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Hardy–Littlewood-type theorems for Fourier transforms in $\mathbb{R}^d$ ☆

Mikhail Dyachenko [a,b], Erlan Nursultanov [c], Sergey Tikhonov [d,e,f], Ferenc Weisz [g,*]

[a] *Moscow State University, Vorobe'vy Gory, 117234, Russia*
[b] *Moscow Center for Fundamental and Applied Mathematics, Russia*
[c] *Lomonosov Moscow State University (Kazakh Branch) and Gumilyov Eurasian National University, Munatpasova 7, 010010 Astana, Kazakhstan*
[d] *Centre de Recerca Matemàtica Campus de Bellaterra, Edifici C, 08193 Bellaterra (Barcelona), Spain*
[e] *ICREA, Pg. Lluís Companys 23, 08010 Barcelona, Spain*
[f] *Universitat Autònoma de Barcelona, Spain*
[g] *Department of Numerical Analysis, Eötvös L. University, H-1117 Budapest, Pázmány P. sétány 1/C, Hungary*

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We obtain Fourier inequalities in the weighted L_p spaces for any $1 < p < \infty$ involving the Hardy–Cesàro and Hardy–Bellman operators. We extend these results to product Hardy spaces for $p \leqslant 1$. Moreover, boundedness of the Hardy-Cesàro and Hardy-Bellman operators in various spaces (Lebesgue, Hardy, BMO) is discussed. One of our main tools is an

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* Corresponding author.
E-mail addresses: dyach@mail.ru (M. Dyachenko), er-nurs@yandex.ru (E. Nursultanov), stikhonov@crm.cat (S. Tikhonov), weisz@inf.elte.hu (F. Weisz).

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appropriate version of the Hardy–Littlewood–Paley inequality $\|\widehat{f}\|_{L_{p',q}} \lesssim \|f\|_{L_{p,q}}$.

1. Introduction

1.1. Fourier inequalities in Lebesgue spaces

For the multi-dimensional Fourier transform $\widehat{f}$, the following Hardy-Littlewood theorem is a counterpart of the Hausdorff–Young inequality

$$\left\|\widehat{f}\right\|_{p'} \lesssim \|f\|_p, \qquad 1 < p \leqslant 2, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

and reads as follows (see [23, Th. 2.2] and [32, Th.2]):

$$\left(\int\limits_{\mathbb{R}^d} (|t_1| \dots |t_d|)^{p-2} \left|\widehat{f}(t)\right|^p dt\right)^{1/p} \lesssim \|f\|_p, \qquad 1 < p \leqslant 2, \tag{1}$$

where $t = (t_1, \dots, t_d) \in \mathbb{R}^d$. In particular, (1) sharpens the well-known inequality (see, e.g., [5, (5.19)] and [7, p.17])

$$\left(\int\limits_{\mathbb{R}^d} |t|^{d(p-2)} \left|\widehat{f}(t)\right|^p dt\right)^{1/p} \lesssim \|f\|_p, \qquad 1 < p \leqslant 2, \tag{2}$$

where $|t|$ denotes the Euclidean norm of t. Throughout the paper, by C and C_p we denote positive constants, which may depend on nonessential parameters and on the dimension. As usual, $F \lesssim G$ stands for $F \leqslant CG$. If $F \lesssim G \lesssim F$, we write $F \asymp G$.

As it is well known, the case $p > 2$ requires special attention. By Q_N define a cube centered at the origin with the edge length N. For $f \in L_p(\mathbb{R}^d)$ we define

$$(\mathfrak{F}_N f)(\xi) := \int\limits_{Q_N} f(x) e^{-i(\xi,x)} dx. \tag{3}$$

As it is well known, if $f \in L_p(\mathbb{R}^d)$, $1 \leqslant p \leqslant 2$, the limit $\lim\limits_{N\to+\infty} (\mathfrak{F}_N f)$ exists in $L_{p'}$ and is called the Fourier transform of f. There are many examples in the literature showing that the Fourier transform of $f \in L_p(\mathbb{R}^d)$, $2 < p < \infty$, is not well-defined in the usual sense; see, e.g., [42, Ch. XVI, §3]. Moreover, one can construct a Carleman-type function

[4, Ch. IV, §16] so that even in the case when the Fourier transform exists, neither inequality (1) nor Hausdorff–Young inequality hold; see Appendix A.

In order to derive the corresponding analogues of inequality (1) in the case $p \geqslant 2$, we define the Hardy-Cesàro and Hardy-Bellman operators. First, let E consist of all n-tuples containing only 0 and 1. Set $\mathcal{H}_\varepsilon := \mathcal{H}_{\varepsilon_d,d} \dots \mathcal{H}_{\varepsilon_2,2}\mathcal{H}_{\varepsilon_1,1}$, where $\varepsilon \in E$ and

$$\left(\mathcal{H}_{\varepsilon_i,i} f\right)(t) := \begin{cases} \frac{1}{t_i}\int\limits_0^{t_i} f(x_1,\dots,x_i,\dots,x_d)\,dx_i, & \text{if} \quad \varepsilon_i = 0 \\ \int\limits_{|t_i|}^{\infty} f(x_1,\dots,\operatorname{sign}(t_i)x_i,\dots,x_d)\,\frac{dx_i}{x_i}, & \text{if} \quad \varepsilon_i = 1. \end{cases}$$

One of the main goals of the paper is to show that certain natural averages of the Fourier transforms, namely Hardy–Cesàro and Hardy–Bellman operators, are not only well-defined for L_p-functions with any $1 < p < \infty$ but the inequalities corresponding to estimates (1) and (2) hold true. We formulate our main result as a Pitt-type inequality with power weights. Let us recall the known Pitt inequality ([23, Th. 2.2]; see also [5,11]).

For $1 < r \leqslant q < \infty$ and $f \in \bigcup_{1\leqslant s\leqslant 2} L_s$,

$$\left(\int\limits_{\mathbb{R}^d} \left((|t_1|\dots|t_d|)^\beta \left|\widehat{f}(t)\right|\right)^q dt\right)^{1/q} \lesssim \left(\int\limits_{\mathbb{R}^d} \left((|x_1|\dots|x_d|)^\alpha \left|f(x)\right|\right)^r dx\right)^{1/r} \tag{4}$$

provided that

$$0 \leqslant \alpha = \frac{1}{r'} - \frac{1}{q} - \beta < \frac{1}{r'} \tag{5}$$

and, additionally,

$$\beta \leqslant 0, \tag{6}$$

which is equivalent to $\frac{1}{r'} - \frac{1}{q} \leqslant \alpha$. For optimality of these conditions see [23, Th. 2.2], [6, Sect. 4], [13, Sects. 2,3].

Now we point out three well-known special cases. For $\alpha = \beta = 0$, $q = r'$, $1 < r \leqslant 2$,

$$\left\|\widehat{f}\right\|_{r'} \lesssim \|f\|_r \tag{7}$$

which is the Hausdorff-Young inequality; for $\alpha = 0$, $\beta = 1 - 2/r$, $1 < r = q \leqslant 2$, we obtain (1), and for $\beta = 0$, $\alpha = 1 - 2/r$, $2 \leqslant r = q < \infty$, we establish the dual to (1), that is,

$$\left\|\widehat{f}\right\|_r \lesssim \left(\int\limits_{\mathbb{R}^d} (|x_1|\dots|x_d|)^{r-2} \left|f(x)\right|^r dx\right)^{1/r}. \tag{8}$$

The latter inequality was also proved in the recent paper [34] provided that $f \in L_1$.

Let $* = (j_1, j_2, ..., j_d)$ be a permutation of $(1, 2, ..., d)$. By $L^*_{p,q}$ we denote the Lorentz space defined by the iterative non-increasing rearrangement $f^{*_{j_1},\dots,*_{j_d}} := \left((f^{*_{j_1}})^{*_{j_2}}\dots\right)^{*_{j_d}}$ taken coordinate-wise; see Section 2 for the precise definition and properties. In the next theorem, we extend inequality (4) not assuming the condition $\beta \leqslant 0$.

Theorem 1. *Let $1 < r \leqslant q < \infty$ and condition (5) hold. Let $* = (j_1, j_2, ..., j_d)$ be a permutation of $(1, 2, ..., d)$. If a measurable function f is such that*

$$\left(\int\limits_{\mathbb{R}^d} \left((|x_1|\dots|x_d|)^\alpha \left|f^{*_{j_1}\cdots *_{j_d}}(x)\right|\right)^r dx\right)^{1/r} < \infty, \tag{9}$$

then, for any $\varepsilon \in E$, $t \in \mathbb{R}^d$, the limit

$$T_\varepsilon f(t) := \lim_{N\to+\infty} (H_\varepsilon \mathfrak{F}_N f)(t)$$

does exist. Moreover, for any $\varepsilon \in E$,

$$\left(\int\limits_{\mathbb{R}^d} \left((|t_1|\dots|t_d|)^\beta \left|T_\varepsilon f(t)\right|\right)^q dt\right)^{1/q} \lesssim \left(\int\limits_{\mathbb{R}^d} \left((|x_1|\dots|x_d|)^\alpha \left|f^{*_{j_1}\cdots *_{j_d}}(x)\right|\right)^r dx\right)^{1/r}. \tag{10}$$

If, additionally,

$$\left(\int\limits_{\mathbb{R}^d} \left((|x_1|\dots|x_d|)^\alpha |f(x)|\right)^r dx\right)^{1/r} < \infty, \tag{11}$$

then

$$\left(\int\limits_{\mathbb{R}^d} \left((|t_1|\dots|t_d|)^\beta \left|T_\varepsilon f(t)\right|\right)^q dt\right)^{1/q} \lesssim \left(\int\limits_{\mathbb{R}^d} \left((|x_1|\dots|x_d|)^\alpha \left|f(x)\right|\right)^r dx\right)^{1/r}. \tag{12}$$

In particular, if $\alpha = 0$, $\beta = 1 - 2/r$, $1 < r = q < \infty$, then

$$\left(\int\limits_{\mathbb{R}^d} (|t_1|\dots|t_d|)^{r-2} \left|T_\varepsilon f(t)\right|^r dt\right)^{1/r} \lesssim \|f\|_r. \tag{13}$$

Note that the right hand side of (10) is $\|f\|_{L^*_{p,r}}$ with $1/p = \alpha + 1/r$.

A counterpart of Theorem 1 – now the condition $0 \leqslant \alpha$ is not assumed but the condition $\beta \leqslant 0$ is fulfilled – is given in the next theorem.

Theorem 2. *Let* $1 < r \leqslant q < \infty$ *and*

$$\frac{1}{r'} - \frac{1}{q} \leqslant \alpha = \frac{1}{r'} - \frac{1}{q} - \beta < \frac{1}{r'}. \tag{14}$$

Suppose that

$$\int_{\mathbb{R}^d} \Big((|x_1| \dots |x_d|)^\alpha \, |f(x)| \Big)^r \, dx < \infty. \tag{15}$$

Then for any $\varepsilon \in E$ *the sequence* $(\mathcal{H}_\varepsilon \mathfrak{F}_N f)_N$ *converges to* $T_\varepsilon f$ *in the weighted Lebesgue norm, i.e.,*

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}^d} \Big((|t_1| \dots |t_d|)^\beta \, |T_\varepsilon f(t) - \mathcal{H}_\varepsilon \mathfrak{F}_N f(t)| \Big)^q \, dt \right)^{1/q} = 0.$$

Moreover,

$$\left(\int_{\mathbb{R}^d} \Big((|t_1| \dots |t_d|)^\beta \, |T_\varepsilon f(t)| \Big)^q \, dt \right)^{1/q} \lesssim \left(\int_{\mathbb{R}^d} \Big((|x_1| \dots |x_d|)^\alpha \, |f(x)| \Big)^r \, dx \right)^{1/r}. \tag{16}$$

In particular, if $\beta = 0$, $\alpha = 1 - 2/r$ *and* $1 < r = q < \infty$, *then the sequence* $(\mathcal{H}_\varepsilon \mathfrak{F}_N f)_N$ *converges to* $T_\varepsilon f$ *in the* L_r *norm and*

$$\|T_\varepsilon f\|_r \lesssim \left(\int_{\mathbb{R}^d} (|x_1| \dots |x_d|)^{r-2} \, |f(x)|^r \, dx \right)^{1/r}. \tag{17}$$

Remark 1. (i) If $\alpha < 1/r'$ as in (5) and (14), then Hölder's inequality and (9) (respectively, (15)) imply that $f \in L_s^{loc}$ for all $1 \leqslant s < r$. Thus $\mathcal{H}_\varepsilon \mathfrak{F}_N f$ is well defined.
(ii) Note that the boundedness of Hardy's operator in L_p (see (23) below) and Hausdorff-Young inequality (7) imply that

$$T_\varepsilon f = \mathcal{H}_\varepsilon \widehat{f} \qquad \text{if } f \in L_p,\ 1 < p \leqslant 2, \text{ or } f \in \mathcal{S}, \tag{18}$$

where $\mathcal{S}$ denotes the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. So T_ε is an extension of the operator $f \mapsto \mathcal{H}_\varepsilon \widehat{f}$. Under the conditions (5), $\beta \leqslant 0$, and (18), we always see that inequality (12) is weaker than (4). This follows from Hardy's inequality for averages (see, e.g., [25])

$$\int_{-\infty}^{\infty} |t|^{\beta q} \left(\frac{1}{|t|} \int_0^t g(s) ds \right)^q dt + \int_{-\infty}^{\infty} |t|^{\beta q} \left(\int_{|t|}^{\infty} \frac{g(s)}{s} ds \right)^q dt \lesssim \int_{-\infty}^{\infty} |t|^{\beta q} g(t)^q \, dt,$$

where $g \geqslant 0$ and $-\frac{1}{q} < \beta < 1 - \frac{1}{q}$, $q > 1$.
(iii) On the other hand, for $\beta > 0$, inequality (12) is an extension of (4). Moreover, (13) extends (1) for the case $p > 2$ while inequality (17) extends (8) for the case $1 < p < 2$ and for wider function spaces. Inequality (17) has been obtained in the one-dimensional case in [12].
(iv) We note that if $1 < r \leqslant 2$ and (15) holds, then $f \in L_r^{loc}$. Assuming $\beta = 0$, $\alpha = 1-2/r$ and $q = r$ in (12), we obtain (17) for $2 \leqslant r < \infty$. On the other hand, letting $\alpha = 0$, $\beta = 1 - 2/r$, $r = q$ in (16), we obtain (13) for $1 < r \leqslant 2$.

