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Globally rigid powers of graphs

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A B S T R A C T

The characterization of rigid graphs in $\mathbb{R}^d$ for $d \geq 3$ is a major open problem in rigidity theory. The same holds for globally rigid graphs. In this paper our goal is to give necessary and/or sufficient conditions for the (global) rigidity of the square G^2 (and more generally, the power G^k) of a graph G in $\mathbb{R}^d$, for some values of k, d. Our work is motivated by some results and conjectures of M. Cheung and W. Whiteley from 2008, the Molecular Theorem of N. Katoh and S. Tanigawa from 2011, which settled the case of rigidity for $k = 2, d = 3$, and the potential applications in molecular conformation and sensor network localization.

We first consider the case when $k = d$ and characterize those graphs G for which G^d is globally rigid in $\mathbb{R}^d$, for all $d \geq 1$, and then focus on the case when $k = d-1$. We provide a new, direct proof for the 3-dimensional bar-and-joint version of the Molecular Theorem ($d = 3$) and a necessary condition for the rigidity of G^{d-1} in $\mathbb{R}^d$, for all $d \geq 3$. We conjecture that this condition is also sufficient.

The global rigidity of square graphs in $\mathbb{R}^3$ is still an open problem. We formulate a Molecular Global Rigidity Conjecture, which proposes a combinatorial characterization of globally rigid square graphs in terms of vertex partitions and edge count conditions. We prove that the condition is necessary. For the general case we give a best possible connectivity based sufficient condition by showing that if G

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is 3-edge-connected then G^{d-1} is globally rigid in $\mathbb{R}^d$, for all $d \geq 3$.
Our results imply affirmative answers to the conjectures of M. Cheung and W. Whiteley in two special cases.

1. Introduction

Informally speaking, a graph G is said to be rigid in $\mathbb{R}^d$ if a bar-and-joint framework (or geometric graph) with underlying graph G has no continuous deformation that preserves the bar (edge) lengths. It is globally rigid, if such a framework has no deformation at all: the edge lengths determine all pairwise distances. (Precise definitions will be given in the next section.)

The characterization of rigid graphs in $\mathbb{R}^d$ for $d \geq 3$ is a major open problem in rigidity theory. The same holds for globally rigid graphs. In this paper our goal is to give necessary and/or sufficient conditions for the (global) rigidity of the square G^2 (and more generally, the power G^k) of a graph G in $\mathbb{R}^d$, for some values of k, d.

The *k'th power* of a graph G is obtained from G by adding a new edge uv for all non-adjacent vertex pairs u, v of G with distance at most k in G. Thus the *square* G^2 of G is obtained from G by adding a new edge uv for all non-adjacent vertex pairs u, v of G with a common neighbour. See Fig. 1. Squares of graphs (sometimes called molecular graphs) and powers of graphs are used e.g. in the study of the (global) rigidity properties of molecules and wireless sensor networks, see e.g. [13,23,32].

The investigation of rigid squares of graphs in $\mathbb{R}^3$ became a central problem in rigidity theory when T. Tay and W. Whiteley proposed the Molecular Conjecture in 1984. The solution (Theorem 1.2 below) was obtained in 2011. The study of (globally) rigid powers of graphs in a more general setting was initiated by M. Cheung and W. Whiteley [3] in their 2008 paper which included several interesting results and conjectures. Our work is motivated by these results, conjectures, and the applications mentioned above.

The following result, and its proof, shows that the required characterization of rigidity (global rigidity, resp.) is not hard to obtain if the power is large enough compared to d.

Proposition 1.1 *([3]). Let G be a graph and let $d \geq 1$ be an integer. Then*
(i) G^d is rigid in $\mathbb{R}^d$ if and only if G is connected,
(ii) G^{d+1} is globally rigid in $\mathbb{R}^d$ if and only if G is connected.

A key conjecture in [3] says that, roughly speaking, one can achieve (global) rigidity in the next dimension by raising the power.

Conjecture 1 *([3]). Suppose that G^k is rigid (resp. globally rigid) in $\mathbb{R}^d$ for some positive integers k, d. Then G^{k+1} is rigid (resp. globally rigid) in $\mathbb{R}^{d+1}$.*

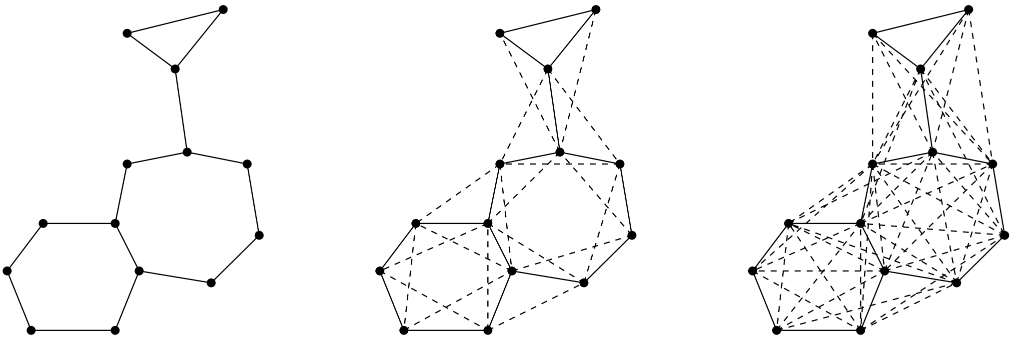

Fig. 1. A graph G, its square G^2, and its cube G^3.

They also posed the following even stronger version.

Conjecture 2 *([3]). Suppose that G^k is rigid in $\mathbb{R}^d$ for some $d \geq 2$. Then G^{k+1} is globally rigid in $\mathbb{R}^{d+1}$.*

The underlying general question is to give a characterization of the rigidity or the global rigidity of G^k in $\mathbb{R}^d$ in terms of a combinatorial property of G. In this paper, our main target is to obtain necessary and/or sufficient conditions for the (global) rigidity of G^k in $\mathbb{R}^d$ for $k = d - 1$ and $k = d$.

A well-known result of this type ($k = 2, d = 3$) follows from the Molecular Theorem due to N. Katoh and S. Tanigawa [22]. Their result, which is about d-dimensional panel-and-hinge frameworks, has the following corollary.

Theorem 1.2 *(Molecular Theorem [22]). Let G be a graph with minimum degree at least two. Then G^2 is rigid in $\mathbb{R}^3$ if and only if $5G$ contains six edge-disjoint spanning trees, where $5G$ is obtained from G by replacing each edge e by five parallel copies of e.*

The new results of this paper are as follows. We first characterize those graphs G for which G^d is globally rigid in $\mathbb{R}^d$, for all $d \geq 1$, and then focus on the case when $k = d-1$. We provide a new, direct proof for the stronger, defect form of Theorem 1.2 and a necessary condition for the rigidity of G^{d-1} in $\mathbb{R}^d$, for arbitrary $d \geq 3$. We conjecture that this condition is also sufficient.

The global rigidity of square graphs in $\mathbb{R}^3$ is still an open problem. We prove a combinatorial necessary condition, in terms of vertex partitions and edge count conditions, and conjecture that it is also sufficient. For the general case we give a best possible connectivity based sufficient condition by showing that if G is 3-edge-connected then G^{d-1} is globally rigid in $\mathbb{R}^d$, for all $d \geq 3$. Our results also imply affirmative answers to Conjectures 1 and 2 in two special cases.

The following table gives a summary of the results on the (global) rigidity of G^k in $\mathbb{R}^d$ for $k \geq d - 1$.

k	rigidity of G^k in $\mathbb{R}^d$	global rigidity of G^k in $\mathbb{R}^d$
$\geq d+1$	Proposition 1.1 [3]	Proposition 1.1 [3]
d	Proposition 1.1 [3]	Theorem 3.5
$d-1$	$d=2$: Geiringer–Laman's theorem $d=3$: Molecular Theorem [22] general: Conjecture 3	$d=2$: Jackson–Jordán's theorem [12] $d=3$: Conjecture 4

2. Preliminaries

2.1. Rigidity and global rigidity

A d-dimensional (bar-and-joint) *framework* is a pair (G, p), where $G = (V, E)$ is a graph[1] and p is a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^d$. Two realizations (G, p) and (G, q) of G are *equivalent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $uv \in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G, p), (G, q) are *congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$.

We say that (G, p) is *globally rigid* in $\mathbb{R}^d$ if every d-dimensional realization of G which is equivalent to (G, p) is congruent to (G, p). The framework (G, p) is *rigid* if there exists an $\epsilon > 0$ such that, if (G, q) is equivalent to (G, p) and $||p(v)-q(v)|| < \epsilon$ for all $v \in V$, then (G, q) is congruent to (G, p). Intuitively, this means that if we think of a d-dimensional framework (G, p) as a collection of bars and joints where points correspond to joints and each edge to a rigid (i.e. fixed length) bar joining its end-points, then the framework is globally rigid if its bar lengths determine the realization up to congruence. It is rigid if every continuous motion of the joints that preserves all bar lengths must preserve all pairwise distances between the joints.

It is a hard problem to decide if a given framework is rigid or globally rigid. We obtain more tractable problems if we consider *generic frameworks* i.e. frameworks in which the set of coordinates of the vertices is algebraically independent over the rationals.

It is known that for every $d \geq 1$ the rigidity (resp. global rigidity) of frameworks in $\mathbb{R}^d$ is a generic property, that is, the rigidity (global rigidity) of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic [2,4,8]. We say that the graph G is *rigid* (resp. *globally rigid*) in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid (resp. globally rigid). The problem of characterizing when a graph is rigid (resp. globally rigid) in $\mathbb{R}^d$ has been solved for $d = 1, 2$. For $d \geq 3$ they remain major open problems in rigidity theory. For a detailed survey of rigid and globally rigid d-dimensional frameworks and graphs, and their applications, we refer the reader to [13,18,21,26,31].

We shall frequently use the following elementary and well-known tools for analyzing the rigidity and global rigidity of graphs.

[1] By a graph we mean a simple graph. If we allow parallel edges, we call it a multigraph.

Lemma 2.1 *(Extension lemma). Let G be a graph obtained from a graph H by adding a new vertex v with k edges incident to v.*

If H is rigid in $\mathbb{R}^d$ and $k \geq d$, then G is rigid in $\mathbb{R}^d$.

If H is globally rigid in $\mathbb{R}^d$ and $k \geq d+1$, then G is globally rigid in $\mathbb{R}^d$.

Lemma 2.2 *(Gluing lemma). Let G_1 and G_2 be graphs with $|V(G_1) \cap V(G_2)| = k$, and let $G = G_1 \cup G_2$.*

If G_1 and G_2 are rigid in $\mathbb{R}^d$ and $k \geq d$, then G is rigid in $\mathbb{R}^d$.

If G_1 and G_2 are globally rigid in $\mathbb{R}^d$ and $k \geq d+1$, then G is globally rigid in $\mathbb{R}^d$.

2.2. Necessary and sufficient conditions for global rigidity

Hendrickson [9] proved two key necessary conditions for the global rigidity of a graph in $\mathbb{R}^d$. We say that G is *redundantly rigid in* $\mathbb{R}^d$ if removing any edge of G results in a rigid graph.

Theorem 2.3 *([9]). Let G be a globally rigid graph in $\mathbb{R}^d$. Then either G is a complete graph on at most $d+1$ vertices, or G has at least $d+2$ vertices and G is*
(i) $(d+1)$-vertex-connected, and
(ii) redundantly rigid in $\mathbb{R}^d$.

The necessary conditions of Theorem 2.3 together are also sufficient to imply the global rigidity of the graph in $\mathbb{R}^d$ for $d = 1, 2$ (see [12]) but this implication is no longer valid in higher dimensions.

