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Sobolev gradient type iterative solution methods for a nonlinear 4th order elastic plate equation

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ABSTRACT

This paper considers the numerical solution of an elastic bending model of a thin plate, based on material nonlinearity, which leads to a nonlinear elliptic 4th order boundary value problem. Our goal is to summarize possible efficient iterative solvers, adapted to this problem, based on a Sobolev space background. It is proved that the proposed methods exhibit robust behaviour, that is, convergence rates are bounded independently of the considered Galerkin discretization subspace.

1. Introduction

In this paper we consider the numerical solution of an elastic bending model of a thin plate. Elasticity theory plays a fundamental role in mechanics, and it has a vast literature, see, e.g., the monographs [2,9,23] and the references therein. In particular, there exist several models of nonlinear elastic plates under different conditions. In this paper we consider a model described in [21, Chap. 9], based on material nonlinearity, which leads to a nonlinear elliptic 4th order boundary value problem (BVP).

The present paper summarizes various types of iterative solvers for Galerkin discretizations of the problem. The presentation is based on a Hilbert space framework. The iterative methods include a Sobolev gradient iteration, Newton's method adapted to the given problem, further, a quasi-Newton method. Each method is based on the weak form of the nonlinear problem in Sobolev space. The Sobolev gradient iteration provides a minimizing sequence for the underlying potential w.r.t. the Sobolev inner product. This idea has been introduced by [22], and has provided a large number of useful applications for various physical and related problems, see, e.g., [24] and the subsequent series of papers [25–27], further, [28,29] and recently [4,16]. A summary of Sobolev gradients and preconditioning is given in the author's book [11, Sec. 7.3]. As described in [22], Sobolev gradients often provide a dramatic improvement of convergence compared to that achieved with Euclidean inner products. The second considered method is the Newton iteration. Finally, the quasi-Newton/variable preconditioning method, based on the author's results [6,11,18] forms an intermediate version between gradient and Newton's method. It offers a trade-off between speed and cost, based on constructing approximate Jacobians via spectral equivalence. Sobolev gradient preconditioning can be used in inner iterations for the Newton and quasi-Newton methods.

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Our paper gives a summary on these iterative solvers with a solid mathematical description for our BVP. We prove that the BVP satisfies the conditions for the convergence of these iterations. Our main goal is to show robust behaviour, that is, convergence rates bounded independently of the considered discretization subspace. This robustness is verified for each of the three considered approaches. A simple numerical experiment is given where Sobolev gradients are coupled with a Fourier method, and illustrates the robustness.

2. Preliminaries

2.1. The nonlinear elastic plate model

The following presentation of the problem is based on [21]. The elastic bending of a thin plane plate $\Omega \subset \mathbf{R}^2$ is described by a fourth order nonlinear Dirichlet boundary value problem, derived from the 3D elasticity system after neglecting the direction orthogonal to the plate. We consider a freely supported plate.

Using the notation $(x, y) \in \mathbf{R}^2$ for the plane variable, the deflection u of the plate $\Omega \subset \mathbf{R}^2$ must satisfy

$$\begin{cases} \frac{\partial^2}{\partial x^2}\left(g(E(D^2u))\left(\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\frac{\partial^2 u}{\partial y^2}\right)\right)+\frac{\partial^2}{\partial x\partial y}\left(g(E(D^2u))\left(\frac{\partial^2 u}{\partial x\partial y}\right)\right) \\ \quad +\frac{\partial^2}{\partial y^2}\left(g(E(D^2u))\left(\frac{\partial^2 u}{\partial y^2}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\right)\right)=\alpha \quad \text{in } \Omega, \\ u_{|\partial\Omega}=\frac{\partial^2 u}{\partial \nu^2}\Big|_{\partial\Omega}=0 \end{cases} \tag{2.1}$$

where $\alpha = \alpha(x, y)$ is proportional to the external normal load per unit area, the scalar nonlinearity g depends on the given material and

$$E(D^2u) = \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x\partial y}\right)^2, \tag{2.2}$$

where

$$D^2u = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x\partial y} \\ \frac{\partial^2 u}{\partial x\partial y} & \frac{\partial^2 u}{\partial y^2} \end{pmatrix} \tag{2.3}$$

is the Hessian of a function $u \in C^2(\Omega)$ or, more generally, $u \in H^2(\Omega)$. The boundary condition expresses that the plate is freely supported at its boundary. The material function $g \in C^1(\mathbf{R}^+)$ satisfies the inequalities

$$0 < \lambda \le g(r) \le \Lambda\,, \tag{2.4}$$

$$0 < \lambda \le (g(r^2)r)' \le \Lambda \tag{2.5}$$

with suitable constants λ, Λ independent of the variable $r > 0$.

Throughout the paper we assume that the boundary $\partial\Omega$ is Lipschitz continuous and piecewise smooth. We note that if the plate is rigidly clamped at the boundary, then the second order boundary condition is replaced by $\frac{\partial u}{\partial \nu}\big|_{\partial\Omega} = 0$, and if free support and clamping are combined, then we have $\frac{\partial u}{\partial \nu}\big|_{\Gamma_1} = 0$, $\frac{\partial^2 u}{\partial \nu^2}\big|_{\Gamma_2} = 0$ for the appropriate portions Γ_1, Γ_2 of the boundary. Most results of this paper can be naturally adapted to these situations as well, see Remarks 3.2 and 4.4 later.

The solution of problem (2.1) is the minimizer of an energy functional in a proper Sobolev space. This property can be exploited in the numerical solution processes, as discussed below in subsection 3.3.

2.2. Some useful iterative methods

This subsection summarizes some iterative methods in an abstract framework in Hilbert spaces. We formulate them under a common set of conditions, which covers our plate equation. Further details on these methods can be found in our monograph [11] and papers [6,18].

Let H be a real Hilbert space and let $F : H \to H$ be a nonlinear operator having a bihemicontinuous Gâteaux derivative F'. The theorems will be based on the following properties:

Assumptions 2.2

(i) (Symmetry.) For any $u \in H$ the operator $F'(u)$ is self-adjoint.

(ii) (Ellipticity.) There exist constants $\Lambda \ge \lambda > 0$ such that

$$\lambda\|h\|^2 \le \langle F'(u)h, h\rangle \le \Lambda\|h\|^2 \qquad (\forall u, h \in H). \tag{2.6}$$

(iii) (Lipschitz continuity.) There exists a constant $L > 0$ such that

$$\|F'(u) - F'(v)\| \le L\|u - v\| \qquad (\forall u,\, h \in H). \tag{2.7}$$

We wish to solve the operator equation

$$F(u) = 0. \tag{2.8}$$

Conditions (i)–(ii) imply that there exists a unique solution $u^* \in H$, see, e.g., [11, Theorem 5.1]. We study three types of iterative methods.

