Discrete Applied Mathematics 322 (2022) 448–464

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Extremal families of redundantly rigid graphs in three dimensions

Tibor Jordán [a,*], Christopher Poston [b], Ryan Roach [c]

[a] Department of Operations Research, ELTE Eötvös Loránd University, and the MTA-ELTE Egerváry Research Group on Combinatorial Optimization, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary
[b] Department of Mathematics, Yale University, USA
[c] Department of Mathematics and Statistics, Washington University in St. Louis, USA

ARTICLE INFO

Article history:
Received 10 July 2021
Received in revised form 1 November 2021
Accepted 4 March 2022
Available online 7 April 2022

Keywords:
Rigid graph
Globally rigid graph
Redundant rigidity
Block and hole graph
Rigid framework

ABSTRACT

A rigid graph G is said to be k-vertex (resp. k-edge) rigid in $\mathbb{R}^d$ if it remains rigid after the removal of less than k vertices (resp. edges). The definition of k-vertex (resp. k-edge) globally rigid graphs in $\mathbb{R}^d$ is similar. We study each of these four versions of redundant (global) rigidity and determine the smallest number of edges in a k-vertex (resp. k-edge) rigid (resp. globally rigid) graph on n vertices in $\mathbb{R}^3$ for all positive integers k, except for four special cases, where we provide a close-to-tight bound.

1. Introduction

We start with an informal definition of rigid and globally rigid graphs and refer the reader to [9,13,17] for more details. A graph $G = (V, E)$ is *rigid* in $\mathbb{R}^d$ if every generic bar-and-joint realization of G in d dimensions, in which vertices correspond to universal joints and edges correspond to rigid bars connecting their end-vertices, is rigid in the sense that it has no continuous deformation that preserves the bar lengths. (A realization is said to be generic if the set of coordinates of the vertices does not satisfy any non-zero polynomial with integer coefficients.) Global rigidity is a stronger property: a graph G is *globally rigid* in $\mathbb{R}^d$ if every generic d-dimensional bar-and-joint realization of G is unique up to congruence: the bar lengths determine all pairwise distances between the joints. Rigid and globally rigid graphs occur in several applications, including sensor network localization [8], molecular conformation [7], formation control [24], and statics [15]. In some applications it is desirable to have a graph which remains rigid or globally rigid even if some joints or bars are removed. This motivates the next definitions.

We say that a graph $G = (V, E)$ is *k-vertex rigid* (resp. *k-vertex globally rigid*) in $\mathbb{R}^d$ if $G - X$ is rigid (resp. globally rigid) for all $X \subseteq V$ with $|X| \leq k - 1$. A graph $G = (V, E)$ on n vertices is said to be *strongly minimally k-vertex rigid* (resp. *strongly minimally k-vertex globally rigid*) in $\mathbb{R}^d$ if it is k-vertex rigid (resp. k-vertex globally rigid) and no graph on n vertices with less than $|E|$ edges satisfies this property. We can define *(strongly minimal) k-edge rigidity* and *k-edge global rigidity* in a similar way, by the deletion of edge sets, rather than vertex sets. It will be convenient to use the following graph parameters. For a graph G we use $R^d_v(G)$ (resp. $R^d_e(G)$) to denote the largest integer ℓ for which G is ℓ-vertex (resp. ℓ-edge) rigid in $\mathbb{R}^d$. The corresponding parameters with respect to global rigidity are denoted by $R^d_{gv}(G)$ and $R^d_{ge}(G)$ (see Fig. 1).

* Corresponding author.
E-mail addresses: tibor.jordan@ttk.elte.hu (T. Jordán), christopher.poston@yale.edu (C. Poston), ryan.roach@wustl.edu (R. Roach).

https://doi.org/10.1016/j.dam.2022.03.006

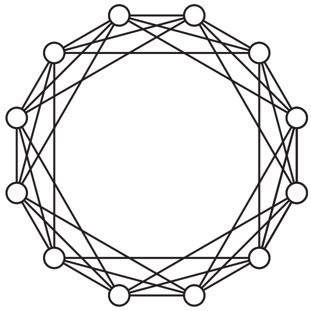

Fig. 1. The cube of a cycle. It is 3-vertex rigid in $\mathbb{R}^3$.

In this paper we investigate the following extremal problem: what is the smallest number of edges in a strongly minimally k-vertex rigid (k-vertex globally rigid, k-edge rigid, k-edge globally rigid, resp.) graph on n vertices in $\mathbb{R}^d$? In $\mathbb{R}^1$ a graph is rigid (resp. globally rigid) if it is connected (resp. 2-connected). Hence the corresponding bounds on the size of strongly minimally rigid and globally rigid graphs follow from basic results on highly connected graphs. The case $d = 2$ requires a different approach. Following the solutions of some special cases (concerning k-vertex rigidity and k-vertex global rigidity in the plane, for $k \leq 3$), a recent paper by the first author [10] gave a complete solution in $\mathbb{R}^2$ by determining the tight bounds for each of the four versions and for all $k \geq 1$.

For $d \geq 3$ Kaszanitzky and Király [14] solved the k-vertex rigid version in the special case when $k = 2$ (for all $d \geq 2$) and for $d = k = 3$. All the other cases remained open. It is worth noting that the characterization of rigid and globally rigid graphs in $\mathbb{R}^d$ is known for $d \leq 2$ and is a major open problem in rigidity theory for $d \geq 3$.

In spite of this fact we shall determine the smallest number of edges in a k-vertex (resp. k-edge) rigid (resp. globally rigid) graph on n vertices in $\mathbb{R}^3$ for all positive integers k, except for four special cases, where we provide a close-to-tight bound. These special cases, which turned out to be the most difficult ones, are 4-vertex rigidity, 3-vertex global rigidity, 2-vertex global rigidity, and 2-edge global rigidity in three-space. See Table 1.

Note that in each of the extremal problems mentioned above, including our new results, the lower and upper bounds and also the exact solutions are valid for "n large enough, depending on k". Here "large enough" typically means some constant times k. It is a natural phenomenon which is present already in the formula for the size of a minimally rigid graph ($k = 1$).

The structure of the paper is as follows. In the next section we collect those previous results that we shall use, including ones that establish connections between the four different parameters we are dealing with. In Sections 3 and 4 we solve the k-vertex and k-edge rigid versions of our problem. In Section 5 we deduce the solutions for vertex and edge redundant global rigidity. In Section 6 we show an additional result that settles a related conjecture concerning the two-dimensional case of our extremal problem. Section 7 contains a few concluding remarks.

2. Preliminary results

The lemma below shows that in the definition of k-vertex (global) rigidity it suffices to consider the removal of vertex sets of cardinality exactly $k - 1$. Note that the corresponding observation for k-edge (global) rigidity is straightforward, since edge addition preserves rigidity as well as global rigidity.

Lemma 2.1 (*[10,14,24]*)**.** *Let $G = (V, E)$ be a graph on $n \geq k + 1$ vertices. Then*
(i) G is k-vertex rigid in $\mathbb{R}^d$ if and only if $G - X$ is rigid in $\mathbb{R}^d$ for all $X \subseteq V$ with $|X| = k - 1$, and
(ii) G is k-vertex globally rigid in $\mathbb{R}^d$ if and only if $G - X$ is globally rigid in $\mathbb{R}^d$ for all $X \subseteq V$ with $|X| = k - 1$.

The four redundancy parameters satisfy the following inequalities. The first one is based on a theorem due to S. Tanigawa [19] which states that 2-vertex rigid graphs are globally rigid.

Lemma 2.2 (*[10]*)**.** *Let $G = (V, E)$ be a k-vertex-rigid graph in $\mathbb{R}^d$ for some $k \geq 2$. Then G is $(k-1)$-vertex globally rigid. Hence for all $d \geq 1$ we have*

$$R^d_{g_v}(G) \geq R^d_v(G) - 1. \quad (1)$$

The next lemma shows that the edge redundancy cannot be smaller than the vertex redundancy.

Lemma 2.3 (*[10,24]*)**.** *Let G be a k-vertex rigid (resp. k-vertex globally rigid) graph on at least $d + k$ (resp. $d + k + 1$) vertices. Then G is k-edge rigid (resp. k-edge globally rigid).*

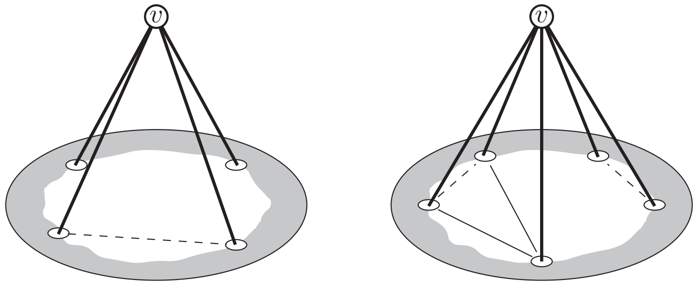

Fig. 2. The 1-extension and the triangle based 2-extension operations.

Hendrickson [6] showed that globally rigid graphs (on at least $d+2$ vertices) are 2-edge-rigid in $\mathbb{R}^d$. Note that 2-edge rigid is the same as *redundantly rigid*, which is also a frequently used term in rigidity theory. This implies the following.

Lemma 2.4 (*[10]*)**.** *Let $G=(V,E)$ be a globally rigid graph in $\mathbb{R}^d$ on $n \geq d+2$ vertices. Then*

$$R_e^d(G) \geq R_{ge}^d(G)+1. \tag{2}$$

The last lemma of this section is implied by the basic fact that a rigid (resp. globally rigid) graph G in $\mathbb{R}^d$ on at least $d+1$ (resp. $d+2$) vertices has minimum degree at least d (resp. $d+1$).

Lemma 2.5 (*[10,14]*)**.** *The minimum degree of a k-vertex or k-edge rigid (resp. k-vertex of k-edge globally rigid) graph on at least $d+k$ (resp. $d+k+1$) vertices is at least $d+k-1$ (resp. $d+k$).*

2.1. Operations

As we noted earlier, the characterization of rigid and globally rigid graphs in $\mathbb{R}^3$ is still an open problem. Verifying that the graphs we define, as well as their subgraphs obtained by removing a certain number of vertices or edges, are indeed rigid or globally rigid is the most difficult part of our solutions. In our proofs we shall rely on sufficient conditions based on various inductive steps, i.e. local graph operations that preserve rigidity and-or global rigidity.