The proofs of Theorems 1 and 2 are based on the next result, which is an extension of the celebrated Hardy–Littlewood–Paley inequality [36, Ch. V] and it is interesting in its own right. Recall that the Hardy–Littlewood–Paley estimate for classical Lorentz spaces states that

$$\big\|\widehat{f}\,\big\|_{p',q} \lesssim \big\|f\big\|_{p,q}, \quad 1 < p < 2, \quad 0 < q \leqslant \infty.$$

Let $\mathcal{N}_{p,q}$ denote the net space; see Section 2 for the precise definition.

Theorem 3. *Let $* = (j_1, j_2, ..., j_d)$ be a permutation of $(1, 2, ..., d)$. If $1 < p < \infty$, $0 < q \leqslant \infty$, and $f \in L^*_{p,q}$, then $\mathfrak{F}_N f \in \mathcal{N}_{p',q}$ and there holds*

$$\big\|\mathfrak{F}_N f\,\big\|_{\mathcal{N}_{p',q}} \lesssim \big\|f\big\|_{L^*_{p,q}}$$

uniformly in $N \in \mathbb{N}$.

1.2. Fourier inequalities in Hardy spaces

The generalization of (2) to Hardy spaces is known (see Taibleson and Weiss [37] and Garcia-Cuerva and Rubio de Francia [18, Corollary 7.23] and also Bownik and Wang [9]):

$$\left(\int_{\mathbb{R}^d} |t|^{d(p-2)} \left|\widehat{f}(t)\right|^p dt\right)^{1/p} \lesssim \|f\|_{H_p(\mathbb{R}^d)}, \qquad 0 < p \leqslant 1. \tag{19}$$

The case $p = 1$ is called Hardy's inequality.

Our second goal is to generalize (1) to Hardy spaces. Here we consider the so called product Hardy spaces $H_p = H_p(\mathbb{R} \times \cdots \times \mathbb{R})$, that is different from $H_p(\mathbb{R}^n)$. See, e.g., [41].

Theorem 4. *If* $0 < p \leqslant 1$ *and* $f \in H_p$*, then*

$$\left(\int_{\mathbb{R}^d} (|t_1| \cdots |t_d|)^{p-2} \left| \widehat{f}(t) \right|^p dt \right)^{1/p} \lesssim \|f\|_{H_p}. \tag{20}$$

Since $\widehat{f}$ is a locally integrable function if $f \in H_p$ with $0 < p \leqslant 1$, the integral in Theorem 4 is well defined. Note that H_p is equivalent to L_p if $1 < p < \infty$. We also extend the inequality of Theorem 1 to Hardy spaces. Let

$$\mathcal{H}f := \mathcal{H}_0 f = \frac{1}{t_1 \cdots t_d} \int_0^{t_1} \cdots \int_0^{t_d} f(x_1, \ldots, x_d)\, dx_1 \ldots dx_d$$

be the Hardy–Cesàro operator.

Theorem 5. *If* $0 < p \leqslant 1$ *and* $f \in H_p \bigcap \bigcup_{1 \leqslant q \leqslant 2} L_q$*, then*

$$\left(\int_{\mathbb{R}^d} (|t_1| \cdots |t_d|)^{p-2} \left| \mathcal{H}\widehat{f}(t) \right|^p dt \right)^{1/p} \lesssim \|f\|_{H_p}. \tag{21}$$

Remark 2. (i) For $p = 1$, Theorem 5 holds for all $f \in H_1$.
(ii) Comparing the left-hand sides in (20) and (21), we first recall the classical reverse Hardy inequality (see [22, Theorem 347]): for any non-negative g,

$$\int_{\mathbb{R}^d} (|t_1| \cdots |t_d|)^{p-2} g^p(t)\, dt \lesssim \int_{\mathbb{R}^d} (|t_1| \cdots |t_d|)^{p-2} \left(\mathcal{H}g(t)\right)^p dt, \qquad 0 < p \leqslant 1. \tag{22}$$

Moreover, the direct Hardy inequality, which is the reverse inequality to (22), holds for $p = 1$ (see [22, Theorem 330]) but does not hold in general for $0 < p < 1$. To show this, consider

$$g(x) = \begin{cases} a_n, & b_n < x < b_n + d_n, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$I_1 = \int_0^\infty t^{p-2} |g(t)|^p\, dt = a_n^p \int_{b_n}^{b_n + d_n} t^{p-2}\, dt$$

and

$$I_2 = \int_0^\infty t^{p-2} |\mathcal{H}g(t)|^p \, dt \geqslant \int_{b_n+d_n}^\infty t^{-2} \, dt \left(\int_0^{b_n+d_n} g(z)dz \right)^p dt = \frac{a_n^p d_n^p}{b_n + d_n}.$$

Letting now $b_n + d_n \asymp b_n \nearrow \infty$, $\frac{b_n}{d_n} \nearrow \infty$, and

$$a_n^p = \left(\int_{b_n}^{b_n+d_n} t^{p-2} \, dt \right)^{-1},$$

we arrive at $a_n^p \asymp \frac{b_n^{2-p}}{d_n}$ and $I_1 \asymp 1$ but $I_2 \asymp (\frac{b_n}{d_n})^{1-p} \nearrow \infty$. In particular, g can be defined by $g(x) = 2^{n(\frac{2}{p}-1)} n^{-\frac{1}{p}}$ for $2^n < x < 2^n + n$ and zero otherwise.

Thus, in the case $\widehat{f}$ is nonnegative, (21) yields (20) for $0 < p < 1$, while for $p = 1$ they are equivalent. Without this condition on the Fourier transform, the left-hand sides of (20) and (21) are not comparable.

(iii) It is easy to see that inequality (22) is no longer true without the condition $g \geqslant 0$. For example, let

$$g(x) = \begin{cases} 0, & 0 < x < 1; \\ a_n, & 2n - 1 < x < 2n; \\ -a_n, & 2n < x < 2n + 1. \end{cases}$$

Then

$$I_1 = \int_0^\infty t^{p-2} |g(t)|^p \, dt \asymp \sum_n a_n^p n^{p-2}$$

but

$$I_2 = \int_0^\infty t^{p-2} |\mathcal{H}g(t)|^p \, dt \asymp \sum_n a_n^p n^{-2}.$$

Taking now $a_n = n^{(1-p)/p}$ for $n < N$ and $a_n = 0$ otherwise, we have $I_2 \asymp 1$ and $I_1 \asymp \ln N$.

Theorem 4 is known in the one-dimensional case, cf. (19) while Theorem 5 is new even in the one-dimensional case. We note that the proof of Theorem 4 was sketched in [24]. However, since we believe it contains some gaps, we present it in Section 6. The dual results to Theorems 4 and 5 are also proved, see Corollary 1 in Section 6.

1.3. Hardy's inequalities for averages

Let us recall the classical Hardy's inequalities. First,

$$\|\mathcal{H}_\varepsilon f\|_p \lesssim \|f\|_p, \quad 1 < p < \infty. \tag{23}$$

If all $\varepsilon_i = 0$ or all $\varepsilon_i = 1$, then even more is true. We just introduced the Hardy–Cesàro operator $\mathcal{H}f$ and now we define the Hardy–Bellman operator by

$$\mathcal{B}f := \mathcal{H}_1 f = \int_{|t_1|}^{\infty} \dots \int_{|t_d|}^{\infty} f(\operatorname{sign}(t_1)x_1, \dots, \operatorname{sign}(t_d)x_d) \frac{dx_1}{x_1} \dots \frac{dx_d}{x_d}.$$

Then

$$\|\mathcal{H}f\|_p \lesssim \|f\|_p \quad (1 < p \leqslant \infty) \quad \text{and} \quad \|\mathcal{B}f\|_p \lesssim \|f\|_p \quad (1 \leqslant p < \infty). \tag{24}$$

The inequalities in (23) and (24) can be proved by iteration using the corresponding one-dimensional results.

Remark 3. If $\widehat{f} \in L_r$ for some $1 < r \leqslant 2$ (say $f \in \mathcal{S}$), then the following estimates sharpen (23) for $1 < p \leqslant 2$:

$$\|\mathcal{H}_\varepsilon f\|_p \lesssim \left(\int_{\mathbb{R}^d} (|t_1| \dots |t_d|)^{p-2} \left|\widehat{f}(t)\right|^p dt \right)^{1/p} \lesssim \|f\|_p.$$

Here the first inequality follows from (17) and the second one from (1). In particular, if

$$|f(x)| \lesssim \sum_{\varepsilon \in E} |\mathcal{H}_\varepsilon f(x)|, \qquad x_i \neq 0 \tag{25}$$

or, more generally, if for some $1 < p \leqslant 2$

$$\left((|x_1| \dots |x_d|)^{-1} \int_x^{2x} \dots \int_x^{2x} |f(t)|^p \, dt \right)^{1/p} \lesssim \sum_{\varepsilon \in E} |\mathcal{H}_\varepsilon f(x)|, \qquad x_i \neq 0, \tag{26}$$

then

$$\|f\|_p \asymp \left(\int_{\mathbb{R}^d} (|t_1| \dots |t_d|)^{p-2} \left|\widehat{f}(t)\right|^p dt \right)^{1/p}.$$

Note that the integral of the p-power of the left hand side of (26) is $\|f\|_p^p$. For example, when $f \geqslant 0$ is non-increasing and even in each direction, then both (25) and (26) hold. Equivalences of this type are usually called Hardy–Littlewood theorems or Boas-type results and they have been previously obtained for monotone or general monotone functions; see [20].

Our third aim in this paper is to investigate Hardy's inequalities (24) for limiting cases. First, it is clear that

$$\|\mathcal{H}f\|_{BMO} \leqslant C\,\|\mathcal{H}f\|_{\infty} \leqslant C\,\|f\|_{\infty}.$$

We also note that the operator $\mathcal{H}$ is bounded in BMO, i.e.,

$$\|\mathcal{H}f\|_{BMO} \leqslant C\,\|f\|_{BMO}, \qquad f \in BMO \bigcap \bigcup_{1<q\leqslant\infty} L_q. \tag{27}$$

See [19,26] for one-dimensional functions. In the multivariate case, this follows from (28) below by duality, cf. Corollary 3.

However, the expected fact that the operator $\mathcal{H}$ is bounded in H_p for $p \leqslant 1$ is not true. For $p = 1$ it is enough to consider the function

$$a(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leqslant x \leqslant 1, \\ -\frac{1}{2}, & \text{if } 1 < x \leqslant 2, \\ 0, & \text{if } x \notin [0,2], \end{cases}$$

which is an atom (see Section 5) and note that $\int_{\mathbb{R}} \mathcal{H}a(x)\,dx = \ln 2 \neq 0$. Hence $\mathcal{H}a \notin H_1$ (see [19]). From this, $\mathcal{H}$ is not bounded in H_p, $p < 1$, since otherwise, by (24) and interpolation, we would obtain boundedness in H_1.

Even though $\mathcal{H}$ is not bounded in H_p, we derive the following weaker result, which can be considered as a generalization of inequality (24).