Theorem 2.1 *(Gradient method). Let Assumptions 2.2 (i)–(ii) hold. Let $u_0 \in H$ be arbitrary. Then the sequence, defined by*

$$u_{n+1} := u_n - \frac{2}{\Lambda + \lambda} F(u_n) \qquad (\forall n \in \mathbb{N}) \tag{2.9}$$

converges to u^*, namely,

$$\|u_n - u^*\| \le \frac{1}{\lambda} \|F(u_0)\| \left(\frac{\Lambda - \lambda}{\Lambda + \lambda} \right)^n \qquad (\forall n \in \mathbb{N}) . \tag{2.10}$$

Moreover, in fact $\|u_n - u^*\| \le \frac{1}{\lambda} \|F(u_n)\|$ and

$$\frac{\|F(u_{n+1})\|}{\|F(u_n)\|} \le \frac{\Lambda - \lambda}{\Lambda + \lambda} . \tag{2.11}$$

Proof. See [11, Theorem 5.4]. ■

Theorem 2.2 *(Newton's method). Let Assumptions 2.2 hold. Let u_0 be in a properly small neighbourhood of u^*. Then the sequence, defined by*

$$u_{n+1} := u_n - F'(u_n)^{-1} F(u_n) \qquad (\forall n \in \mathbb{N}), \tag{2.12}$$

converges to u^*, that is, $\|u_n - u^*\| \le \frac{1}{\lambda} \|F(u_n)\| \to 0$ and

$$\|F(u_{n+1})\| \le \frac{L}{2\lambda^2} \|F(u_n)\|^2 \qquad (n \in \mathbb{N}).$$

Proof. It follows from [11, Theorem 5.9]. ■

The formulation of the third theorem uses the energy $*$-norm $\|v\|_* := \langle F'(u^*)^{-1} v, v \rangle^{1/2}$, which is equivalent to the original norm, due to assumption (ii).

Theorem 2.3 *(Quasi-Newton/variable preconditioning method). Let Assumptions 2.2 hold. Let u_0 be in a properly small neighbourhood of u^*, and let the sequence (u_n) be defined by*

$$u_{n+1} := u_n - \frac{2}{M_n + m_n} B_n^{-1} F(u_n) \qquad (n \in \mathbb{N}), \tag{2.13}$$

where $0 < m_n \le M_n$ and the self-adjoint linear operators $B_n : H \to H$ satisfy

$$m_n \langle B_n h, h \rangle \le \langle F'(u_n) h, h \rangle \le M_n \langle B_n h, h \rangle \qquad (n \in \mathbb{N}, \ h \in H). \tag{2.14}$$

We require (m_n) to be positively bounded from below and (M_n) bounded from above. Then (u_n) converges to u^*, that is, $\|u_n - u^*\| \le \frac{1}{\lambda} \|F(u_n)\| \to 0$ and

$$\limsup \frac{\|F(u_{n+1})\|_*}{\|F(u_n)\|_*} \le \limsup \frac{M_n - m_n}{M_n + m_n} < 1. \tag{2.15}$$

Proof. It follows from [11, Theorems 5.15 and 5.16]. ■

We remark that the preconditioner B_n should be chosen such that m_n and M_n are closer than the original spectral bounds λ and Λ, since it is a natural expectation to overperform the simple iteration.

Remark 2.1.

(a) Conditions (i)–(ii) imply that F has a uniformly convex potential and the solution of (2.8) is its minimizer. The sequence (2.9) defines a gradient method, which is the Sobolev gradient iteration in the case of weak forms of elliptic PDEs. The other theorems also involve descent methods, that is, the potential has decreasing values along these sequences.

(b) The quasi-Newton method is an intermediate version between the simple but slow gradient method and the fast but more costly Newton's method. The variable preconditioners B_n can be chosen cheaper than the full derivative operators while still allowing fast convergence, up to superlinear (when the lim sup value in (2.15) is zero).
(c) The above versions of Theorems 2.2–2.3 formulate local convergence, which can be extended to global by proper damping, but this is not formulated here for simplicity, see [11, Chapter 5] for details. The asymptotic convergence results of the damped versions are the same as above.

3. Weak formulation

3.1. A concise formulation

Let us introduce the following notations: for $u \in H^2(\Omega)$ we set the modified Hessian

$$\tilde{D}^2 u = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\partial^2 u}{\partial y^2} & \frac{1}{2}\frac{\partial^2 u}{\partial x \partial y} \\ \frac{1}{2}\frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \end{pmatrix},$$

and for any symmetric matrix-valued function $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in C^2(\Omega, \mathbf{R}^{2\times 2})$ let

$$\operatorname{div}^2 \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \frac{\partial^2 a}{\partial x^2} + 2\frac{\partial^2 b}{\partial x \partial y} + \frac{\partial^2 d}{\partial y^2}. \tag{3.1}$$

Then problem (2.1) is written briefly as

$$\begin{cases} \operatorname{div}^2 \left(g(E(D^2 u))\, \tilde{D}^2 u \right) = \alpha \\ u_{|\partial\Omega} = \frac{\partial^2 u}{\partial \nu^2}\Big|_{\partial\Omega} = 0. \end{cases} \tag{3.2}$$

3.2. Weak solution

The weak formulation of problem (3.2) reads as follows. We work in the Hilbert space

$$V := H^2(\Omega) \cap H_0^1(\Omega) = \{u \in H^2(\Omega) : \, u_{|\partial\Omega} = 0\} \tag{3.3}$$

endowed with the inner product and induced norm

$$\langle u, v \rangle_V := \int_\Omega D^2 u : D^2 v, \qquad \|u\|_V^2 = \int_\Omega |D^2 u|^2, \tag{3.4}$$

where the elementwise matrix product : and induced norm are defined by

$$P : Q = \sum_{i,k=1}^{2} P_{ik} Q_{ik}, \qquad |P|^2 = \sum_{i,k=1}^{2} P_{ik}^2 \qquad (P, Q \in \mathbf{R}^{2\times 2}), \tag{3.5}$$

respectively. (The latter is the Frobenius matrix norm.) For regular functions, the weak formulation of (3.2) is obtained via multiplying problem (3.2) by $v \in V$, integration and the divergence theorem. In this way we obtain the following problem: find $u \in V$ such that

$$\int_\Omega g(E(D^2 u))\, \tilde{D}^2 u : D^2 v = \int_\Omega \alpha v \qquad (\forall v \in V). \tag{3.6}$$

Further, observe that

$$\tilde{D}^2 u = \frac{1}{2}\left(D^2 u + \Delta u \, I \right)$$

(where $I \in \mathbf{R}^{2\times 2}$ is the identity matrix) and $I : D^2 v = \Delta v$, which yield that

$$\tilde{D}^2 u : D^2 v = \frac{1}{2}(D^2 u : D^2 v + \Delta u \, \Delta v).$$

Hence the weak formulation (3.6) can be also presented as

$$\frac{1}{2}\int_\Omega g(E(D^2 u))\, (D^2 u : D^2 v + \Delta u \, \Delta v) = \int_\Omega \alpha v \qquad (\forall v \in V). \tag{3.7}$$

Remark 3.1. It is easy to check that

$$\operatorname{div}^2 \tilde{D}^2 u = \operatorname{div}^2 D^2 u = \Delta^2 u,$$

which is the biharmonic operator. Hence, for the constant coefficient $g(r) \equiv 1$ we obtain that equation (3.2) becomes the biharmonic equation $\Delta^2 u = \alpha$. Analogously, in weak form,

$$\int_\Omega \tilde{D}^2 u : D^2 v = \int_\Omega D^2 u : D^2 v \qquad (\forall v \in V), \tag{3.8}$$

and both sides above equal the integral of $\Delta^2 u\, v$ for regular functions.