The (d-dimensional) 0-*extension* operation adds a new vertex v to a graph as well as d new edges incident with v. The (d-dimensional) 1-*extension* operation removes an edge v_iv_j and adds a new vertex v as well as a set of $d+1$ new edges incident with v which includes vv_i and vv_j. See Fig. 2. The first two statements of the next lemma are well-known, see e.g. [17]. The third one is based on a result due to Connelly [2]. We shall use this lemma several times, without explicitly referring to it.

Lemma 2.6. *Let G be a graph and $d \geq 1$ be an integer. Then*
(i) if G is rigid in $\mathbb{R}^d$ and G' is obtained from G by a 0-extension or 1-extension then G' is rigid in $\mathbb{R}^d$,
(ii) if G is 2-edge rigid in $\mathbb{R}^d$ and G' is obtained from G by a 1-extension then G' is 2-edge rigid in $\mathbb{R}^d$,
(iii) if G is globally rigid in $\mathbb{R}^d$ on at least $d+2$ vertices and G' is obtained from G by a 1-extension then G' is globally rigid in $\mathbb{R}^d$.

The (d-dimensional) *2-extension* operation removes two disjoint edges v_iv_j and v_qv_r from a graph and adds a new vertex v, along with a set of $d+2$ edges incident with v including vv_i, vv_j, vv_q, and vv_r. The next lemma is folklore, a proof can be found e.g. in [14].

Lemma 2.7. *Let G be a graph and suppose that the vertices v_i, v_j, v_s form a triangle and v_qv_r is an edge disjoint from this triangle. Then if G is rigid in $\mathbb{R}^3$ and G' is obtained from G by a 2-extension on edges v_iv_j, v_qv_r, then G' is rigid in $\mathbb{R}^3$.*

We shall refer to the operation described in Lemma 2.7 as *triangle based 2-extension*. See Fig. 2.

Let G be a graph and let uv, vw be a pair of incident edges in G. Let E_{uw}^v be the set of the remaining edges incident with v and let $E_{uw}^v = F \cup F'$ be a bipartition of E_{uw}^v. The *vertex splitting* operation (at v, on edges uv, vw) adds a new vertex v' to the graph, adds the new edges uv', $v'w$, vv', and then replaces every edge xv in F' by an edge xv'. The edges in F stay connected to v. See Fig. 3.

A similar operation is *extended vertex splitting*: it picks three edges uv, vw, vz, partitions the set E_{uwz}^v of the remaining edges incident with v into two parts $E_{uwz}^v = F \cup F'$, adds a new vertex v' to the graph, adds the new edges uv', $v'w$, $v'z$, and then replaces every edge xv in F' by an edge xv'. The next theorem is due to Whiteley.

Theorem 2.8 (*[21], [22, Theorem 9.3.7]*)**.** *If G is rigid in $\mathbb{R}^3$ and G' is obtained from G by a vertex splitting or an extended vertex splitting operation then G' is also rigid in $\mathbb{R}^3$.*

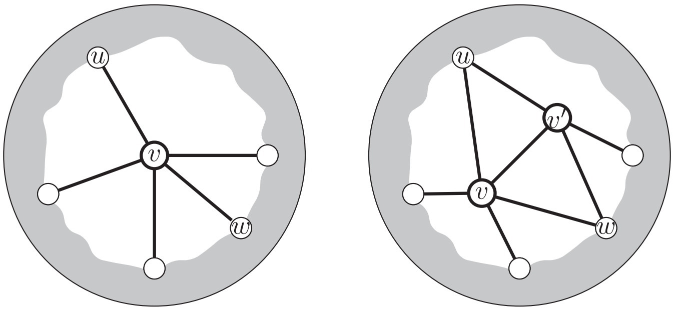

Fig. 3. The vertex splitting operation.

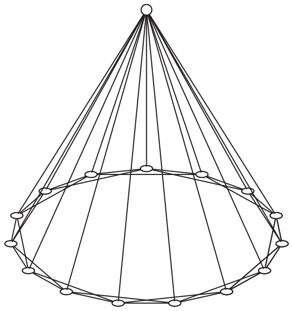

Fig. 4. Coned, Squared Cycle.

2.2. Coning

There is another operation that we shall use to transfer (global) rigidity to higher dimensions. The *cone* of a graph G is obtained from G by adding a new vertex v and new edges from v to every vertex of G. See Fig. 4. Whiteley [23] (resp. Connelly and Whiteley [3]) proved that a graph G is rigid (resp. globally rigid) in $\mathbb{R}^d$ if and only if the cone of G is rigid (resp. globally rigid) in $\mathbb{R}^{d+1}$. We shall refer to these results as the coning theorem(s).

We shall need the following new result, which shows that redundant (global) rigidity can also be transferred to higher dimensions by coning, in a certain sense.

Theorem 2.9. *Let G be a graph on n vertices. Then*
(i) if G is k-vertex globally rigid in $\mathbb{R}^d$ and $n \geq d+k+1$ then the cone of G is k-edge globally rigid in $\mathbb{R}^{d+1}$,
(ii) if G is k-vertex rigid in $\mathbb{R}^d$ and $n \geq d+k$ then the cone of G is k-edge rigid in $\mathbb{R}^{d+1}$.

Proof. We prove (i). Let G be a k-vertex globally rigid graph in $\mathbb{R}^d$. Let H denote the cone of G. Choose a set F of $k-1$ edges in H. We have to show that $H-F$ is globally rigid in $\mathbb{R}^{d+1}$.

If no edge in F is incident with v then $H-F$ is the cone of $G-F$. Now $G-F$ is globally rigid in $\mathbb{R}^d$ by Lemma 2.3 and our assumption on G. Hence $H-F$, and so H is globally rigid in $\mathbb{R}^{d+1}$ by the globally rigid coning theorem.

Next suppose that the set of edges of F incident with v, denoted by F', is not empty. Let T be the set of the end-vertices of the edges in F' different from v and let $t = |T| = |F'|$.

Since G is k-vertex globally rigid in $\mathbb{R}^d$, the graph $G-T$ is $(k-t)$-vertex globally rigid in $\mathbb{R}^d$. Hence it is also $(k-t)$-edge globally rigid by Lemma 2.3. This implies, by using $|F-F'| = k-1-t$, that $G-T-(F-F')$ is globally rigid in $\mathbb{R}^d$. Thus $H-T-(F-F')$ is globally rigid in $\mathbb{R}^{d+1}$ by the globally rigid coning theorem.

By Lemma 2.5 each vertex of G has degree at least $d+k$. Moreover, each vertex $w \in T$ has at most $t-1$ neighbours in T (in G as well as in H), and has at most $k-1-t$ edges in F that connect it to a vertex in $V(G)-T$. Thus there exist at least $d+k-(t-1)-(k-1-t) = d+2$ edges from w to $V(G)-T$ in $H-T-F$.

Therefore we can add the vertices of T to $H-T-F$ one by one, without using edges from F, preserving global rigidity in $\mathbb{R}^{d+1}$. Hence $H-F$ is globally rigid in $\mathbb{R}^{d+1}$. This completes the proof.

The proof of (ii) is very similar: it can be obtained by replacing global rigidity by rigidity, and the degree lower bound $d+k$ by $(d-1)+k$ in the proof above. □

2.3. Lower bounds

There are three natural lower bounds for the size of a k-vertex (k-edge) rigid (globally rigid) graph. The first bound (see e.g. [14]) works for each of the four versions and comes from Lemma 2.5: the number of edges in a k-vertex (k-edge) rigid graph on $n \geq d+k$ vertices is at least

$$\left\lceil \frac{n(d+k-1)}{2} \right\rceil \tag{3}$$

and the number of edges in a k-vertex (k-edge) globally rigid graph on $n \geq d+k+1$ vertices is at least

$$\left\lceil \frac{n(d+k)}{2} \right\rceil \tag{4}$$

The other bounds use the next two well-known inequalities.

Lemma 2.10. *Let $G=(V,E)$ be a rigid graph in $\mathbb{R}^d$ with $|V| \geq d+1$. Then $|E| \geq d|V| - \binom{d+1}{2}$.*

Lemma 2.11. *Let $G=(V,E)$ be a globally rigid graph in $\mathbb{R}^d$ with $|V| \geq d+2$. Then $|E| \geq d|V| - \binom{d+1}{2} + 1$.*

Rigid graphs for which equality holds in Lemma 2.10 are called *minimally rigid* in $\mathbb{R}^d$. Minimally rigid graphs exist for all d and $n \geq d+1$. There exist globally rigid graphs, for every d and $n \geq d+2$, that satisfy the bound of Lemma 2.11 with equality (see e.g. [10]). Hence the tight bounds in Lemmas 2.10 and 2.11 give rise to the tight bounds for our extremal problems in the special case $k=1$.

We also have the following corollaries for edge-redundancy. The number of edges in a k-edge rigid graph in $\mathbb{R}^d$ on $n \geq d+1$ vertices is at least

$$dn - \binom{d+1}{2} + (k-1). \tag{5}$$

The number of edges in a k-edge globally rigid graph in $\mathbb{R}^d$ on $n \geq d+2$ vertices is at least

$$dn - \binom{d+1}{2} + k. \tag{6}$$

The third bound, for vertex redundancy, is based on [14, Theorem 5], which works for k-vertex rigidity for all d and k. The next lemma improves the corresponding lower bound of [14] by one.

