \cdot \cdots \cdot \left(2^{|P_{vj}|} - 2\right).$$

Meanwhile, there are also many choices of $\mathcal{P}_u$, and $\mathcal{P}_v$ satisfying the above two conditions, and the number of them is

$$\sum_{i=1}^{s-1} \sum_{j=1}^{s-i} C_s^i C_{s-i}^j.$$

So, the sum of the approximation accuracies of all undefinable sets is equal to

$$\sum_{i=1}^{s-1} \sum_{j=1}^{s-i} C_s^i C_{s-i}^j \left(\prod_{k=1}^{j} \left(2^{|P_{vk}|} - 2\right) \right)$$

$$\frac{|P_{u1}| + |P_{u2}| + \cdots + |P_{ui}|}{(|P_{u1}| + |P_{u2}| + \cdots + |P_{ui}|) + \left(|P_{v1}| + |P_{v2}| + \cdots + |P_{vj}|\right)}.$$

In addition, for each $X \in \mathcal{DS}_A$, we have that $\alpha_A(X) = 1$, then the sum of the approximation accuracies of all definable sets is

$$|\mathcal{DS}_A| = 2^s.$$

Finally, according to Definition 3.1, one can find that

$$\alpha_A\left(2^{OB}\right) = \left(2^s + \sum_{i=1}^{s-1} \sum_{j=1}^{s-i} C_s^i C_{s-i}^j \left(\prod_{k=1}^{j} \left(2^{|P_{vk}|} - 2\right) \right) / |2^{OB}| \right.$$

$$\left. \frac{|P_{u1}| + |P_{u2}| + \cdots + |P_{ui}|}{(|P_{u1}| + |P_{u2}| + \cdots + |P_{ui}|) + \left(|P_{v1}| + |P_{v2}| + \cdots + |P_{vj}|\right)} \right). \square$$

In what follows, we explore other important properties of the relative average approximation accuracy.

Proposition 3.3 *In an information table shown in Equation (1), $A \subseteq AT$ is a subset of attributes, then*

$$\frac{1}{2^{|OB|-1}} \leq \alpha_A\left(2^{OB}\right) \leq 1.$$

Proof. Let us first prove that the minimum value of α_A (2^{OB}) is $\frac{1}{2^{|OB|-1}}$.

When $OB/E_A = \{OB\}$, it is obvious that the value of α_A (2^{OB}) is minimum. Meanwhile, α_A $(\varnothing) = \alpha_A$ $(OB) = 1$ and for each $\varnothing \neq X \subset OB$, $\alpha_A(X) = 0$. Then, from the Definition 3.1, $\alpha_A\left(2^{OB}\right) = \frac{1}{2^{|OB|-1}}$.

Next, we prove that the maximum value of α_A (2^{OB}) is equal to 1.

It is not difficult to see that when OB/E_A consists of singleton subsets of OB, α_A (2^{OB}) takes the maximum. And for $\forall X \subseteq OB$, we have $\alpha_A(X) = 1$. Therefore, α_A $(2^{OB}) = 1$.□

From Proposition 3.3, the value of α_A (2^{OB}) is bound within the limits of $\frac{1}{2^{|OB|-1}}$ and 1, which represent the finest (singleton blocks) and coarsest (one block) partitions respectively.

Definition 3.2. A partition OB/E is a refinement of another partition OB/E', denoted by $OB/E \sqsubseteq OB/E'$, if every equivalence class of OB/E is contained in some equivalence class of OB/E'. If $OB/E \sqsubseteq OB/E'$ and $OB/E \neq OB/E'$, then OB/E is a proper refinement of OB/E' and written by $OB/E \sqsubset OB/E'$.

Let $OB/E' = \{\{x\} | x \in OB\}$, and $OB/E'' = \{OB\}$, for any partition OB/E, we have $OB/E' \sqsubseteq OB/E \sqsubseteq OB/E''$. Then, we have the following conclusion about the relative average approximation accuracy.