In general, problem (3.2) is a nonlinear counterpart of the biharmonic problem. Moreover, typically the function g is (properly approximated as) constant for small values of r, which corresponds to the linear elasticity model (with validity for small second derivatives). Then there is a nonlinear behaviour according to (2.4)–(2.5) on a larger range of r.

The theoretical background for (3.6) uses nonlinear operators in Hilbert space, following the concise treatment in [11]. Let us define the operator $A : V \to V$ via the formula

$$\langle A(u), v\rangle_V = \int_\Omega \Big(g(E(D^2 u))\, \tilde{D}^2 u : D^2 v - \alpha v \Big) \qquad (u, v \in V). \tag{3.9}$$

Then problem (3.6) coincides with the equation $A(u) = 0$.

Proposition 3.1. *If conditions (2.4)–(2.5) hold, then operator A satisfies items (i)–(ii) of Assumptions 2.2 in V, namely, for any $u \in V$ the operator $A'(u)$ is self-adjoint, further,*

$$\lambda \|h\|_V^2 \le \langle A'(u)h, h\rangle_V \le \Lambda \|h\|_V^2 \qquad (\forall u, h \in V). \tag{3.10}$$

Proof. We follow [11, Remark 6.1]. Let us define the bilinear mapping $[\,.\,,\,.\,] : V \times V \to L^1(\Omega)$ as

$$[u, v] := \tilde{D}^2 u : D^2 v = D^2 u : \tilde{D}^2 v = [v, u]. \tag{3.11}$$

It is easy to see that

$$[u, u] = E(D^2 u) \tag{3.12}$$

defined in (2.2), hence operator A has the structure

$$\langle A(u), v\rangle_V = \int_\Omega \Big(g([u,u])\, [u, v] - \alpha v \Big) \qquad (u, v \in V).$$

Hence, as seen in [11, Remark 6.1], the derivatives of such operators have the structure

$$\langle A'(u)h, v\rangle_V = \int_\Omega \Big(g([u,u])\, [h, v] + 2g'([u,u])\, [u, h][u, v] \Big) \qquad (u, v \in V), \tag{3.13}$$

which shows that $A'(u)$ is self-adjoint, in particular,

$$\langle A'(u)h, h\rangle_V = \int_\Omega \Big(g([u,u])\, [h, h] + 2g'([u,u])\, [u, h]^2 \Big) \qquad (u, v \in V), \tag{3.14}$$

further, using the notations

$$p(r^2) = \min\{g(r^2),\ (g(r^2)r)'\}, \qquad q(r^2) = \max\{g(r^2),\ (g(r^2)r)'\} \qquad (r \ge 0) \tag{3.15}$$

and conditions (2.4)–(2.5), we obtain

$$\lambda \int_\Omega [h, h] \le \int_\Omega p([u,u])\, [h, h] \le \langle A'(u)h, h\rangle_V \le \int_\Omega q([u,u])\, [h, h] \le \Lambda \int_\Omega [h, h]. \tag{3.16}$$

Using (3.8) and (3.4), we have

$$\int_\Omega [h, h] = \|h\|_V^2, \tag{3.17}$$

hence (3.16) becomes (3.10). ■

Corollary 3.1. *Problem (3.2) has a unique weak solution $u^* \in V$.*

Proof. Proposition 3.1 implies that equation $A(u) = 0$ has a unique solution, see, e.g., [11, Theorem 5.1]. As seen above in (3.9), $A(u) = 0$ is equivalent to (3.6). ■

Remark 3.2. The problem can be studied similarly for other boundary conditions in (3.2), for instance, if the plate is clamped on the boundary, or if it is clamped on a portion $\Gamma_1 \subset \partial\Omega$ and freely supported on the remaining portion $\Gamma_2 := \partial\Omega \setminus \Gamma_1$:

$$u_{|\partial\Omega} = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0; \qquad \text{or} \qquad u_{|\partial\Omega} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\Gamma_1} = 0, \quad \frac{\partial^2 u}{\partial \nu^2}\Big|_{\Gamma_2} = 0,$$

respectively. Then Proposition 3.1 and Corollary 3.1 can be repeated in the corresponding Sobolev space, which takes into account the essential boundary conditions; for the above examples, it is $H_0^2(\Omega)$ in the first case and $H^2_{\Gamma_1}(\Omega) := \{u \in H^2(\Omega) \cap H_0^1(\Omega) : \frac{\partial u}{\partial \nu}\big|_{\Gamma_1} = 0\}$ in the second case.

Finally, note that since $\Omega \subset \mathbf{R}^2$ and $\partial\Omega$ is Lipschitz continuous, one has the embedding and corresponding estimate

$$V \subset H^2(\Omega) \subset C(\overline{\Omega}), \qquad \|u\|_{max} \le C_\Omega \|u\|_V \quad (\forall u \in V) \tag{3.18}$$

for some constant $C_\Omega > 0$ independent of u, see [1]. Hence the weak solution is continuous: $u^* \in C(\overline{\Omega})$.

3.3. Sobolev gradient formulation

The unique weak solution, that is, the solution of (3.6) is the minimizer of the functional $\phi : V \to \mathbf{R}$,

$$\phi(u) = \frac{1}{2} \int_\Omega G(E(D^2 u)) - \int_\Omega \alpha u\,,$$

where $G'(r) = g(r)$ ($\forall r \ge 0$). The Sobolev gradient of this functional represents the gradient w.r.t. the H^2-inner product (3.4), which is the operator A:

$$\phi'(u)v = \langle A(u), v\rangle_V \qquad (\forall u, v \in V). \tag{3.19}$$

As seen above, A has a Gateaux derivative which is self-adjoint and uniformly elliptic.

On the other hand, representing the gradient w.r.t. the L^2-inner product would yield

$$\phi'(u)v = \langle T(u), v\rangle_{L^2} \qquad (\forall u, v \in D(T)), \tag{3.20}$$

where

$$T(u) := \operatorname{div}^2 \Big(g(E(D^2 u))\, \tilde{D}^2 u \Big) - \alpha, \tag{3.21}$$

which is a non-continuous differential operator in strong form. As discussed in [11, Section 5.3.1], in such situations the condition number of A is Λ/λ, whereas the condition number of T is $+\infty$. This has a basic influence on convergence properties. As described in [22], Sobolev gradients provide a dramatic improvement of convergence of descent methods compared to Euclidean inner products.