Lemma 2.12. *Let $G=(V,E)$ be a 4-vertex rigid graph in $\mathbb{R}^3$ on $|V| \geq 15$ vertices. Then $|E| \geq 3|V|+5$.*

Proof. By Lemma 2.10 a rigid graph on at least three vertices satisfies $|E| \geq 3|V|-6$, and hence the sum of the degrees of its vertices is at least $6|V|-12$. Thus the maximum degree of G is at least six, whenever $|V| \geq 13$. Let us remove a maximum degree vertex v_1 from G, then remove a maximum degree vertex v_2 of (the rigid graph) $G-v_1$, and repeat this once more by removing a maximum degree vertex v_3 of $G-v_1-v_2$. Since G is 4-vertex rigid and $|V| \geq 15$, the maximum vertex degree in graphs $G-v_1$ and $G-v_1-v_2$ is at least six, and the resulting graph, denoted by H, is rigid. Thus $|E(H)| \geq 3|V(H)|-6 = 3|V|-15$. Since we removed at least six edges when we removed v_1, v_2, and v_3, we have $|E| \geq 3|V|-15+18 = 3|V|+3$.

The last inequality shows that the maximum degree of G is in fact at least seven. This can be used to strengthen the above argument and deduce that $|E| \geq 3|V|+4$.

Suppose that equality holds and G has exactly $3|V|+4$ edges. Then by rereading the above arguments we obtain that the maximum degree of G is equal to seven, and the vertices of degree seven are pairwise adjacent. Let us assume that v_1, v_2, v_3 are pairwise non-adjacent. Then the number of edges from $\{v_1, v_2, v_3\}$ to $V(H)$ is equal to 19. Since the total degree of H is $6|V(H)|-12$, these edges make the degree of at least seven vertices of H equal to seven in G. But it is impossible, since (as v_1 also has degree seven) the graph cannot have eight pairwise adjacent vertices of degree seven. Similar arguments can be used in the remaining cases to show that we cannot have equality. Thus $|E| \geq 3|V|+5$, as claimed. □

A proof similar to the first part of the proof of Lemma 2.12 and Lemma 2.11 give the following bounds.

Lemma 2.13. *Let $G=(V,E)$ be a 2-vertex (resp. 3-vertex) globally rigid graph in $\mathbb{R}^3$ on $|V| \geq 13$ vertices. Then $|E| \geq 3|V|-2$ (resp. $|E| \geq 3|V|+2$).*

We close this subsection by pointing out an interesting phenomenon concerning the tight bounds of our problems (in every dimension d). In the case of k-vertex (global) rigidity there seems to be a threshold value k_0 such that the tight bounds for $k < k_0$ are equal to $d|V| + c(d,k)$ for some constant c depending only on the redundancy k and the dimension d. On the other hand, if $k \geq k_0$, then the degree lower bounds (3) and (4) are tight. In the case of k-edge (global) rigidity

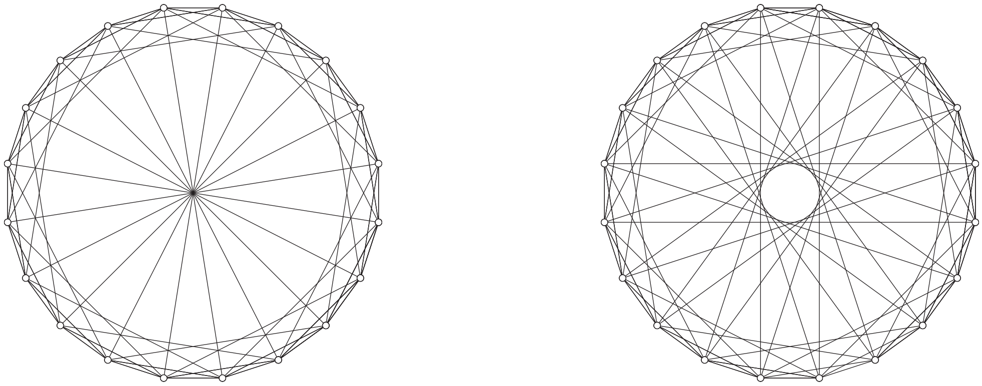

Fig. 5. The graphs L_{20}^5 and D_{20}^6.

there is a similar value ℓ_0 such that the tight bounds for $k < \ell_0$ are equal to the corresponding lower bounds (5) and (6), while for $k \geq \ell_0$ the tight bound matches the degree lower bounds (3) and (4).

We call the set of values below k_0 (resp. l_0) the *lower range*, and the rest the *upper range*, whenever these threshold values exist. In [10] it was shown that the upper and lower ranges indeed exist in each of the four versions of the problem for $d = 2$. In this paper we extend this result to $d = 3$.

Concerning the applications of our extremal constructions the values k in the lower range have a remarkable property: there exist graphs (frameworks, formations, networks) of redundancy k which need only a constant number of extra edges (bars, connections, measurements) compared to a minimally (globally) rigid graph.

3. Vertex-redundant rigidity in $\mathbb{R}^3$

The tight bounds for the size of strongly minimally k-vertex rigid graphs in $\mathbb{R}^3$ on n vertices are known for $k = 1, 2, 3$. The case $k = 1$ is easy: the minimally rigid graphs are the extremal graphs and the bound is $3n-6$, for $n \geq 3$. For $k = 2, 3$ Kaszanitzky and Király [14] showed that the bounds are $3n - 3$ and $3n$ respectively, for n sufficiently large.

In this section we determine the exact bounds for all $k \geq 5$ and give a close-to-tight upper bound for $k = 4$.

3.1. k-Vertex rigidity for $k \geq 5$

The *r'th power* of a graph G, denoted by G^r, is obtained from G by adding all edges uv, for which u and v are non-adjacent vertices of G whose distance is at most r in G. In our constructions we shall frequently use powers of cycles.

In this subsection we analyse two families of graphs and show that each graph in these families is k-vertex rigid. Let C_n be a cycle on n vertices, where n is even. It will be convenient to say that an edge on the vertex set of C_n is of *length m* if it connects two vertices of the cycle which are at distance m in C_n. An edge of length $\frac{n}{2}$ is a *longest diagonal*. The *second longest diagonals* are the edges of length $\frac{n}{2} - 1$. Let $k \geq 5$ be an integer. For odd values of k the graph L_n^k is obtained from $C_n^{(k-1)/2}$ by adding all edges of length $\frac{k+3}{2}$ as well as all longest diagonals. For even values of k the graph D_n^k is obtained from $C_n^{(k-2)/2}$ by adding all edges of length $\frac{k+2}{2}$ as well as all second longest diagonals. See Fig. 5.

Note that the graphs L_n^k and D_n^k are both $(k+2)$-regular and for $k \geq 7$ they contain C_n^3 as a spanning subgraph. It is worth mentioning that the graph obtained from $C_n^{(k+1)/2}$ by adding all longest diagonals is *not* k-vertex rigid in $\mathbb{R}^3$. It will be convenient to deal with the cases $k \geq 7$ separately, although the proof for $k = 5, 6$ is similar.

Theorem 3.1. *Let $k \geq 7$ and let $n \geq 12k$ be even. Then the graphs L_n^k (for k odd) and D_n^k (for k even) are k-vertex rigid in $\mathbb{R}^3$.*

Proof. Let $G = (V, E)$ denote the graph in question (which is L_n^k or D_n^k, depending on the parity of k) and let $S \subseteq V$ be a set of $k - 1$ vertices. By Lemma 2.1 it suffices to show that $H = G - S$ is rigid.

Since $n \geq 12k$, the vertex set V contains k pairwise disjoint subsets such that each subset corresponds to a pair of opposite intervals (i.e. consecutive vertices of the underlying cycle) of size six. By using that $|S| = k - 1$, we can deduce that S is disjoint from at least one of these subsets, which implies that H contains two intervals I_1, I_2 of size six each, positioned exactly opposite each other on the cycle. Furthermore, since C_n^3 is a spanning subgraph of G, the subgraphs $H[I_1]$ and $H[I_2]$ are both rigid: they can be obtained from a triangle graph by 0-extensions. Due to the existence of the (second) longest diagonals, these intervals are connected by six disjoint edges in H, which implies that $H[I_1 \cup I_2]$ is also rigid.

Let us extend I_1 and I_2 to maximal intervals in H and let $J \subset V(H)$ be the union of these maximal intervals. We define the ends of J naturally, denoting them by t_1, t_2, t_3, and t_4, see Fig. 6. It is possible that the two maximal intervals are the

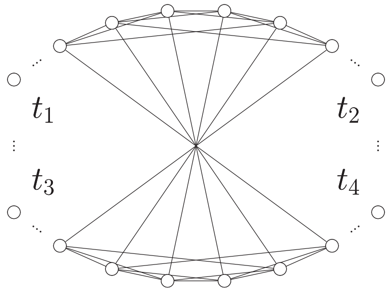

Fig. 6. The rigid substructure J, whose ends appear as t_i, $i = 1, \ldots, 4$.

same, in which case J itself is an interval with only two ends, denoted by t_2 and t_4. A similar argument, using 0-extensions, shows that $H[J]$ is rigid. We shall show that $H[J]$ can be extended further to a rigid spanning subgraph of H.

The vertices of the set S that we removed from G are distributed between the two pairs of ends. The vertices of $V - J$ next to the ends are all in S by the definition of J. Let S_{in} and S_{out} denote the set of vertices of S between t_1 and t_3 (resp. t_2 and t_4).

Case 1: k is odd.

Recall that if k is odd, we have $G = L_n^k$. Since $|S| = k - 1$, we may suppose, without loss of generality, that $|S_{in}| \leq \frac{k-1}{2}$. Let us add the vertices of H between t_1 and t_3 one by one, as long as we can, by applying 0-extensions, moving outward from t_1 towards t_3, and using edges of length at most $\frac{k+3}{2}$. Suppose we get stuck at some vertex v. Since G contains $\frac{k+1}{2}$ edges of length at most $\frac{k+3}{2}$ going backwards, S must contain at least $\frac{k+3}{2} - 3$ out of the $\frac{k+3}{2}$ vertices that precede v on the cycle in order to prevent a 0-extension. If this happens then let us add vertices moving from t_3 towards t_1 in a similar fashion. If we get stuck again, then we can conclude that S contains at least $2(\frac{k+3}{2} - 3) = k - 3$ vertices in total between t_1 and t_3. But this is impossible, as $|S_{in}| \leq \frac{k-1}{2} < k - 3$, whenever $k \geq 7$. Therefore we can add all vertices of H between t_1 and t_3, preserving rigidity. By the same argument we can add the vertices between t_2 and t_4 as well and obtain a rigid spanning subgraph of H, provided $|S_{out}| \leq k-4$ holds. So we may assume that $|S_{out}| \geq k-3$. We introduce three subcases.