Theorem 6. *If $0 < p \leqslant 1$ and $f \in H_p \bigcap \bigcup_{1<q\leqslant 2} L_q$, then*

$$\|\mathcal{H}f\|_p \lesssim \|f\|_{H_p}.$$

For the operator $\mathcal{B}$ the situation is symmetric. It is easy to see that $\mathcal{B}$ is trivially bounded from H_1 to L_1 since

$$\|\mathcal{B}f\|_1 \leqslant C\,\|f\|_1 \leqslant C\,\|f\|_{H_1}.$$

In fact, it turns out that the operator $\mathcal{B}$ is bounded in H_p, i.e.,

$$\|\mathcal{B}f\|_{H_p} \lesssim \|f\|_{H_p}, \qquad 0 < p \leqslant 1, f \in H_p \bigcap \bigcup_{1\leqslant q\leqslant 2} L_q \tag{28}$$

(see [16,19,28,29] for one-dimensional functions and [17,40] for the multivariate case) but it is not bounded from BMO to BMO [19]. We obtain the following weaker estimate

$$\|\mathcal{B}f\|_{BMO} \leqslant C\,\|f\|_\infty\,, \qquad f\in L_\infty \bigcap \bigcup_{1\leqslant q<\infty} L_q;$$

see Corollary 2 below.

1.4. Structure of the paper

The paper is organized as follows. In Section 2 we introduce the Lorentz and net spaces. Section 3 contains the extension of the Hardy–Littlewood–Paley inequality to the range $1<p<\infty$ with the help of the net spaces. Here we prove Theorem 7 for the anisotropic Lorentz spaces, which implies Theorem 3. Section 4 is devoted to the proofs of Theorems 1 and 2. In Sections 5 and 6 we discuss the needed properties of the product Hardy spaces and prove Theorems 4–6, correspondingly.

2. Lorentz and net spaces

The L_p space is equipped with the quasi-norm

$$\|f\|_p := \left(\int_{\mathbb{R}^d} |f(x)|^p\,dx\right)^{1/p}, \qquad 0<p<\infty,$$

with the usual modification for $p=\infty$. Here we integrate with respect to the Lebesgue measure. For n quasi-normed spaces $X_1,\dots,X_d$ of one-dimensional functions, let us denote by $(X_1,\dots,X_d)$ the space consisting of n-dimensional measurable functions for which

$$\|f\|_{(X_1,\dots,X_d)} := \Big\|\dots\|f\|_{X_1}\dots\Big\|_{X_d} < \infty,$$

where the X_j norm is taken with respect the j-th variable.

The non-increasing rearrangement of a one-dimensional measurable function f is given by

$$f^*(t) := \inf\left\{\rho : |\{|f|>\rho\}| \leqslant t\right\}.$$

For a multi-dimensional measurable function and for fixed variables $y_1,\dots,y_{i-1},y_{i+1},\dots,y_d$, by $f^{*_i}(y_1,\dots,y_{i-1},\cdot,y_{i+1},\dots,y_d)$, we denote the non-increasing rearrangement with respect to the i-th variable ($i=1,\dots,n$). Let $*=(j_1,j_2,...,j_d)$ be a permutation of $(1,2,...,d)$. Applying the non-increasing rearrangement in all variables consecutively, we obtain

$$f^* = f^{*_{j_1},\dots,*_{j_d}} := \left(\left(f^{*_{j_1}}\right)^{*_{j_2}}\dots\right)^{*_{j_d}}.$$

Various function spaces defined with the help of iterative rearrangements were considered in many papers, see e.g. [1–3,8,31].

Let $\mathbf{p} = (p_1, \dots, p_d)$ and $\mathbf{q} = (q_1, \dots, q_d)$ with $0 < p_j < \infty$ and $0 < q_j \leqslant \infty$, $j = 1, 2 \dots n$. Let $* = (j_1, j_2, ..., j_d)$ be a permutation of $(1, 2, ..., d)$. In this case we will write $0 < \mathbf{p} < \infty$ and $0 < \mathbf{q} \leqslant \infty$. The Lorentz space $L^*_{\mathbf{p},\mathbf{q}}(\mathbb{R}^d)$ consists of all measurable functions f for which

$$\|f\|_{L^*_{\mathbf{p},\mathbf{q}}} := \left(\int_0^\infty \dots \left(\int_0^\infty \left(t_1^{\frac{1}{p_1}} \dots t_2^{\frac{1}{p_d}} f^{*_{j_1} \dots *_{j_d}}(t_1, \dots, t_d) \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \dots \frac{dt_d}{t_d} \right)^{\frac{1}{q_d}} < \infty,$$

where in the case $q_i = \infty$ the integral $\left(\int_0^\infty |g(t_i)|^{q_i} \frac{dt_i}{t_i} \right)^{\frac{1}{q_i}}$ is understood as $\sup_{t_i > 0} |g(t_i)|$.

In higher dimensions, this definition is different (see [38,31]) from the usual definition of Lorentz spaces, while, in the one-dimensional case, $L^*_{p,q}$ is the same as the classical Lorentz spaces $L_{p,q}$. Note also that the space $L^*_{\mathbf{p},\mathbf{q}}$ with $\mathbf{p} = \mathbf{q}$ does not coincide with the mixed Lebesgue space $(L_{p_1}, ..., L_{p_d})$. However, if $p_i = q_i = p,\ i = 1, 2, ..., n$, then $L^*_{\mathbf{p},\mathbf{q}} = L_p$.

We now define the net spaces [33] (see also [30,31]). Let us denote by M the collection of all rectangles $I = I_1 \times ... \times I_d$ of positive measure with sides parallel to the axes. For a measurable function f defined on $\mathbb{R}^d$, we define the average function by

$$\overline{f}(t_1, ..., t_d) := \sup_{I \in M, |I_i| \geqslant t_i} \frac{1}{|I_1| \dots |I_d|} \left| \int_I f(x)\, dx \right|.$$

A measurable function belongs to the net space $\mathcal{N}_{\mathbf{p},\mathbf{q}}(M)$ if

$$\|f\|_{\mathcal{N}_{\mathbf{p},\mathbf{q}}} = \left(\int_0^\infty \dots \left(\int_0^\infty \left(t_1^{\frac{1}{p_1}} \dots t_d^{\frac{1}{p_d}} \overline{f}(t_1, \dots, t_d) \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \dots \frac{dt_d}{t_d} \right)^{\frac{1}{q_d}} < \infty$$

for $0 < \mathbf{p}, \mathbf{q} \leqslant \infty$. Moreover, $\mathcal{N}_{\mathbf{p},\mathbf{q}}$ is a normed linear space.

For $p_i = p,\ q_i = q,\ i = 1, 2, ..., n$, we also use the notation

$$\|f\|_{\mathcal{N}_{p,q}} = \left(\int_0^\infty \dots \int_0^\infty \left((t_1 \dots t_d)^{\frac{1}{p}} \overline{f}(t_1, \dots, t_d) \right)^q \frac{dt_1}{t_1} \dots \frac{dt_d}{t_d} \right)^{\frac{1}{q}}$$

and, similarly,

$$\|f\|_{L^*_{p,q}} = \left(\int_0^\infty \dots \int_0^\infty \left((t_1 \dots t_d)^{\frac{1}{p}} f^{*_{j_1} \dots *_{j_d}}(t_1, \dots, t_d) \right)^q \frac{dt_1}{t_1} \dots \frac{dt_d}{t_d} \right)^{\frac{1}{q}}.$$

The next result follows easily from the monotonicity of $\overline{f}(t_1, ..., t_d)$ and the following Hardy's inequality: $\sum_{k=1}^{\infty} 2^{k\alpha} \left(\sum_{m=k}^{\infty} a_m\right)^q \asymp \sum_{k=1}^{\infty} 2^{k\alpha} a_k^q$ with $a_k \geqslant 0$ and $\alpha, q > 0$.

Lemma 1. *Let* $0 < p < \infty,\ 0 < q \leqslant \infty$, *then*

$$\|f\|_{\mathcal{N}_{p,q}} \asymp \left(\sum_{k\in\mathbb{Z}^d} \left(2^{\frac{k_1+\ldots k_d}{p}} \overline{f}(2^{k_1}, ..., 2^{k_d}) \right)^q \right)^{\frac{1}{q}}$$
$$\asymp \left(\sum_{k\in\mathbb{Z}^d} \left(2^{\frac{k_1+\ldots k_d}{p}} \sum_{m_d=k_d}^{\infty} \cdots \sum_{m_1=k_1}^{\infty} \overline{f}(2^{m_1}, ..., 2^{m_d}) \right)^q \right)^{\frac{1}{q}}.$$

3. Hardy–Littlewood–Paley inequality for $1 < p < \infty$

We start with the following extension of the Hardy–Littlewood–Paley inequality. For the case $p_i = p,\ q_i = q,\ i = 1, 2, ..., n$, we recover Theorem 3.

Theorem 7. *Let* $* = (j_1, j_2, ..., j_d)$ *be a permutation of* $(1, 2, ..., d)$. *If* $1 < \mathbf{p} < \infty,\ 0 < \mathbf{q} \leqslant \infty$ *and* $f \in L^*_{\mathbf{p},\mathbf{q}}$, *then* $\mathfrak{F}_N f \in \mathcal{N}_{\mathbf{p}',\mathbf{q}}$ *and there holds, uniformly in* $N \in \mathbb{N}$,

$$\left\|\mathfrak{F}_N f\right\|_{\mathcal{N}_{\mathbf{p}',\mathbf{q}}} \lesssim \left\|f\right\|_{L^*_{\mathbf{p},\mathbf{q}}}. \tag{29}$$

In particular, for $1 < p < \infty$,

$$\left(\int_0^{\infty} \cdots \int_0^{\infty} (t_1 \ldots t_d)^{p-2} \left|\overline{\mathfrak{F}_N f}(t_1, ..., t_d)\right|^p \, dt \right)^{1/p} \lesssim \|f\|_{L_p}. \tag{30}$$

In order to prove this result, we obtain the following interpolation theorem, which, in turn, is based on Theorems 1 and 2 in [31].

Lemma 2. *Let* $0 < \mathbf{p}_0 = (p_1^0, ..., p_d^0) < \mathbf{p_1} = (p_1^1, ..., p_d^1) < \infty,\ 0 < \mathbf{q}_0 = (q_1^0, ..., q_d^0), \mathbf{q_1} = (q_1^1, ..., q_d^1) < \infty,\ q_i^0 \neq q_i^1,\ \ i = 1, ..., n$, *and* $0 < \mathbf{r} = (r_1, ..., r_d) \leqslant \infty$. *Let* $* = (j_1, j_2, ..., j_d)$ *be a permutation of* $(1, 2, ..., d)$. *Suppose* $\mathbf{p}_\varepsilon = (p_1^{\varepsilon_1}, ..., p_d^{\varepsilon_d})$ *and* $\mathbf{q}_\varepsilon = (q_1^{\varepsilon_1}, ..., q_d^{\varepsilon_d})$. *If* T *is a linear operator such that for all* $\varepsilon \in E$,

$$T : (L_{p_{j_1}^{\varepsilon_{j_1}},1}, ..., L_{p_{j_d}^{\varepsilon_{j_d}},1}) \;\to\; \mathcal{N}_{\mathbf{q}_\varepsilon,\infty},$$

then

$$T : L^*_{\mathbf{p},\mathbf{r}} \to N_{\mathbf{q},\mathbf{r}}, \tag{31}$$

where $\bar{\theta} = (\theta_1, ..., \theta_d) \in (0, 1)^d$ *and*

$$\frac{1}{\mathbf{p}} = \frac{1-\bar{\theta}}{\mathbf{p_0}} + \frac{\bar{\theta}}{\mathbf{p_1}}, \quad \frac{1}{\mathbf{q}} = \frac{1-\bar{\theta}}{\mathbf{q_0}} + \frac{\bar{\theta}}{\mathbf{q_1}}.$$

Proof. By Lemma 4 (c) in [31], we have that

$$T : (\mathbf{A_0}, \mathbf{A_1})^*_{\bar{\theta},\mathbf{r}} \to (\mathbf{B_0}, \mathbf{B_1})^*_{\bar{\theta},\mathbf{r}},$$

where

$$\mathbf{A_0} = (L_{p^0_{j_1},1}, ..., L_{p^0_{j_d},1}), \qquad \mathbf{A_1} = (L_{p^1_{j_1},1}, ..., L_{p^1_{j_d},1}),$$

and

$$\mathbf{B_0} = \mathcal{N}_{(q^0_1,...,q^0_d),(\infty,...,\infty)}, \qquad \mathbf{B_1} = \mathcal{N}_{(q^1_1,...,q^1_d),(\infty,...,\infty)}.$$