We also have two sufficient conditions that work in all dimensions and provide another link between rigidity and global rigidity. We say that graph $G = (V, E)$ is *vertex-redundantly rigid* in $\mathbb{R}^d$ if $G - v$ is rigid in $\mathbb{R}^d$ for all $v \in V$.

Theorem 2.4 *([27]). If G is vertex-redundantly rigid in $\mathbb{R}^d$ then it is globally rigid in $\mathbb{R}^d$.*

For some graph H and $X \subseteq V(H)$ let $H + K(X)$ denote the graph obtained from H by adding new edges connecting all pairs of non-adjacent vertices of X. The *neighbour set* of X consists of those vertices in $V(H) - X$ which are connected to X by an edge. It is denoted by $N_H(X)$. If $X = \{v\}$ then we simply write $N_H(v)$.

Theorem 2.5 *([27]). Let v be a vertex of degree at least $d+1$ in graph G for some integer $d \geq 1$. Suppose that $G - v$ is rigid and $G - v + K(N_G(v))$ is globally rigid in $\mathbb{R}^d$. Then G is globally rigid in $\mathbb{R}^d$.*

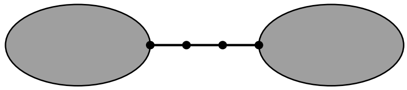

Fig. 2. A separating 4-chain.

3. The global rigidity of G^d in $\mathbb{R}^d$

In this section we characterize, for every positive integer d, those graphs G for which G^d is globally rigid in $\mathbb{R}^d$. Up to dimension three the following previous results provide necessary and sufficient conditions. The next lemma is well-known and not hard to prove.

Lemma 3.1. *Let G be a connected graph. Then G^1 is globally rigid in $\mathbb{R}^1$ if and only if G is 2-vertex-connected.*

The next two theorems were announced in [3]. The corresponding proofs (for weaker versions) appeared in [1].

Theorem 3.2. *[3] Let G be a connected graph. Then G^2 is globally rigid in $\mathbb{R}^2$ if and only if for every separating edge e in G one of the two components of $G - e$ is a single vertex.*

Theorem 3.3. *[3] Let G be a connected graph. Then G^3 is globally rigid in $\mathbb{R}^3$ if and only if for every separating vertex v of degree two in G one of the two components of $G - v$ is a single vertex.*

We shall unify and extend these results to all d. A *k-chain* in a graph G is a path with k vertices for which every internal vertex has degree two in G. A k-chain P with end-vertices v_1, v_2 is said to be *separating* if G can be obtained from P and two disjoint connected graphs H_1, H_2, on at least two vertices, by identifying a vertex of H_i and v_i, for $i = 1, 2$. See Fig. 2.

For example, a separating 1-chain is a cut-vertex of G. A separating 2-chain corresponds to a cut-edge e of G for which each component of $G - e$ has at least two vertices. The middle vertex of a separating 3-chain is a cut-vertex v of degree two in G for which both components of $G - v$ are non-trivial.

It is clear that if G^d is globally rigid in $\mathbb{R}^d$ then G has no separating d-chain P, since the vertex set of P is a vertex separator in G^d of size d (and hence G^d does not satisfy the necessary connectivity condition of d-dimensional global rigidity, cf. Theorem 2.3). For $d \leq 3$ there are no other obstacles by Lemma 3.1, and Theorems 3.2 and 3.3. We shall prove that the same holds for all d. It will be convenient to have the following lemma, which settles a special case.

Lemma 3.4. *Let G_1, G_2 be two disjoint graphs and $v_i \in V(G_i)$, $i = 1, 2$. Let H be the graph obtained from G_1 and G_2 by identifying the vertices v_1, v_2. Suppose that G_1^d and*

G_2^d are globally rigid in $\mathbb{R}^d$ for some $d \geq 2$. Then H^d is globally rigid in $\mathbb{R}^d$ if and only if H has no separating d-chains.

Proof. As we noted above, necessity is clear. To prove sufficiency, suppose that H has no separating d-chains. Let $v \in V(H)$ be the (cut-)vertex created by the identification of v_1 and v_2.

Pick vertices $w_i \in V(G_i) - v_i$ for which the distance $\mathrm{dist}_{G_i}(v_i, w_i)$ to v_i in G_i is as large as possible. If

$$\mathrm{dist}_{G_1}(v_1, w_1) + \mathrm{dist}_{G_2}(v_2, w_2) \leq d \tag{1}$$

then each vertex of G_1 is connected to each vertex of G_2 in H^d and hence all pairwise distances are fixed in H^d. Thus H^d is globally rigid in $\mathbb{R}^d$, as required. In what follows suppose that (1) does not hold. Then, without loss of generality, we may assume $\mathrm{dist}_{G_1}(v_1, w_1) \geq \lceil \frac{d+1}{2} \rceil$.

For $j \geq 0$, let $U_j = \{u \in V(G_2) : \mathrm{dist}_{G_2}(u, v_2) = j\}$ and put $q = \max\{\mathrm{dist}_{G_2}(u, v_2) : u \in V(G_2)\}$. Let $I_0 = G_1^d$ and I_j be the subgraph of H^d induced by $V(G_1) \cup \bigcup_{i \leq j} U_i$ for $j = 1, \ldots, q$. Then $H^d = I_q$ holds. We shall inductively show the global rigidity of I_j for all $j = 0, 1, \ldots, q$. By the assumption of the lemma I_0 is globally rigid.

Let $r = d - \mathrm{dist}_{G_1}(w_1, v_1)$. We first prove the global rigidity of I_j for any j with $j \leq r$. (This case occurs only when $r \geq 0$.) Since $2r \leq 2d - 2\lceil \frac{d+1}{2} \rceil \leq d - 1$, any pair of vertices in $\bigcup_{j \leq r} U_j$ is within distance d in H, and any pair consisting of a vertex in G_1 and a vertex in $\bigcup_{j \leq r} U_j$ is also within distance d in H by $r + \mathrm{dist}_{G_1}(w_1, v_1) = d$. Thus the global rigidity of G_1^d implies the global rigidity of I_j for any $j \leq r$.

Hence we may focus on the case when $j > r$, i.e., $j + \mathrm{dist}_{G_1}(w_1, v_1) \geq d + 1$. Take any vertex $u \in U_j$. By $j + \mathrm{dist}_{G_1}(w_1, v_1) \geq d + 1$, a shortest path P_u between u and w_1 in H has length at least $d + 1$. Let $S_u = (u = x_0, x_1, \ldots, x_d)$ be the subpath of P_u starting from u and having length d and let $S_u^- = (x_1, \ldots, x_d)$ be the subpath of S_u on d vertices.

Since $V(S_u^-) \subset V(I_{j-1})$ and $|V(S_u^-)| = d$, I_j contains d edges from u to d vertices in I_{j-1}. We show that there is at least one vertex in $V(I_{j-1}) - V(S_u^-)$ whose distance from u is at most d.

If an internal vertex of S_u is incident to a vertex $a \in V(G_1) \setminus V(S_u)$, then we are done. So assume this is not the case. Then, from the fact that H has no separating d-chains and that S_u^- forms a path with d vertices in H, an internal vertex of S_u^- must have degree at least three in H. In other words, there is a vertex $a \in V(G_2) \setminus V(S_u)$ incident to an internal vertex of S_u^-. Since a is incident to an internal vertex of S_u^-, $\mathrm{dist}_H(v_1, a) \leq j - 1$ and $\mathrm{dist}_H(u, a) \leq d$ holds. Thus I_j contains $d + 1$ edges between u and $V(S_u^-) \cup \{a\} \subseteq V(I_{j-1})$, implying the global rigidity of I_j by the Extension lemma.

Hence $H^d = I_q$ is globally rigid in $\mathbb{R}^d$, as required. □

Theorem 3.5. *Let G be a connected graph. Then G^d is globally rigid in $\mathbb{R}^d$ if and only if G does not contain a separating d-chain.*

Proof. Necessity is clear, as we noted above. We prove sufficiency.

Claim 3.6. *If G is 2-vertex-connected then G^d is globally rigid in $\mathbb{R}^d$.*

Proof. If G is 2-vertex-connected then $G - v$ is connected for all $v \in V$, which implies, by Proposition 1.1(i), that $(G - v)^d$ is rigid. Since $(G - v)^d \subseteq G^d - v$, it follows that G^d is vertex-redundantly rigid. Hence G^d is globally rigid by Theorem 2.4. □

We prove the theorem by induction on $|V|$. We may assume that G has at least one cut-vertex. Let W be an end-block (i.e. an edge or maximal 2-connected subgraph, which is incident with exactly one cut-vertex) of G. Suppose that W is connected to the rest of the graph along the cut-vertex v. Since W is 2-connected (or is isomorphic to K_2), W^d is globally rigid by Claim 3.6.

Now focus on $J = G - (V(W) - v)$, the graph obtained by detaching W along vertex v. If J has no separating d-chains then J^d is globally rigid by induction. Hence G^d is globally rigid by Lemma 3.4.

Next suppose that J has a separating d-chain $P = (x_1, x_2, \dots x_d)$. Since G has no separating d-chains, by our assumption, and W has only one attachment vertex v in J, it follows that v is an internal vertex of P, i.e. $v = x_i$ for some $2 \leq i \leq d - 1$. Thus v is a cut-vertex in G, along which three subgraphs of G are merged: a connected subgraph J_1 of J that contains the subpath $(x_1, x_2, \dots x_i = v)$ of P, a connected subgraph J_2 of J that contains the subpath $(v = x_i, x_{i+1}, \dots x_d)$ of P, and W, which also includes v.

Now detach J_2 from G along v. We claim that both graphs obtained by this operation are free of separating d-chains. First consider J_2. Suppose it has a separating d-chain P. The key observation is that such a d-chain cannot be eliminated by attaching a subgraph along a leaf vertex. Since v is a leaf in J_2, that would mean G has a separating d-chain, a contradiction.

Next consider the union of J_1 and W, denoted by K. Suppose it has a separating d-chain P'. The key observation here is that such a d-chain can be eliminated by attaching a subgraph along one vertex only if it is attached to some internal vertex w of P'. Observe that an internal vertex of P' has degree two in K and is not incident with a leaf in K. But v is part of an end-block W in K (note that W is an end-block in K, too) and hence either it has degree at least three in K or is incident with a leaf in K, a contradiction.

The theorem now follows from Lemma 3.4, applied to K and J_2. □

Observe that if a graph G contains no separating d-chain then it does not contain separating $(d + 1)$-chains either. Thus the theorem implies that if G^d is globally rigid in $\mathbb{R}^d$ then G^{d+1} is globally rigid in $\mathbb{R}^{d+1}$. Therefore we can use it to verify the case $k = d$ of the globally rigid version of Conjecture 1. Note also that it is easy to test, in polynomial time, whether a graph G has a separating d-chain.

We can also deduce that Conjecture 2, in its most general form, is false. Let $k = d \geq 2$. By Proposition 1.1, for a connected graph G we have that G^d is rigid in $\mathbb{R}^d$. However, (the easy direction of) Theorem 3.5 shows that there exist connected graphs G for which G^{d+1} is not globally rigid in $\mathbb{R}^{d+1}$: for example, a path on at least $d+3$ vertices.

By rereading the previous proofs and observing that the extension operation preserves vertex-redundant rigidity (as well as global rigidity), we can deduce that for every graph G the d'th power G^d is globally rigid in $\mathbb{R}^d$ if and only if it is vertex-redundantly rigid in $\mathbb{R}^d$, for all $d \geq 1$. It may be interesting to find further families of graphs for which vertex-redundant rigidity is equivalent to global rigidity.