Subcase 1.1: $|S_{out}| = k - 3$.

In this case $|S_{in}| = 2$. We have at most two vertices between t_2 and t_4, call them u and v, which are connected to S_{in} by a longest diagonal. Let us follow the same strategy and try to add the missing vertices of H by 0-extensions. For all vertices of H between t_2 and t_4, except for u and v, we can now use the longest diagonals, too, when we attempt to add the vertex by a 0-extension. Thus, by using the previous argument and the assumption $|S_{out}| = k-3$, we can deduce that the only way to get stuck from both directions is to get stuck at u and v, in such a way that we have exactly $\frac{k-3}{2}$ vertices of S between t_2 and v, and the same number between t_4 and u.

In this case we can proceed as follows. Let J' denote the set of vertices already in the rigid subgraph we have constructed. We continue adding vertices from v towards u, but when we add v, we also add a temporary edge vw, where w is a neighbour of u in J', to make the 0-extension work. After that we add the remaining vertices by 1-extensions, migrating the temporary edge to the next vertex in every iteration. At the end it aligns with an edge of H. Therefore we obtain a rigid spanning subgraph of H, as required.

Subcase 1.2: $|S_{out}| = k - 2$.

In this case $|S_{in}| = 1$. Hence there is at most one vertex between t_2 and t_4, call it v, which is connected to a vertex of S by a longest diagonal. At every other vertex we can use the longest diagonal, too, when we attempt to add it by a 0-extension. Thus if we get stuck at some vertex, which is different from v, then there must be at least $\frac{k+3}{2} - 2$ vertices of S in the set of the $\frac{k+3}{2}$ vertices preceding it. Since $2(\frac{k+3}{2} - 2) = k - 1 > k - 2 = |S_{out}|$, it follows that if we get stuck then it happens at a pair u, v, where we have $\frac{k+3}{2} - 2$ vertices of S preceding u, and we have $\frac{k+3}{2} - 3$ vertices of S preceding v on the other side. Then we proceed from u by adding a temporary edge, like in the previous subcase, and obtain a rigid spanning subgraph of H.

Subcase 1.3: $|S_{out}| = k - 1$.

In this case we can use the longest diagonals at each vertex, showing that if we get stuck at a pair u, v, then we must have exactly $\frac{k+3}{2} - 2 = \frac{k-1}{2}$ vertices of S preceding each of them from the appropriate direction. Then we proceed from u to v, by first adding a temporary edge uw, where w is a neighbour of v in J', and then applying 1-extensions, migrating the temporary edge until it aligns with an edge of H. Note that here we also need the fact that for the vertex u' right after u we can use the edge of length $(k + 3)/2$ going back, since its other end-vertex cannot be in S (for otherwise u could have been added by a 0-extension). This completes the proof of Case 1, it follows that H is indeed rigid in $\mathbb{R}^3$.

Case 2: k is even.

Now we consider the second family of graphs, D_n^k, for even $k \geq 8$. As in Case 1, we attempt to add the non-removed vertices between t_1 and t_3 (and then between t_2 and t_4) by 0-extensions. We may again suppose that $|S_{in}| \leq \frac{k}{2} - 1$. If S

contains at most $\frac{k}{2} - 3$ vertices out of the $\frac{k}{2} + 1$ vertices preceding the next vertex v, we can add v by a 0-extension and continue with the next vertex. Since $\frac{k}{2} - 1 < k - 4$ for $k \geq 8$, it follows that we cannot get stuck from both directions. Thus we can add all non-removed vertices between t_1 and t_3, preserving rigidity. Similarly, if $|S_{out}| \leq k - 5$, then we can add all vertices of H and conclude that H is rigid. So we may assume that $|S_{out}| \geq k - 4$. We introduce two subcases. In each of these subcases we shall use the fact that when we attempt to add a new vertex v between t_2 and t_4 then we can also use one or two second longest diagonals, leading to the opposite side, unless it leads to a vertex of S. This means we must have even more vertices of S preceding v if we get stuck: at least $\frac{k}{2} - 1$, or even $\frac{k}{2}$.

Subcase 2.1: $k - 4 \leq |S_{out}| \leq k - 2$.

If $|S_{out}| = k - 4$, we have three (resp. two, one for $|S_{out}| = k - 3, k - 2$) vertices of S on the opposite side. So the only way to get stuck from both directions, say at vertices u and v, is if the three (resp. two, one) vertices of S_{in} hit each of the four (resp. two, one) second longest diagonals incident with u and v. Then there is exactly one vertex, call it w, between u and v, so we can add w first by a 0-extension and only then u and v.

Subcase 2.2: $|S_{out}| = k - 1$.

Here, both second longest diagonals are available for every vertex coming from both t_2 and t_4; we may use 0-extensions from at least one direction and attach every non-removed vertex. This completes the proof of Case 2. The theorem follows. □

Next we consider the cases $k = 5, 6$. Our approach is similar to that of Theorem 3.1 but the lack of edges with length 3 requires a different argument to show that the initial rigid structure spanned by two opposite intervals exists and also makes the case analysis slightly more complicated. We shall follow the notation introduced in the proof of Theorem 3.1 wherever it is possible.

Theorem 3.2. *Let $n \geq 82$ be even. Then the graph L_n^5 is 5-vertex rigid and D_n^6 is 6-vertex rigid in $\mathbb{R}^3$.*

Proof. The lower bound on n ensures the existence of two exactly opposite intervals I_1 and I_2 of size at least eight in H. These two intervals are connected by (at least) eight (second) longest diagonals in L_n^5 as well as in D_n^6. First we show that $H[I_1 \cup I_2]$ is rigid.

Consider L_n^5. We begin by constructing each interval from right to left, but starting with the second vertex (the first will be added last). We insert a temporary edge connecting the second vertex to the fifth in each. Then each subsequent vertex is added using a 0-extension connecting it to the first-, second-, and fourth-previous vertices. We now have two disjoint rigid structures on seven vertices each; we may connect them to form a single rigid structure using the six middle longest diagonals. Lastly, we attach the first vertex of each interval using a 1-extension involving a longest diagonal and edges of distance 1, 2, and 4; the temporary edges form a three-cycle with the edges of distance 1 and 4, so they are removed. The argument for D_n^6 is similar. Having established that this subgraph is rigid, we define J, the ends t_i, $1 \leq i \leq 4$, as well as S_{in}, S_{out} as above. Note that we can again use 0-extensions to show that $H[J]$ is rigid.

Case 1: $k = 5$.

Recall that in L_n^5 each vertex is incident with edges of length 1,2, and 4 (in both directions along the cycle), and a longest diagonal. We have $|S| = 4$. We may assume that $|S_{out}| \geq 2$ which leads to three subcases.

Subcase 1.1: $|S_{out}| = 2$.

Now $|S_{in}| = 2$. Let $S_{in} = \{u_S, v_S\}$. First suppose that we have at most three vertices on the cycle between u_S and v_S. If they are next to each other on the cycle then there are no vertices between them to attach, so we are done. If there is one vertex between them, it may be attached by a 0-extension using edges of length 2 and one of length 4. Finally, if there are 2 or 3 vertices between them, then it is easy to check that we can add them by two or three 0-extensions. So in the rest of this subcase we may assume that u_S and v_S are separated by at least four vertices on the cycle.

Let $m \geq 4$ denote the number of vertices between u_S and v_S. We extend J by adding vertices one by one, as before. We proceed outward from t_1. Let u_S be the vertex next to t_1. The operations we perform depend on m mod 4; the goal is to use three temporary edges so that we can use 1-extensions to ultimately place them on the three vertices leading up to the second removed vertex v_S, as temporary edges attaching them to the first, second, and third vertices in I_2 respectively (counting inward from t_3). Then we will be done, since these edges of length 4 already exist in H.

If $m \equiv 0 \mod 4$, then the first vertex after u_S is attached using a 0-extension incorporating the usual edges of length 2 and 4, along with a temporary edge connecting it to the second vertex in I_2. The next vertex is added using a 0-extension with a temporary edge connecting it to the 1st vertex in I_2. Then the third vertex after the removed one is attached by a 1-extension with edges length 1 and 4, along with one of length 2 and a temporary edge connecting to the second vertex in I_2 (this removes the first temporary edge). Finally, the fourth vertex is added using a 0-extension with a temporary edge connecting to the third vertex in I_2. Then the remaining vertices are added in groups of size four by using 0- and 1-extensions so that the end-vertices of the temporary edges are moved towards v_S keeping the same pattern. A similar strategy works for $m \equiv i \mod 4$, $1 \leq i \leq 3$. We omit the details.

The temporary edges now align with edges in H, implying that we have attached every non-removed vertex between t_1 and t_3 while preserving rigidity. Since $|S_{out}| = 2$, the same process may be followed for the vertices between t_2 and t_4, resulting in a rigid spanning subgraph of H. This completes the proof of Subcase 1.1.

Subcase 1.2: $|S_{out}| = 3$.

In this case $|S_{in}| = 1$ and hence there are no vertices to attach between t_1 and t_3, so we can focus on the other side. Furthermore, we have at most one vertex between t_2 and t_4, say u, whose longest diagonal leads to the single vertex of S on the other side. Observe that due to the existence of longest diagonals, the two vertices of S next to t_2 and t_4 cannot block the 0-extensions at a vertex v ($v \neq u$), and hence the vertices may now be added starting from t_2 or t_4 using 0-extensions until either the third vertex of S_{out}, call it s, is encountered, or we reach u. Then we add a temporary edge connecting u to the last vertex added from the opposite end (if the other two removed vertices are neighbours, we connect it to the vertex distance 4 away in J). Then proceed by 1-extensions until the temporary edge aligns with an edge of H and all non-removed vertices, including the vertices from u up to s, are included.

Subcase 1.3: $|S_{out}| = 4$.