Taking into account [31, Theorem 1], we obtain

$$\left(\mathcal{N}_{(q^0_1,...,q^0_d),(\infty,...,\infty)}, \mathcal{N}_{(q^1_1,...,q^1_d),(\infty,...,\infty)}\right)^*_{\bar{\theta},\mathbf{r}} \hookrightarrow \mathcal{N}_{\mathbf{q},\mathbf{r}},$$

where $\frac{1}{\mathbf{q}} = \frac{1-\bar{\theta}}{\mathbf{q_0}} + \frac{\bar{\theta}}{\mathbf{q_1}}$, $\bar{\theta} = (\theta_1, ..., \theta_d)$ with $0 < \theta_i < 1$. Finally, [31, Theorem 2] implies that

$$L^*_{\mathbf{p},\mathbf{r}} \hookrightarrow (\mathbf{A_0}, \mathbf{A_1})^*_{\bar{\theta},\mathbf{r}}.$$

Combining the above estimates, we arrive at (31). □

Proof of Theorem 7. We estimate the $\mathcal{N}_{\mathbf{p}',\infty}$-norm of $\widehat{f}$ as follows:

$$\begin{aligned}
\|\mathfrak{F}_N f\|_{\mathcal{N}_{\mathbf{p}',\infty}} &\leqslant \sup_{I\in M} |I_1|^{-1/p_1} \dots |I_d|^{-1/p_d} \left| \int_{I_1} \dots \int_{I_d} \mathfrak{F}_N f(\xi_1, ..., \xi_d)\, d\xi \right| \\
&\leqslant \sup_{I\in M} \int_{Q_N} |f(x)| \prod_{i=1}^{d} \left(|I_i|^{-1/p_i} \left| \int_{I_i} e^{-\imath \xi_i x_i}\, d\xi_i \right| \right) dx \\
&\leqslant \sup_{I\in M} \int_{\mathbb{R}^d} |f(x)| \prod_{i=1}^{d} \left(|I_i|^{-1/p_i} \left| \int_{I_i} e^{-\imath \xi_i x_i}\, d\xi_i \right| \right) dx.
\end{aligned}$$

Let

$$\varphi_i(x) = |I_i|^{-1/p_i} \left| \int_{I_i} e^{-\imath \xi_i \cdot x_i}\, d\xi_i \right|, \qquad i = 1, \dots, n.$$

If $I_i = [a_i, b_i]$, then

$$\left|\int\limits_{I_i} e^{-\imath\xi_i\cdot x_i}\,d\xi_i\right| = \frac{2}{|x_i|}\left|\sin((b_i-a_i)x_i/2)\right| \leqslant 2\min\left(\frac{b_i-a_i}{2}, \frac{1}{|x_i|}\right)$$

and so

$$\varphi_i^*(t_i) \leqslant \frac{4}{(b_i-a_i)^{1/p_i}}\min\left(\frac{b_i-a_i}{2}, \frac{1}{t_i}\right) \leqslant 2t_i^{-1/p_i'}.$$

Using the Hardy–Littlewood–Pólya inequality for rearrangements (see, e.g., [5, p. 7])

$$\int\limits_{-\infty}^{\infty} g(x)\varphi(x)\,dx \leqslant \int\limits_{0}^{\infty} g^*(t)\varphi^*(t)\,dt$$

n-times, we conclude that

$$\begin{aligned}\|\mathfrak{F}_N f\|_{\mathcal{N}_{\mathbf{p}',\infty}} &\leqslant 2^d \int\limits_0^\infty t_d^{-1/p_d'}\left(\cdots\int\limits_0^\infty t_{j_1}^{-1/p_{j_1}'} f^{*_{j_1}}(\cdot, t_{j_1}, \cdot)\,\frac{dt_{j_1}}{t_{j_1}}\cdots\right)_{t_{j_d}}^{*_{j_d}} \frac{dt_{j_d}}{t_{j_d}} \\ &= 2^d \left\|\cdots\|f\|_{L_{p_{j_1},1}}\cdots\right\|_{L_{p_{j_d},1}} = 2^d\,\|f\|_{(L_{p_{j_1},1},\dots,L_{p_{j_d},1})}.\end{aligned}$$

Using this inequality for $1 < \mathbf{p}_0 < \mathbf{p} < \mathbf{p}_1 < \infty$ and Lemma 2, we derive that

$$\|\mathfrak{F}_N f\|_{\mathcal{N}_{\mathbf{p}',\mathbf{q}}} \lesssim \|f\|_{L^*_{\mathbf{p},\mathbf{q}}}.$$

Setting $p = p_i = q_i,\ i = 1, ..., n$ in (29), we immediately obtain inequality (30). $\square$

4. Proofs of Theorems 1 and 2

First we prove the following lemmas.

Lemma 3. *For a locally integrable function f and for $I = I_1 \times ... \times I_d \in M$ with $|I_i| \leqslant t_i$, we have*

$$\frac{1}{t_1\cdots t_d}\left|\int\limits_I f(x)\,dx\right| \leqslant C_d\,\overline{f}\left(\frac{t_1}{2}, \cdots, \frac{t_d}{2}\right).$$

Proof. Let $I = I_1 \times ... \times I_d = I_1 \times I'$, where $I' \subset \mathbb{R}^{n-1}$. Consider a corresponding rectangular parallelepiped $J_1 \times I'$, where J_1 is defined as follows.

If $|I_1| \geqslant \frac{t_1}{2}$, then define $J_1 := I_1$. In other cases, let $I_1 = [a, b]$, $|I_1| = b - a < \frac{t_1}{2}$ and $Q_1 = [a,\, a + t_1]$. If

$$\left|\int\limits_{I_1\times I'} f(x)dx\right| \leqslant 2\left|\int\limits_{Q_1\times I'} f(x)dx\right|,$$

then set $J_1 := Q_1$. If

$$\left|\int\limits_{I_1\times I'} f(x)dx\right| > 2\left|\int\limits_{Q_1\times I'} f(x)dx\right|,$$

then set $J_1 := [b, a+t_1]$. Hence,

$$\begin{aligned}\left|\int\limits_{J_1\times I'} f(x)dx\right| &= \left|\int\limits_{Q_1\times I'} f(x)dx - \int\limits_{I_1\times I'} f(x)dx\right| \\ &\geqslant \left|\int\limits_{I_1\times I'} f(x)dx\right| - \left|\int\limits_{Q_1\times I'} f(x)dx\right| \geqslant \frac{1}{2}\left|\int\limits_{I_1\times I'} f(x)dx\right|.\end{aligned}$$

Taking into account that $t \geqslant |J_1| \geqslant \frac{t_1}{2}$, we derive that

$$\left|\int\limits_{I_1\times I'} f(x)dx\right| \leqslant 2\left|\int\limits_{J_1\times I'} f(x)dx\right|.$$

In this way, in n steps, we get the parallelepiped $J = J_1 \times ... \times J_d$ such that $\frac{t_i}{2} \leqslant |J_i| \leqslant t_i$ and

$$\left|\int\limits_{I} f(x)dx\right| \leqslant 2^d\left|\int\limits_{J} f(x)dx\right|.$$

Finally, we have

$$\frac{1}{t_1\cdots t_d}\left|\int\limits_{I} f(x)\,dx\right| \leqslant 2^d\frac{1}{t_1\cdots t_d}\left|\int\limits_{J} f(x)\,dx\right| \leqslant 2^d\,\overline{f}(\frac{t_1}{2},\cdots,\frac{t_d}{2})$$

which shows the lemma. □

Lemma 4. *For a locally integrable function f satisfying*

$$\sum_{m_d=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\overline{f}(2^{m_1},\ldots,2^{m_d}) < \infty,$$

for any $k \in \mathbb{Z}^d$ and $\varepsilon \in E$, there holds

$$\sup_{\substack{2^{k_i} \leqslant |t_i| \leqslant 2^{k_i+1} \\ i=1,\cdots,n}} |\mathcal{H}_\varepsilon f(t)| \leqslant 2^d \sum_{m_d=k_d}^{\infty} \ldots \sum_{m_1=k_1}^{\infty} \overline{f}(2^{m_1-1}, \ldots, 2^{m_d-1}).$$

Proof. For $t \in \mathbb{R}^d$, the operator H_ε can be represented as

$$\mathcal{H}_\varepsilon f(t) = \int_0^\infty \ldots \int_0^\infty f(t_1\varepsilon_1 + x_1 \operatorname{sgn} t_1, \ldots, t_d\varepsilon_d + x_d \operatorname{sgn} t_d) \prod_{i=1}^d \psi_{i,\varepsilon_i}(x_i) dx,$$

with

$$\psi_{i,0}(x_i) = \begin{cases} \frac{1}{t_i}, & \text{if } 0 \leqslant x_i \leqslant |t_i|, \\ 0, & \text{if } x_i > |t_i|, \end{cases} \qquad \text{and} \qquad \psi_{i,1}(x_i) = \frac{1}{|t_i| + x_i}.$$

Let $2^{k_i} \leqslant |t_i| < 2^{k_i+1}$, $i = 1, \ldots, n$, then

$$\mathcal{H}_\varepsilon f(t) = \sum_{m_d=k_d}^{\infty} \ldots \sum_{m_1=k_1}^{\infty} \int_{2^{m_d}-2^{k_d}}^{2^{m_d+1}-2^{k_d}} \ldots \int_{2^{m_1}-2^{k_1}}^{2^{m_1+1}-2^{k_1}} f(t_1\varepsilon_1 + x_1 \operatorname{sgn} t_1, \ldots, t_d\varepsilon_d + x_d \operatorname{sgn} t_d) \prod_{i=1}^d \psi_{i,\varepsilon_i}(x_i) dx.$$

Then the mean value theorem gives

$$\mathcal{H}_\varepsilon f(t) = \sum_{m_d=k_d}^{\infty} \ldots \sum_{m_1=k_1}^{\infty} \prod_{i=1}^d \psi_{i,\varepsilon_i}(2^{m_i} - 2^{k_i}) \int_{I_m} f(x) dx,$$

where $I_m = I_{m_1} \times \ldots \times I_{m_d}$ is a parallelepiped such that $|I_{m_i}| \leqslant 2^{m_i}$. Since

$$\psi_{i,\epsilon_i}(2^{m_i} - 2^{k_i}) \leqslant 2^{-m_i} \qquad (i = 1, \ldots, n),$$

Lemma 3 completes the proof:

$$\begin{aligned} |\mathcal{H}_\varepsilon f(t)| &= \left| \sum_{m_d=k_d}^{\infty} \ldots \sum_{m_1=k_1}^{\infty} \prod_{i=1}^d \psi_{i,\varepsilon_i}(2^{m_i} - 2^{k_i}) \int_{I_m} f(x) dx \right| \\ &\leqslant \sum_{m_d=k_d}^{\infty} \ldots \sum_{m_1=k_1}^{\infty} 2^{-m_1 - \ldots - m_d} \left| \int_{I_m} f(x) dx \right| \\ &\leqslant 2^d \sum_{m_d=k_d}^{\infty} \ldots \sum_{m_1=k_1}^{\infty} \overline{f}(2^{m_1-1}, ..., 2^{m_d-1}). \quad \square \end{aligned}$$

Lemma 5. *Let $0 < p < \infty$ and $0 < q \leqslant \infty$. Then for any Cauchy sequence $\{f_m\}_m$ from $\mathcal{N}_{p,q}$ and any $\varepsilon \in E$ there exists the limit*

$$\lim_{m\to\infty} (\mathcal{H}_\varepsilon f_m)(t), \qquad t \in \mathbb{R}^d.$$

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{N}_{p,q}$. Let $\varepsilon > 0$, there exists N such that for $m > N$ and $r \in \mathbb{N}$ one has

$$\|f_m - f_{m+r}\|_{\mathcal{N}_{p,q}} < \varepsilon. \tag{32}$$

Then, by $\mathcal{N}_{p,q} \hookrightarrow \mathcal{N}_{p,\infty}$, we derive that

$$\sup_{e\in M} \frac{1}{|e|^{\frac{1}{p'}}} \left| \int_e (f_m - f_{m+r})\, d\mu \right| \lesssim \varepsilon, \qquad m > N.$$

Thus, for any $e \in M$, $\left\{\int_e f_m d\mu\right\}_{m=1}^{\infty}$ is the Cauchy sequence, which implies that there exists

$$\lim_{n\to\infty} \int_e f_m d\mu. \tag{33}$$

Since $f_m \in \mathcal{N}_{p,q}$, Lemma 1 yields that

$$\sum_{m_d=1}^{\infty} \dots \sum_{m_1=1}^{\infty} \overline{f}(2^{m_1}, ..., 2^{m_d}) < \infty.$$