A concise reason why the Sobolev gradient $A(u)$ performs substantially better than the L^2-gradient $T(u)$ is the following product representation: if $u \in H^4(\Omega)$ and satisfies the boundary condition of (3.2), then

$$A(u) = \left(\Delta^2\right)^{-1} T(u),$$

which follows from (3.19)–(3.20) and the fact that the inner product (3.4) is generated by the operator Δ^2. That is, the Sobolev gradient is obtained from the L^2-gradient by regularizing with the inverse of the biharmonic operator.

3.4. Galerkin discretization

In general, a Galerkin discretization is defined using a finite dimensional subspace $V_N \subset V$ and projecting the weak form of equation (3.6) into V_N. That is, we look for $u^N \in V_N$ such that

$$\int_\Omega g(E(D^2 u^N))\, \tilde{D}^2 u^N : D^2 v^N = \int_\Omega \alpha v^N \qquad (\forall v^N \in V_N). \tag{3.22}$$

A widespread way of discretization is the finite element method, where V_N is a finite element subspace of V. Since this implies $V_N \subset H^2(\Omega)$, one has to use special finite elements to satisfy this inclusion: well-known choices are the Argyris or Bell elements on triangular meshes and Bogner-Fox-Schmit elements on rectangular meshes, see, e.g., [8,30]. We note that the latter are both cheaper and simpler to implement, hence the limitation caused by rectangular cells has been circumvented in different ways [5,14]. To relax the requirement $V_N \subset H^2(\Omega)$, mixed or nonconforming finite elements are also used [19,20], but these are out of the scope of this paper. Finally, we note that V_N can be a Fourier subspace, which will be discussed in subsection 4.4 in the context of the biharmonic problem.

For later use, we note that in the above realizations the condition $V_N \subset H^2(\Omega)$ is satisfied in the stronger way

$$V_N \subset W^{2,\infty}(\Omega), \tag{3.23}$$

that is, the second partial derivatives are not only in $L^2(\Omega)$ but they are in fact bounded.

The convergence of the Galerkin method is ensured by Céa's lemma [8], since Proposition (3.1) implies that A is uniformly monotone and Lipschitz continuous.

3.5. Further operator properties

In what follows, we consider the solution of the projected equation (3.22) in the subspace V_N. The formula defines an operator $F : V_N \to V_N$:

$$\langle F(u), v\rangle_V = \int_\Omega \Big(g(E(D^2u))\, \tilde{D}^2 u : D^2 v \,-\, \alpha v \Big) \qquad (u, v \in V_N). \tag{3.24}$$

(For simplicity we drop here the superscript N for elements of V_N.) Here F is the suitable projection of the original operator A into V_N, in fact,

$$F = P_{V_N} A_{|\, V_N}$$

where $P_{V_N} : V \to V_N$ is the orthogonal projection of V to V_N. Then problem (3.22) is equivalent to equation

$$F(u) = 0$$

in V_N.

The operator F inherits the properties of A, since Proposition 3.1 can be repeated in the space V_N instead of V. Hence, from (3.13), with notation (3.11),

$$\langle F'(u)h, v\rangle_V = \int_\Omega \Big(g([u,u])\,[h,v] + 2g'([u,u])\,[u,h][u,v] \Big) \qquad (u, h, v \in V_N), \tag{3.25}$$

and thus for any $u \in V_N$ the operator $F'(u)$ is self-adjoint, further, in analogy with (3.16),

$$\lambda \|h\|_V^2 \le \int_\Omega p([u,u])\,[h,h] \le \langle F'(u)h, h\rangle_V \le \int_\Omega q([u,u])\,[h,h] \le \Lambda \|h\|_V^2 \qquad (u, h \in V_N) \tag{3.26}$$

and hence

$$\lambda \|h\|_V^2 \le \langle F'(u)h, h\rangle_V \le \Lambda \|h\|_V^2 \qquad (\forall u, h \in V_N). \tag{3.27}$$

Consequently, analogously to (3.2), the projected equation (3.22) has a unique weak solution $u^N \in V_N$.

Our goal will be to apply the iterative methods of subsection 2.2. In order to ensure the Lipschitz continuity of F', we impose a further condition in addition to (2.4)–(2.5): we assume that $g \in C^2(\mathbf{R}^+)$ and satisfies

$$\big| \big(g(r^2) r\big)'' \big| \le L_1 \qquad (r \ge 0) \tag{3.28}$$

for some constant $L_1 > 0$. Further, using (3.23) and that V_N is finite dimensional, there exists a constant $C_N > 0$ such that

$$\sup_\Omega v^2 \le C_N \int_\Omega v^2 \qquad (\forall v \in V_N). \tag{3.29}$$

Proposition 3.2. *If (2.4)–(2.5) and (3.28) hold, then operator F has a bounded second derivative: there exists $L > 0$ such that*

$$\big\| F''(u) \big\| \le L \qquad (\forall u \in V_N). \tag{3.30}$$

Proof. The proof is an analogue of [17, Prop. 3.1], now for an implicitly defined mapping with matrix arguments. From (3.28) we have

$$\left|\left(g(r^2)r\right)''\right| = |6\,g'(r^2)r + 4\,g''(r^2)r^3\,| \le L_1 \qquad (r \ge 0). \tag{3.31}$$

Based on the structure (3.14), inherited by F' in V_N, the operator F has a second derivative whose symmetric triadic form satisfies

$$F''(u)(h,h,h) = \int_\Omega \Big(6g'([u,u])\,[u,h]\,[h,h] \;+\; 4g''([u,u])\,[u,h]^3\Big) \qquad (\forall u, h \in V_N).$$

Hence, using the notation and inequality

$$[[h]] := [h,h]^{1/2} = (E(D^2h))^{1/2}, \qquad |[u,h]| \le [[u]][[h]] \qquad (\forall u, h \in V_N),$$

respectively, we obtain

$$\begin{aligned} |F''(u)(h,h,h)| &\le \int_\Omega \max\Big\{\,6\,|g'(r^2)r|_{|r=[[u]]},\; \big|6\,g'(r^2)r + 4\,g''(r^2)r^3|_{|r=[[u]]}\,\big|\,\Big\}\,|[[h]]|^3 \\ &\le \max\{K_1, L_1\} \int_\Omega |[[h]]|^3 \end{aligned}$$

where L_1 is from (3.31) and $K_1 := \sup 6\,|g'(r^2)r|$. Denoting $L_g := \max\{K_1, L_1\} < \infty$, we have

$$|F''(u)(h,h,h)| \le L_g \int_\Omega |[[h]]|^3 \qquad (\forall h \in V_N). \tag{3.32}$$