We may now use longest diagonals to attach every vertex since $S_{in} = \emptyset$. Let us denote the vertices of S_{out}, in order as encountered travelling from t_2 to t_4, by s_1, s_2, s_3, and s_4. Due to the longest diagonals, we may attach the vertices between s_1 and s_2 and those between s_4 and s_3 by 0-extensions without any trouble. So it remains to add the vertices between s_2 and s_3. If s_1 and s_2 are separated by at least three vertices on the cycle then we can simply continue attaching the vertices which occur after s_2 and up to s_3 by 0-extensions, completing the rigid spanning subgraph of H. It is easy to check that if s_1 and s_2 are separated by at most two vertices then we can proceed by using only a single temporary edge, which is attached to the first vertex after s_3 coming from t_2 (or to the third one after s_3, if s_3 and s_4 are neighbours). Subsequent 1-extensions are used to attach the remaining vertices between s_2 and s_3, moving the temporary edge until it is of length 4 or 2. It follows that H is rigid in $\mathbb{R}^3$ in this subcase, too. With this final subcase the argument for $k = 5$ is complete.

Case 2: $k = 6$.

Now we consider D_n^6, in which each vertex is incident with edges of length 1,2, and 4, as well as two second longest diagonals. We have $|S| = 5$. We may suppose that $|S_{in}| \leq 2$. By the analysis of Case 1 we conclude that all non-removed vertices between t_1 and t_3 can be added preserving rigidity. It remains to attach the non-removed vertices between t_2 and t_4. We have $|S_{out}| \geq 3$.

Subcase 2.1: $|S_{out}| = 3$.

We may now use the second longest diagonals when we perform the 0-extensions. We attach vertices from two directions, starting from t_2 and t_4. If at least one second longest diagonal is available at each vertex, we can simply add all vertices by a 0-extension and complete the process. Otherwise there is a single vertex, call it v, for which both of the two incident second longest diagonals lead to S_{in}. Then at least one second longest diagonal is available at every other vertex and it is easy to see that we can again add all vertices but v by 0-extensions, and then complete the process by adding v.

Subcase 2.2: $|S_{out}| = 4$.

In this case $|S_{in}| = 1$, and hence all but two vertices (say u, v) have two second longest diagonals available and u and v also have at least one. So we can keep on adding the non-removed vertices by 0-extensions, attempting to extend the rigid subgraph from both directions, unless we are in the unique situation where we can get stuck: only three vertices u, w, v remain, in this order on the cycle, and one of the second longest diagonals incident with u and v are blocked. Then we add w by a 0-extension, and then the other two vertices.

Subcase 2.3: $|S_{out}| = 5$.

Here the two long diagonals are available for every vertex to be attached, so we may simply use 0-extensions to attach past two vertices of S from both directions until we attach all the non-removed vertices and form a rigid spanning subgraph of H. This shows that H is rigid and completes the proof. □

As a corollary we obtain the following result.

Theorem 3.3. *Let $k \geq 5$ and let $n \geq 12k + 10$ be even. Then the number of edges in a strongly minimally k-vertex rigid graph on n vertices in $\mathbb{R}^3$ is equal to $\lceil \frac{(k+2)n}{2} \rceil$.*

We may observe that, strictly speaking, Theorem 3.3 does not give a complete solution to the extremal problem since it does not cover the case when n is odd. It may be possible to extend the construction and the proof for odd values of n but we do not attempt to work out the details in this paper. It is perhaps not a major shortcoming since Theorem 3.3 gives the tight bound for infinitely many values of n. Furthermore, it also gives the "asymptotic" answer for all n, due to the fact that adding a new vertex of degree $d + k - 1$ preserves k-vertex rigidity. We can apply this operation to any extremal graph on n vertices, with n even, to obtain an almost extremal graph for n odd.

Theorem 3.3 is one of the key results in the sense that it will easily imply the solutions to the k-edge rigid, k-vertex globally rigid, and k-edge globally rigid versions in the upper range. Our previous remark on the parity of n applies to each of these corollaries.

3.2. 4-Vertex rigidity

Recall the lower bound $3n + 5$ from Lemma 2.12 for this version of our extremal problem. We also have an upper bound for the size of a strongly minimally 4-vertex connected graph on n vertices, for n sufficiently large: we can show that for every $n \geq 44$ there exists a graph on n vertices and with $3n + 20$ edges which is 4-vertex rigid in $\mathbb{R}^3$. Since the proof of this result is based on a lengthy case analysis and the constant term is probably not tight, we refer the reader to the Appendix of [11] for a proof sketch. It remains an open problem to determine the tight bound with respect to 4-vertex rigidity.

Theorem 3.4. *Let G be a strongly minimally 4-vertex rigid graph on $n \geq 44$ vertices in $\mathbb{R}^3$. Then G has at most $3n+20$ edges.*

4. Edge-redundant rigidity in $\mathbb{R}^3$

The k-edge rigid version of our extremal problem has not been studied before. In this section we give a complete solution, for all $k \geq 1$. The cases $k = 1, 2$ are easy: the tight bounds are $3n - 6$ and $3n - 5$, respectively. The extremal graphs for $k = 1$ are the minimally rigid graphs. For $k = 2$ we can construct an extremal graph for all $n \geq 5$ by applying a sequence of 1-extensions to K_5, c.f. Lemma 2.6. The case $k = 3$ is more difficult.

4.1. 3-Edge rigidity

In this subsection we show an infinite family of 3-edge rigid graphs in $\mathbb{R}^3$ in which each member G_n, on $n \geq 14$ vertices, has $3n - 4$ edges, matching the lower bound from (5). We shall need the following definitions and previous results. A *triangulation* is a maximal planar graph. A *braced triangulation* is a graph obtained from a triangulation by adding a non-empty set of new edges, called the *bracing edges*. Consider a planar embedding of a triangulation $G = (V, E)$. A cycle C of G divides the plane into two parts and hence it determines two subgraphs of G that share the edges and vertices of C. Such a subgraph is called a *disc*. We say that it is *bounded* by C, or that C is its *boundary cycle*. The *interior* of a disc consists of the vertices and edges of the disc that do not belong to its boundary cycle. Two discs are *internally disjoint* if their common edges or vertices, if they exist, are part of their boundary cycles. Let $\mathcal{C} = \mathcal{B} \cup \mathcal{H}$ be a designated set of pairwise internally disjoint discs (bounded by cycles of length at least four in G). Then the *block-and-hole graph* G', with block-and-hole set $\mathcal{C}$, is obtained from G by removing the interiors of the discs in $\mathcal{C}$ and then making each subgraph induced by the vertex set of the boundary cycle C of some disc in $\mathcal{B}$ rigid (i.e. a block) by adding new edges connecting vertices of $V(C)$. See [4,5] for more details. We shall only consider block-and-hole graphs with at most one block and at most two holes, each of size at most five. The following result is a corollary of a celebrated theorem due to Cauchy from 1813.

Theorem 4.1 (*Cauchy*)**.** *Every triangulation is rigid in $\mathbb{R}^3$.*

The next result, for a single bracing edge, is due to Whiteley. The general case is from [12].

Theorem 4.2 (*[12,20]*)**.** *Every 4-connected braced triangulation is 2-edge rigid in $\mathbb{R}^3$.*

Let G be a block-and-hole graph with block-and-hole set $\mathcal{C} = \mathcal{B} \cup \mathcal{H}$ and let $\mathcal{B}'$ and $\mathcal{H}'$ be subsets of the blocks and holes, respectively, from $\mathcal{C}$. Let $\mathcal{C}'$ denote their union. The *index* of $\mathcal{C}'$ is defined to be

$$ind(\mathcal{C}') = \sum_{B \in \mathcal{B}'} (|B| - 3) - \sum_{H \in \mathcal{H}'} (|H| - 3)$$

A block-and-hole graph G is said to satisfy the *girth inequalities* if, for every cycle C in G and every planar realization of G,

$$|V(C)| \geq |ind(\mathcal{C}')| + 3,$$

where $\mathcal{C}'$ is the collection of blocks and holes of G which lie inside C.

Theorem 4.3 (*[4, Theorem 46]*)**.** *Let G be a block-and-hole graph with a single block. Then G is minimally rigid in $\mathbb{R}^3$ if and only if G satisfies the girth inequalities.*

The inverse operation of vertex splitting is *edge contraction*. It identifies two adjacent vertices u, v with exactly two common neighbours (and removes the resulting extra copies of parallel edges and the loop). It takes a triangulation to a smaller triangulation. Note that an edge in a triangulation is contractible if and only if it belongs to exactly two triangles. If it belongs to three or more triangles then one of them is a non-facial triangle (in every planar embedding) whose vertex set forms a 3-vertex separator. Hence every edge in a 4-connected triangulation is contractible.

We shall also use the following observations: (i) let G be a k-connected graph and let G' be obtained from G by a vertex splitting operation. If G' has minimum degree at least k then G' is also k-connected; (ii) the edge contraction operation decreases the vertex-connectivity of the graph by at most one.

Now we are ready to describe our construction and analyse its edge redundancy. Let G be a 5-connected triangulation with a fixed planar embedding. Choose a five-cycle C in G whose interior contains two edges. Thus it bounds three triangular faces. Let s, z, y, x, t be the vertices of the cycle, labelled clock-wise, and let sx, sy be the edges (diagonals) inside C. Let h be the common neighbour of x and y different from s. We shall assume that the degrees of the vertices t, x, y, z, and h are equal to five[1]. Add two bracing edges $q = ty$, $r = xz$ to obtain graph H. With these edges $V(C)$ induces a minimally rigid graph B (isomorphic to K_5 minus an edge) in H.

[1] It is not hard to show that such a 5-connected triangulation with the required configuration (five-cycle and corresponding degree five vertices) exists for $n = 12$ and for all $n \geq 14$. The smallest 5-connected triangulation is the icosahedron. A complete description of these triangulations, in terms of an inductive construction, is known [1]. One can use this result. A different argument is based on the observation that, given any 5-connected triangulation and five-cycle C, we can apply appropriate vertex splitting operations (preserving planarity and 5-connectivity), at most one at each designated vertex, to make sure that the degrees of t, x, y, z, h are equal to five in the resulting graph. We omit the details.