Let $t \in \mathbb{R}^d$ and $2^{k_i} \leqslant t_i < 2^{k_i+1}$. By Lemma 4, there is N such that

$$|\left(\mathcal{H}_\varepsilon(f_m - f_{m+r})\right)(t)| \leqslant 2^{d+1} \sum_{m_d=k_d-1}^{N} \dots \sum_{m_1=k_1-1}^{N} \overline{(f_m - f_{m+r})}(2^{m_1}, ..., 2^{m_d}).$$

From the definition of $\overline{f}(t_1, ..., t_d)$, there are $Q_m \in M$ satisfying

$$|\left(\mathcal{H}_\varepsilon(f_m - f_{m+r})\right)(t)| \leqslant 2^{d+2} \sum_{m_d=k_d-1}^{N} \dots \sum_{m_1=k_1-1}^{N} \frac{1}{|Q_m|} \left| \int_{Q_m} f_m(x) - f_{m+r}(x) dx \right|.$$

Finally, taking into account that the limit (33) exists, the sequence $\{(\mathcal{H}_\varepsilon f_m)(t)\}_m$ is a Cauchy sequence and, therefore, convergent. □

Proof of Theorem 1. Denote $\frac{1}{p} = \alpha + \frac{1}{r}$. Then we have that $1 < p \leqslant r$ and $\beta = \frac{1}{p'} - \frac{1}{q}$. Let $* = (j_1, j_2, ..., j_d)$ be a permutation of $(1, 2, ..., d)$. Due to the embedding $L^*_{p,r} \subset L^*_{p,q}$, taking into account $\frac{1}{p} - \frac{1}{r} \geqslant 0$ and condition (9), we have

$$\|f\|_{L^*_{p,q}} \lesssim \|f\|_{L^*_{p,r}} = \left(\int_{\mathbb{R}^d} \left((|t_1| \dots |t_d|)^{\frac{1}{p} - \frac{1}{r}} f^{*_{j_1} \dots *_{j_d}}(t) \right)^r dt \right)^{1/r}. \tag{34}$$

By Theorem 3 and (34),

$$\|\mathfrak{F}_N f - \mathfrak{F}_{N+r} f\|_{\mathcal{N}_{p',q}}$$
$$\lesssim \left(\int_{\mathbb{R}^d} \left((|t_1| \dots |t_d|)^{\alpha} |(f\chi_{Q_{N+r} \setminus Q_N})(t)| \right)^r dt \right)^{1/r} \to 0 \quad \text{as} \quad N \to +\infty.$$

Thus, $\{\mathfrak{F}_N f\}$ is a Cauchy sequence in $\mathcal{N}_{p',q}$, and using Lemma 5, there exists

$$\lim_{N \to +\infty} (\mathcal{H}_\varepsilon \mathfrak{F}_N f)(t).$$

Discretizing the integral yields

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left((|t_1| \dots |t_d|)^{\beta} \left| (\mathcal{H}_\varepsilon \mathfrak{F}_N f)(t) \right| \right)^q dt \\
&= \sum_{\delta \in E} \int_{\mathbb{R}^d} \left((|t_1| \dots |t_d|)^{\frac{1}{p'} - \frac{1}{q}} \left| (\mathcal{H}_\varepsilon \mathfrak{F}_N f)\left((-1)^{\delta_1} t_1, \dots, (-1)^{\delta_d} t_d\right) \right| \right)^q dt \\
&= \sum_{\delta \in E} \sum_{k_d = -\infty}^{\infty} \dots \sum_{k_1 = -\infty}^{\infty} \\
&\quad \int_{2^{k_d}}^{2^{k_d+1}} \dots \int_{2^{k_1}}^{2^{k_1+1}} \left((|t_1| \dots |t_d|)^{\frac{1}{p'} - \frac{1}{q}} \left| (\mathcal{H}_\varepsilon \mathfrak{F}_N f)\left((-1)^{\delta_1} t_1, \dots, (-1)^{\delta_d} t_d\right) \right| \right)^q dt.
\end{aligned}$$

Using Lemmas 4, 1, Theorem 3 and (34), we continue as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left((|t_1| \dots |t_d|)^{\beta} \left| (\mathcal{H}_\varepsilon \mathfrak{F}_N f)(t) \right| \right)^q dt \\
&\lesssim \sum_{k_d = -\infty}^{\infty} \dots \sum_{k_1 = -\infty}^{\infty} \left(2^{\frac{k_1 + \dots + k_d}{p'}} \sum_{m_d = k_d - 1}^{\infty} \dots \sum_{m_1 = k_1 - 1}^{\infty} \overline{\mathfrak{F}_N f}(2^{m_1}, \dots, 2^{m_d}) \right)^q \\
&\asymp \|\mathfrak{F}_N f\|^q_{\mathcal{N}_{p',q}} \lesssim \|f\|^q_{L^*_{p,q}}.
\end{aligned}$$

Now (10) follows from (34) and from Fatou's lemma.

If, additionally, (11) holds, then we can use the Hardy–Littlewood–Pólya inequality for rearrangements (see, e.g., [5, p. 7])

$$\int_0^\infty g^*(t)\frac{1}{(1/\varphi)^*(t)}\,dt \leqslant \int_{-\infty}^\infty g(x)\varphi(x)\,dx$$

to obtain

$$\left(\int_{\mathbb{R}^d}\Big((|t_1|\dots|t_d|)^{\frac{1}{p}-\frac{1}{r}}f^{*_{j_1}\dots *_{j_d}}(t)\Big)^r\,dt\right)^{1/r} \lesssim \left(\int_{\mathbb{R}^d}\big((|t_1|\dots|t_d|)^{\alpha}|f(t)|\big)^r\,dt\right)^{1/r} < \infty,$$

which completes the proof. □

Proof of Theorem 2. Since by (24), $\mathcal{H}_\varepsilon$ in bounded on L_s for $1 < s < \infty$, we have

$$\langle \mathcal{H}_\varepsilon f, g\rangle = \langle f, \mathcal{H}_{1-\varepsilon} g\rangle \tag{35}$$

for $f \in L_s$ and $g \in \mathcal{S}$. Suppose now that f satisfies condition (15). Then, by Remark 1, $f \in L_s^{loc}$ for some $1 < s < \min(r, 2)$ and so $\mathfrak{F}_N f \in L_{s'}$. Thus (35) can be applied for $\mathfrak{F}_N f$.

Let us introduce the weighted Lebesgue space $L_q(\mathbb{R}^d, \beta)$ by the norm

$$\|f\|_{L_q(\beta)} := \left(\int_{\mathbb{R}^d}\big((|t_1|\dots|t_d|)^{\beta}\,|f(t)|\big)^q\,dt\right)^{1/q}.$$

Since the Schwartz space $\mathcal{S}$ is dense in $L_{q'}(-\beta)$, we have

$$\begin{aligned}\|\mathcal{H}_\varepsilon \mathfrak{F}_N f\|_{L_q(\beta)} &= \sup_{\|g\|_{L_{q'}(-\beta)}\leqslant 1}\langle \mathcal{H}_\varepsilon \mathfrak{F}_N f, g\rangle \\ &= \sup_{\|g\|_{L_{q'}(-\beta)}\leqslant 1}\langle \mathfrak{F}_N f, \mathcal{H}_{1-\varepsilon} g\rangle = \sup_{\|g\|_{L_{q'}(-\beta)}\leqslant 1}\langle f\chi_{Q_N}, \widehat{\mathcal{H}_{1-\varepsilon} g}\rangle,\end{aligned}$$

where $g \in \mathcal{S}$. Now we show that $\widehat{\mathcal{H}_{1-\varepsilon} g} = \mathcal{H}_\varepsilon \widehat{g}$. It is enough to prove this for one dimension and for the operator $\mathcal{H}$. Indeed, since $\mathcal{H}g \in L_s$ for a given $1 < s \leqslant 2$,

$$\begin{aligned}\widehat{\mathcal{H}g}(y) &= \lim_{N\to\infty}\mathfrak{F}_N(\mathcal{H}g)(y) = \lim_{N\to\infty}\int_{Q_N}\frac{1}{x}\int_0^x g(z)\,dz\,e^{-ixy}\,dx \\ &= \lim_{N\to\infty}\int_{Q_N}\int_0^1 g(\xi x)\,d\xi\,e^{-ixy}\,dx \\ &= \lim_{N\to\infty}\int_0^1\int_{Q_N} g(\xi x)\,e^{-ixy}\,dx\,d\xi\end{aligned}$$

$$= \int_0^1 \int_{\mathbb{R}} g(\xi x)\, e^{-ixy}\, dx\, d\xi, \tag{36}$$

where the limit denotes the $L_{s'}$-limit. The last equation comes from

$$\left\| \int_0^1 \left(\int_{\mathbb{R}} - \int_{Q_N} \right) g(\xi x)\, e^{-ix\cdot}\, dx\, d\xi \right\|_{s'} \leqslant \int_0^1 \left\| \left(\int_{\mathbb{R}} - \int_{Q_N} \right) g(\xi x)\, e^{-ix\cdot}\, dx \right\|_{s'} d\xi \to 0 \tag{37}$$

as $N \to \infty$. Indeed, by Hausdorff-Young inequality,

$$\left\| \left(\int_{\mathbb{R}} - \int_{Q_N} \right) g(\xi x)\, e^{-ix\cdot}\, dx \right\|_{s'} = \left\| \widehat{G\chi_{\mathbb{R}\setminus Q_N}} \right\|_{s'} \leqslant \left\| G\chi_{\mathbb{R}\setminus Q_N} \right\|_s \to 0$$

as $N \to \infty$ and $\xi \in (0,1)$, where $G(x) := g(\xi x)$. On the other hand

$$\left\| G\chi_{\mathbb{R}\setminus Q_N} \right\|_s \leqslant \|G\|_s = \xi^{-1/q} \|g\|_s$$

which is integrable on $(0,1)$ with respect to ξ. Now Lebesgue dominated convergence theorem implies (37). Changing the variables in (36), we get that

$$\widehat{\mathcal{H}g}(y) = \int_0^1 \int_{\mathbb{R}} g(s)\, e^{-iys/\xi}\, ds\, \frac{d\xi}{\xi} = \int_0^1 \widehat{g}(y/\xi)\, \frac{d\xi}{\xi} = \mathcal{B}\widehat{g}(y).$$

Using this and Hölder's inequality, we arrive at

$$\begin{aligned}
&\|\mathcal{H}_\varepsilon \mathfrak{F}_N f\|_{L_q(\beta)} \\
&= \sup_{\|g\|_{L_{q'}(-\beta)} \leqslant 1} \langle f\chi_{Q_N}, \mathcal{H}_\varepsilon \widehat{g} \rangle \\
&\leqslant \sup_{\|g\|_{L_{q'}(-\beta)} \leqslant 1} \int_{\mathbb{R}^d} (|t_1| \dots |t_d|)^\alpha \, |f(t)\chi_{Q_N}(t)| \, (|t_1| \dots |t_d|)^{-\alpha} \, |\mathcal{H}_\varepsilon \widehat{g}(t)| \, dt \\
&\leqslant \sup_{\|g\|_{L_{q'}(-\beta)} \leqslant 1} \left(\int_{\mathbb{R}^d} \left((|t_1| \dots |t_d|)^\alpha \, |f(t)\chi_{Q_N}(t)| \right)^r dt \right)^{1/r} \\
&\qquad \left(\int_{\mathbb{R}^d} \left((|t_1| \dots |t_d|)^{-\alpha} \, |\mathcal{H}_\varepsilon \widehat{g}(t)| \right)^{r'} dt \right)^{1/r'}.
\end{aligned}$$

Now, by Theorem 1, there holds

$$\|\mathcal{H}_\varepsilon \mathfrak{F}_N f\|_{L_q(\beta)} \lesssim \left(\int_{\mathbb{R}^d} \left((|t_1| \dots |t_d|)^\alpha |f(t)\chi_{Q_N}(t)| \right)^r dt \right)^{1/r}.$$

Finally, the theorem follows from a density argument. □

5. Hardy spaces and atoms

Let us choose a one-dimensional Schwartz function ϕ such that $\int_{\mathbb{R}} \phi \, dx \neq 0$. Then we say that a tempered distribution f is in the product Hardy space $H_p = H_p(\mathbb{R} \times \cdots \times \mathbb{R})$ $(0 < p < \infty)$ if

$$\|f\|_{H_p} := \left\| \sup_{t_1 > 0, \dots, t_d > 0} |(f * (\phi_{t_1} \otimes \cdots \otimes \phi_{t_d})| \right\|_p < \infty,$$

where $*$ denotes the convolution, $\phi_s(y) := s^{-1}\phi(y/s)$ $(s > 0, y \in \mathbb{R})$ and

$$(\phi_{t_1} \otimes \cdots \otimes \phi_{t_d})(x) := \prod_{j=1}^d \phi_{t_j}(x_j), \qquad x \in \mathbb{R}^d.$$

It is known that different Schwartz functions yield equivalent norms. Moreover, H_p is equivalent to L_p for $1 < p < \infty$. For more about Hardy spaces see [36,21].