Proposition 3.4 *In an information table shown in Equation (1), A, $B \subseteq AT$ are two subsets of attributes, if $OB/E_A \sqsubseteq OB/E_B$, then*

$$\alpha_B\left(2^{OB}\right) \leq \alpha_A\left(2^{OB}\right).$$

Proof. If $OB/E_A = OB/E_B$, the equation α_B $(2^{OB}) = \alpha_A$ (2^{OB}) clearly holds. If $OB/E_A \sqsubset OB/E_B$, there exists $X \in OB/E_B$ such that

$$X = \cup_{i=1}^{v} X_i,$$

where $X_i \in OB/E_A$, $i = 1, 2, \ldots, v$. By Definition 2.2, for each X_i, $i \in \{1, 2, \ldots, v\}$, the inequality

$$\alpha_B(X_i) < \alpha_A(X_i)$$

obviously holds.

In summary, we have

$$\alpha_B\left(2^{OB}\right) \leq \alpha_A\left(2^{OB}\right). \square$$

Definition 3.3 For two partitions OB/E and OB/E', if there exists a bijection f: $OB/E \rightarrow OB/E'$ such that $\forall X \in OB/E$, $|f(X)| = |X|$, then the two partitions are size-isomorphic.

Based on Definition 3.3, an equivalence relation on all partitions of the universe is introduced and studied [23, 24]. If partitions OB/E and OB/E' are size-isomorphic, then the following result holds.

Proposition 3.5 *In an information table shown in Equation (1), $A, B \subseteq AT$ are two subsets of attributes, if OB/E_A and OB/E_B are size-isomorphic, then we have*

$$\alpha_A\left(2^{OB}\right) = \alpha_B\left(2^{OB}\right).$$

Proof. It is immediate by Definitions 3.1 and 3.3.□

Example 3.2 In an information table shown in Equation (1), $A, B \subseteq AT$ are two subsets of attributes, suppose

$$OB = \{x_1, x_2, x_3, x_4\},$$

$$OB/E_A = \{\{x_1, x_2\}, \{x_3\}, \{x_4\}\},$$

$$OB/E_B = \{\{x_1, x_3\}, \{x_2\}, \{x_4\}\}.$$

By Definition 3.3, OB/E_A is size-isomorphic to OB/E_B. Hence, according to Proposition 3.5, we have $\alpha_A\ (2^{OB}) = \alpha_B\ (2^{OB}) = 0.625$.

4 | THE ABSOLUTE AVERAGE APPROXIMATION ACCURACY

In the above section, the relative average approximation accuracy is defined based on all sets in the universe. According to the classical rough set model, any definable set is accurately described, that is, the approximation accuracy of each definable set is 1. One of the main tasks of rough set is to use definable sets to approximate the undefinable sets. Thus, it is more meaningful to discuss the accuracy of the approximate description of the undefined sets. Next, the absolute average approximate accuracy and roughness are proposed as follows.

Definition 4.1 In an information table shown in Equation (1), $A \subseteq AT$ is a subset of attributes and $|\mathcal{UDS}_A| > 0$, then

$$\alpha_A(\mathcal{UDS}_A) = \frac{\sum_{X \in \mathcal{UDS}_A} \alpha_A(X)}{|\mathcal{UDS}_A|},$$

$$\beta_A(\mathcal{UDS}_A) = 1 - \alpha_A(\mathcal{UDS}_A)$$

are the absolute average approximation accuracy and the absolute average approximation roughness respectively.

Example 4.1 (Continued from Example 3.1) From $OB = \{x_1, x_2, x_3\}$ and $OB/E_A = \{\{x_1\}, \{x_2, x_3\}\}$, it can be obtained that $\mathcal{UDS}_A = \{X_1, X_2, X_3, X_4\}$.