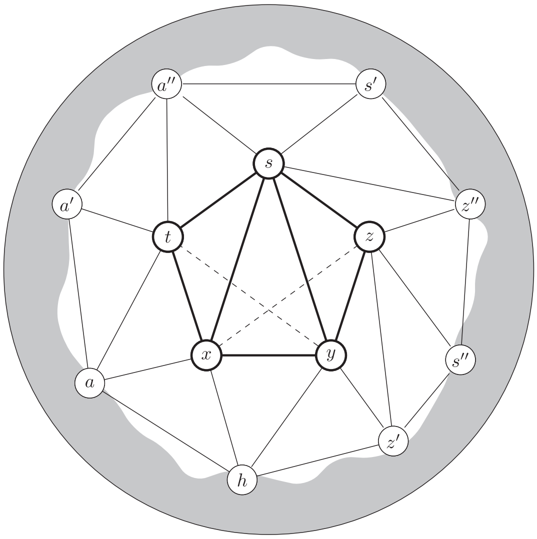

Fig. 7. The block and its neighbourhood.

Note that the 5-connectivity of G implies that $V(C)$ induces no further edges in H. See Fig. 7.

The resulting graph H has $3n-4$ edges. It is a block-and-hole graph with a single 5-block, B, and no holes.

Theorem 4.4. *H is 3-edge-rigid in $\mathbb{R}^3$.*

Proof. We have to show that $H' = H - e - f$ is rigid for all pairs e, f of edges of H. We have to consider several cases depending on the locations of e and f with respect to C, its diagonals, and the two bracing edges. We start with the case when the removal of e and f does not change the block and hence we can directly use the characterization of (minimally) rigid block and hole graphs.

Case 1: e and f are disjoint from the edges of the block B.

In this case H' is a block-and-hole graph with a 5-block B and either one 5-hole (if e, f belong to the boundary of the same face) or two 4-holes. Since H is 5-connected, a simple calculation shows that H' satisfies the girth inequalities. Hence H' is (minimally) rigid by Theorem 4.3, as required.

Case 2: e or f is equal to q or r (i.e. we remove at least one of the two bracing edges).

By symmetry we may suppose that $e = q$. Then $H - e$ is a 5-connected braced triangulation, which is 2-edge rigid by Theorem 4.2. Therefore H' is rigid.

Case 3: e or f is equal to sx or sy (i.e. we remove at least one of the two diagonals).

By symmetry we may suppose that $f = sy$. Then $G - f + r$ is also a triangulation, and hence $H - f$ is a 4-connected braced triangulation. Since $H - f$ is 2-edge-rigid by Theorem 4.2, it follows that H' is rigid. Thus we are done in Case 3.

In what follows it remains to consider the situation where e and f are different from the diagonals and the bracing edges, and at least one of them, say e, is an edge of the cycle C. To deal with these cases we shall mostly use local modifications within the block and the adjacent faces and then apply various rigidity preserving operations in order to show that H' is rigid. We shall need to refer to the neighbours of vertices t, x, y, in the specific order they occur in the planar embedding. See Fig. 7 for the notation.

Case 4: $e = st$.

The edge xt belongs to two triangular faces in G, in which the third vertex is s and a, respectively. First suppose f is disjoint from the boundaries of these two faces. Then contract the edge xt in H' to obtain a 4-connected braced triangulation $\bar{H}$. Let w be the new vertex. The graph $\bar{H}$ is 2-edge rigid by Theorem 4.2. Hence $\bar{H} - f$ is rigid. Now perform a suitable vertex splitting operation at w on edges aw, wy in $\bar{H} - f$ to obtain H'. By Theorem 2.8 H' is rigid, as required.

Next suppose f is one of the edges ta, tx, xa (we have already dealt with the case when $f = sx$ is a diagonal). If $f = ta$ then t has degree four in H'. Consider $H' - t + xa''$. This graph is a block-and-hole graph with one 4-block and one 4-hole. It is 4-connected, and hence rigid by Theorem 4.3. Since we can recover H' by a 1-extension operation on edge xa'' from this graph, it follows that H' is rigid. A similar argument works when $f = tx$.

It remains to consider the subcase when $f = xa$. Recall that t, x, y, and z are all degree-five vertices and refer to Fig. 7 for the labels of the vertices around t, x, y.

Let G'' be obtained from G by deleting the vertices t, x and adding the edges $sa', a'y, hz$. Observe that G'' is a block-and-hole graph with a single 4-block (on h, y, z, z') and a single 4-hole (on h, y, a', a). We claim that each cycle separating the block from the hole has length at least four. Indeed, otherwise – since the block and the hole share the vertices h, y – a potential separating three-cycle would include a vertex which is a common neighbour of h and y in G''. This vertex can be s or a'. However, this would mean that $\{y, s, h\}$ or $\{a', t, x, h\}$ are separators in G, contradicting 5-connectivity. This verifies the claim. Now Theorem 4.3 implies that G'' is rigid.

In order to obtain H' from G'' we first apply a triangle based 2-extension to G'' to create the graph G', where the triangle is on vertices h, y, z, and the two removed edges are hz, sa'. We call the new vertex x. Since this operation preserves rigidity, G' is rigid. Next we perform a vertex splitting in G' at a', on edges $aa', a'a''$. We call the new vertex t and perform the split in such a way that t gets connected to x and y (in addition to a, a', a''). The resulting graph is isomorphic to H', and it is rigid by Theorem 2.8, as required.

Case 5: $e = tx$.

First we consider the subcases when f is also on the cycle. By Case 4 and symmetry we may assume that f is different from st, sz. Thus we have two subcases of this type.

If $f = yz$ then contract xy in H' to obtain the graph $\bar{G}$. Denote the new vertex by w. Notice that $\bar{G}$ is a triangulation, so it is rigid by Theorem 4.1. By applying a suitable vertex splitting operation at w in $\bar{G}$ we obtain H'. Hence H' is rigid by Theorem 2.8.

If $f = xy$ then contract zy to obtain a 4-connected braced triangulation $\bar{H}$. Denote the new vertex by w. Theorem 4.2 implies that $\bar{H}$ is 2-edge rigid. Hence $\bar{H} - e$ is rigid. By applying a suitable vertex splitting operation at w in $\bar{H} - e$ on edges sw, wz' we obtain H'. Thus H' is rigid by Theorem 2.8.

Next we deal with the subcase when f is not on the cycle. Suppose that f is different from ta'', sa''. Contract st in H' to obtain a 4-connected braced triangulation $\bar{H}$, in which the new vertex is w. Since $\bar{H}$ is 2-rigid by Theorem 4.2, it follows that $\bar{H} - f$ is rigid. Then a suitable vertex splitting operation at w can be used to obtain H'. Thus H' is rigid by Theorem 2.8.

It remains to consider the subcases when the previous argument does not work: when f and st belong to the same triangle in $H - e$. Let us suppose $f = ta''$. Then the graph $H' - t + a's$ is a 4-connected block-and-hole graph with one 4-hole and one 4-block. Thus it is rigid by Theorem 4.3. From this graph we can regain H' by a 1-extension operation, showing that H' is rigid.

Now suppose $f = sa''$. Contract the edge tx in H to obtain a 4-connected braced triangulation $\bar{H}$. It is 2-rigid by Theorem 4.2, so $\bar{H} - sa''$ is rigid. We can then obtain H' by a suitable extended vertex splitting operation, which shows that H' is rigid.

Case 6: $e = xy$.

By Cases 4, 5, and by symmetry, we may assume that f is not on the cycle. Suppose that f is different from xa, ta.

Then contract tx in H to obtain a 4-connected braced triangulation $\bar{H}$. The graph $\bar{H}$ is 2-rigid by Theorem 4.2. Hence $\bar{H} - f$ is rigid. We can then obtain H' by a suitable vertex splitting operation, showing that H' is rigid.

It remains to consider the subcases when the previous argument does not work: when f and tx belong to the same triangle in $H - e$. Let us suppose $f = xa$. Delete vertex x and add an edge tz to obtain $\bar{H}$. It is a 4-connected block-and-hole graph with one 4-block and one 4-hole. Thus it is rigid. By applying a 1-extension to $\bar{H}$ we obtain H'. Hence H' is rigid.

Finally suppose that $f = ta$. In this subcase we shall use the fact that vertex h is a degree-five vertex of H.

Let $\bar{H} = H - h - f + az'$. Observe that $\bar{H}$ is a block-and-hole graph, with one 5-block (x, t, s, z, y) and two 4-holes (a, a', t, x) and (a, x, y, z'), which satisfies the girth inequalities (by the 5-connectivity of H). Thus it is rigid. Then apply a triangle based 2-extension on the triangle a, h', z' and edge xy so that the edges az' and xy are removed. Here h' is the fifth neighbour of h which does not appear in Fig. 7. This operation creates H', and hence H' is rigid. This completes the proof. □

The lower bound $3n - 4$ and the previous construction implies:

Theorem 4.5. *Let G be a strongly minimally 3-edge rigid graph in $\mathbb{R}^3$ on $n \geq 14$ vertices. Then G has $3n - 4$ edges.*

Theorem 4.4 corresponds to a special case of a much more general conjecture: Whiteley [20] conjectured that every 5-connected braced triangulation with at least two bracing edges is 3-edge-rigid.

We remark that a completely different construction gives rise to a family of 3-edge rigid graphs with $3n - 1$ edges: consider the cone of a strongly minimally 3-vertex rigid graph G' in $\mathbb{R}^2$. Such a graph G' on $n' = n - 1 \geq 8$ vertices has $2n' + 2$ edges, as it was shown in [16]. Hence the cone has $3n - 1$ edges. The cone is 3-edge rigid in $\mathbb{R}^3$ by Theorem 2.9(ii).

4.2. 4-Edge-rigidity

Before we show the solution to the 4-edge rigid version, we prove a result that we shall use in other constructions, too. Let C_n^3 denote the cube of the cycle on n vertices. See Fig. 1.