Denoting by $|\Omega|$ the area of Ω, and using (3.23), (3.29) and (3.17), respectively, we have

$$\int_\Omega |[[h]]|^3 \le |\Omega| \sup_\Omega |[[h]]|^3 \le |\Omega|\, C_N^{3/2} \Big(\int_\Omega [[h]]^2\Big)^{3/2} = |\Omega|\, C_N^{3/2} \|h\|_V^3 \qquad (\forall h \in V_N). \tag{3.33}$$

Since $F''(u)$ is a symmetric trilinear form, (3.32) and (3.33) imply that (3.30) holds with $L := L_g\,|\Omega| C_N^{3/2}$. ■

Corollary 3.2. *If (2.4)–(2.5) and (3.28) hold, then operator F is Lipschitz continuous with Lipschitz constant L from (3.30).*

Remark 3.3. Condition (3.28) does not require much more than (2.4)–(2.5), we essentially exclude large oscillations in the gradient. For instance, if g approaches the limit Λ, arising in (2.4), at a power order: $g(r^2) = \Lambda + cr^{-\beta}$ for some $\beta > 0$, when $r \ge r_0$, then $\left(g(r^2)r\right)'' = const. \cdot r^{-1-\beta}$ and hence remains bounded for $r \ge r_0$. Similarly, condition (3.28) was shown in [17] to hold for functions of the form $r \mapsto \frac{r^k+\varrho}{r^k+\tau}$, since the latter often arise in interpolating nonlinearities which have two-sided bounds.

4. Iterative solution algorithms

4.1. Sobolev gradient iteration

First, the gradient method (2.9) will be applied in our Sobolev subspace V_N for problem (3.22).

Theorem 4.1. *Let (2.4)–(2.5) hold. Then for any $u_0 \in V_N$, the sequence $(u_n) \subset V_N$ defined by*

$$u_{n+1} := u_n - \frac{2}{\Lambda + \lambda}\, z_n \qquad (n \in \mathbb{N}), \tag{4.1}$$

where $z_n \in V_N$ is the solution of the linear problem

$$\int_\Omega D^2 z_n : D^2 v = \int_\Omega \Big(g(E(D^2 u_n))\, \tilde{D}^2 u_n : D^2 v \;-\; \alpha v\Big) \qquad (\forall v \in V_N), \tag{4.2}$$

converges linearly to the solution u^N of (3.22), namely,

$$\|u_n - u^N\|_V \leq \frac{1}{\lambda}\|F(u_0)\|_V \left(\frac{\Lambda-\lambda}{\Lambda+\lambda}\right)^n \qquad (n \in \mathbb{N}), \tag{4.3}$$

where the constants Λ and λ are from (2.4)–(2.5). Moreover, in fact $\|u_n - u^N\|_V \leq \frac{1}{\lambda}\|F(u_n)\|_V$ and

$$\frac{\|F(u_{n+1})\|_V}{\|F(u_n)\|_V} \leq \frac{\Lambda-\lambda}{\Lambda+\lambda}. \tag{4.4}$$

Proof. We will apply Theorem 2.1 in V_N. Using (3.4) and (3.24), equation (4.2) becomes

$$\langle z_n, v\rangle_V = \langle F(u_n), v\rangle_V \quad (\forall v \in V_N), \qquad \text{that is,} \quad z_n = F(u_n). \tag{4.5}$$

Hence (4.1) is equivalent to (2.9). We have seen in subsection 3.5 (see (3.27)) that the operator F satisfies Assumptions 2.2 (i)–(ii). Hence our statements (4.3)–(4.4) follow from Theorem 2.1 in V_N. ■

The linear auxiliary problems (4.2) are the weak formulations, corresponding to the subspace V_N, of the problems

$$\begin{cases} \Delta^2 z_n = r_n \\ z_{n|\partial\Omega} = \frac{\partial^2 z_n}{\partial \nu^2}\Big|_{\partial\Omega} = 0 \end{cases} \tag{4.6}$$

where, using (3.21), the r.h.s. $r_n := T(u_n) - \alpha$ is the residual corresponding to the previous iterate u_n. That is, we have to solve auxiliary biharmonic problems in course of the iteration, which is standard linear problem and several solvers are available. One of the possibilities will be mentioned in subsection 4.4 below. We note that the function z_n realizes the Sobolev gradient at u_n w.r.t. the inner product (3.4).

Remark 4.1. Preconditioned gradient (steepest descent) iterations and their linear convergence under biharmonic-related preconditioning operators have been analyzed for other fourth order problems, such as arising in thin film models or crystal equations, in [7,12,13,31].

4.2. Newton's method

Now we apply the Newton iteration (2.12) for our problem (3.22). As in general, we thus obtain faster convergence, but at the price of a significant extra cost.

Theorem 4.2. *Let (2.4)–(2.5) and (3.28) hold, and let u_0 be in a properly small neighbourhood of u^N. For given u_n ($n \in \mathbb{N}$), let $z_n \in V_N$ be the solution of the linear problem*

$$\int_\Omega \left(J_g(\tilde{D}^2 u_n)\,\tilde{D}^2 z_n\right) : D^2 v = \int_\Omega \Big(g(E(D^2 u_n))\,\tilde{D}^2 u_n : D^2 v \;-\; \alpha v\Big) \quad (\forall v \in V_N), \tag{4.7}$$

where the notation

$$\left(J_g(\tilde{D}^2 u_n)\,\tilde{D}^2 z_n\right) : D^2 v := g(E(D^2 u_n)\,\tilde{D}^2 z_n : D^2 v \;+\; 2g'(E(D^2 u_n)\,(\tilde{D}^2 u_n : D^2 z_n)(\tilde{D}^2 u_n : D^2 v)\big) \tag{4.8}$$

is used, and let

$$u_{n+1} := u_n - z_n. \tag{4.9}$$

Then $\|u_n - u^N\|_V \leq \frac{1}{\lambda}\|F(u_n)\|_V \to 0$ and

$$\|F(u_{n+1})\|_V \leq \frac{L}{2\lambda^2}\|F(u_n)\|_V^2 \qquad (n \in \mathbb{N}), \tag{4.10}$$

where the constants are from (2.4)–(2.5) and (3.28).