By a *dyadic interval* we mean one of the form $(k2^{-n}, (k+1)2^{-n})$. For each dyadic interval I let I^r $(r \in \mathbb{N})$ be the dyadic interval for which $I \subset I^r$ and $|I^r| = 2^r|I|$. If $R := I_1 \times \cdots \times I_d$ is a dyadic rectangle, then set $R^r := I_1^r \times \cdots \times I_d^r$.

For each dyadic interval I we define $\overline{I} := \left\{ x \in \mathbb{R} : |x| \in \left(|I|^{-1}, \infty \right) \right\}$. Obviously, $I \subset J$ implies $\overline{I} \subset \overline{J}$. For a dyadic rectangle $R = I_1 \times \cdots \times I_d$ let $\overline{R} = \overline{I_1} \times \cdots \times \overline{I_d}$. If $F \subset \mathbb{R}^d$ is a measurable set, then let

$$\overline{F} := \bigcup_{R \subset F, R \text{ is dyadic}} \overline{R}.$$

It is clear that $F_1 \subset F_2$ implies $\overline{F_1} \subset \overline{F_2}$.

Let us introduce the concept of simple p-atoms. A function $a \in L_2$ is called a *simple p-atom* if there exist $I_i \subset \mathbb{R}$ dyadic intervals, $i = 1, \dots, j$ for some $1 \leqslant j \leqslant d-1$, such that

(i) $\operatorname{supp} a \subset I_1 \times \ldots \times I_j \times A$ for some open bounded set $A \subset \mathbb{R}^{d-j}$,

(ii)

$$\|a\|_2 \leqslant (|I_1| \cdots |I_j||A|)^{1/2 - 1/p},$$

(iii)
$$\int_{\mathbb{R}} a(x)x_i^k\,dx_i = \int_A a\,d\lambda = 0$$

for all $i = 1,\ldots,j$, $k = 0,\ldots,N = \lfloor 2/p - 3/2 \rfloor$ and almost every fixed $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_d$.

If $j = d-1$, we may suppose that $A = I_d$ is also a dyadic interval. Of course if $a \in L_2$ satisfies these conditions for another subset of $\{1,\ldots,d\}$ than $\{1,\ldots,j\}$, then it is also called simple p-atom.

Although not every function in H_p can be decomposed into simple p-atoms [10], the following result holds.

Lemma 6. *Let η be a measure on the Lebesgue measurable sets of $\mathbb{R}^d$ satisfying*

$$\eta(\overline{F}) \leqslant C|F| \qquad \text{for all open bounded } F \subset \mathbb{R}^d. \tag{38}$$

Let $0 < p \leqslant 1$, $V : L_q \to L_s$ be a bounded linear operator for some $1 \leqslant q \leqslant 2$, $1 \leqslant s \leqslant \infty$ and

$$Tf(t) = \left(\prod_{j=1}^d t_j\right)^i Vf(t), \qquad t \in \mathbb{R}^d, i = 0,1.$$

Suppose that there exist $\eta_1,\ldots,\eta_d > 0$ such that for every simple p-atom a and for every $r_1\ldots,r_d \in \mathbb{P}$,

$$\int_{\left(\mathbb{R}\setminus\overline{I_1^{r_1}}\right)\times\cdots\times\left(\mathbb{R}\setminus\overline{I_j^{r_j}}\right)} \int_{\overline{A}} |Ta|^p\,d\eta \lesssim 2^{-\eta_1 r_1}\cdots 2^{-\eta_j r_j}, \tag{39}$$

where $I_1\times\ldots\times I_j\times A$ is the support of a. If $j = d-1$ and $A = I_d$ is a dyadic interval, then we also assume that

$$\int_{\left(\mathbb{R}\setminus\overline{I_1^{r_1}}\right)\times\cdots\times\left(\mathbb{R}\setminus\overline{I_{d-1}^{r_{d-1}}}\right)} \int_{\left(\overline{I_d}\right)^c} |Ta|^p\,d\eta \lesssim 2^{-\eta_1 r_1}\cdots 2^{-\eta_{d-1} r_{d-1}}. \tag{40}$$

If T is bounded from $L_2(\mathbb{R}^d)$ to $L_2(\mathbb{R}^d,\eta)$, then

$$\|Tf\|_{L_p(\mathbb{R}^d,\eta)} \lesssim \|f\|_{H_p}, \qquad f \in H_p \cap L_q. \tag{41}$$

If $\lim_{k\to\infty} f_k = f$ in H_p-norm implies that $\lim_{k\to\infty} Vf_k = Vf$ in the sense of tempered distributions, then (41) holds for all $f \in H_p$.

Note that $H_p \cap L_q$ is dense in H_p. We omit the proof because it is exactly the same as those of Theorems 3.6.12 and 1.8.1 in [41] (see also [39]). The only difference is that we have to apply (38). In [41], we supposed that $\overline{F} = F$ and η is the Lebesgue measure λ. For $d = 2$ and $\overline{F} = F$, $\eta = \lambda$, the lemma was shown in Fefferman [14] in a different version. However, that version does not hold for higher dimensions. For $d \geqslant 3$, the present lemma is due to the last author [41]. Applying Lemma 6, we can prove Theorems 4 and 5.

6. Proofs of Theorems 4–6

Proof of Theorem 4. Let us introduce the measure

$$\eta(A) = \int_A \prod_{j=1}^{d} t_j^{-2}\, dt, \qquad A \subset \mathbb{R}^d, \tag{42}$$

and the operator

$$Tf(t) = \left(\prod_{j=1}^{d} t_j \right) \widehat{f}(t), \qquad t \in \mathbb{R}^d.$$

We say that $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $m = (m_1, \ldots, m_d) \in \mathbb{N}^d$ are incomparable if neither $n \leqslant m$ nor $m \leqslant n$ hold. Let us denote by $\mathcal{F}_n$ $(n \in \mathbb{Z}^d)$ the set of all dyadic rectangles

$$I = (k_1 2^{-n_1}, (k_1+1) 2^{-n_1}) \times \cdots \times (k_d 2^{-n_d}, (k_d+1) 2^{-n_d}), \qquad k \in \mathbb{N}^d.$$

For $I \in \mathcal{F}_n$ $(n \in \mathbb{Z}^d)$, let

$$I_0 = (0, 2^{-n_1}) \times \cdots \times (0, 2^{-n_d}).$$

Since F is bounded, if $I \subset F$, $I \in \mathcal{F}_n$ is a dyadic rectangle, then $n_1, \ldots, n_d$ are bounded from below. Thus there are only finitely many dyadic rectangles $I^{(j)} \subset F$, $j = 1, \ldots, N$ such that $I^{(j)} \in \mathcal{F}_{n^{(j)}}$ and $n^{(1)}, \ldots, n^{(N)}$ are incomparable vectors. It is easy to see that

$$\overline{I^{(j)}} = (2^{n_1^{(j)}}, \infty) \times \cdots \times (2^{n_d^{(j)}}, \infty), \qquad j = 1, \ldots, N$$

and

$$\overline{F} = \bigcup_{j=1}^{N} \overline{I^{(j)}} = \bigcup_{j=1}^{N} \overline{I_0^{(j)}}.$$

For $I^{(j)} \in \mathcal{F}_{n^{(j)}}$, the union $\cup_{j=1}^{N} I^{(j)}$ has minimal measure if $I^{(j)} \cap I^{(k)} \neq \emptyset$ for all $j \neq k$, more exactly,

$$\left|\bigcup_{j=1}^{N} I_0^{(j)}\right| \leqslant \left|\bigcup_{j=1}^{N} I^{(j)}\right|.$$

Indeed, if $I^{(j)} \cap I^{(k)} \neq \emptyset$, then the set $I_0^{(j)} \cup I_0^{(k)}$ arises from the set $I^{(j)} \cup I^{(k)}$ $(j \neq k)$ by a dyadic translation. By the same dyadic translation, we get $I_0^{(j)} \cap I_0^{(k)}$ from $I^{(j)} \cap I^{(k)}$ and the intersections have equal measures. If $I^{(j)} \cap I^{(k)} = \emptyset$, then the set $I_0^{(j)} \cup I_0^{(k)}$ arises from the set $I^{(j)} \cup I^{(k)}$ $(j \neq k)$ by two dyadic translations. The same holds for more than two dyadic rectangles. So the corresponding set to $I_0^{(j)} \cap I_0^{(k)}$ is counted only once in the measure of the union $\cup_{j=1}^N I_0^{(j)}$ and at most once in $\cup_{j=1}^N I^{(j)}$. By the substitution $1/t_j = x_j$,

$$\eta(\overline{F}) = \eta\left(\bigcup_{j=1}^{N} \overline{I_0^{(j)}}\right) = \int\limits_{\bigcup_{j=1}^{N} \overline{I_0^{(j)}}} \prod_{j=1}^{d} t_j^{-2}\, dt$$

$$= \int\limits_{\bigcup_{j=1}^{N} I_0^{(j)}} 1\, dx = \left|\bigcup_{j=1}^{N} I_0^{(j)}\right| \leqslant \left|\bigcup_{j=1}^{N} I^{(j)}\right| \leqslant |F|,$$

which is exactly (38).

Now we are going to prove (39). Choose a simple p-atom a with support $R = I_1 \times \ldots \times I_j \times A$ for some open bounded set $A \subset \mathbb{R}^{d-j}$ and for some $1 \leqslant j \leqslant d-1$, where we may suppose that $I_k = (0, 2^{-K_k})$ $(K_k \in \mathbb{Z}, k = 1, \ldots, j)$. Note that

$$\int\limits_{\left(\mathbb{R}\setminus\overline{I_1^{r_1}}\right)\times\cdots\times\left(\mathbb{R}\setminus\overline{I_j^{r_j}}\right)} \int\limits_{A} |Ta|^p\, d\eta = \int\limits_{-2^{K_1-r_1}}^{2^{K_1-r_1}} \cdots \int\limits_{-2^{K_j-r_j}}^{2^{K_j-r_j}} \int\limits_{A} (|t_1|\cdots|t_d|)^{p-2}\, |\widehat{a}(t)|^p\, dt.$$

By the definition of the atom,

$$|\widehat{a}(t)| = \left|\int\limits_{I_1} \cdots \int\limits_{I_j} \int\limits_{A} a(x) \prod_{k=1}^{d} e^{-\imath t_k x_k}\, dx\right|$$

$$= \Bigg|\int\limits_{I_1} \cdots \int\limits_{I_j} \int\limits_{A} a(x)$$

$$\left(\prod_{k=1}^{j}\left(e^{-\imath t_k x_k} - \sum_{i=0}^{N} \frac{(-\imath t_k x_k)^i}{i!}\right)\right)\left(\prod_{k=j+1}^{d} e^{-\imath t_k x_k}\right) dx\Bigg|$$

$$\leqslant \int\limits_{I_1} \cdots \int\limits_{I_j} \left(\prod_{k=1}^{j}\left|e^{-\imath t_k x_k} - \sum_{j=0}^{N} \frac{(-\imath t_k x_k)^i}{i!}\right|\right)$$

$$\left|\int\limits_A a(x)\left(\prod_{k=j+1}^d e^{-\imath t_k x_k}\right)\,dx_{j+1}\cdots dx_d\right|\,dx_1\cdots dx_j,$$

where $N=\lfloor 2/p-3/2\rfloor$. Using Taylor's formula,

$$\begin{aligned}|\widehat{a}(t)| &\leqslant C\int\limits_{I_1}\cdots\int\limits_{I_j}\left(\prod_{k=1}^j|t_k x_k|\right)^{N+1} \\ &\qquad \left|\int\limits_A a(x)\left(\prod_{k=j+1}^d e^{-\imath t_k x_k}\right)\,dx_{j+1}\cdots dx_d\right|\,dx_1\cdots dx_j \\ &\leqslant C\left(\prod_{k=1}^j 2^{-K_k}\right)^{N+1}\left(\prod_{k=1}^j|t_k|\right)^{N+1} \\ &\qquad \int\limits_{I_1}\cdots\int\limits_{I_j}\left|\int\limits_A a(x)\left(\prod_{k=j+1}^d e^{-\imath t_k x_k}\right)\,dx_{j+1}\cdots dx_d\right|\,dx_1\cdots dx_j.\end{aligned} \tag{43}$$