According to

$$\alpha_A(X_1) = \alpha_A(X_2) = 0,\ \alpha_A(X_3) = \alpha_A(X_4) = \frac{1}{3},$$

we have

$$\alpha_A(\mathcal{UDS}_A) = (\alpha_A(X_1) + \alpha_A(X_2) + \alpha_A(X_3) + \alpha_A(X_4))/4 = \frac{1}{6}.$$

$$\beta_A(\mathcal{UDS}_A) = 1 - \alpha_A(UDS_A) = 1 - \frac{1}{6} = \frac{5}{6}.$$

Similar to Proposition 3.1, there is a result about the absolute average approximate roughness as follows.

Proposition 4.1 *In an information table shown in Equation (1), $A \subseteq AT$ is a subset of attributes, then*

$$\beta_A(\mathcal{UDS}_A) = \frac{\sum_{X \in \mathcal{UDS}_A} \beta_A(X)}{|\mathcal{UDS}_A|}.$$

Proof. By Definition 4.1, we have

$$\begin{aligned}\beta_A(\mathcal{UDS}_A) &= 1 - \alpha_A(\mathcal{UDS}_A)\\ &= 1 - \frac{\sum_{X \in \mathcal{UDS}_A} \alpha_A(X)}{|\mathcal{UDS}_A|}\\ &= \frac{|\mathcal{UDS}_A| - \sum_{X \in \mathcal{UDS}_A} \alpha_A(X)}{|\mathcal{UDS}_A|}\\ &= \frac{\sum_{X \in \mathcal{UDS}_A} (1 - \alpha_A(X))}{|\mathcal{UDS}_A|}\\ &= \frac{\sum_{X \in \mathcal{UDS}_A} \beta_A(X)}{|\mathcal{UDS}_A|}.\square\end{aligned}$$

Proposition 4.1 shows that the absolute average approximation roughness in Definition 4.1 can be rearranged as the arithmetic mean of the approximation roughnesses of all undefinable sets in the universe. Next, we mainly focus on the properties of absolute average approximation accuracy.

We know that for each $X \subseteq OB$, the relation $0 \le \alpha_A(X) \le 1$ holds. At the same time, for the absolute average approximation accuracy $\alpha_A(\mathcal{UDS}_A)$, we have the same conclusion.

Proposition 4.2 *In an information table shown in Equation (1), $A \subseteq AT$ is a subset of attributes, then we have*

$$0 \le \alpha_A(\mathcal{UDS}_A) \le 1.$$

Proof. Firstly, if $|OB/E_A| = |OB|$, then $\alpha_A(\mathcal{UDS}_A) = 1$. Secondly, when $|OB/E_A| = 1$, $\alpha_A(\mathcal{UDS}_A)$ will take the minimum value 0.□

Similar to Proposition 3.4, we guess that if $OB/E_A \sqsubseteq OB/E_B$, then the inequality $\alpha_B(\mathcal{UDS}_B) \le \alpha_A(\mathcal{UDS}_A)$ should hold. In order to verify it, we first prove the following lemma.

Lemma 4.1 *In an information table shown in Equation (1), $A \subseteq AT$ is a subset of attributes. Suppose $|OB| = n \geq 4$, and $OB/E_A = \{P_1, P_2, \ldots, P_s\}$, where $s \geq 2$ and $n > s$, then we have*

$$\alpha_A\left(2^{OB}\right) > \frac{2^s + 1}{2^n}.$$

Proof. From Definition 3.1, we have

$$\begin{aligned}\alpha_A\left(2^{OB}\right) &= \frac{\sum_{X \in 2^{OB}} \alpha_A(X)}{|2^{OB}|} \\ &= \frac{\sum_{X \in \mathcal{UDS}_A} \alpha_A(X) + \sum_{X \in \mathcal{DS}_A} \alpha_A(X)}{|2^{OB}|} \\ &= \frac{\sum_{X \in \mathcal{UDS}_A} \alpha_A(X) + 2^s}{2^n}\end{aligned}$$

Next, we will prove that $\sum_{X \in \mathcal{UDS}_A} \alpha_A(X) > 1$. So we will discuss it in two cases:

Case a: Suppose that there are two sets $P = \{x_1, x_2, \ldots, x_u\}$, $P' = \{x'_1, x'_2, \ldots, x'_v\}$ in OB/E_A such that $u \geq 2$ and $v \geq 2$. Let

$$X_1 = \{x_1\} \cup P', X_2 = \{x_2\} \cup P', X_3 = \{x'_1\} \cup P,$$

then

$$\sum_{X \in \mathcal{UDS}_A} \alpha_A(X) > \sum_{i=1}^{3} \alpha_A(X_i) = \tfrac{u+2v}{u+v} > 1.$$

Case b: Suppose there is only one set in OB/E_A whose cardinality is greater than or equal to 2. Without losing generality, let

$$OB/E_A = \{P_1, P_2, \ldots, P_s\},$$

where

$$P_1 = \{x_1\}, P_2 = \{x_2\}, \ldots, P_{s-1} = \{x_{s-1}\},$$

$$P_s = \{x_s, x_{s+1}, \ldots, x_n\}, and |P_s| = n - s + 1 \geq 2.$$

Here, we prove it based on two cases:

Case b_1: If $s - 1 \geq \frac{n}{2}$, let

$$X_1 = \{x_1, x_2, \ldots, x_{s-1}, x_s\}, X_2 = \{x_1, x_2, \ldots, x_{s-1}, x_{s+1}\},$$

$$X_3 = \{x_1, x_s\}$$

then

$$\sum_{X \in \mathcal{UDS}_A} \alpha_A(X) > \sum_{i=1}^{3} \alpha_A(X_i) > \tfrac{s+1}{n} + \tfrac{s+1}{n} + \tfrac{1}{n} > 1.$$

Case b_2: If $s - 1 < \frac{n}{2}$, that is, $n - s + 1 > \frac{n}{2}$. Next, the proof will be based on two cases:

Case b_{21}: Suppose $s - 1 = 1$, then $OB/E_A = \{P_1, P_2\}$, that is,

$$P_1 = \{x_1\}, P_2 = \{x_2, x_3, \ldots, x_n\}.$$

Let

$$X_1 = \{x_1, x_2\}, X_2 = \{x_1, x_3\}, \ldots, X_{n-1} = \{x_1, x_n\},$$

$$X_n = \{x_1, x_2, x_3\}, X_{n+1} = \{x_1, x_2, x_4\},$$

we have

$$\sum_{X \in \mathcal{UDS}_A} \alpha_A(X) \geq \sum_{i=1}^{n+1} \alpha_A(X_i) = \tfrac{n+1}{n} > 1.$$

Case b_{22}: Suppose $s - 1 \geq 2$, one can find that

$$P_1 = \{x_1\}, P_2 = \{x_2\}, \ldots, P_{s-1} = \{x_{s-1}\},$$

$$P_s = \{x_s, x_{s+1}, \ldots, x_n\}.$$

Let

$$X_1 = \{x_1, x_2, x_s\}, X_2 = \{x_1, x_2, x_{s+1}\}, \ldots,$$

$$X_{n-s+1} = \{x_1, x_2, x_n\}.$$

Because for each X_i, we have

$$\alpha_A(X_i) = \frac{2}{n - s + 3} \geq \frac{2}{n}, i = 1, 2, \ldots, n - s + 1.$$

Then

$$\sum_{X \in \mathcal{UDS}_A} \alpha_A(X) > \sum_{i=1}^{n-s+1} \alpha_A(X_i) \geq \tfrac{2(n-s+1)}{n} > 1.$$

To sum up, the inequality $\sum_{X \in \mathcal{UDS}_A} \alpha_A(X) > 1$ is always true. This means $\alpha_A\left(2^{OB}\right) > \frac{2^s+1}{2^n}$ holds.□

Proposition 4.3 *In an information table shown in Equation (1), A, $B \subseteq AT$ are two subsets of attributes, if $OB/E_A \sqsubseteq OB/E_B$, and $|OB/E_A| = s < |OB| = n$ (where $n \geq 4$, $s \geq 2$), then*

$$\alpha_B(\mathcal{UDS}_B) \leq \alpha_A(\mathcal{UDS}_A).$$

Proof. When $OB/E_A = OB/E_B$, the equation $\alpha_B(\mathcal{UDS}_B) = \alpha_A(\mathcal{UDS}_A)$ clearly holds.