Proof. We will apply Theorem 2.2 in V_N. Using (3.11), (3.12) and (3.25), equation (4.7)–(4.8) becomes

$$\langle F'(u_n) z_n, v\rangle_V = \langle F(u_n), v\rangle_V \qquad (\forall v \in V_N), \tag{4.11}$$

that is, $z_n = F'(u_n)^{-1}F(u_n)$. Hence (4.9) is equivalent to (2.12). We have seen in subsection 3.5 (see (3.27) and Corollary 3.2) that the operator F satisfies Assumptions 2.2. Hence, again, our statements follow from Theorem 2.2 in V_N. ■

We note that the linear auxiliary problems (4.7)–(4.8) are the weak formulations of the problems

$$\begin{cases} \operatorname{div}^2\big(J_g(\tilde D^2 u_n)\,\tilde D^2 z_n\big) = r_n \\ \quad z_{n|\partial\Omega} = \frac{\partial^2 z_n}{\partial \nu^2}\big|_{\partial\Omega} = 0, \end{cases} \tag{4.12}$$

where $r_n := T(u_n) - \alpha$. Here we can formally write

$$J_g(\tilde D^2 u_n) = g(E(D^2 u_n)\,I \,+\, 2g'(E(D^2 u_n)\,\tilde D^2 u_n \oplus \tilde D^2 u_n\,, \tag{4.13}$$

this is the tensor in $\mathbf{R}^{N^4}$ that generates the bilinear form (4.8) on $N \times N$ matrices. It is computable from the previous iterate u_n, hence problem (4.12) is a linear fourth order BVP for z_n with a tensor coefficient.

Remark 4.2 *(Inner iterations).* A widespread possibility to solve linear problems like (4.12) is to apply an inner iteration, since the matrix of the system is sparse. For such problems, one may propose the preconditioned conjugate gradient method, see, e.g., [15, Chap. 2.3], that is, to apply the conjugate gradient method to the formally preconditioned form of (4.11):

$$P^{-1}F'(u_n)z_n = P^{-1}F(u_n),$$

where P is a preconditioner to be chosen. An efficient choice for P can be the biharmonic operator (more precisely, its projection into the subspace V_N). In this case one has to solve ultimately the same type of standard linear problems as in (4.6).

4.3. Quasi-Newton/variable preconditioning method

As seen above, the Newton iteration requires a rather complicated and costly update (4.8) of the linearized operators. The main idea of the quasi-Newton/variable preconditioning method in our context is that we only use linearized operators with scalar coefficients, such that they are spectrally equivalent to the exact linearizations. Hence the construction of the iteration is much simpler, but a fast convergence can be retained.

The construction relies on the scalar functions p and q:

$$p(r^2) = \min\{g(r^2),\ (g(r^2)r)'\}, \qquad q(r^2) = \max\{g(r^2),\ (g(r^2)r)'\} \qquad (r \ge 0) \tag{4.14}$$

introduced in (3.15). As seen in (3.26),

$$\int_\Omega p([u,u])\,[h,h] \le \langle F'(u)h,h\rangle_V \le \int_\Omega q([u,u])\,[h,h] \qquad (\forall u,h \in V_N). \tag{4.15}$$

We introduce a new scalar function w satisfying

$$p \le w \le q \qquad \text{on } \mathbf{R}^+. \tag{4.16}$$

This will be the scalar coefficient in the linearized operator.

Theorem 4.3. *Let (2.4)–(2.5) and (3.28) hold, and let u_0 be in a properly small neighbourhood of u^N. For given u_n ($n \in \mathbb{N}$), we calculate*

$$M_n := \sup_\Omega \frac{q(E(D^2u_n))}{w(E(D^2u_n))}, \qquad m_n := \inf_\Omega \frac{p(E(D^2u_n))}{w(E(D^2u_n))}\,, \tag{4.17}$$

we let $z_n \in V_N$ be the solution of the linear problem

$$\int_\Omega w(E(D^2u_n))\,\tilde D^2 z_n : D^2 v = \int_\Omega \Big(g(E(D^2u_n))\,\tilde D^2 u_n : D^2 v \,-\, \alpha v\Big) \quad (\forall v \in V_N), \tag{4.18}$$

and define

$$u_{n+1} := u_n - \frac{2}{M_n + m_n}\, z_n\,. \tag{4.19}$$

Then $\|u_n - u^*\|_V \le \frac{1}{\lambda}\|F(u_n)\|_V \to 0$ and

$$\limsup \frac{\|F(u_{n+1})\|_*}{\|F(u_n)\|_*} \le \limsup \frac{M_n - m_n}{M_n + m_n} < 1. \tag{4.20}$$

Proof. We will apply Theorem 2.3 in V_N. We have seen in subsection 3.5 (see (3.27) and Corollary 3.2) that the operator F satisfies Assumptions 2.2. Now we must define the linear operators B_n in V_N and check (2.14). For given u_n ($n \in \mathbb{N}$), let B_n be defined via the formula

$$\langle B_n h, v\rangle_V = \int_\Omega w(E(D^2 u_n))\, \tilde{D}^2 h : D^2 v \qquad (h, v \in V_N). \tag{4.21}$$

Then B_n is self-adjoint. Further, from (3.11) and (3.12), for any $u, h \in V_N$,

$$\langle B_n h, h\rangle_V = \int_\Omega w(E(D^2 u_n))\, \tilde{D}^2 h : D^2 h = \int_\Omega w([u_n, u_n])\, [h, h]$$

hence (4.15) and (4.17) imply

$$\langle F'(u_n)h, h\rangle_V \le \sup_\Omega \frac{q([u_n, u_n])}{w([u_n, u_n])} \int_\Omega w([u_n, u_n])\, [h, h] = M_n \langle B_n h, h\rangle_V\,,$$

$$\langle F'(u_n)h, h\rangle_V \ge \sup_\Omega \frac{p([u_n, u_n])}{w([u_n, u_n])} \int_\Omega w([u_n, u_n])\, [h, h] = m_n \langle B_n h, h\rangle_V\,,$$

that is, (2.14) holds. To verify the requirement that (m_n) is positively bounded from below and (M_n) bounded from above, just note that by assumptions (2.4)–(2.5) and (4.16), each function p, w and q is positively bounded from both below and above. Now, using (4.21) and (3.24), equation (4.18) becomes

$$\langle B_n z_n, v\rangle_V = \langle F(u_n), v\rangle_V \qquad (\forall v \in V_N),$$

that is, $z_n = B_n^{-1} F(u_n)$. Hence (4.19) is equivalent to (2.13). Thus, altogether, our statements follow from Theorem 2.3 in V_N. ■

Remark 4.3.

(i) The function z_n realizes the Sobolev gradient at u_n w.r.t. a weighted variable inner product in which (3.4) is modified with the weight function $w_n := w(E(D^2 u_n))$. If we had $w \equiv 1$ then we would reobtain the standard gradient method of subsection 4.1. The introduction of the weight function w_n only causes a small extra computational effort, since the problems (4.18) only differ from the biharmonic problems in the scalar coefficients w_n, which is a great simplification compared to the tensor coefficient (4.13) of the full Newton's method. On the other hand, this scalar modification still gives a much better approximation of the linearized operator compared to the biharmonic operator.