4. The rigidity of G^{d-1} in $\mathbb{R}^d$

The characterization of the graphs G for which G^{d-1} is rigid in $\mathbb{R}^d$ is a challenging problem. For $d = 2$ it amounts to finding the characterization of rigid graphs in $\mathbb{R}^2$, which is the celebrated Geiringer-Laman theorem, see e.g. [26]. The 3-dimensional case is Theorem 1.2, which follows from the Molecular Theorem [22], whose proof is given in terms of d-dimensional hinge-coplanar body-hinge frameworks. Whiteley [29] pointed out that the 3-dimensional case has equivalent forms in terms of "molecular graphs" and squares of graphs, which can be used to deduce Theorem 1.2. Further results and a proof for the easier direction of Theorem 1.2, in terms of squares of graphs, have been obtained in [15,16]. Since the proof of the Molecular Theorem is more general and rather lengthy, and most of its applications are in terms of bar-and-joint frameworks and squares of graphs, a shorter direct proof of (a strengthening of) Theorem 1.2 may be of interest. In this section we provide such a proof. The cases $d \geq 4$ remain open. We shall close this section with a conjectured characterization for the general case and the proof of necessity.

4.1. A new proof of the Molecular Theorem

In order to state a refined, stronger version of Theorem 1.2 we need a few more definitions. We first recall the three-dimensional versions of some basic notions of rigidity theory in Section 4.1.1, and then introduce some combinatorial results on tree-connectivity in Section 4.1.2. We then state the defect form of the Molecular Theorem in Section 4.1.3.

4.1.1. Degree of freedom of graphs

The *rigidity matrix* $R(G,p)$ of a 3-dimensional realization (G,p) of graph $G=(V,E)$ is a matrix of size $|E| \times 3|V|$. For each edge $v_iv_j \in E$ the entries in the row corresponding to edge v_iv_j are defined as follows: the three columns corresponding to the vertex v_i (resp. v_j) contain the three coordinates of $p(v_i)-p(v_j)$ (resp. $p(v_j)-p(v_i)$), the remaining entries are zeros. Suppose that G has at least three vertices. Then the rank of the rigidity matrix of a realization of G cannot be more than $3|V|-6$. We say that (G,p) is *infinitesimally rigid* if the rank of $R(G,p)$ is equal to $3|V|-6$. The rank of the rigidity matrix is the same for all generic realizations of G. We denote this rank by $r(G)$ and call it the *rank* of G. If G has at least three vertices then G is rigid if and only if $r(G) = 3|V| - 6$. See [26] for more details on infinitesimal rigidity of d-dimensional frameworks.

More generally, we shall call the number $3|V| - 6 - r(G)$ the *degree of freedom* of G and denote it by $\mathrm{dof}(G)$. (The degree of freedom of a framework is defined analogously, by replacing the rank of the graph by the rank of its rigidity matrix.) This number measures the flexibility of a generic realization of the graph. We say that an edge uv with $u, v \in V(G)$ is in the *closure* of a graph G, denoted by $\mathrm{cl}(G)$, if $\mathrm{dof}(G+uv) = \mathrm{dof}(G)$ holds.

We shall use the following observation in the proof of the Molecular Theorem.

Lemma 4.1. *Let (G, p) be a framework in $\mathbb{R}^3$ for which $(G - u, p)$ is infinitesimally rigid for some $u \in V(G)$ and p is injective. Suppose that $p(v_1), p(v_2), p(v_3)$ are collinear for $v_1, v_2, v_3 \in V(G) \setminus \{u\}$, and $uv_1, uv_2 \in E(G)$. Then* rank $R(G + uv_3, p) =$ rank $R(G, p)$.

Proof. The statement is obvious if (G, p) is infinitesimally rigid. Suppose that it is not the case. Then, since $(G - u, p)$ is infinitesimally rigid and $uv_1, uv_2 \in E(G)$, (G, p) has one degree of freedom, and any nontrivial infinitesimal motion fixing $(G - u, p)$ is an infinitesimal rotation about the line through $p(v_1)$ and $p(v_2)$. Since $p(v_1), p(v_2), p(v_3)$ are collinear, this infinitesimal rotation also satisfies the edge-constraint given by uv_3. □

4.1.2. Tree-connectivity

Let $H = (V, E)$ be a multigraph. For a partition $\mathcal{P}$ of V let $E_H(\mathcal{P})$ denote the set, and $e_H(\mathcal{P})$ the number of edges of H connecting distinct members of $\mathcal{P}$. For a partition $\mathcal{P}$ of V let

$$\mathrm{def}_H(\mathcal{P}) = 6|\mathcal{P}| - 6 - 5e_H(\mathcal{P}) \tag{2}$$

and let $\mathrm{def}(H) := \max\{\mathrm{def}_H(\mathcal{P}) : \mathcal{P} \text{ is a partition of } V\}$. Note that $\mathrm{def}(H) \geq 0$, since $\mathrm{def}_H(\{V\}) = 0$. A partition $\mathcal{P}$ with $\mathrm{def}_H(\mathcal{P}) = \mathrm{def}(H)$ is called *tight*. We say that H is $\frac{6}{5}$*-tree-connected* if $def(H) = 0$. If $\mathrm{def}_H(\mathcal{P}) \leq -1$ for all partitions $\mathcal{P}$ with $|\mathcal{P}| \geq 2$ then H is called *highly* $\frac{6}{5}$*-tree-connected.*

For a graph H and positive integer k the multigraph obtained from H by replacing each edge e of H by k parallel copies of e is denoted by kH. A theorem of Nash-Williams [25] and Tutte [28] implies that H is $\frac{6}{5}$-tree-connected (resp. highly $\frac{6}{5}$-tree-connected) if and only if $5H$ (resp. $5H - e$, for all $e \in E(5H)$) contains six pairwise edge-disjoint spanning trees. It can also be deduced from this result that $\mathrm{def}(H)$ is equal to the minimum number of edges which have to be added to $5H$ in order to obtain a graph which has six pairwise edge-disjoint spanning trees.

4.1.3. The defect form of the Molecular Theorem

Theorem 1.2 asserts that G^2 is rigid in $\mathbb{R}^3$ if and only if $\mathrm{def}(G) = 0$. The following stronger form provides the exact relationship between the degree of freedom of G^2 and the

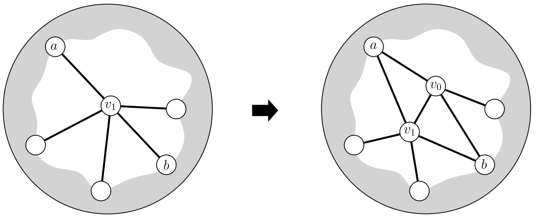

Fig. 3. The vertex splitting operation.

deficiency of G. Thus it implies that $\mathrm{dof}(G^2)$ can be expressed by a purely combinatorial parameter of G that is defined by counting edges between members of partitions of V. This parameter will also show up in our conjectured characterization for globally rigid squares in the next section.

Theorem 4.2 *(Molecular Theorem (defect form) [22]). Let G be a graph with minimum degree at least two. Then*

$$\mathrm{dof}(G^2) = \mathrm{def}(G).$$

It is easy to extend Theorem 4.2 to the case where G may have vertices of degree one, see [16, Lemma 4.2].

Before presenting a complete proof of Theorem 4.2 in Section 4.1.5, we need to recall a few more standard tools from rigidity theory in the next subsection.

4.1.4. Basic operations

In the proof of Theorem 4.2, we shall apply two well-known operations of rigidity theory: vertex splitting and 1-extension. We shall only use the 3-dimensional versions, which are defined as follows. Let $G = (V, E)$ be a graph. Given a vertex $v_1 \in V$ and a partition $\{U_{01}, U_0, U_1\}$ of $N_G(v_1)$ with $|U_{01}| = 2$, the *vertex splitting operation at v_1 with respect to $\{U_{01}, U_0, U_1\}$* removes the edges connecting v_1 to U_0 and inserts a new vertex v_0 as well as new edges between v_0 and $\{v_1\} \cup U_{01} \cup U_0$. If $U_{01} = \{a, b\}$, the operation is said to be a *vertex splitting along v_1a and v_1b*. See Fig. 3. Whiteley proved that the vertex splitting operation preserves the rigidity of a graph. In fact his proof implies the following stronger, infinitesimally rigid version.

Theorem 4.3 *([30]). Let (G, p) be an infinitesimally rigid framework in $\mathbb{R}^3$, and let H be a graph obtained from G by a vertex splitting operation at vertex v_1 along v_1a and v_1b. Let v_0 be the new vertex created by the operation, and let d be a vector in $\mathbb{R}^3$. Suppose that the vectors $\{d, p(a) - p(v_1), p(b) - p(v_1)\}$ are linearly independent. Then the map p can be extended from $V(G)$ to $V(H)$ by specifying $p(v_0)$ such that $p(v_0) = p(v_1) + td$ for some nonzero scalar $t \in \mathbb{R}$ and so that (H, p) is infinitesimally rigid in $\mathbb{R}^3$.*

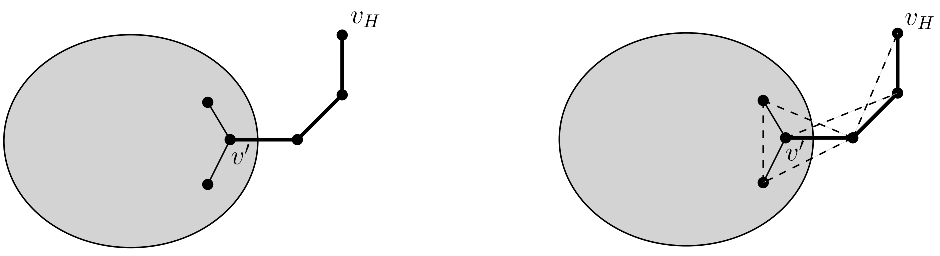

Fig. 4. The graph on the left has a leg with three (thick) edges. The figure on the right also shows the additional (dashed) edges incident with the internal vertices of the leg in its square graph.

This result has the following corollaries: if G is rigid and H is obtained from G by a vertex splitting operation then $\mathrm{dof}(H) \leq \mathrm{dof}(G)$. In particular, if G is rigid then so is H. We shall simply refer to Theorem 4.3 when we use these corollaries.

Another well-known operation, that can be performed on a graph G or a framework (G, p), is called 1-*extension*. This operation removes an edge xy from the graph and adds a new vertex v, along with four edges vx, vy, vw, vz, where w and z are different vertices of $V(G) - \{x, y\}$. When it is applied to a realization of G, the position $p(v)$ of the new vertex should be on the line through $p(x), p(y)$. It is known that 1-extension preserves the rigidity of a graph G, and – assuming that the four neighbours of v are not coplanar – the infinitesimal rigidity of a framework on G in $\mathbb{R}^3$ [31, Theorem 9.2.2].

Variations of the next lemma have been used earlier in the solutions of different rigidity problems, see e.g. [6].

Lemma 4.4 *(Rigid Substitution lemma). Let G be a graph and let K be a rigid subgraph of G on vertex set $X \subseteq V(G)$ with $|X| \geq 3$. Suppose that H is obtained from G by replacing the subgraph K by another rigid graph whose vertex set contains X. Then* $\mathrm{dof}(G) = \mathrm{dof}(H)$.

4.1.5. Proof of Theorem 4.2

We begin with two lemmas. A non-trivial path $P = (v_1, v_2, \ldots v_l)$ is called a *leg* in graph G if $d_G(v_1) = 1$, $d_G(v_i) = 2$ for all $2 \leq i \leq l-1$, and $d_G(v_l) \geq 3$. See Fig. 4. We omit the proof the next simple lemma (cf. [16, Lemma 4.2]).