Theorem 4.6. *Let Q be a graph obtained from C_n^3 by deleting a vertex v and an edge e, for some $n \geq 10$. Then Q can be obtained from K_5 by a sequence of 1-extensions and edge additions.*

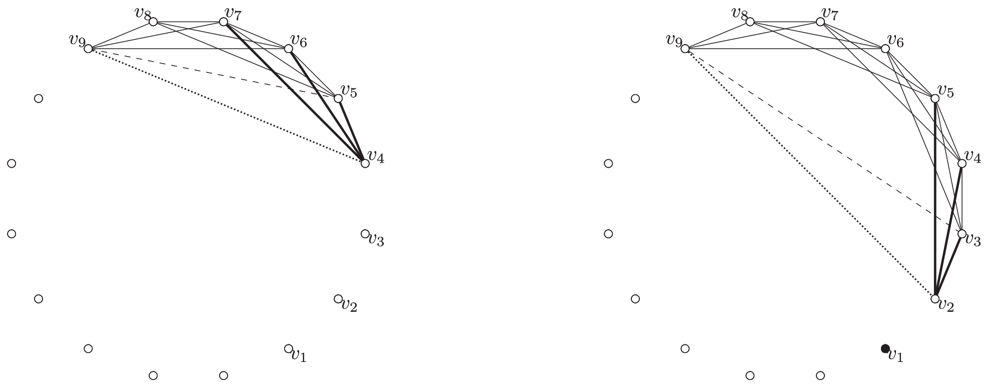

Fig. 8. Migrating the temporary edge towards v_2.

Proof. We may suppose that e is disjoint from v. First suppose that v is not a vertex of the shortest path between the end-vertices of e in C_n. By symmetry, and using that $n \geq 10$ we may assume that $v = v_1$ and $e = v_iv_j$ with $i < j$ and $j \geq 7$. Consider the set X of five vertices $\{v_{j-1}, v_{j-2}, \ldots, v_{j-5}\}$ that precede v_j on C_n. This set, which must include v_i, will be the vertex set of the base K_5. The graph $Q[X]$ is isomorphic to K_5 minus an edge.

Add this missing edge $v_{j-1}v_{j-5}$ to $Q[X]$. We call it a temporary edge. Then add the vertices $v_{j-6}, v_{j-7}, \ldots, v_2$, in this order, by a sequence of 1-extensions which involves the temporary edge and two edges of Q. This can be done, and in the resulting graph the temporary edge will connect v_{j-1} and v_2, see Fig. 8.

Then add the vertices $v_j, v_{j+1}, \ldots, v_n$, in this order, by a sequence of 1-extensions in such a way that when v_j is added then the 1-extension deletes $e = v_iv_j$ and adds another temporary edge from v_j to v_2. If $j < n$ then the remaining 1-extensions are performed in such a way that the temporary edge furthest from v_n is involved, along with two edges from Q. This way the end-vertices of the two temporary edges will move towards v_n and will end up in the positions $v_{n-1}v_2$ and v_nv_2. But these edges exists in Q. Hence the resulting graph is a spanning subgraph of Q and our construction shows that Q can indeed be obtained from K_5 by a sequence of 1-extensions and edge additions.

Next suppose that $v = v_1$ is a vertex of the shortest path between the end-vertices of e in C_n. In this case it is easy to check that a similar proof works: we start with a K_5 on the five vertices $\{v_n, v_{n-1}, \ldots, v_{n-4}\}$, with one temporary edge, and then add the remaining vertices by 1-extensions. By adding an appropriate second temporary edge right before the last vertex, v_2, is added, the construction can be completed so that both temporary edges align with edges of Q. □

Theorem 4.7. *The graph C_n^3 is 4-edge rigid in $\mathbb{R}^3$ for $n \geq 10$.*

Proof. Let E, V denote the edge set and the vertex set of C_n^3, respectively. It suffices to show that $C_n^3 - F$ is rigid for all $F \subseteq E$ with $|F| = 3$. Consider a fixed triple $F = \{e, f, g\}$. Let $f = uv$.

The graph $Q = C_n^3 - v - e$ can be obtained from K_5 by a sequence of 1-extensions and edge additions by Theorem 4.6. Thus Q is 2-edge rigid by Lemma 2.6, which implies that $Q - g$ is rigid. Since the degree of v in C_n^3 is equal to six, v is connected to $Q - g$ by at least three edges, which are different from f. Hence $C_n^3 - F$ can be obtained from $Q - g$ by a 0-extension (and possibly some edge additions). Therefore it is rigid, as required. □

The degree lower bound (3) and Theorem 4.7 implies:

Theorem 4.8. *The number of edges in a strongly minimally 4-edge rigid graph on $n \geq 10$ vertices in $\mathbb{R}^3$ is equal to $3n$.*

4.3. k-edge rigidity for $k \geq 5$

The tight bounds for $k \geq 5$ follow from the lower bound (3), Lemma 2.3, and Theorem 3.3.

Theorem 4.9. *Let $k \geq 5$ and $n \geq 12k + 10$ be even. Then the number of edges in a strongly minimally k-edge rigid graph on n vertices in $\mathbb{R}^3$ is equal to $\lceil \frac{(k+2)n}{2} \rceil$.*

The previous results for $k \leq 4$ and Theorem 4.9 show that the lower and upper ranges indeed exist. The upper range starts at $k = 5$.

5. Vertex and edge redundant global rigidity in $\mathbb{R}^3$

The extremal problems for k-vertex and k-edge global rigidity in $\mathbb{R}^3$ have not been studied before, except for $k = 1$, where the tight bound (in both cases) is $3n - 5$, as we noted earlier. In this section we deduce the tight bounds for almost all cases.

5.1. Vertex redundant global rigidity

Lemma 2.13 implies that the number of edges in a 2-vertex globally rigid graph on n vertices is at least $3n-2$. The next construction comes very close to this bound.

Theorem 5.1. *Let e be an edge of C_n^3, $n \geq 5$. Then $C_n^3 - e$ is 2-vertex globally rigid in $\mathbb{R}^3$.*

Proof. Let v be a vertex of C_n^3 and let $Q = C_n^3 - e - v$. The graph Q can be obtained from K_5 by a sequence of 1-extensions and edge additions by Theorem 4.6. Thus it follows from Lemma 2.6 that Q is globally rigid. Since the choice of v is arbitrary, the theorem follows. □

As a corollary, we obtain:

Theorem 5.2. *The number of edges in a strongly minimally 2-vertex globally rigid graph on n vertices in $\mathbb{R}^3$ is at most $3n-1$.*

Thus the tight bound for the size of a strongly minimally 2-vertex globally rigid graph is either $3n-2$ or $3n-1$. We believe that the graph obtained from C_n^3 by removing two disjoint edges is 2-vertex globally rigid in $\mathbb{R}^3$, and hence the tight bound is $3n-2$. Even though our computational experiments suggest that these graphs are indeed globally rigid, we have not yet found a proof.

Finding the tight bound for 3-vertex global rigidity remains an open problem, too. A close-to-tight bound follows from Theorem 3.4 and Lemma 2.2.

Theorem 5.3. *Let G be a strongly minimally 3-vertex globally rigid graph on $n \geq 44$ vertices in $\mathbb{R}^3$. Then G has at most $3n+20$ edges.*

For $k \geq 4$, however, we have the exact result. It follows from the degree lower bound (4), Lemma 2.2, and Theorem 3.3.

Theorem 5.4. *Let $k \geq 4$ and $n \geq 12(k+1)+10$ be even. Then the number of edges in a strongly minimally k-vertex globally rigid graph on n vertices in $\mathbb{R}^3$ is equal to $\lceil \frac{(k+3)n}{2} \rceil$.*

5.2. Edge-redundant global rigidity

In the case of 2-edge global rigidity we have an almost tight bound. Let C_n^2 denote the square of a cycle on n vertices.

Theorem 5.5. *Let H_n be the cone of C_n^2, $n \geq 5$. Then H_n is 2-edge globally rigid in $\mathbb{R}^3$.*

Proof. It is known that C_n^2 is 2-vertex globally rigid in $\mathbb{R}^2$ for $n \geq 5$, see [18]. The theorem follows from Lemma 2.2. □

The number of edges in the cone of the square of a cycle on n vertices in total is $3n-3$. Therefore we have the following upper bound.

Theorem 5.6. *The number of edges in a strongly minimally 2-edge globally rigid graph on $n \geq 5$ vertices in $\mathbb{R}^3$ is at most $3n-3$.*

By (6) the number of edges in a 2-edge globally rigid graph on n vertices is at least $3n-4$. Thus the tight bound for the size of a strongly minimally 2-edge globally rigid graph is either $3n-4$ or $3n-3$.

Determining the best possible bound remains an open question. We conjecture that the $3n-4$ is the right number. We note that it is conjectured in [12] that every 5-connected braced triangulation on n vertices with at least $3n-4$ edges is 2-edge globally rigid in $\mathbb{R}^3$. The truth of this conjecture would imply that $3n-4$ is indeed tight.

For $k \geq 3$ we can deduce the exact bound. In the case of 3-edge global rigidity we have the following construction.

Theorem 5.7. *The graph C_n^3, for $n \geq 6$, is 3-edge globally rigid in $\mathbb{R}^3$.*

Proof. Let E, V denote the edge set and the vertex set of C_n^3, respectively. It suffices to show that $C_n^3 - F$ is globally rigid for all $F \subseteq E$ with $|F| = 2$. We shall prove that $C_n^3 - F$ is in fact 2-vertex rigid for all $F \subseteq E$ with $|F| = 2$. This implies that it is globally rigid by Lemma 2.2.