(ii) Some practical choices of w are as follows:
- $w(s) = g(s)$ ($s \in \mathbf{R}^+$), which is the simplest choice, moreover, in this case the iteration reduces to the "frozen coefficient method" or "Kachanov's method";
- $w(s) = \sqrt{q(s)/p(s)}$ ($s \in \mathbf{R}^+$), which gives an optimal rate since the role of m_n and M_n becomes reciprocal;
- w is a piecewise constant function, following the construction described in [11, Subsec. 8.2.7] for second order problems, which yields a simple updating of the linearized operator.

Furthermore, although the cost of evaluating the minima and maxima in (4.17) is not large, one can avoid this by estimating in advance as

$$M_n \le \sup_{s\ge 0} q(s)/w(s), \qquad m_n \ge \inf_{s\ge 0} p(s)/w(s)\,.$$

(iii) The local convergence in Theorem 4.3 can be extended to global by proper damping, see Remark 2.1, item (c).

(iv) Similarly as mentioned in Remark 4.2 for Newton's method, a widespread possibility to solve the linearized problems is to apply an inner preconditioned conjugate gradient iteration. Now the formally preconditioned form of the linear problems is

$$P^{-1} B_n z_n = P^{-1} F(u_n). \tag{4.22}$$

Again, the preconditioner P can be chosen as the biharmonic operator, leading to the ultimate linear problems as in (4.6).

Remark 4.4. For each of the three methods of this chapter (Sobolev gradient method, Newton's method, Quasi-Newton/variable preconditioning), the following observations hold:

(i) The embedding estimate (3.18) yields that the obtained convergence $\|u_n - u^*\|_V \to 0$ implies uniform convergence too:

$$\|u_n - u^*\|_{max} \le C_\Omega \|u_n - u^*\|_V \to 0\,.$$

(ii) The convergence results also hold if problem (3.2) is endowed with other boundary conditions, e.g., involving clamping. Then, as mentioned in Remark 3.2, one just works in the corresponding other Sobolev space.

4.4. Realization using Fourier's method

As seen above, the iterations often reduce our task to solving a sequence of linear problems arising as the projection of biharmonic problems

$$\begin{cases} \Delta^2 z = r \\ z_{|\partial\Omega} = \frac{\partial^2 z}{\partial\nu^2}\big|_{\partial\Omega} = 0 \end{cases} \tag{4.23}$$

into a given subspace V_N. When the eigenfunctions $e_{k\ell}$ and eigenvalues $\lambda_{k\ell}$ are known, then the solution of problem (4.2) in the subspace defined by $k, \ell \le N$ can be obtained as follows:

$$z = \sum_{i,j=1}^{N} \frac{c_{ij}}{\lambda_{ij}} e_{ij}\,, \qquad \text{where} \qquad c_{k\ell} := \int_\Omega \Big(g(E(D^2 u_n))\, \tilde{D}^2 u_n : D^2 e_{k\ell} - \alpha e_{k\ell} \Big) \qquad (k, \ell = 1, \dots, N).$$

In a similar way, the above method can be used in the inner iterations mentioned in Remarks 4.2 and 4.3, when the preconditioner P in the conjugate gradient method arises from the biharmonic operator. If Ω is an irregular domain, then one may apply the fictitious domain method, in which one replaces the original PDE on Ω by a proper modified problem in a simpler-shaped larger domain $D \supset \Omega$, see, e.g. [3] for this idea for biharmonic problems. In our context, D should be a domain where the eigenfunctions and eigenvalues can be simply determined.

4.5. Numerical illustration

In this subsection we give a simple numerical example, which will illustrate that the convergence factor is uniform, that is, bounded independently of the subspace V_N, as predicted by the theory. We will apply the Sobolev gradient method combined with Fourier's method on a square.

Let us consider problem (3.2) on the domain $\Omega = [0, \pi]^2$ with the following nonlinearity:

$$g(s) = 1.02\Big/\Big(1 + \sqrt{1 - \tfrac{s}{3}}\Big) \;\text{ if } 0 \le s \le S_0 = 2.76; \quad g(s) = g(S_0) \approx 0.7951 \;\text{ if } s \ge S_0 = 2.76.$$

This function is taken from [10], and corresponds to a situation when an elastic model has its domain of validity up to $s \le S_0$, that is, the solutions within the model must satisfy $E(D^2 u) \le S_0$. The extension of g for values $s > S_0$ is only done so that the operator T is defined everywhere. The bounds in (2.4)–(2.5) for this function are

$$\lambda = 0.51, \quad \Lambda = 2.81. \tag{4.24}$$

We apply the Sobolev gradient iteration (4.1). To solve the linear problems (4.2), we use Fourier's method as described in subsection 4.4. Now the eigenvalues and eigenfunctions are

$$\lambda_{k\ell} = \big(k^2 + \ell^2\big)^2, \qquad u_{k\ell} = \tfrac{2}{\pi} \sin kx \sin \ell y \qquad (k, \ell \in \mathbb{N}^+),$$

respectively. The method uses truncated Fourier series, that is, trigonometric polynomials of some degree N. We will consider the values $N = 3$, 6, 12.

In our experiments we used different source functions α, which describe the external load:

- α_1 is a model of positive point source: $\alpha_1 \equiv \varrho > 0$ on a small subdomain Ω_ε of area $\varepsilon = 10^{-2}$, and $\alpha_1 \equiv 0$ otherwise. (The sign $\varrho > 0$ means that the force acts downwards.)
- α_2 is a model of two positive point sources in the same sense as above.
- α_3 is a model of two point sources, of which one is positive and one is negative (i.e. the latter force acts upwards).

The convergence estimation for the Sobolev gradient iteration is described by Theorem 4.1. We have

$$\|u_n - u^N\|_V \le \tfrac{1}{\lambda} \|F(u_n)\|_V \le \text{const.} \cdot Q^n \qquad (n \in \mathbb{N}) \tag{4.25}$$

in the Sobolev norm, where $Q = \frac{\Lambda - \lambda}{\Lambda + \lambda}$. We expect a convergence factor $Q \approx 0.7$, since the ideal value corresponding to (4.24) is 0.6929. By Remark 4.4 (i), we also have uniform convergence, so the estimates with const. $\cdot\, Q^n$ in (4.25) also hold in maximum norm.