Lemma 4.5. *Let G be a graph and $P = (v_1, v_2, \ldots v_l)$ be a leg in G. Let H be the graph obtained from G by removing all the vertices of P except v_l. Then* $\mathrm{dof}(G^2) = \mathrm{dof}(H^2) + l - 2$ *and* $\mathrm{def}(G) = \mathrm{def}(H) + l - 1$.

More general versions of the next lemma appeared in [17,22]. We give a proof for completeness. A subgraph H of G is said to be *proper* if $E(H) \neq \emptyset$ and $H \neq G$.

Lemma 4.6. *Let $G = (V, E)$ be a graph with minimum degree at least two, and suppose that G has no proper $\frac{6}{5}$-tree-connected subgraph. Then G has two vertices of degree two which are adjacent.*

Proof. It follows from known results concerning highly tree-connected graphs (see e.g. [14]) that if $5(G-e)$ has at least $6|V|-6$ edges for some $e \in E$ then $G-e$ contains a $\frac{6}{5}$-tree-connected subgraph, which is proper in G. Hence we have

$$5|E| \leq 6|V| - 2. \tag{3}$$

Let $n = |V|$, $m = |E|$, and let n_i be the number of vertices of degree i in G. Since the minimum degree of G is at least two, we have $n = \sum_{i \geq 2} n_i$ and $2m = \sum_{i \geq 2} i n_i$. Suppose that G has no adjacent vertices of degree two. Then $2n_2 \leq \sum_{i \geq 3} i n_i$. Combining this with $n = \sum_{i \geq 2} n_i$, we get

$$2n \leq \sum_{i \geq 3} (i+2) n_i. \tag{4}$$

On the other hand, by $2m = \sum_{i \geq 2} i n_i$ and $n = \sum_{i \geq 2} n_i$, we have $2m = 2n + \sum_{i \geq 3}(i-2)n_i$. Combining this with (4), $10m = 10n + \sum_{i \geq 3} 5(i-2)n_i \geq 10n + \sum_{i \geq 3}(i+2)n_i \geq 12n$, implying $5m \geq 6n$. This contradicts (3). □

It was proved in [16, Theorem 4.1], by a direct argument using squares of graphs, that for a graph G with minimum degree at least two we have $\mathrm{dof}(G^2) \geq \mathrm{def}(G)$. Thus it suffices to prove the next result (which was the missing part before the paper by Katoh and Tanigawa [22]) in order to complete a new direct proof of Theorem 4.2.

Theorem 4.7. *Let G be a graph with minimum degree at least two. Then*

$$\mathrm{dof}(G^2) \leq \mathrm{def}(G).$$

Proof. Suppose, for a contradiction, that the assertion is false and let G be a smallest counter-example. Let $\mathrm{def}(G) = k$, for some integer $k \geq 0$. Then $\mathrm{dof}(G^2) > k$.

Claim 4.8. *G has no proper $\frac{6}{5}$-tree-connected subgraph H.*

Proof. Suppose that G has such a subgraph H. We may assume that H is a proper $\frac{6}{5}$-tree-connected subgraph of G for which $|V(H)|$ is maximal. The minimum degree of H is at least two, and hence we have $\mathrm{dof}(H^2) \leq \mathrm{def}(H) = 0$ by induction. Thus H^2 is rigid. If $V(H) = V(G)$, then G^2 is rigid as well, which gives $k < \mathrm{dof}(G^2) = 0$, a contradiction. So we must have $V(H) \neq V(G)$. Hence we can also assume that H is an induced subgraph of G. Let G/H be the graph obtained from G by contracting the vertices of H into one vertex v_H. Since the contraction of a vertex set cannot increase the deficiency of a graph, we have $\mathrm{def}(G/H) \leq \mathrm{def}(G) = k$. Furthermore, the maximality of H implies that G/H is simple. Note that it has fewer edges than G. By induction $\mathrm{dof}((G/H)^2) \leq k$.

Let $X = N_G(V(H)) = N_{G/H}(v_H)$ and $Y = N_G(V(H) \cup X)$ be the sets of neighbours and second neighbours of v_H in G/H. Since H is $\frac{6}{5}$-tree-connected, each vertex has degree at least two in H. Hence, for each $u \in X$, G^2 contains at least three edges between u and $V(H)$. Therefore $V(H) \cup X$ induces a rigid subgraph in G^2 that we denote by K. We shall consider two cases separately, depending on whether X has at least two vertices or not.

First suppose that $|X| \leq 1$. Suppose that $|X| = 1$ (the case when X is empty is similar, but simpler). In this case v_H has degree one in G/H. Let P be a maximal path in G/H starting with v_H for which each internal vertex has degree two in G/H. Since G has minimum degree two, the other end-vertex v' of P has $d_{G/H}(v') = d_G(v') \geq 3$. Thus P is a leg. Let $\ell = |V(P)|$ and $J = G - (V(H) \cup (V(P) - \{v'\}))$. By induction, we have $\mathrm{dof}(J^2) \leq \mathrm{def}(J)$. Furthermore, Lemma 4.5 gives that $\mathrm{dof}((G/H)^2) = \mathrm{dof}(J^2) + \ell - 2$ and $\mathrm{def}(G/H) = \mathrm{def}(J) + \ell - 1$. By observing that G^2 can be obtained from $(G/H)^2$ by gluing a rigid graph (namely, K) along an edge (the first edge of P), we obtain $\mathrm{dof}(G^2) = \mathrm{dof}((G/H)^2) + 1$. By putting these inequalities together we get $\mathrm{dof}(G^2) \leq \mathrm{def}(G/H) = k$, a contradiction.

It remains to consider the case when $|X| \geq 2$. Let $I = N_G(V - V(H))$ with $I = \{v_1, v_2, \ldots, v_t\}$. Our strategy is to construct a graph G' from $(G/H)^2$ by a sequence of vertex splitting operations that replace v_H by a set of t vertices that we shall identify with the vertices of I in a natural way, such that

(i) $I \cup X$ induces a rigid subgraph in G', and
(ii) the set of edges between Y and I in G' is the same as that in G^2.

This will allow us to apply the Rigid Substitution lemma to argue that replacing G' by K does not change the degree of freedom (which will give the desired contradiction).

We need one more observation. Let $X_i = N_G(v_i) \cap X$, and $Y_i = N_G(X_i) \cap Y$ for $1 \leq i \leq t$. We claim that

$$X_i \cap X_j = \emptyset \text{ and } Y_i \cap Y_j = \emptyset \tag{5}$$

for $1 \leq i \neq j \leq t$. To see this, suppose that $u \in X_i \cap X_j$. Then $V(H) \cup \{u\}$ induces a $\frac{6}{5}$-tree-connected subgraph and hence the maximality of H implies that $V(G) = K = V(H) \cup \{u\}$. Thus G^2 is rigid, a contradiction. We obtain $Y_i \cap Y_j = \emptyset$ by a similar argument. The definition of I, the minimum degree condition on G, the maximality of H, and (5) imply that $\{X_1, \ldots, X_t\}$ and $\{Y_1, \ldots, Y_t\}$ are partitions of X and Y, respectively. Moreover, $X_i \neq \emptyset$ for $1 \leq i \leq t$.

Next we describe the algorithm that constructs G'.

- Initially, let $G' = (G/H)^2$, $v_t = v_H$, and pick $x_1, x_2 \in X$.
- For $i = 1, 2, \ldots, t-1$, do the following:
  - Apply a vertex splitting operation in G' at v_t along the edges x_1v_t, x_2v_t. Denote the two vertices created by the split by v_i and v_t. Perform the operation in such a

way that each vertex in Y_i gets connected to v_i and each vertex in $\bigcup_{j>i} Y_j$ remains connected to v_t.

In the resulting graph G' we identify I with $\{v_1, v_2, ..., v_t\}$. Since $X \cup \{v_H\}$ induces a rigid subgraph in $(G/H)^2$ and the vertex splitting operation preserves rigidity by Theorem 4.3, property (i) follows. The construction implies property (ii).

To complete the proof of this case replace the rigid subgraph of G' on vertex set $I \cup X$ by K (keeping the common vertices $I \cup X$ fixed) to obtain a graph G''. By (ii), the graph G'' is isomorphic to G^2. Since the vertex splitting operations do not increase the degree of freedom and by the Rigid Substitution lemma, we have $\mathrm{dof}(G^2) = \mathrm{dof}(G'') \leq \mathrm{dof}((G/H)^2) \leq k$, a contradiction. This completes the proof of the claim. □

We henceforth assume that G has no proper $\frac{6}{5}$-tree-connected subgraph. Then G has two adjacent vertices x, y of degree two by Lemma 4.6. At this point it is convenient to directly verify the theorem in the special case when G is isomorphic to a cycle C_k of length $k \geq 3$: it is not hard to check that we have $\mathrm{dof}(C_k^2) = \mathrm{def}(C_k) = \max\{k - 6; 0\}$. (The expression for the degree of freedom of a cycle can also be deduced from a result of [10].) So we may suppose that G is not a cycle.

Let $N_G(x) = \{a, y\}$ and $N_G(y) = \{b, x\}$. Since G has minimum degree at least two and G is not a cycle, we may suppose that $d_G(b) \geq 3$. From the fact that G has no proper $\frac{6}{5}$-tree-connected subgraph, we can also observe that $a \neq b, ab \notin E(G)$, and that G/xy is simple. Since contraction cannot increase the deficiency, we have $\mathrm{def}(G/xy) \leq \mathrm{def}(G) = k$. Hence $\mathrm{dof}((G/xy)^2) \leq k$ follows by induction. For simplicity we denote the vertex of G/xy obtained by the contraction of xy by y.

Claim 4.9. *$ab \notin \mathrm{cl}(G^2)$ and $\mathrm{dof}((G/xy)^2 - ab) > k$.*

Proof. Observe that $G^2 + ab$ is obtained from $(G/xy)^2$ by a vertex splitting at y along ay and by. Hence, if $ab \in \mathrm{cl}(G^2)$, then $\mathrm{dof}(G^2) = \mathrm{dof}(G^2 + ab) = \mathrm{dof}((G/xy)^2) \leq k$, contradicting $\mathrm{dof}(G^2) > k$.

Similarly, observe that G^2 is obtained from $(G/xy)^2 - ab$ by a vertex splitting at y along ay and by. Hence, $\mathrm{dof}((G/xy)^2 - ab) \geq \mathrm{dof}(G^2) > k$. □

Let $H = G - x - y$. Let H' be the graph obtained from $H^2 + ab$ by a 1-extension by splitting ab with new vertex y and adding two new edges between y and $N_G(b) \setminus \{y\}$.

We next verify two inequalities. The first one asserts that

$$\mathrm{dof}(H' + ab) \geq k + 1. \tag{6}$$

To see this, suppose that $\mathrm{dof}(H' + ab) \leq k$ holds. Since a spanning subgraph of G^2 can be obtained from $H' + ab$ by applying a 1-extension which splits the edge ab by the new vertex x, we have $\mathrm{dof}(G^2) \leq k$, a contradiction. The second one claims that

$$\mathrm{dof}(H' + ab) \leq \mathrm{dof}(H^2) - 2. \tag{7}$$

We can see this as follows. By $H^2 \subseteq G^2$ and Claim 4.9, we have $ab \notin \mathrm{cl}(H^2)$. Hence $\mathrm{dof}(H') \leq \mathrm{dof}(H^2 + ab) = \mathrm{dof}(H^2) - 1$. Observe that H' is a subgraph of $(G/xy)^2$. Since $\mathrm{dof}((G/xy)^2) \leq k$, Claim 4.9 implies that $ab \notin \mathrm{cl}(H')$, implying (7).