When $OB/E_A \sqsubset OB/E_B$, according to Proposition 5.1, let

$$f(s) = \frac{2^n}{2^n - 2^s}\left[K - \frac{2^s}{2^n}\right] = \frac{K \cdot 2^n - 2^s}{2^n - 2^s} \quad (\ K \text{ is a constant }).$$

If function $f(s)$ is monotonically increasing, then the relation $\alpha_B(\mathcal{UDS}_B) < \alpha_A(\mathcal{UDS}_A)$ must hold.

Based on Lemma 4.1, we have $K > \frac{2^s+1}{2^n}$. Next, let us calculate the derivative of $f(s)$.

$$\begin{aligned}f'(s) &= \left(\frac{K \cdot 2^n - 2^s}{2^n - 2^s}\right)' \\ &= \frac{ln2 \cdot 2^s \cdot (K \cdot 2^n - 2^s - 1)}{(2^n - 2^s)^2} \\ &> \frac{ln2 \cdot 2^s \cdot \left(\frac{2^s+1}{2^n} \cdot 2^n - 2^s - 1\right)}{(2^n - 2^s)^2} = 0.\end{aligned}$$

From the knowledge of calculus, because the derivative of $f(s)$ is greater than zero, then $f(s)$ is a monotonically increasing function. Hence, if $OB/E_A \sqsubset OB/E_B$, we have $\alpha_B(\mathcal{UDS}_B) < \alpha_A(\mathcal{UDS}_A)$. □

Example 4.2 In an information table shown in Equation (1), where $OB = \{x_1, x_2, x_3, x_4, x_5\}$, $A, B \subseteq AT$ are two subsets of attributes, and $OB/E_A \sqsubset OB/E_B$. Let

$$OB/E_A = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5\}\},$$

$$OB/E_B = \{\{x_1, x_2, x_3, x_4\}, \{x_5\}\}.$$

Thus, by Definition 4.1, we have $\alpha_A(\mathcal{UDS}_A) = 0.27, \alpha_B(\mathcal{UDS}_B) = 0.10$, that is, $\alpha_B(\mathcal{UDS}_B) < \alpha_A(\mathcal{UDS}_A)$.

Proposition 4.4 *In an information table shown in Equation (1), $A, B \subseteq AT$ are two subsets of attributes, if OB/E_A and OB/E_B are size-isomorphic, then we have*

$$\alpha_A(\mathcal{UDS}_A) = \alpha_B(\mathcal{UDS}_B).$$

Proof. It is immediate by Propositions 3.3 and 4.1.□

To facilitate understanding, comparative analysis of traditional approximation accuracy and average approximation accuracy is listed as follows.

(1) The traditional approximation accuracy is a local concept. Obviously, when calculating the approximate precision of a target concept, only the equivalent classes associated with X are involved. However, the average approximation accuracy is a global concept. For any target concept, all equivalent classes participate in the calculation of its average approximation accuracy.
(2) For a rough set model induced from the information table, the traditional approximation accuracy will change with the change of target concept. While, the average approximation accuracy is a constant.
(3) The traditional approximation accuracy cannot measure the ability of a rough set model to describe all the knowledge in the information table. However, the average approximation accuracy can well reflect the level of a rough set model to approximate all target concepts.

5 | THE RELATIONSHIP BETWEEN TWO KINDS OF AVERAGE APPROXIMATION ACCURACY

In order to understand these two average approximation accuracies more comprehensively and apply them more conveniently, here, we explore the relationship between relative and absolute average approximation accuracies. According to Definitions 3.1 and 4.1, the relative average approximation accuracy is the average value of the approximation accuracies of all sets in the universe, while the absolute average approximate accuracy is the arithmetic mean of the approximation accuracies of all undefinable sets in an information table. Although the relative and absolute average approximate accuracies are different, there is still a close relationship between them.