Table 1
Errors in maximum norm and convergence factors for different values of N under the source function α_1 with a general nonlinearity.

	e_n			$Q_n := e_{n+1}/e_n$		
n	$N = 3$	$N = 6$	$N = 12$	$N = 3$	$N = 6$	$N = 12$
1	0.003677	0.003858	0.003597	–	–	–
2	0.002651	0.002759	0.002531	0.7211	0.7153	0.7036
3	0.001912	0.001974	0.001781	0.7211	0.7153	0.7037
4	0.001378	0.001412	0.001253	0.7211	0.7153	0.7037
5	0.000994	0.001010	0.000882	0.7211	0.7153	0.7037
6	0.000717	0.000722	0.000621	0.7211	0.7153	0.7046
7	0.000517	0.000517	0.000439	0.7211	0.7153	0.7064
8	0.000373	0.000370	0.000311	0.7211	0.7153	0.7091
9	0.000269	0.000264	0.000221	0.7211	0.7153	0.7102
10	0.000194	0.000189	0.000157	0.7211	0.7153	0.7103
11	0.000140	0.000135	0.000112	0.7211	0.7154	0.7104
12	0.000101	0.000097	0.000079	0.7211	0.7154	0.7119
13	0.000073	0.000069	0.000057	0.7211	0.7154	0.7121
14	0.000052	0.000050	0.000040	0.7211	0.7154	0.7122
15	0.000038	0.000035	0.000029	0.7211	0.7154	0.7130
16	0.000027	0.000025	0.000020	0.7211	0.7154	0.7135
17	0.000020	0.000018	0.000015	0.7211	0.7154	0.7136
18	0.000014	0.000013	0.000010	0.7211	0.7155	0.7137
19	0.000010	0.000009	0.000007	0.7211	0.7158	0.7138
20	0.000007	0.000007	0.000005	0.7211	0.7158	0.7139

Table 2
Errors in maximum norm and convergence factors for different values of N with a mild nonlinearity.

	e_n			$Q_n := e_{n+1}/e_n$		
n	$N = 3$	$N = 6$	$N = 12$	$N = 3$	$N = 6$	$N = 12$
1	0.000729	0.000741	0.000647	–	–	–
2	0.000279	0.000276	0.000228	0.3832	0.3724	0.3529
3	0.000107	0.000102	0.000082	0.3832	0.3724	0.3603
4	0.000041	0.000038	0.000029	0.3832	0.3723	0.3641
5	0.000015	0.000014	0.000011	0.3832	0.3723	0.3657
6	0.000006	0.000005	0.000004	0.3831	0.3723	0.3663
7	0.000002	0.000002	0.000002	0.3831	0.3723	0.3675
8	0.000001	0.000001	0.000001	0.3831	0.3723	0.3689

The exact solution and the approximate solution $u_N \in V_N$ are unknown. Hence to follow (4.25), we can only measure the values of the norm of $F(u_n)$. We are interested in uniform convergence. Taking into account (4.5), we thus study the errors

$$e_n := \|F(u_n)\|_{max} = \|z_n\|_{max} \qquad (n = 1, 2, \ldots).$$

Below we briefly summarize some results. First the errors and convergence factors are given in tables, then finally the graphs of the obtained solutions are shown (Figs. 4.1–4.3) when the maximum-norm error is $O(10^{-6})$.

Table 1 shows the results for the source function α_1 mentioned above. The errors e_n and the relative convergence factors (the ratio of consecutive errors) for the values $N = 3$, 6, 12 are presented. The latter values are in accordance with the expected factor $Q \approx 0.7$. Furthermore, the obtained values behave uniformly, that is, they are bounded independently of the subspaces V_N as expected.

The results for the source functions α_2 and α_3 were very much similar, hence those results are not shown here.

Table 2 refers to a case of small point source, when the density ϱ in the definition of α_1 is significantly decreased. Then the solution u is closer to zero, hence the range of the values where the nonlinearity g is applied is smaller and the effective values of λ and Λ are tighter. We observe that the convergence factor now is much less than the general bound around 0.7.

4.6. Concluding remarks

The paper has given a detailed description of three iterative processes for fourth order nonlinear plate problems. The importance of the results lies in the Sobolev space theory as a rigorous mathematical background for the study of the iterative methods, also yielding robust convergence. In particular, the quasi-Newton/variable preconditioning method, based on the author's earlier results for second order problems, has been extended to the present situation as a new application of Sobolev gradients, providing a potential of cost reduction for the overall iteration. Numerical tests have been given for

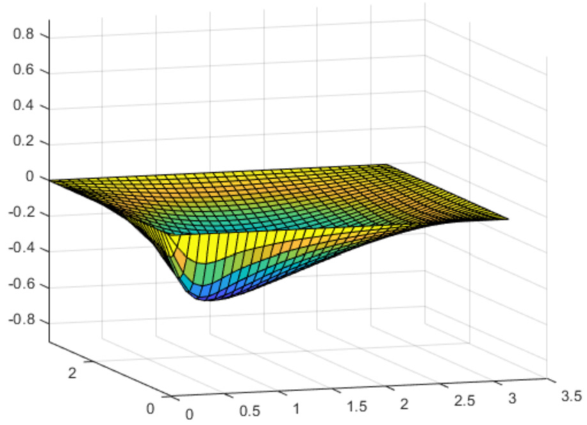

Fig. 4.1. The solution under a one-point source function.

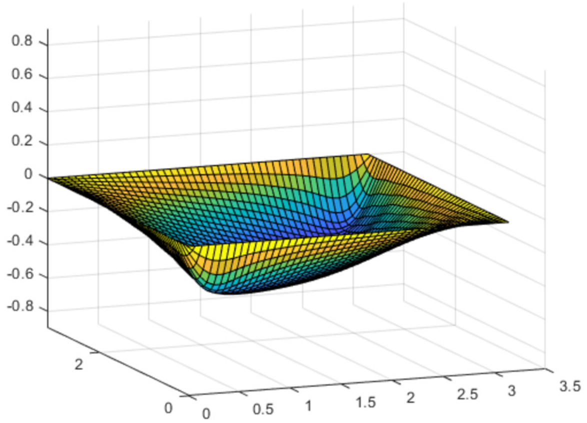

Fig. 4.2. The solution under a two-point source function.

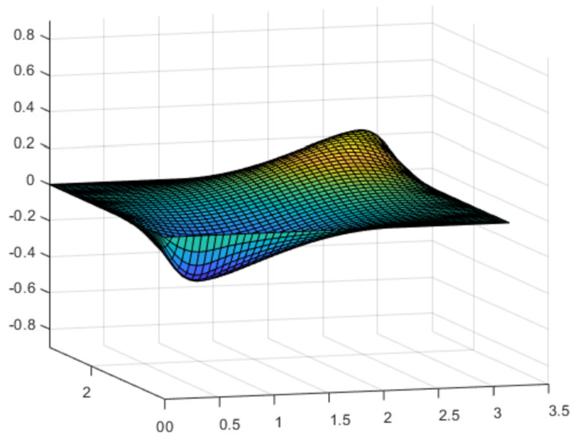

Fig. 4.3. The solution under a sign-changing two-point source function.

the Sobolev gradient iteration combined with a Fourier method, and have illustrated the robustness of convergence, that is, convergence factors bounded independently of the dimension of the discretization subspace. Future research directions may involve the combination of finite element discretizations, mentioned in subsection 3.4, with the studied iterative methods, furthermore, the quasi-Newton/variable preconditioning idea of our paper might be applied to fourth or sixth order problems mentioned in Remark 4.1.

CRediT authorship contribution statement

I am the sole author of the paper, and hence there have been no diverse contributions.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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