Claim 4.10. *$d_G(a) \geq 3$.*

Proof. Suppose that this is not the case. Let S be the maximal path consisting of degree two vertices that contains a, x, y in G. Then, in H, $S \setminus \{x, y\}$ and the vertex incident to $S \setminus \{x, y\}$ form a leg. So we can use Lemma 4.5 to get

$$\mathrm{dof}(H^2) = \mathrm{dof}((G - S)^2) + (|S \setminus \{x, y\}| + 1 - 2) = \mathrm{dof}((G - S)^2) + |S| - 3.$$

On the other hand, $G - S$ is obtained from G by removing $|S|$ vertices and $|S| + 1$ edges, and so

$$\mathrm{def}(G - S) \leq \mathrm{def}(G) - 6|S| + 5(|S| + 1) = k - |S| + 5.$$

By induction, $\mathrm{def}(G - S) = \mathrm{dof}((G - S)^2)$. By combining these equations we obtain $\mathrm{dof}(H^2) \leq k + 2$, which is a contradiction, as (6) and (7) together imply that $\mathrm{dof}(H^2) \geq k + 3$. $\square$

Claim 4.11. *$\mathrm{dof}(H^2) = k + 3$ and $\mathrm{dof}(H' + ab) = k + 1$.*

Proof. Since H is obtained from G by removing x and y and the three edges incident to them, we have $\mathrm{def}(H) + 2 \cdot 6 - 3 \cdot 5 \leq \mathrm{def}(G)$, which gives $\mathrm{def}(H) \leq k + 3$. It follows from Claim 4.10 that the degree of a in H is at least two, and hence we can use induction to obtain

$$\mathrm{dof}(H^2) \leq k + 3. \tag{8}$$

By using the inequalities (6), (7), and (8) we can deduce that equality must hold everywhere, from which the statement of the claim follows. $\square$

Consider $H' + ab$. See Fig. 5(a). $H' + ab$ is a subgraph of $(G/xy)^2$, and $\mathrm{dof}(H' + ab) = k + 1 > k \geq \mathrm{dof}((G/xy)^2)$ by Claim 4.11. Hence $(G/xy)^2$ contains an edge e with $e \notin \mathrm{cl}(H' + ab)$. Since every edge in $E((G/xy)^2) \setminus E(H' + ab)$ is incident to y, e is incident to y. Note also that, in $H' + ab$, $N_G(b) \cup \{b\}$ induces a rigid subgraph, and hence every edge between y and $N_G(b)$ belongs to $\mathrm{cl}(H' + ab)$. Hence, e connects y and a neighbour c of a.

Let $G_1 = H' + ab + e$. See Fig. 5(b). We have $\mathrm{dof}(G_1) = k$. Next we apply a 1-extension so that we split the edge e in G_1 by adding a new vertex x' and two new edges $x'a$ and

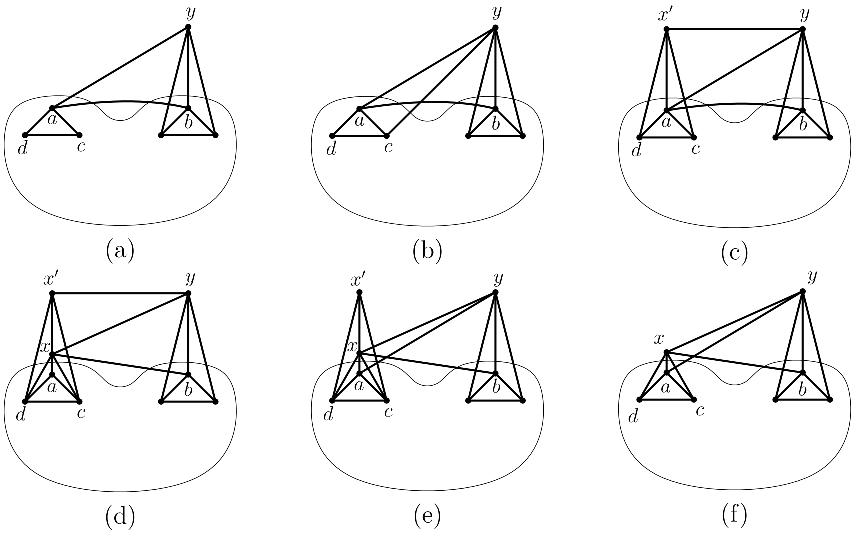

Fig. 5. (a) $H' + ab$, (b) G_1, (c) G_2, (d) G_3, (e) G_4, (f) $G_4 - x' = G^2$.

$x'd$ for some $d \in N_G(a) \setminus \{x, c\}$. (This is possible by Claim 4.10.) Let G_2 be the resulting graph. See Fig. 5(c). Let (G_2, p) be a generic realization of G_2. We perform the vertex splitting operation at a with respect to the partition $\{\{c, d\}, \{x', y, b\}, N_G(a) \setminus \{x, c, d\}\}$ of $N_{G_2}(a)$ to get a new framework (G_3, p'). Let x be the new vertex obtained by the vertex splitting as in Fig. 5(d). We perform this vertex splitting operation such that p' is an extension of p and $p'(x)$ is in the interior of the line segment between $p(a)$ and $p(x')$. By Theorem 4.3, we can do this without increasing the degree of freedom, i.e., $\mathrm{dof}(G_3, p') \leq k$.

Let $G_4 = G_3 - yx' + ya$. See Fig. 5(e). In (G_4, p'), $\{x', x, a, c, d\}$ induces a rigid subframework. Since $p(x'), p(x), p(a)$ are collinear, we have $\mathrm{rank}\ R(G_4, p') = \mathrm{rank}\ R(G_4 + yx', p') \geq \mathrm{rank}\ R(G_3, p')$ by Lemma 4.1. Hence, $\mathrm{dof}(G_4) \leq k$. Note that x' has degree three in G_4. Thus $\mathrm{dof}(G_4 - x') \leq k$. Finally, by observing that $G_4 - x' \subseteq G^2$, we obtain $\mathrm{dof}(G^2) \leq k$. This final contradiction completes the proof of the theorem. $\square$

4.2. A conjecture for the d-dimensional version

Let $G = (V, E)$ be a graph. Recall that a k-chain in G is a path with k vertices for which every internal vertex has degree two in G (and the end-vertices of the path are distinct). For a vertex $v \in V$ and non-negative integer k let $N_G^{\leq k}(v) := \{u \in V : \mathrm{dist}_G(u, v) \leq k\}$ denote the set of vertices u for which a shortest path from v to u in G has at most k edges.

Lemma 4.12. *Let v be a vertex in G with $d_G(v) \geq 3$ and $d \geq 3$. Then the subgraph of G^{d-1} induced by the vertex set $N_G^{\leq d-2}(v)$ is rigid in $\mathbb{R}^d$.*

Proof. Let $X = N_G(v) \cup \{v\}$ and $Y = N_G^{\leq d-2}(v)$. We have $X \subseteq Y$. Consider a maximal subset $X' \subseteq Y$ for which $X \subseteq X'$ and the subgraph of G^{d-1} induced by X' is rigid in $\mathbb{R}^d$. Since X induces a complete (and hence rigid) subgraph in G^{d-1}, such a set indeed exists. We are done if $X' = Y$ holds, so we may assume that there is a vertex $w \in Y - X'$. We can also assume that there is an edge from w to X' in G. Let $Z = X' \cup \{w\}$. First suppose that for all $x \in X'$ the distance from w to x in $G[Z]$, i.e. the subgraph of G induced by Z, is at most $d-1$. Then w is connected to each vertex of X' in G^{d-1}, implying that Z induces a rigid subgraph in G^{d-1}. This contradicts the choice of X'. Next suppose that there is a vertex $q \in X'$ for which a shortest path P from w to q in $G[Z]$ has at least d edges. Since v has at least three neighbours in G, there is a vertex $r \in X$ which does not belong to P. It follows that w is connected to $d-1$ vertices of P as well as to vertex r in G^{d-1}. We can now use the Extension lemma to deduce that Z induces a rigid subgraph of G^{d-1}, contradicting the maximality of X'. $\square$

Before formulating our conjecture we recall the definition of another well-studied structure of rigidity theory. A d-dimensional *body-hinge framework* is a structure consisting of rigid bodies in d-space in which some pairs of bodies are connected by a hinge. Each hinge corresponds to a $(d-2)$-dimensional affine subspace which restricts the relative motion of the corresponding bodies to a rotation about the hinge. The framework is rigid if every such motion preserves the distances between all pairs of points belonging to different rigid bodies, i.e. the motion extends to an isometry of $\mathbb{R}^d$. In the underlying multigraph of a body-hinge framework the vertices correspond to the bodies and the edges correspond to the hinges.

We next define a graph $B_G = (\mathcal{X}_G, \mathcal{E})$ in which the vertices correspond to certain subsets of V. Formally, the vertex set of B_G is

$$\mathcal{X}_G := \{V(C) : C \text{ is a } d\text{-chain in } G\} \cup \{N^{\leq d-2}(v) : v \in V, d_G(v) \geq 3\},$$

and there is an edge connecting two vertices $X_1, X_2 \in \mathcal{X}_G$ if $|X_1 \cap X_2| \geq d-1$. Moreover, if $|X_1 \cap X_2| \geq d$, then B_G contains two copies of the edge between X_1 and X_2. Note that if C is a d-chain then $V(C)$ induces a complete (and hence rigid) subgraph of G^{d-1}. Furthermore, for each vertex $v \in V$ with $d_G(v) \geq 3$ the set $N^{\leq d-2}(v)$ induces a rigid subgraph in G^{d-1} by Lemma 4.12. Thus we may think of the vertices of B_G as d-dimensional rigid bodies (as subgraphs of G^{d-1}). In this sense each edge of B_G represents a hinge between two such bodies (and the existence of parallel edges shows that the union of the corresponding bodies is rigid).

With this definition, we can now formulate our conjecture. It is not hard to see that when $d = 3$ the condition is equivalent to that of Theorem 1.2.

Conjecture 3. *Let $d \geq 3$ be an integer and $G = (V, E)$ be a connected graph with at least $d+1$ vertices. Then G^{d-1} is rigid in $\mathbb{R}^d$ if and only if $\left(\binom{d+1}{2} - 1\right) B_G$ contains $\binom{d+1}{2}$ edge-disjoint spanning trees.*

We sketch the proof of necessity. Consider the union of the complete graphs $K(X)$ on vertex sets X, over all $X \in \mathcal{X}_G$, and let H be the resulting graph on V. Let (H, p) be a generic d-dimensional realization of H. This realization determines a (non-generic) body-hinge framework in $\mathbb{R}^d$ by regarding each $K(X)$ as a body and the intersection $K(X_1 \cap X_2)$ with $|X_1 \cap X_2| = d-1$ as a hinge. If $|X_1 \cap X_2| \geq d$ holds then $K(X_1) \cup K(X_2)$ is rigid, which is represented by the two copies of the edge (i.e. two hinges) between X_1 and X_2. In this sense the underlying multigraph of this d-dimensional body-hinge structure is exactly B_G. Hence, if $\left(\binom{d+1}{2} - 1\right) B_G$ does not contain $\binom{d+1}{2}$ edge-disjoint spanning trees, then we can use a theorem of Tay and Whiteley (see e.g. [31]) to deduce that any body-hinge framework with underlying multigraph B_G is infinitesimally flexible. This implies that there is a map $\dot{p} : V \to \mathbb{R}^d$ such that the restriction of $\dot{p}$ to each $X \in \mathcal{X}_G$ is an infinitesimal congruence on X but $\dot{p}$ itself is not an infinitesimal congruence. Thus, by using that every edge in H is induced by some $X \in \mathcal{X}_G$, we obtain that $\dot{p}$ is a nontrivial infinitesimal motion of (H, p).