Proposition 5.1 *In an information table shown in Equation (1), $A \subseteq AT$ is a subset of attributes. Suppose $|OB| = n > |OB/E_A| = s$, then we have*

$$\alpha_A(\mathcal{UDS}_A) = \frac{2^n}{2^n - 2^s}\left[\alpha_A\left(2^{OB}\right) - \frac{2^s}{2^n}\right].$$

Proof.

$$\begin{aligned}\alpha_A\left(2^{OB}\right) &= \frac{\sum_{X\in 2^{OB}}\alpha_A(X)}{|2^{OB}|}\\ &= \frac{\sum_{X\in\mathcal{UDS}_A}\alpha_A(X) + \sum_{X\in\mathcal{DS}_A}\alpha_A(X)}{2^n}\\ &= \frac{\sum_{X\in\mathcal{UDS}_A}\alpha_A(X) + 2^s}{2^n}\\ &= \frac{\sum_{X\in\mathcal{UDS}_A}\alpha_A(X)}{2^n - 2^s}\cdot\frac{2^n - 2^s}{2^n} + \frac{2^s}{2^n}\\ &= \alpha_A(\mathcal{UDS}_A)\cdot\frac{2^n - 2^s}{2^n} + \frac{2^s}{2^n}.\end{aligned}$$

Hence, it can be obtained that

$$\alpha_A(\mathcal{UDS}_A) = \frac{2^n}{2^n - 2^s}\left[\alpha_A\left(2^{OB}\right) - \frac{2^s}{2^n}\right].□$$

For the relative and absolute average approximation accuracies, if one is known, then the other can be obtained according to Proposition 5.1.

Example 5.1 From Example 3.1, we have $n = 3$, $s = 2$, and $\alpha_A\left(2^{OB}\right) = \frac{5}{8}$. According to Proposition 5.1, it can be seen that

$$\alpha_A(\mathcal{UDS}_A) = \frac{2^3}{2^3 - 2^2}\left(\frac{7}{12} - \frac{2^2}{2^3}\right) = \frac{1}{6}.$$

Proposition 5.2 *In an information table shown in Equation (1), $A \subseteq AT$ is a subset of attributes, then we have*

$$\alpha_A(\mathcal{UDS}_A) \le \alpha_A\left(2^{OB}\right).$$

Proof. Suppose $|OB| = n$, $|OB/E_A| = s$. If $n > s$, from Proposition 5.1, we have

$$\begin{aligned}\alpha_A\left(2^{OB}\right) - \alpha_A(\mathcal{UDS}_A) &= \alpha_A\left(2^{OB}\right)\\ &\quad - \frac{2^n}{2^n - 2^s}\left[\alpha_A\left(2^{OB}\right) - \frac{2^s}{2^n}\right]\\ &= \frac{2^s\left(1 - \alpha_A\left(2^{OB}\right)\right)}{2^n - 2^s} > 0.\end{aligned}$$

If $n = s$, we know that $\alpha_A\left(2^{OB}\right) = 1$ and $\alpha_A(\mathcal{UDS}_A) = 1$. To sum up, the inequality $\alpha_A(\mathcal{UDS}_A) \le \alpha_A\left(2^{OB}\right)$ holds.□

To facilitate understanding, comparative analysis of relative average approximation accuracy and absolute average approximation accuracy is shown as follows.

(1) The relative average approximation accuracy is the average of the approximation accuracies of all subsets of the universe in the information table. The absolute average approximate accuracy is the average of the approximation accuracies of all undefined sets in the information table.
(2) These two approximation accuracies are closely linked by the formula shown in Proposition 5.1.
(3) The absolute average approximation accuracy is less than or equal to the relative average approximation accuracy.

6 | THE APPLICATION OF THE AVERAGE APPROXIMATION ACCURACY IN INCOMPLETE INFORMATION TABLES

In many cases, due to the constraints of technology, cognition, cost and other factors, some attribute values are missing or unknown, so we only get an incomplete information table. How to properly deal with these missing attribute values for data mining is a very important research topic.