Then $Np+2p-1>0$ and

$$\begin{aligned}&\int\limits_{\left(\mathbb{R}\setminus\overline{I_1^{r_1}}\right)\times\cdots\times\left(\mathbb{R}\setminus\overline{I_j^{r_j}}\right)}\int\limits_{\overline{A}}|Ta|^p\,d\eta \\ &\lesssim\left(\prod_{k=1}^j 2^{-K_k}\right)^{(N+1)p}\int\limits_{-2^{K_1-r_1}}^{2^{K_1-r_1}}\cdots\int\limits_{-2^{K_j-r_j}}^{2^{K_j-r_j}}\left(\prod_{k=1}^j|t_k|\right)^{(N+1)p+p-2}\int\limits_{\overline{A}}\left(\prod_{k=j+1}^d|t_k|\right)^{p-2} \\ &\qquad\left(\int\limits_{I_1}\cdots\int\limits_{I_j}\left|\int\limits_A a(x)\left(\prod_{k=j+1}^d e^{-\imath t_k x_k}\right)\,dx_{j+1}\cdots dx_d\right|\,dx_1\cdots dx_j\right)^p\,dt \\ &\lesssim\left(\prod_{k=1}^j 2^{-r_k}\right)^{Np+2p-1}\left(\prod_{k=1}^j 2^{K_k}\right)^{p-1}\int\limits_{\overline{A}}\left(\prod_{k=j+1}^d|t_k|\right)^{p-2} \\ &\qquad\left(\int\limits_{I_1}\cdots\int\limits_{I_j}\left|\int\limits_A a(x)\left(\prod_{k=j+1}^d e^{-\imath t_k x_k}\right)\,dx_{j+1}\cdots dx_d\right|\,dx_1\cdots dx_j\right)^p\,dt_{j+1}\cdots dt_d.\end{aligned}$$

By Hölder's inequality,

$$\int\limits_{\left(\mathbb{R}\setminus\overline{I_1^{r_1}}\right)\times\cdots\times\left(\mathbb{R}\setminus\overline{I_j^{r_j}}\right)}\int\limits_{\overline{A}}|Ta|^p\,d\eta$$

$$\lesssim \left(\prod_{k=1}^{j} 2^{-r_k}\right)^{Np+2p-1} \left(\prod_{k=1}^{j} 2^{K_k}\right)^{p-1} \left(\int_{\overline{A}} \left(\prod_{k=j+1}^{d} |t_k|\right)^{-2} dt_{j+1}\cdots dt_d\right)^{(2-p)/2}$$

$$\left(\int_{\overline{A}} \left(\int_{I_1}\cdots\int_{I_j} \left|\int_{A} a(x) \left(\prod_{k=j+1}^{d} e^{-\imath t_k x_k}\right) dx_{j+1}\cdots dx_d\right| dx_1\cdots dx_j\right)^2 dt_{j+1}\cdots dt_d\right)^{p/2}$$

$$\lesssim \left(\prod_{k=1}^{j} 2^{-r_k}\right)^{Np+2p-1} \left(\prod_{k=1}^{j} 2^{K_k}\right)^{p-1} \eta\left(\overline{A}\right)^{(2-p)/2}$$

$$\left(\int_{\overline{A}} \left(\int_{I_1}\cdots\int_{I_j} \left|\int_{A} a(x) \left(\prod_{k=j+1}^{d} e^{-\imath t_k x_k}\right) dx_{j+1}\cdots dx_d\right| dx_1\cdots dx_j\right)^2 dt_{j+1}\cdots dt_d\right)^{p/2}.$$

In the next step, we use Hölder's inequality and Plancherel's theorem and (38) to obtain

$$\int\limits_{\left(\mathbb{R}\setminus\overline{I_1^{r_1}}\right)\times\cdots\times\left(\mathbb{R}\setminus\overline{I_j^{r_j}}\right)} \int_{\overline{A}} |Ta|^p \, d\eta$$

$$\lesssim \left(\prod_{k=1}^{j} 2^{-r_k}\right)^{Np+2p-1} \left(\prod_{k=1}^{j} 2^{K_k}\right)^{p-1} |A|^{(2-p)/2} \left(\prod_{k=1}^{j} |I_k|\right)^{p/2}$$

$$\left(\int_{\overline{A}}\int_{I_1}\cdots\int_{I_j} \left|\int_{A} a(x) \left(\prod_{k=j+1}^{d} e^{-\imath t_k x_k}\right) dx_{j+1}\cdots dx_d\right|^2 dx_1\cdots dx_j\, dt_{j+1}\cdots dt_d\right)^{p/2}$$

$$\lesssim \left(\prod_{k=1}^{j} 2^{-r_k}\right)^{Np+2p-1} \left(\prod_{k=1}^{j} 2^{K_k}\right)^{p/2-1} |A|^{1-p/2}$$

$$\left(\int_{\overline{A}}\int_{I_1}\cdots\int_{I_j} \left|\int_{A} a(x) \left(\prod_{k=j+1}^{d} e^{-\imath t_k x_k}\right) dx_{j+1}\cdots dx_d\right|^2 dx_1\cdots dx_j\, dt_{j+1}\cdots dt_d\right)^{p/2}$$

$$\lesssim \left(\prod_{k=1}^{j} 2^{-r_k}\right)^{Np+2p-1} \left(\prod_{k=1}^{j} 2^{K_k}\right)^{p/2-1} |A|^{1-p/2} \left(\int_{I_1}\cdots\int_{I_j}\int_{A} |a(x)|^2 \, dx\right)^{p/2}.$$

Taking into account (ii) of the definition of the atom, we conclude

$$\int\limits_{\left(\mathbb{R}\setminus\overline{I_1^{r_1}}\right)\times\cdots\times\left(\mathbb{R}\setminus\overline{I_j^{r_j}}\right)} \int\limits_{A} |Ta|^p \, d\eta \lesssim \left(\prod_{k=1}^{j} 2^{-r_k}\right)^{Np+2p-1}.$$

Since $Np+2p-1>0$, (39) holds.

To prove (40), we obtain

$$\int\limits_{\left(\mathbb{R}\setminus\overline{I_1^{r_1}}\right)\times\cdots\times\left(\mathbb{R}\setminus\overline{I_{d-1}^{r_{d-1}}}\right)} \int\limits_{\mathbb{R}\setminus\overline{I_d}} |Ta|^p \, d\eta$$
$$= \int\limits_{-2^{K_1-r_1}}^{2^{K_1-r_1}} \cdots \int\limits_{-2^{K_{d-1}-r_{d-1}}}^{2^{K_{d-1}-r_{d-1}}} \int\limits_{-2^{K_d}}^{2^{K_d}} (|t_1|\cdots|t_d|)^{p-2} \, |\widehat{a}(t)|^p \; dt.$$

where $I_d = (0, 2^{-K_d})$ $(K_d \in \mathbb{Z})$. Similarly to (43),

$$\begin{aligned}
|\widehat{a}(t)| &= \left| \int\limits_{I_1} \cdots \int\limits_{I_d} a(x) \prod_{k=1}^{d} \left(e^{-\imath t_k x_k} - \sum_{i=0}^{N} \frac{(-\imath t_k x_k)^i}{i!} \right) dx \right| \\
&\leqslant C \int\limits_{I_1} \cdots \int\limits_{I_d} \left(\prod_{k=1}^{d} |t_k x_k| \right)^{N+1} |a(x)| \, dx \\
&\lesssim \left(\prod_{k=1}^{d} 2^{-K_k} \right)^{N+1} \left(\prod_{k=1}^{d} |t_k| \right)^{N+1} \left(\prod_{k=1}^{d} |I_k| \right)^{1/2} \left(\int\limits_{I_1} \cdots \int\limits_{I_d} |a(x)|^2 \, dx \right)^{1/2} \\
&\lesssim \left(\prod_{k=1}^{d} |t_k| \right)^{N+1} \left(\prod_{k=1}^{d} 2^{-K_k} \right)^{N+2-1/p}. \qquad (44)
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int\limits_{\left(\mathbb{R}\setminus\overline{I_1^{r_1}}\right)\times\cdots\times\left(\mathbb{R}\setminus\overline{I_{d-1}^{r_{d-1}}}\right)} \int\limits_{\mathbb{R}\setminus\overline{I_d}} |Ta|^p \, d\eta \\
&\lesssim \left(\prod_{k=1}^{d} 2^{-K_k} \right)^{Np+2p-1} \int\limits_{-2^{K_1-r_1}}^{2^{K_1-r_1}} \cdots \int\limits_{-2^{K_{d-1}-r_{d-1}}}^{2^{K_{d-1}-r_{d-1}}} \int\limits_{-2^{K_d}}^{2^{K_d}} \left(\prod_{k=1}^{d} |t_k| \right)^{(N+1)p+p-2} dt \\
&\lesssim \left(\prod_{k=1}^{d-1} 2^{-r_k} \right)^{Np+2p-1}.
\end{aligned}$$

If $\lim_{k\to\infty} f_k = f$ in H_p-norm, then the convergence holds also in the sense of tempered distributions and then $\lim_{k\to\infty} \widehat{f_k} = \widehat{f}$ in the sense of tempered distributions. By Lemma 6, this completes the proof of Theorem 4. $\square$

Proof of Theorem 5. The proof is similar but slightly more advanced than that of Theorem 4. We use the measure defined in (42) and introduce the operator

$$Tf(t) = \left(\prod_{j=1}^{d} t_j\right) \mathcal{H}\widehat{f}(t), \qquad t \in \mathbb{R}^d.$$

Inequality (43) implies that

$$\begin{aligned}
|\mathcal{H}\widehat{a}(t)| &\leqslant C \left(\prod_{k=1}^{j} 2^{-K_k}\right)^{N+1} \left(\prod_{k=1}^{j} |t_k|\right)^{N+1} \left(\prod_{k=j+1}^{d} |t_k|\right)^{-1} \int_0^{|t_{j+1}|} \cdots \int_0^{|t_d|} \int_{I_1} \cdots \int_{I_j} \\
&\quad \left| \int_A a(x) \left(\prod_{k=j+1}^{d} e^{-\imath u_k x_k}\right) dx_{j+1} \cdots dx_d \right| dx_1 \cdots dx_j \, du_{j+1} \cdots du_d \\
&= C \left(\prod_{k=1}^{j} 2^{-K_k}\right)^{N+1} \left(\prod_{k=1}^{j} |t_k|\right)^{N+1} \\
&\quad \int_{I_1} \cdots \int_{I_j} \mathcal{H}_{j+1,\dots,d} |\mathcal{F}_{j+1,\dots,d} a(x_1, \dots, x_j, t_{j+1}, \dots, t_d)| \, dx_1 \cdots dx_j,
\end{aligned}$$

where

$$\mathcal{F}_{j+1,\dots,d} a(x_1, \dots, x_j, t_{j+1}, \dots, t_d) := \int_{\mathbb{R}^{d-j}} a(x) \prod_{k=j+1}^{d} e^{-\imath t_k x_k} \, dx_{j+1} \dots dx_d$$

and

$$\begin{aligned}
&\mathcal{H}_{j+1,\dots,d} f(x_1, \dots, x_j, t_{j+1}, \dots, t_d) \\
&\quad := \frac{1}{t_{j+1} \cdots t_d} \int_0^{t_{j+1}} \cdots \int_0^{t_d} f(x_1, \dots, x_j, u_{j+1}, \dots, u_d) \, du_{j+1} \cdots du_d,
\end{aligned}$$

$(t_k \neq 0, k = j+1, \dots, d)$. Then

$$\int_{\left(\mathbb{R}\setminus\overline{I_1^{r_1}}\right)\times\cdots\times\left(\mathbb{R}\setminus\overline{I_j^{r_j}}\right)} \int_{\overline{A}} |Ta|^p \, d\eta$$

$$= \int_{-2^{K_1-r_1}}^{2^{K_1-r_1}} \cdots \int_{-2^{K_j-r_j}}^{2^{K_j-r_j}} \int_{\overline{A}} (|t_1|\cdots|t_d|)^{p-2} \left|\mathcal{H}\widehat{a}(t)\right|^p \, dt$$

$$\lesssim \left(\prod_{k=1}^{j} 2^{-K_k}\right)^{(N+1)p} \int_{-2^{K_1-r_1}}^{2^{K_1-r_1}} \cdots \int_{-2^{K_j-r_j}}^{2^{K_j-r_j}} \left(\prod_{k=1}^{j} |t_k|\right)^{(N+1)p+p-2}$$

$$\int_{\overline{A}} \left(\prod_{k=j+1}^{d} |t_k|\right)^{p-2}$$

$$\left(\int_{I_1}\cdots\int_{I_j} \mathcal{H}_{j+1,\ldots,d}|\mathcal{F}_{j+1,\ldots,d}a(x_1,\ldots,x_j,t_{j+1},\ldots,t_d)|\, dx_1\cdots dx_j\right)^p \, dt$$

$$\lesssim \left(\prod_{k=1}^{j} 2^{-r_k}\right)^{Np+2p-1} \left(\prod_{k=1}^{j} 2^{K_k}\right)^{p-1} \int_{\overline{A}} \left(\prod_{k=j+1}^{d} |t_k|\right)^{p-2}$$

$$\left(\int_{I_1}\cdots\int_{I_j} \mathcal{H}_{j+1,\ldots,d}|\mathcal{F}_{j+1,\ldots,d}a(x_1,\ldots,x_j,t_{j+1},\ldots,t_d)|\, dx_1\cdots dx_j\right)^p \, dt_{j+1}\cdots dt_d.$$