We claim that $\dot{p}$ is a nontrivial infinitesimal motion of (G^{d-1}, p). This can be checked by observing $G^{d-1} \subseteq H$. To see this, consider any edge uv in G^{d-1}. Then $\mathrm{dist}_G(u, v) \leq d - 1$. If G has a vertex w of degree at least three such that $u, v \in N^{\leq d-2}(w)$, then $uv \in E(H)$ holds. Hence, suppose there is no such vertex w. Then, by $\mathrm{dist}_G(u, v) \leq d-1$, every path between u and v forms a chain. Since $|V| \geq d + 1$, there is a d-chain that contains u and v, implying $uv \in E(H)$. This implies the claim and completes the proof of necessity.

5. Global rigidity of squares of graphs

Finding the characterization of those graphs G for which G^{d-1} is globally rigid in $\mathbb{R}^d$ seems to be a rather difficult problem. Obtaining the counterpart of the Molecular Theorem ($d = 3$) is already challenging. In this section we focus on the global rigidity of G^2 in $\mathbb{R}^3$, and offer the Molecular Global Rigidity Conjecture. We also give a proof of the necessity of the conjectured condition along with some further remarks and examples. Our conjecture is as follows.

Conjecture 4 *(Molecular Global Rigidity Conjecture). Let G be a graph on at least five vertices with minimum degree at least two. Then G^2 is globally rigid in $\mathbb{R}^3$ if and only if G^2 is 4-connected and G is highly $\frac{6}{5}$-tree-connected.*

We note that similar, but somewhat weaker or incomplete versions of Conjecture 4 appeared earlier in [5,19,21].

Before proving the "only if" direction we give some examples that illustrate the difficulties and the connections to Theorem 1.2. It is clear from Theorem 2.3 that the 4-connectivity of G^2 is a necessary condition. The graph in Fig. 6 shows that this condition cannot be omitted even if G is highly $\frac{6}{5}$-tree connected.

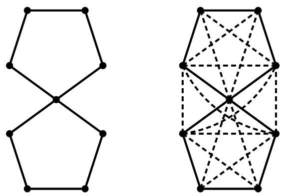

Fig. 6. A graph G for which G is highly $\frac{6}{5}$-tree connected but G^2 is not 4-connected. The vertex of degree four is sticky.

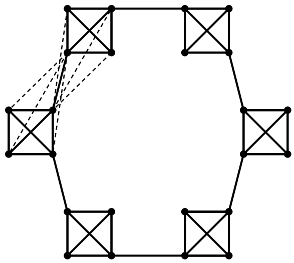

Fig. 7. A graph G (solid edges) for which G is not highly $\frac{6}{5}$-tree connected, but G^2 is redundantly rigid.

Theorem 2.3 also implies that if G^2 is globally rigid then it is redundantly rigid. Characterizing the redundant rigidity of the square of a graph in $\mathbb{R}^3$ is also an open problem. It was pointed out in [15] that the high $\frac{6}{5}$-tree-connectivity of G is, in general, not sufficient to guarantee the redundant rigidity of G^2. On the other hand, the body-hinge version of Theorem 1.2 and the characterization of globally rigid body-hinge graphs in [20] may suggest that the high $\frac{6}{5}$-tree-connectivity of G is a necessary condition for redundant rigidity. The graph in Fig. 7 shows that it is not the case.

Note that the square of graph G in Fig. 7 is 4-connected. However, as it will follow from the main theorem of this section, G^2 is not globally rigid. Hence G^2 is another example which shows that the necessary conditions in Hendrickson's theorem are not sufficient to imply global rigidity in $\mathbb{R}^3$. (See [20] for a more detailed discussion on such examples.)

In the rest of this section we verify the "only if" direction of Conjecture 4. We start with a simple lemma. A cut-vertex v of a connected graph G is called *sticky* if there is a connected component C of $G - v$ on at least three vertices, for which the number of edges connecting C to v is exactly two. (See Fig. 6 for an example.) Since we only need the easier "only if" direction of the lemma, the proof is omitted.

Lemma 5.1. *Let G be a graph with minimum degree at least two. Then G^2 is 4-connected if and only if*
(i) G is 2-edge-connected, and
(ii) G has no sticky cut-vertex.

Let $G = (V, E)$ be a graph. Following [11] we define a *cover* of G as a collection $\mathcal{X}$ of subsets of V, each of size at least two, such that $\bigcup_{X\in\mathcal{X}} E(X) = E$. A cover $\mathcal{X} = \{X_1, X_2, ..., X_m\}$ is 2-*thin* if $|X_i \cap X_j| \leq 2$ for all $1 \leq i < j \leq m$. For $X_i \in \mathcal{X}$ let $f(X_i) = 1$ if $|X_i| = 2$ and $f(X_i) = 3|X_i| - 6$ if $|X_i| \geq 3$. Let $H(\mathcal{X})$ be the set of all pairs of vertices uv such that $X_i \cap X_j = \{u, v\}$ for some $1 \leq i < j \leq m$. For each $uv \in H(\mathcal{X})$ let $h(uv)$ be the number of sets X_i in $\mathcal{X}$ with $\{u, v\} \subseteq X_i$ and put

$$\mathrm{val}(\mathcal{X}) = \sum_{X\in\mathcal{X}} f(X) - \sum_{uv\in H(\mathcal{X})} (h(uv) - 1).$$

We say that a 2-thin cover $\mathcal{X}$ of graph $G = (V, E)$ is *independent* if the rows of the rigidity matrix of a generic realization of the graph $(V, H(\mathcal{X}))$ are linearly independent.

Lemma 5.2 *([11]). Let G be a graph and let $\mathcal{X}$ be an independent 2-thin cover of G. Then $r(G) \leq \mathrm{val}(\mathcal{X})$.*

The next two lemmas follow from the proof of [16, Theorem 3.4] and [16, Lemma 3.2], respectively. Recall (2), and the definition of a tight partition.

Lemma 5.3 *([16]). Let $G = (V, E)$ be a graph with minimum degree at least two. Suppose that $\mathcal{P} = \{P_1, P_2, ..., P_t\}$ is a tight partition of V. Let $X_i = P_i \cup N_G(P_i)$ for $1 \leq i \leq t$ and let $\mathcal{X} = \{X_1, \ldots, X_t\}$. Then $\mathcal{X}$ is an independent 2-thin cover of G^2. Furthermore, we have $H(\mathcal{X}) = \{uv : uv \in E_G(\mathcal{P})\}$, $h(uv) = 2$ for all $uv \in H(\mathcal{X})$, $|X_i| \geq 3$, and $|N_G(P_i)| = d_G(P_i)$ for $1 \leq i \leq t$.*

Lemma 5.4 *([16]). Let G be a graph with minimum degree at least two. The (multi)graph obtained from G by contracting the members of a tight partition has no cycles of length at most five.*

We are ready to state the main result of this section.

Theorem 5.5. *Let $G = (V, E)$ be a graph with minimum degree at least two and $|V| \geq 5$. Suppose that G^2 is globally rigid in $\mathbb{R}^3$. Then G^2 is 4-connected and G is highly $\frac{6}{5}$-tree-connected.*

Proof. The necessity of 4-connectivity follows from Theorem 2.3. Since global rigidity implies rigidity, we may assume, by (the easier direction of) Theorem 1.2, that G is $\frac{6}{5}$-tree-connected. For a contradiction suppose that G is not highly $\frac{6}{5}$-tree-connected. Let $H = 5G$. Then there is a tight partition $\mathcal{P} = \{P_1, P_2, ..., P_t\}$ of V with $t \geq 2$, which satisfies

$$e_H(\mathcal{P}) = 5e_G(\mathcal{P}) = 6(t - 1). \tag{9}$$

Equality (9) implies that there exists a member of $\mathcal{P}$, call it P_1, with $d_G(P_1) = 2$. Let $e = x_1x_2$ and $f = y_1y_2$ be the edges incident with P_1 in G, with $x_1, y_1 \in P_1$.

Suppose that e and f have a vertex v in common. Then $v \in P_1$ must hold by Lemma 5.4. A similar argument shows that $|V - P_1| \geq 3$. Then $P_1 = \{v\}$ follows, since otherwise v is a sticky cut-vertex, contradicting the 4-connectivity of G^2 and Lemma 5.1.

So either (i) P_1 is a singleton or (ii) the four vertices x_i, y_i, $i = 1, 2$ are pairwise distinct. In the rest of the proof we shall consider a special cover of G^2 and an associated upper bound on the rank of G^2. By slightly refining and modifying an analysis of [16] we shall deduce that, roughly speaking, there is an edge induced by $P_1 \cup N_G(P_1)$ in G^2 which is not redundant. It will contradict the fact that G^2 is globally rigid.

Let $X_i = P_i \cup N_G(P_i)$ for $1 \leq i \leq t$ and let $\mathcal{X} = \{X_1, \ldots, X_t\}$. By Lemma 5.3, $\mathcal{X}$ is an independent 2-thin cover of G^2 and we have

$$\mathrm{val}(\mathcal{X}) = \sum_{i=1}^{t} f(X_i) - \sum_{uv \in H(\mathcal{X})} (h(uv) - 1) = \sum_{i=1}^{t} (3|X_i| - 6) - |E_G(\mathcal{P})| =$$

$$= \sum_{i=1}^{t} 3(|P_i| + d_G(P_i)) - |E_G(\mathcal{P})| - 6t = 3|V| + 6|E_G(\mathcal{P})| - |E_G(\mathcal{P})| - 6t =$$

$$= 3|V| + 5e_G(\mathcal{P}) - 6t = 3|V| - 6.$$

First we consider case (i) when P_1 is a singleton. Then X_1 induces a complete graph on three vertices in G^2, namely, a triangle with edges $e = x_1x_2$, $f = y_1y_2$, and a third edge q. We have $x_1 = y_1$ and the two hinges in X_1 correspond to e and f. Consider the cover $\mathcal{X}' = \{\mathcal{X} - X_1\} \cup \{x_1, x_2\} \cup \{y_1, y_2\}$. Observe that $\mathcal{X}'$ is a cover of $G^2 - q$ and the hinge sets of $\mathcal{X}$ and $\mathcal{X}'$ are the same. Furthermore, by inspecting the count above we observe that $\mathrm{val}(\mathcal{X}') = \mathrm{val}(\mathcal{X}) - 1$. By using Lemma 5.2 this gives $r(G^2 - q) \leq \mathrm{val}(\mathcal{X}') < \mathrm{val}(\mathcal{X}) = 3|V| - 6$. Hence q is not redundant in G^2, contradicting the fact that G^2 is globally rigid (and Theorem 2.3).

It remains to consider case (ii), when the vertices x_i, y_i, $i = 1, 2$ are pairwise distinct. In this case we apply a similar argument after slightly modifying the graph and the cover.