At present, there are three main methods to deal with these unknown attribute values. The first one is to delete the missing data directly to get a complete data set. However, this method is not applicable when there are many missing data. In the second way, authors usually induce the binary relations, and study the incomplete information table by means of rough set theory [25–31]. Sometimes, in order to avoid the waste of human and material resources, it is necessary to take measures to reasonably estimate the missing data. Hence, the third method is to estimate the missing data according to specific principles, and replace the missing data with the estimated values, so as to get a complete information table [32–35].

Suppose the information table shown in Equation (1) is incomplete. For any $x \in OB$ and any $a \in AT$, $f_a(x) = *$ means that the attribute value of object x under attribute a is unknown. For any $a \in AT$, let $V_a = \{f_a(x) | f_a(x) \neq *, x \in OB\}$. Here, V_a is allowed to contain the same elements, $|V_a|$ is the number of elements in V_a, and the number of the same elements needs to be calculated repeatedly. The probability $p_a(x, y)$ of objects x and y taking the same attribute value under attribute a is usually defined as follows [36]:

$$p_a(x,y) = \begin{cases} |f_a(y)|/|V_a| & (f_a(x) = *) \wedge (f_a(y) \neq *), \\ |f_a(x)|/|V_a| & (f_a(y) = *) \wedge (f_a(x) \neq *), \\ 1 & (f_a(x) = *) \wedge (f_a(y) = *), \\ 1 & (f_a(x) \neq *) \wedge (f_a(y) \neq *) \\ & \wedge (f_a(x) = f_a(y)), \\ 0 & (f_a(x) \neq *) \wedge (f_a(y) \neq *) \\ & \wedge (f_a(x) \neq f_a(y)). \end{cases}$$

Based on $p_a(x, y)$, two rules to fill the missing attribute values are introduced as follows [36]:

Lemma 6.1 *Suppose the information table shown in Equation (1) is incomplete. For any $a \in A \subseteq AT$ and any $x \in OB$, if $f_a(x) = *$, let $Y_0 = \{y_0 \in OB | p_a(x, y_0) = max_{y \in OB} p_a(x, y)\}$, then $*$ can take any value in the set $\{f_a(y_0) | y_0 \in Y_0\}$.*

Lemma 6.2 *Suppose the information table shown in Equation (1) is incomplete. For any $a \in A \subseteq AT$ and any $x \in OB$, if $f_a(x) = *$, let $Y_0 = \{y_0 \in OB | p_a(x, y_0) = max_{y \in OB} p_a(x, y)\}$, $|Y_0| = r (r \geq 2)$, and $y_i \in Y_0$, $i = 1, 2, \ldots, r$. At the same time, let $Y^* = \{y^* \in OB | \prod_{a \in A} p_a(x, y^*) = max_{1 \leq i \leq r} \{\prod_{a \in A} p_a(x, y_i)\}\}$, then $*$ can take any value in the set $\{f_a(y^*) | y^* \in Y^*\}$.*

Next, a simplified incomplete information table is employed. The application of average approximation accuracy in filling missing attribute values is preliminarily explored.

Example 6.1 Here is a toy incomplete information table, as shown in Table 1. According to Lemmas 6.1 and 6.2, we have $*_1 = 0$ or 1, $*_2 = 0$ or 1, and $*_3 = 0$. Although both $*_1$ and $*_2$ are given different values, these values are all reasonable.

At the same time, we naturally hope that the data information contained in an information table should be as detailed as possible, that is, the definable sets in the information table can describe the undefined sets as accurately as possible. Therefore, in the sense of absolute average approximate accuracy, we try to further investigate the values of $*_1$ and $*_2$ so that the information table has stronger ability to express knowledge.

Let attribute set $A = \{a_1, a_2, a_3\}$, we discuss this problem in four cases.

Case 1: If $*_1 = 0$, $*_2 = 0$, $*_3 = 0$, we have

$$OB/E_A = \{\{x_1\}, \{x_2, x_3\}, \{x_4, x_5\}, \{x_6\}\}.$$

Then, $\alpha_A(\mathcal{UDS}_A) = 0.3625$.