By Hölder's inequality,

$$\int_{\left(\mathbb{R}\setminus\overline{I_1^{r_1}}\right)\times\cdots\times\left(\mathbb{R}\setminus\overline{I_j^{r_j}}\right)} \int_{\overline{A}} |Ta|^p \, d\eta$$

$$\lesssim \left(\prod_{k=1}^{j} 2^{-r_k}\right)^{Np+2p-1} \left(\prod_{k=1}^{j} 2^{K_k}\right)^{p-1} \left(\int_{\overline{A}} \prod_{k=j+1}^{d} |t_k|^{-2} \, dt_{j+1}\cdots dt_d\right)^{(2-p)/2}$$

$$\left(\int_{\overline{A}} \left(\int_{I_1}\cdots\right.\right.$$

$$\left.\left.\int_{I_j} \mathcal{H}_{j+1,\ldots,d}|\mathcal{F}_{j+1,\ldots,d}a(x_1,\ldots,x_j,t_{j+1},\ldots,t_d)|\, dx_1\cdots dx_j\right)^2 dt_{j+1}\cdots dt_d\right)^{p/2}$$

$$\lesssim \left(\prod_{k=1}^{j} 2^{-r_k}\right)^{Np+2p-1} \left(\prod_{k=1}^{j} 2^{K_k}\right)^{p-1} \eta\left(\overline{A}\right)^{(2-p)/2}$$

$$\left(\int\limits_{\overline{A}}\left(\int\limits_{I_1}\cdots\right.\right.$$

$$\left.\left.\int\limits_{I_j}\mathcal{H}_{j+1,\dots,d}|\mathcal{F}_{j+1,\dots,d}a(x_1,\dots,x_j,t_{j+1},\dots,t_d)|\,dx_1\cdots dx_j\right)^2 dt_{j+1}\cdots dt_d\right)^{p/2}. \tag{45}$$

Taking into account (24) and Plancherel's theorem, we conclude that

$$\begin{aligned}
&\int\limits_{\overline{A}}\left(\int\limits_{I_1}\cdots\int\limits_{I_j}\mathcal{H}_{j+1,\dots,d}|\mathcal{F}_{j+1,\dots,d}a(x_1,\dots,x_j,t_{j+1},\dots,t_d)|\,dx_1\cdots dx_j\right)^2 dt_{j+1}\cdots dt_d \\
&\leqslant |I_1|\cdots|I_j|\int\limits_{I_1}\cdots\int\limits_{I_j}\int\limits_{\mathbb{R}^{d-j}} \\
&\qquad\Big(\mathcal{H}_{j+1,\dots,d}|\mathcal{F}_{j+1,\dots,d}a(x_1,\dots,x_j,t_{j+1},\dots,t_d)|\Big)^2 dt_{j+1}\cdots dt_d\,dx_1\cdots dx_j \\
&\leqslant C|I_1|\cdots|I_j|\int\limits_{I_1}\cdots\int\limits_{I_j}\int\limits_{\mathbb{R}^{d-j}} \\
&\qquad|\mathcal{F}_{j+1,\dots,d}a(x_1,\dots,x_j,t_{j+1},\dots,t_d)|^2\,dt_{j+1}\cdots dt_d\,dx_1\cdots dx_j \\
&\leqslant C|I_1|\cdots|I_j|\int\limits_{I_1}\cdots\int\limits_{I_j}\int\limits_{A}|a(x)|^2\,dx.
\end{aligned}$$

Substituting this into (45) and using (38), we can see that

$$\begin{aligned}
&\int\limits_{\left(\mathbb{R}\setminus\overline{I_1^{r_1}}\right)\times\cdots\times\left(\mathbb{R}\setminus\overline{I_j^{r_j}}\right)}\ \int\limits_{\overline{A}}|Ta|^p\,d\eta \\
&\lesssim\left(\prod_{k=1}^{j}2^{-r_k}\right)^{Np+2p-1}\left(\prod_{k=1}^{j}2^{K_k}\right)^{p/2-1}|A|^{1-p/2}\left(\int\limits_{I_1}\cdots\int\limits_{I_j}\int\limits_{A}|a(x)|^2\,dx\right)^{p/2} \\
&\lesssim 2\left(\prod_{k=1}^{j}2^{-r_k}\right)^{Np+2p-1}.
\end{aligned}$$

Using (44), we remark that

$$|\mathcal{H}\widehat{a}(t)|\lesssim\left(\prod_{k=1}^{d}|t_k|\right)^{N+1}\left(\prod_{k=1}^{d}2^{-K_k}\right)^{N+2-1/p}.$$

Hence, the estimate

$$\begin{aligned}
&\int\limits_{\left(\mathbb{R}\setminus\overline{I_1^{r_1}}\right)\times\cdots\times\left(\mathbb{R}\setminus\overline{I_j^{r_j}}\right)} \int\limits_{\overline{A}} |Ta|^p \, d\eta \\
&= \int\limits_{-2^{K_1-r_1}}^{2^{K_1-r_1}} \cdots \int\limits_{-2^{K_{d-1}-r_{d-1}}}^{2^{K_{d-1}-r_{d-1}}} \int\limits_{-2^{K_3}}^{2^{K_3}} (|t_1|\cdots|t_d|)^{p-2} \, |\mathcal{H}\widehat{a}(t)|^p \, dt \\
&\lesssim \left(\prod_{k=1}^{d} 2^{-K_k}\right)^{Np+2p-1} \int\limits_{-2^{K_1-r_1}}^{2^{K_1-r_1}} \cdots \int\limits_{-2^{K_{d-1}-r_{d-1}}}^{2^{K_{d-1}-r_{d-1}}} \int\limits_{-2^{K_3}}^{2^{K_3}} (|t_1|\cdots|t_d|)^{(N+1)p+p-2} \, dt \\
&\lesssim \left(\prod_{k=1}^{d-1} 2^{-r_k}\right)^{Np+2p-1}
\end{aligned}$$

can be proved as in Theorem 4. Since $Vf := \mathcal{H}\widehat{f}$ is bounded from L_q to $L_{q'}$ for all $1 \leqslant q \leqslant 2$, Lemma 6 finishes the proof. □

Let us denote by BMO the dual space of H_1 (see [10]). Note that this space is different from the usual $BMO(\mathbb{R}^d)$ space, that is the dual of $H_1(\mathbb{R}^d)$ (see Fefferman and Stein [15]). Similarly to Theorem 2, by a duality argument, one can obtain the following corollary.

Corollary 1. *If $f \in L_1^{loc}$ and*

$$\sup_{t\in\mathbb{R}^n} (|t_1|\ldots|t_d|\,|f(t)|) < \infty,$$

then, for all $N \in \mathbb{N}$,

$$\|\mathfrak{F}_N f\|_{BMO} \lesssim \sup_{t\in\mathbb{R}^n} (|t_1|\ldots|t_d|\,|f(t)|), \tag{46}$$

where $\mathfrak{F}_N f$ was defined in (3).

Note that inequality (27) implies that

$$\|\mathcal{H}\mathfrak{F}_N f\|_{BMO} \lesssim \sup_{t\in\mathbb{R}^n} (|t_1|\ldots|t_d|\,|f(t)|).$$

Note that the similar result to (46) for Walsh-Fourier coefficients of one-dimensional functions was proved by Ladhawala [27]. Moreover, for the Fourier series $f(x) \sim \sum_{n=0}^{\infty} a_d e^{\imath nx}$ with non-negative coefficients a_n the corresponding result follows from a characterization of BMO due to Fefferman (see [35]).

Proof of Theorem 6. We use the original version of Lemma 6, with $i = 0$, η the Lebesgue measure and $\overline{F} = F$. It is easy to see that if a is a simple p-atom with support R (a dyadic rectangle), then $\operatorname{supp} \mathcal{H}a \subset R$. This means that the integrals in (39) and (40) are 0. Since $\mathcal{H}$ is bounded on L_q for all $1 < q < \infty$, Lemma 6 completes the proof. □

It is known that the operator $\mathcal{B}$ is not bounded from BMO to BMO (see Golubov [19]) but the following weaker result holds true.

Corollary 2. *If* $f \in L_\infty \bigcap \bigcup_{1\leqslant q<\infty} L_q$, *then*

$$\|\mathcal{B}f\|_{BMO} \leqslant C\,\|f\|_\infty .$$

Proof. Since $\mathcal{B}$ in bounded on L_q for $1 \leqslant q < \infty$, we have

$$\langle \mathcal{B}f, g\rangle = \langle f, \mathcal{H}g\rangle,$$

where $g \in \mathcal{S}$. We have by Theorem 6,

$$\begin{aligned}\|\mathcal{B}f\|_{BMO} &= \sup_{\|g\|_{H_1}\leqslant 1} \langle \mathcal{B}f, g\rangle = \sup_{\|g\|_{H_1}\leqslant 1} \langle f, \mathcal{H}g\rangle \\ &\leqslant \sup_{\|g\|_{H_1}\leqslant 1} \|f\|_\infty \|\mathcal{H}g\|_1 \leqslant C\|f\|_\infty. \quad \square\end{aligned}$$

However, the operator $\mathcal{H}$ is bounded on BMO.

Corollary 3. *If* $f \in BMO \bigcap \bigcup_{1<q\leqslant\infty} L_q$, *then*

$$\|\mathcal{H}f\|_{BMO} \leqslant C\,\|f\|_{BMO} .$$

Proof. Inequality (28) implies that

$$\begin{aligned}\|\mathcal{H}f\|_{BMO} &= \sup_{\|g\|_{H_1}\leqslant 1} \langle \mathcal{H}f, g\rangle = \sup_{\|g\|_{H_1}\leqslant 1} \langle f, \mathcal{B}g\rangle \\ &\leqslant \sup_{\|g\|_{H_1}\leqslant 1} \|f\|_{BMO} \|\mathcal{B}g\|_{H_1} \leqslant C\|f\|_{BMO},\end{aligned}$$

where $g \in \mathcal{S}$. In the second equality we used that $\mathcal{H}$ in bounded on L_q for $1 < q \leqslant \infty$. □

Data availability

No data was used for the research described in the article.

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Appendix A. Carleman-type result for Fourier transform

Example. There is a continuous function F from $\cap_{p>1} L_p(\mathbb{R})$ such that $\widehat{F} \notin L_p$, $p < 2$ and

$$\int_{\mathbb{R}} |x|^{p-2} \big|\widehat{F}(x)\big|^p dx = \infty, \qquad p > 2.$$

Indeed, define

$$g(x) = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{\sqrt{n+1}\ln^2(n+2)} \chi_{(n-\frac{1}{2}, n+\frac{1}{2})}(x),$$

where $\{\varepsilon_n\}_{n=0}^{\infty}$ is the Rudin–Shapiro sequence.

First, it is easy to see that $g \in L_2(\mathbb{R})$, $g \notin L_p(\mathbb{R})$, $1 < p < 2$, and $\int_{\mathbb{R}} |x|^{p-2} |g(x)|^p dx = \infty$, $p > 2$. Second,

$$\int_{-n-\frac{1}{2}}^{n+\frac{1}{2}} e^{-itx} g(x) dx = \frac{2\sin\frac{t}{2}}{t} \sum_{k=0}^{n} \frac{e^{-ikt}\varepsilon_k}{\sqrt{k+1}\ln^2(k+2)} =: h(t) f_n(t).$$

Applying Abel's transformation, we obtain

$$f_n(t) = \sum_{k=0}^{n-1} \Big(\frac{1}{\sqrt{k+1}\ln^2(k+2)} - \frac{1}{\sqrt{k+2}\ln^2(k+3)}\Big) \sum_{r=0}^{k} e^{-irt}\varepsilon_r$$
$$+ \frac{1}{\sqrt{n+1}\ln^2(n+2)} \sum_{r=0}^{n} e^{-irt}\varepsilon_r =: \sum_{k=0}^{n-1} a_k P_k(t) + \frac{1}{\sqrt{n+1}\ln^2(n+2)} P_n(t).$$

Since $|P_k(t)| \leqslant 5\sqrt{k+1}$ and $a_k \leqslant \frac{C}{(k+1)\sqrt{k+1}\ln^2(k+2)}$, we have that $f_n \to f$ uniformly, where f is continuous and bounded on $\mathbb{R}$, and $\hat{g} = hf \in L_p(\mathbb{R})$ for any $1 < p < \infty$. Finally, we put $F := \hat{g}$.

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