Let $W = P_1 - \{x_1, y_1\}$ and $G^* = G^2 - W + K(\{x_1, x_2, y_1, y_2\})$. Let $\mathcal{X}^* = \{\mathcal{X} - X_1\} \cup \{x_1, x_2\} \cup \{y_1, y_2\} \cup \{x_1, y_1\} \cup \{x_1, y_2\} \cup \{x_2, y_1\}$. Note that $\mathcal{X}^*$ covers $G^* - x_2y_2$, and the hinge sets of $\mathcal{X}^*$ and $\mathcal{X}$ are the same. (To see this note that x_1y_2 and y_1x_2 cannot be hinges by the no short cycle property.) A count similar to that of the previous case gives that $\mathrm{val}(\mathcal{X}^*) < 3|V(G^*)| - 6$. By using Lemma 5.2 this gives $r(G^* - x_2y_2) < 3|V(G^*)| - 6$, which implies that x_2y_2 is not redundant in G^*. Thus G^* is not globally rigid by Theorem 2.3. Since G^2 is obtained from G^* by attaching the set W of vertices along a complete subgraph (and possibly deleting some edges), G^2 is not globally rigid. This contradiction completes the proof. $\square$

6. A sufficient edge-connectivity condition

In this section we prove that if G is 3-edge-connected then G^{d-1} is globally rigid in $\mathbb{R}^d$, for all $d \geq 3$. This is the strongest possible sufficient condition in terms of the edge- or vertex-connectivity of G, which follows from the fact that for a cycle C_n on n vertices, with n large enough, C_n^{d-1} is not even rigid in $\mathbb{R}^d$.

In $\mathbb{R}^3$ a weaker sufficient condition was obtained by Gortler, Gotsman, Liu, and Thurston [7], who proved that the square of a 4-vertex-connected graph is globally rigid[2] in $\mathbb{R}^3$.

The main result of this section is as follows.

Theorem 6.1. *Let $G = (V, E)$ be a 3-edge-connected graph and let $d \geq 3$. Then G^{d-1} is globally rigid in $\mathbb{R}^d$ for all $d \geq 3$.*

In some parts of the proof of Theorem 6.1 the cases $d = 3$ and $d \geq 4$ are completely separated, while in some other parts the proofs of the higher dimensional versions are substantially more complicated. Thus the reader may find it useful to first read the proof by assuming that $d = 3$.

Proof of Theorem 6.1. We shall prove the theorem by induction on $|V|$. The smallest 3-edge-connected graph is K_4, for which the statement is obvious. Thus we may assume that $|V| \geq 5$. We may also assume that G is minimally 3-edge-connected, that is, $G - e$ is not 3-edge-connected for all $e \in E$. Therefore, by a well-known result of Lick [24], G has a vertex v with $d(v) = 3$. Our basic strategy is to apply Theorem 2.5 at v. To this end, we claim that $G^{d-1} - v$ is rigid in $\mathbb{R}^d$. Since the proof method depends on whether $d = 3$ or $d > 3$, we give it in two separate claims.

Claim 6.2. *Let v be a vertex of degree three in G. Then $G^2 - v$ is rigid in $\mathbb{R}^3$.*

Proof. We show that $(G - v)^2$ is rigid. Since $(G - v)^2$ is a spanning subgraph of $G^2 - v$, this will imply the claim. Let $H = G - v$. For a contradiction suppose that H^2 is not rigid. Now the minimum degree of H is at least two, so we can use Theorem 1.2 to deduce that there is a partition $\mathcal{P} = \{X_1, X_2, ..., X_t\}$ of $V(H)$ with $t \geq 2$ for which

$$5e_H(\mathcal{P}) \leq 6t - 7. \tag{10}$$

Since G is 3-edge-connected and $d_G(v) = 3$, we have $d_H(X_i) \geq 2$ for all $1 \leq i \leq t$, and $d_H(X_i) \geq 3$ for all but at most three members of $\mathcal{P}$. This implies

[2] A different proof for this result is as follows: if G is 4-vertex-connected then $G - v$ is 3-vertex-connected for all $v \in V(G)$. Thus $5(G - v)$ is 15-edge-connected, which implies, by the results of Nash-Williams and Tutte [25,28], that it contains 6 edge-disjoint spanning trees. Hence $(G - v)^2$ (and also $G^2 - v$) is rigid for all $v \in V(G)$ by Theorem 1.2. Therefore G^2 is globally rigid in $\mathbb{R}^3$ by Theorem 2.4.

$$5e_H(\mathcal{P}) = 5\left(\frac{\sum_i d_H(X_i)}{2}\right) \geq \frac{5(3t-3)}{2} = \frac{15t-15}{2} \tag{11}$$

It is easy to check that we cannot have (10) and (11) at the same time. This contradiction shows that H^2 is rigid in $\mathbb{R}^3$, as claimed. □

Next we deduce the same conclusion for $d \geq 4$ by using a different approach.

Claim 6.3. *Let v be a vertex of degree three in G. Suppose that $d \geq 4$. Then $G^{d-1} - v$ is rigid in $\mathbb{R}^d$.*

Proof. Let $H = G - v$. Since H^{d-1} is a spanning subgraph of $G^{d-1} - v$, it suffices to show that H^{d-1} is rigid. Let X be the set of vertices of degree two in H and let $Y = V(H) - X$. Since G is 3-edge-connected, $|X| \leq 3$. Consider a vertex $u \in Y$. Then $N_H^{\leq d-2}(u)$ induces a rigid subgraph in H^{d-1} by Lemma 4.12. We denote this subgraph of H^{d-1} by B_u.

If $V(B_u) = V(H)$ for some $u \in Y$ then we are done. So we may assume that for every $u \in Y$ there is a vertex w with $\mathrm{dist}_H(u, w) > d - 2$. We claim that for all $u_1, u_2 \in Y$ with $u_1u_2 \in E(H)$ we have

$$|N_H^{\leq d-2}(u_1) \cap N_H^{\leq d-2}(u_2)| \geq d. \tag{12}$$

To see this, let s be the farthest point in H from u_1, and take a shortest path $P = w_1, w_2, \dots, w_k$ starting from $w_1 = u_1$ and ending at $w_k = s$. Since $\mathrm{dist}_H(u_1, s) > d - 2$, we can take the subpath $P' = w_1, w_2, \dots, w_{d-1}$ of length $d - 2$. Note that $V(P') \subseteq N_H^{\leq d-2}(u_1)$. Note also that, since u_1 has degree at least three in H, there is at least one vertex $t \in N_H(u_1) \setminus (V(P') \cup \{u_2\})$. This vertex t belongs to $N_H^{\leq d-2}(u_1) \cap N_H^{\leq d-2}(u_2)$ by the assumption $d \geq 4$. If u_2 is not on P', then $\{u_2, t, w_1 = u_1, w_2, \dots, w_{d-2}\}$ is in $N_H^{\leq d-2}(u_1) \cap N_H^{\leq d-2}(u_2)$. On the other hand, if u_2 is on P', then $\{t, w_1 = u_1, w_2, \dots, w_{d-2}, w_{d-1}\}$ is in $N_H^{\leq d-2}(u_1) \cap N_H^{\leq d-2}(u_2)$. Hence, (12) follows.

(12) implies that $B_{u_1} \cup B_{u_2}$ is rigid in $\mathbb{R}^d$ (by the Gluing lemma). Thus, for each connected component C of $H - X$, $\bigcup_{u \in C} B_u$ forms a rigid subgraph of H^{d-1}.

Since G is 3-edge-connected, each vertex in X is adjacent to a vertex in Y in H. If $H - X$ is connected, then $\bigcup_{u \in Y} B_u$ is a rigid subgraph of H^{d-1} spanning $V(H)$, implying the rigidity of H. Hence we can assume that $H - X$ is not connected. By the fact that G is 3-edge-connected, v has degree three in G, and X is the set of degree two vertices in H, it can be checked that $H - X$ consists of two connected components C_1 and C_2 and X consists of three vertices w_1, w_2, w_3, each of which is adjacent to each component C_i in H. In other words, H has three paths $a_iw_ib_i$ with $a_i \in C_1, w_i \in X, b_i \in C_2$ for $i = 1, 2, 3$.

Note that all of a_i, w_i, b_i for $i = 1, 2, 3$ are contained in both $\bigcup_{u \in C_1} B_u$ and $\bigcup_{u \in C_2} B_u$, and hence their intersection has size at least 5. In fact, we further have $N_{C_2}^{\leq d-4}(b_i) \subseteq N_H^{\leq d-2}(a_i)$, which implies that $\left|(\bigcup_{u \in C_1} B_u) \cap (\bigcup_{u \in C_2} B_u)\right| \geq 5 + (d - 4) = d + 1$ or

$V(C_2) \subseteq \bigcup_{u\in C_1} B_u$. In either case, $\bigcup_{u\in C_1} B_u \cup \bigcup_{u\in C_2} B_u$ is a rigid spanning subgraph of H. This completes the proof. $\square$

Let v be a vertex of degree three in G and suppose that $G - v + K(N_G(v))$ is 3-edge-connected. In this case we can use the following argument to show that G^{d-1} is globally rigid in $\mathbb{R}^d$. Consider the graph $J := G^{d-1} - v + K(N_{G^{d-1}}(v))$. The vertex set $N_{G^{d-1}}(v)$ consists of the three neighbours of v in G as well as the second (third, and so on, up to $d-1$) neighbours of v in G. Since the vertices of $N_{G^{d-1}}(v)$ are pairwise adjacent in J, it follows that $(G - v + K(N_G(v)))^{d-1}$ is a spanning subgraph of J. Now $(G - v + K(N_G(v)))^{d-1}$ is globally rigid in $\mathbb{R}^d$ by induction (since $G - v + K(N_G(v))$ is 3-edge-connected), implying that so is J. Claims 6.2 and 6.3 show that $G^{d-1} - v$ is rigid in $\mathbb{R}^d$.

If the degree of v in G^{d-1} is at least $d+1$ then the global rigidity of G^{d-1} follows immediately from Theorem 2.5. Otherwise, when $d_{G^{d-1}}(v) \leq d$, the fact that v has degree three in G implies that the shortest u-v path in G has length at most $d-2$ for all $u \in V$. Hence v is connected to every other vertex in G^{d-1}. Then $G^{d-1} - u$ is rigid in $\mathbb{R}^d$ for all $u \in V$: it follows from the observation that for every $u \in V$ the graph $G^{d-1} - u$ contains a subgraph isomorphic to $G^{d-1} - v$, which is rigid in $\mathbb{R}^d$. Therefore G^{d-1} is vertex-redundantly rigid, and hence globally rigid in $\mathbb{R}^d$ by Theorem 2.4.

This argument shows that in the rest of the proof we may assume that

$$\text{for all } v \in V \text{ with } d(v) = 3 \text{ the graph } G - v + K(N_G(v)) \text{ is not 3-edge-connected.} \quad (13)$$

We say that two adjacent degree three vertices v, v' in G are *partners* if $N_G(v') = (N_G(v) - v') \cup \{v\}$. Note that it is indeed a symmetric relation.

Claim 6.4. *Let v be a vertex of degree three in G. Then either*
(i) G has a cut-vertex, or
(ii) v has a partner.

Proof. Let $H = G - v + K(N_G(v))$ and let T be the triangle on $N_G(v)$ in H. By (13) the graph H can be separated by removing a set F of at most two edges. The edge cut F must intersect the edge set of T, for otherwise it is also an edge cut in G, which is not possible, since G is 3-edge-connected. It follows that F consists of two edges e, f of T, with a common end-vertex v', say. The two connected components of $H - \{e, f\}$ define a bipartition of $V(G) - v$. If there are more vertices on the v'-side of this bipartition then v' is a cut-vertex in G. If not, then v' satisfies $N_G(v') = (N_G(v) - v') \cup \{v\}$. So v' is a partner of v. $\square$

Claim 6.5. *Let v and v' be partners in G and let $N_G(v) - v' = \{a, b\}$. Then either*
(i) G has a cut-vertex, or
(ii) ab is not an edge in G.