Case 2: If $*_1 = 0$, $*_2 = 1$, $*_3 = 0$, we have

$$OB/E_A = \{\{x_1, x_4\}, \{x_2, x_3\}, \{x_5\}, \{x_6\}\}.$$

Then, $\alpha_A(\mathcal{UDS}_A) = 0.3625$.

Case 3: If $*_1 = 1$, $*_2 = 0$, $*_3 = 0$, we have

$$OB/E_A = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4, x_5\}, \{x_6\}\}.$$

TABLE 1 An incomplete information table.

OB	a_1	a_2	a_3
x_1	1	0	0
x_2	0	1	1
x_3	$*_1$	1	1
x_4	$*_2$	0	0
x_5	0	0	0
x_6	1	$*_3$	1

Then, $\alpha_A(\mathcal{UDS}_A) = 0.4625$.

Case 4: If $*_1 = 1$, $*_2 = 1$, $*_3 = 0$, we have

$$OB/E_A = \{\{x_1, x_4\}, \{x_2\}, \{x_3\}, \{x_5\}, \{x_6\}\}.$$

Then, $\alpha_A(\mathcal{UDS}_A) = 0.4625$.

Based on Lemmas 6.1, 6.2 and the above analysis, we know that $*_1 = 1$, $*_2 = 0$,$*_3 = 0$ and $*_1 = 1$, $*_2 = 1$,$*_3 = 0$ are not only reasonable, but the corresponding complete information Tables 2 and 3 also have stronger knowledge representation ability. Therefore, $*_1 = 1$, $*_2 = 0$,$*_3 = 0$ or $*_1 = 1$, $*_2 = 1$,$*_3 = 0$ is the better choice.

7 | CONCLUSION

In order to measure the ability of a rough set model to describe knowledge more objectively and comprehensively, the relative and the absolute average approximation accuracies are introduced. We study many important mathematical properties of these two average approximation accuracies. In addition, we establish the relationship between the two kinds of approximation accuracies. Finally, with the help of the idea of average approximation accuracy, we can better fill the missing data in the incomplete information table.

It should be admitted that we only propose and study the average approximation accuracy from the perspective of theoretical expansion in this paper. However, it is necessary to further explore the average approximation accuracy from the aspects of calculation and application. Therefore, there is still much work to be done on the average approximation accuracy in the future. For example, more effective algorithms for calculating the average approximation accuracy need to be studied and designed. According to the average approximation accuracy, the faster ways to fill in the missing attribute values in the complex incomplete information tables need to be explored. Attribute reduction based on average approximation accuracy should also be studied. Finally, it is also necessary to carry out simulation experiments related to the average approximation accuracy.

TABLE 2 A complete information table based on $*_1 = 1$, $*_2 = 0$, and $*_3 = 0$.

OB	a_1	a_2	a_3
x_1	1	0	0
x_2	0	1	1
x_3	1	1	1
x_4	0	0	0
x_5	0	0	0
x_6	1	0	1

TABLE 3 A complete information table based on $*_1 = 1$, $*_2 = 1$, and $*_3 = 0$.

OB	a_1	a_2	a_3
x_1	1	0	0
x_2	0	1	1
x_3	1	1	1
x_4	1	0	0
x_5	0	0	0
x_6	1	0	1

ACKNOWLEDGEMENTS

This work is partially supported by the National Natural Science Foundation of China (No. 61976254) and the Natural Science Foundation of Fujian Province (Nos. 2020J01707, 2020J01710).

CONFLICT OF INTEREST STATEMENT

The authors declared that they have no conflicts of interest to this work.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ORCID

Qingzhao Kong https://orcid.org/0000-0002-9823-3827

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How to cite this article: Kong, Q., et al.: Two kinds of average approximation accuracy. CAAI Trans. Intell. Technol. 1–10 (2023). https://doi.org/10.1049/cit2.12222