Proof. Let $e = ab$. Suppose that $e \in E(G)$. First consider the case when a and b are in the same connected component of $G - \{v, v'\} - e$. Then there exist four edge-disjoint paths in G from a to b, and hence $G - e$ is 3-edge-connected. This contradicts the minimality of G. Next suppose that a and b are in different connected components of $G - \{v, v'\} - e$. Then, since G has at least five vertices, at least one of a, b is a cut-vertex in G. □

Consider two degree three vertices v and v', which are partners, and let $N_G(v) - v' = \{a, b\}$ and $e = ab$. Suppose that G has no cut-vertices and that $Q = G - \{v, v'\} + e$ is 3-edge-connected. Note that Q is simple by Claim 6.5. Then Q^{d-1} is globally rigid by induction. Let $\bar{Q}$ be obtained from Q^{d-1} by adding v' and all edges from v' to $N_{G^{d-1}}(v') - \{v\}$. Observe that $\bar{Q}$ is a spanning subgraph of $G^{d-1} - v + K(N_{G^{d-1}}(v))$. Moreover, since a and b have at least three neighbours in Q, v' is connected to all, or to at least $d + 1$ vertices of Q^{d-1} in $\bar{Q}$. (To see this consider a shortest path P from v' to some vertex x in $G - v$ which is farthest from v'. Suppose it contains b. If P has less than d vertices then v' is connected to all vertices in the power. Otherwise v' is connected to $d - 1$ vertices of P, a, and another neighbour of b.) Thus $\bar{Q}$, and hence also $G^{d-1} - v + K(N_{G^{d-1}}(v))$, is globally rigid in $\mathbb{R}^d$. So in this case the theorem follows from Claims 6.2, 6.3, and Theorem 2.5. In what follows we may therefore assume that if G has no cut-vertices then no partners satisfy that $Q = G - \{v, v'\} + e$ is 3-edge-connected (using the previous notation).

Claim 6.6. *There is a cut-vertex in G.*

Proof. Suppose that there is no cut-vertex in G. Then each degree three vertex has a partner by Claim 6.4. Consider two degree three vertices v, v', which are partners, and let $N_G(v) - v' = \{a, b\}$. The edge $e = ab$ is not present in G by Claim 6.5(ii). This implies that the partner of each degree three vertex is unique. It also follows that the degree of a (and b) is at least four in G.

So we obtain that the degree three vertices of G can be partitioned into pairs, so that the vertices in each pair are partners. Replace each pair v, v' by a pair of parallel edges connecting the vertices of $N_G(v) - v'$ (which is equal to $N_G(v') - v$). Let H be the resulting multigraph.

Our assumption given right before the claim, saying that $G - \{v, v'\} + ab$ is not 3-edge-connected for all pairs of partners, implies that there is no other pair u, u' of partners with $N_G(u) - u' = \{a, b\}$. Since G has at least five vertices, it also implies that H has at least three vertices.

We claim that H is minimally 3-edge-connected. To see this first suppose that there is an edge-cut F of size at most two in H. Then, since G is 3-edge-connected, F must contain a pair e, e' of parallel edges in H. Then at least one of the end-vertices of e is a cut-vertex in H. It follows from the construction of H that this vertex is also a cut-vertex in G, which is a contradiction.

Minimality can be seen as follows. Our assumption saying that for all pairs v, v' of partners $G - \{v, v'\} + e$ is not 3-edge-connected implies that the two parallel edges e, e' are both critical: removing one of them destroys the 3-edge-connectivity of H. For an edge f in H which is also an edge of G the minimality of G implies that f belongs to an edge cut of size three in G. If the edges of F are all present in H then F verifies that f is critical in H, too. Otherwise, if F contains an edge incident with a degree three vertex v in G, then, since $N_G(v) \cup \{v\}$ induces a 2-edge-connected subgraph in G, F contains two edges from this subgraph. But then f, together with the two parallel edges on the common neighbours of v and its partner give rise to a 3-edge-cut containing f in H.

Now we can use the fact that H is minimally 3-edge-connected to deduce that there is a vertex w in H with $d_H(w) = 3$. Since our construction of H from G preserves the vertex degrees, and the end-vertices of the added parallel edge pairs are of degree at least four in G, we also have $d_G(w) = 3$. But in this case H would not contain w, a contradiction. This proves the claim. $\square$

By Claim 6.6 G has a cut-vertex v. It means $G = G_1 \cup G_2$, with $V(G_1) \cap V(G_2) = \{v\}$. Now G_1, G_2 are also 3-edge-connected and hence, by induction, G_1^{d-1} and G_2^{d-1} are globally rigid in $\mathbb{R}^d$. Furthermore, v has at least three neighbours a_i, b_i, c_i in G_i for each $i = 1, 2$. For $j = 0, \dots, d-2$, let $H_{1,j}$ be the subgraph of G^{d-1} induced by $V(G_1)$ and $N_{G_2}^{\leq j}(v)$. The graphs $H_{2,j}$ are defined similarly, by interchanging the role of G_1 and G_2.

Claim 6.7. *For $i = 1, 2$ and $j = 0, \dots, d-2$, $H_{i,j}$ is globally rigid.*

Proof. By symmetry it suffices to consider the case $i = 1$. We shall show that $H_{1,j}$ is globally rigid by induction on j. The base case $j = 0$ follows from the fact that $H_{1,0} = G_1^{d-1}$.

Consider the case when $1 \leq j \leq d-2$. Let $G_{1,j}$ be the subgraph of G induced by $V(H_{1,j-1})$. Suppose that the diameter of $G_{1,j}$ is at most $d-3$. Then $H_{1,j}$ is a complete graph, which implies the claim. So we may assume that the diameter of $G_{1,j}$ is at least $d-2$. We split the proof into two cases depending on the size of $V(H_{1,j-1})$.

The first case is when $|V(H_{1,j-1})| \leq d$. A shortest path in G between any two vertices of $H_{1,j-1}$ misses at least two vertices from the set $\{a_1, a_2, b_1, b_2, c_1, c_2\}$ if $j \geq 2$. Since the diameter of $G_{1,j-1}$ is at least $d-2$, we have $|V(H_{1,j-1})| \geq d+1$ if $j \geq 2$. Hence we must have $j = 1$. Then, a shortest path between any two vertices in $G_{1,j-1}$ misses at least one vertex from the set $\{a_1, b_1, c_1, v\}$. So, by the assumption $|V(H_{1,j-1})| \leq d$, the diameter of $G_{1,j-1}$ is $d-2$. Since $j = 1$, any pair of vertices in $G_{1,j}$ has distance at most $d-1$. (Note that, by $j = 1$, any two vertices in $V(G_{1,j}) \setminus V(G_{1,j-1})$ have distance at most two in $G_{1,j}$.) So $H_{1,j}$ is a complete graph, completing the proof in the first case.

The second case is when $|V(H_{1,j-1})| \geq d+1$. Since $H_{1,j-1}$ is globally rigid by induction, it suffices to show that, for any $u \in V(H_{1,j}) \setminus V(H_{1,j-1})$, there are at least $d+1$ edges between u and $V(H_{1,j-1})$ in $H_{1,j}$ (by the Extension lemma). Consider a vertex $u \in V(H_{1,j}) \setminus V(H_{1,j-1})$. Let G_u be the subgraph of G induced by $V(H_{1,j-1})$ and u.

If $\mathrm{dist}_{G_u}(u,w) \le d-1$ holds for all $w \in V(H_{1,j-1})$, then there are at least $d+1$ edges between u and $V(H_{1,j-1})$ by the assumption $|V(H_{1,j-1})| \ge d+1$. If $\mathrm{dist}_{G_u}(u,w) \ge d$ holds for some $w \in V(G_{1,j-1})$, then consider a shortest path P between u and w in G_u. Let P' be the subpath of P of length $d-1$ starting at u. Then P' misses at least two vertices from the set $\{a_1, b_1, c_1\}$. Hence, $|(V(P') \setminus \{u\}) \cup \{a_1, b_1, c_1\}| \ge d+1$. Since each vertex of $(V(P') \setminus \{u\}) \cup \{a_1, b_1, c_1\}$ is within distance $d-1$ from u by $j \le d-2$, there are at least $d+1$ edges between u and $V(H_{1,j-1})$, as required. □

By Claim 6.7, $G^{d-1}_{i,d-2}$ is globally rigid for $i = 1, 2$. Since v has degree at least six in G, $|V(G^{d-1}_{1,d-2}) \cap V(G^{d-1}_{2,d-2})| \ge 2(d-2) + 5 \ge d+1$ holds, unless $G^{d-1}_{1,d-2} \subseteq G^{d-1}_{2,d-2}$ or $G^{d-1}_{2,d-2} \subseteq G^{d-1}_{1,d-2}$. Thus, by the Gluing lemma, it follows that $G^{d-1}_{1,d-2} \cup G^{d-1}_{2,d-2}$ is globally rigid. Since $G^{d-1}_{1,d-2} \cup G^{d-1}_{2,d-2}$ is a spanning subgraph of G^{d-1}, it follows that G^{d-1} is globally rigid, as required. □

We close this section by deducing some corollaries in the three-dimensional case. First we observe that it is easy to extend the theorem to the case when the graph is obtained from a 3-edge-connected graph by attaching some leaves.

Theorem 6.8. *Let $G = (V, E)$ be a connected graph and let $L = \{v \in V : d(v) = 1\}$. If $G - L$ is 3-edge-connected then G^2 is globally rigid in $\mathbb{R}^3$.*

Proof. Since G is simple, $G - L$ is either a single vertex or it has at least four vertices. In the former case G^2 is complete, and hence it is globally rigid in $\mathbb{R}^3$. In the latter case G^2 can be obtained from $(G-L)^2$ by a sequence of vertex additions so that each new vertex is connected to at least four vertices. By Theorem 6.1 $(G-L)^2$ is globally rigid in $\mathbb{R}^3$. Thus G^2 is also globally rigid in $\mathbb{R}^3$. □

Next we verify Conjecture 2 in the case when $k = 1$, $d = 2$.

Theorem 6.9. *Let $G = (V, E)$ be a rigid graph in $\mathbb{R}^2$. Then G^2 is globally rigid in $\mathbb{R}^3$.*

Proof. It suffices to verify global rigidity for the squares of minimally rigid graphs. We do this by induction on $|V|$.

The statement is obvious for $|V| \le 4$, so we may assume that $|V| \ge 5$. Now G is 2-connected and each vertex has degree at least two in G. Furthermore, since $|E| = 2|V| - 3$, G has a vertex v of degree at most three. If there exists a vertex with $d(v) = 2$ then $G - v$ is minimally rigid and hence $(G-v)^2$ is globally rigid in $\mathbb{R}^3$ by induction. It is easy to check that v is connected to at least four vertices of $G - v$ in G^2, for otherwise G has a cut-vertex. This implies that G^2 is globally rigid in $\mathbb{R}^3$.

So we may assume that the minimum degree of G is equal to three. Then G is 3-edge-connected (see e.g. [12]). Thus G^2 is globally rigid in $\mathbb{R}^3$ by Theorem 6.1. □

Another corollary of Theorem 6.1 is that if a graph G with $|V| \geq 4$ has at least $2|V|-3$ edges then G^2 has a globally rigid subgraph on at least four vertices in $\mathbb{R}^3$.

7. Concluding remarks

We conclude the paper by recalling another conjecture. It was conjectured in [5] that for every d there is an integer c_d such that every c_d-connected graph is globally rigid in $\mathbb{R}^d$. The special case of this conjecture, when the graph is a square and $d = 3$, is still open.

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