Consider a fixed pair $F = \{e, f\}$ of edges and a vertex v. The graph $Q = C_n^3 - v - e$ can be obtained from K_5 by a sequence of 1-extensions and edge additions by Theorem 4.6. Thus Q is 2-edge rigid by Lemma 2.6(ii), which implies that $Q - f = C_n^3 - F - v$ is rigid, as claimed. □

By comparing Theorem 5.7 and the lower bound (4), we have:

Theorem 5.8. *The number of edges in a strongly minimally 3-edge globally rigid graph on $n \geq 6$ vertices in $\mathbb{R}^3$ is equal to $3n$.*

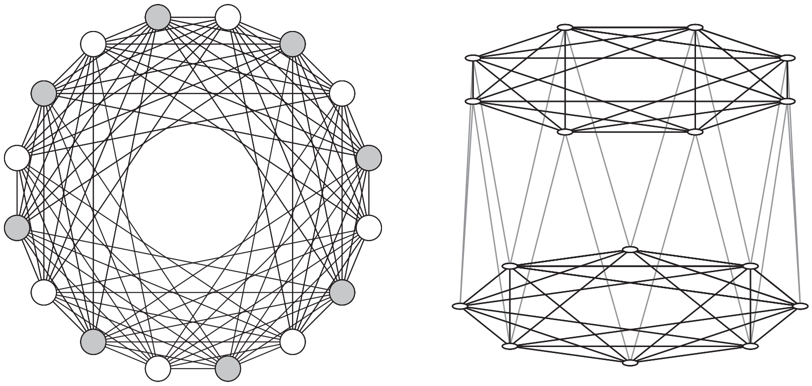

Fig. 9. Two drawings of the graph T_{16}^3. Even and odd vertices are denoted by empty and filled circles, respectively, on the left. The corresponding partition into two C_8^3's is shown on the right, with lighter lines for the matching edges.

For $k \geq 4$ Theorem 5.4 and Lemma 2.3 imply the following bound.

Theorem 5.9. *Let $k \geq 4$ and $n \geq 12(k+1)+10$ be even. Then the number of edges in a strongly minimally k-edge globally rigid graph on n vertices in $\mathbb{R}^3$ is equal to $\lceil \frac{(k+3)n}{2} \rceil$.*

6. One more result

In this section we consider our extremal problem for k-vertex rigidity in the plane in the case when $k \geq 5$ is odd. A solution to this problem was given in [10], where strongly minimally k-vertex rigid graphs on n vertices in $\mathbb{R}^2$ were constructed for every (sufficiently large) odd n, for every odd $k \geq 5$. Here we give a different construction, which works for n even, complementing the result of [10]. It also gives an affirmative answer to a conjecture from [14].

We shall need the next lemma, already used in [10].

Lemma 6.1. *Let G_1 and G_2 be two disjoint rigid graphs in $\mathbb{R}^2$ and let $F = \{e, f, g\}$ be a set of three edges connecting G_1 to G_2. If e and f are disjoint, then $G_1 \cup G_2 \cup F$ is rigid in $\mathbb{R}^2$.*

We shall also use the following tight bound on the vertex redundancy of powers of cycles in the plane.[2]

Lemma 6.2 ([10])**.** *Let $k \geq 2$ and $n \geq max\{2k+2, 4k-5\}$. Then*

$$R_v^2(C_n^k) = 2k - 2.$$

Let T_n^k be the graph obtained from C_n by adding all edges vu for which the distance from v to u in C_n is an even integer less than or equal to $2k$. Note that T_n^k is $(2k+2)$-regular if $n \geq 4k+2$.

Theorem 6.3. *Let $k \geq 2$ and let $n \geq 4k+6$ be even. Then*

$$R_v^2(T_n^k) = 2k + 1.$$

Proof. First observe that T_n^k can be obtained by taking two disjoint copies of $C_{\frac{n}{2}}^k$ (spanned by the vertices with odd, resp. even indices in C_n) and connecting them by n edges (the edge set of C_n). The connecting edges can be partitioned into two matchings. See Fig. 9. The two copies of $C_{\frac{n}{2}}^k$ will be denoted by G_1 and G_2. Let S be a vertex set with $|S| = 2k$. Let $S_i = G_i \cap S$, $i = 1, 2$. We have to show that $T_n^k - S$ is rigid in $\mathbb{R}^2$.

Case 1: $k \geq 3$.

First suppose that $|V(G_i) \cap S| \leq 2k - 3$ for $i = 1, 2$. Then $G_i - S$ is rigid for $i = 1, 2$, by Lemma 6.2. Since $n \geq 4k+6$, there exists a set F of three disjoint edges between $G_1 - S$ and $G_2 - S$. By Lemma 6.1 this implies that $(G_1 \cup G_2 \cup F) - S$ is rigid. Since this graph is a spanning subgraph of $T_n^k - S$, it follows that $T_n^k - S$ is rigid, as required.

Next suppose that S contains at least $2k-2$ vertices from, say, G_1. Since $|S_2| \leq 2$ and $k \geq 3$, Lemma 6.2 implies that $G_2 - S_2$ is rigid. Thus if $G_1 - S_1$ is also rigid, we can apply the argument of the previous paragraph to deduce that $T_n^k - S$ is rigid. So we may assume that $G_1 - S_1$ is not rigid.

[2] We studied the 3-dimensional case and verified the corresponding tight bound: for $k \geq 3$ and $n \geq 6k - 10$ we have $R_v^3(C_n^k) = 2k - 3$. We omit the details.

Since $|S_2| \le 2$ and the edge set connecting G_1 and G_2 is a 2-regular bipartite graph, there are at most $2|S_2| \le 4$ vertices in $G_1 - S_1$ with less than two neighbours in $G_2 - S_2$. Let Q be this set of vertices. Since $G_2 - S_2$ is rigid, we can attach all vertices in $G_1 - S_1 - Q$ to $G_2 - S_2$ by 0-extensions, using edges of $T_n^k - S$ (these edges belong to the underlying cycle C_n). Let the resulting rigid subgraph of $T_n^k - S$ be called G.

It remains to add the vertices of Q to G, preserving rigidity. If $S_2 = \emptyset$ then $Q = \emptyset$ and there is nothing to prove. If $|S_2| = 1$ then $|S_1| = 2k - 1$, $|Q| \le 2$, and each vertex in Q has a neighbour in $G_2 - S_2$. Since G_2 is $2k$-regular, this implies that the vertices of Q can be added by 0-extensions, using edges of $T_n^k - S$. The final subcase to consider is when $|S_2| = 2$, $|S_1| = 2k - 2$, and $|Q| \in \{3, 4\}$. If $|Q| = 3$ then the vertices of S_2 are consecutive in the cycle underlying G_2 and the vertices of Q, call them a, b, c, are consecutive in the cycle underlying G_1. Now a and c have at least one neighbour in $G_2 - S$ and at least one of them has a neighbour in $G_1 - S$, for otherwise the k vertices preceding a and the k vertices following c are all in S_1, which is impossible as $|S_1| = 2k - 2$ and $n \ge 2k + 4$. Hence we can add one of them, say a, by a 0-extension. Then we can add c and b, in this order, by two more 0-extensions. This completes the argument in this subcase.

If $|Q| = 4$ then each vertex in Q has a neighbour in $G_2 - S_2$. Furthermore, Q consists of two pairs of consecutive vertices in the cycle underlying G_1. A proof similar to that of the previous subcase shows that we can add the vertices of Q to G by four 0-extensions.

Case 2: $k = 2$.

Now $|S| = 4$, and G_1, G_2 are isomorphic to the square of a cycle. We may assume that $|S_2| \le |S_1|$. If $G_2 - S_2$ is rigid then the proof of Case 1 works without any changes. So we may assume that $|S_1| = |S_2| = 2$ and $G_i - S_i$ is non-rigid for $i = 1, 2$. It follows that S_i is a non-adjacent pair of vertices in the cycle underlying G_i, $i = 1, 2$.

Let us consider $H := T_n^2 - S$ and the cycle C_n underlying T_n^2. Recall that we have added the edges of length 2 and 4 to C_n to obtain T_n^2. By our choice of $n \ge 4k + 6 = 14$, there exists an interval I of size at least 3 in H. We may assume that I is the largest interval. It is easy to see that $H[I]$ is rigid. By removing the vertices of S the cycle is split into two, three, or four intervals. One of them is I. Since the vertices of S_i, $i = 1, 2$, are non-adjacent on the cycle of G_i, each interval contains at least two vertices. If we have only two intervals then S consists of two pairs of consecutive vertices of C_n. Then four edges of T_n^2 connect the two intervals and hence their union induces a rigid spanning subgraph of H by Lemma 6.1. So $T_n^2 - S$ is rigid, as required. It remains to consider the cases when we have three or four intervals.

First suppose that we have three intervals. Then we have a subset $S' \subset S$ which consists of two consecutive vertices of C_n. We shall prove that the vertices of $H - I$ can be added to I by a sequence of extensions, using edges of H. If S' neighbours I and $|I| \ge 3$ then we add the vertices cyclically, away from I, via 0-extensions, starting at the opposite end of I. In the case when $|I| = 2$ we need 1-extensions, too. If S' does not neighbour I, in each of the other two intervals, we add the vertices cyclically, away from I, via 0-extensions. It is easy to see that it is indeed possible to perform these additions in T_n^2, as claimed.

Finally, suppose that we have four intervals. As in the previous case, we add the vertices in the intervals that neighbour I cyclically, away from I, via 0-extensions. Since each interval has size at least two, we can use Lemma 6.1 to conclude that the fourth interval can also be added preserving rigidity. It follows that H is rigid. This completes the proof. □

Since T_n^k is a $(2k+2)$-regular $(2k+1)$-vertex rigid graph in $\mathbb{R}^2$, it is strongly minimally $(2k+1)$-vertex rigid (assuming n is even). The special case of this corollary for $k = 2$ was conjectured in [14, Conjecture 3].

7. Concluding remarks

In this paper we have determined the size of the strongly minimally k-vertex and k-edge (globally) rigid graphs in $\mathbb{R}^3$ on n vertices for all k (and n large enough) and for each of the four versions with the exception of four special cases. In these special cases we obtained close-to-tight bounds. See Table 1. We conjecture that $\epsilon_2 = \epsilon_4 = 0$, that is, our lower bounds are in fact tight in these two special cases as well.

Our results demonstrated that in each of the four versions of the 3-dimensional problem there is indeed a bipartition into lower and upper ranges: if k is in the upper range, the extremal value is given by the degree lower bound, while for values in the lower range the tight bound is $3|V|$ plus some constant. Table 1 shows that the upper range starts at $k = 5$ for vertex and edge rigidity, and at $k = 4$ for vertex and edge global rigidity. We conjecture that the lower and upper ranges (which have been previously shown to exist in $\mathbb{R}^d$ for $d = 1, 2$, too) also exist for all $d \ge 4$, with threshold values $d + 2$ and $d + 1$, following the pattern of the cases $d = 2, 3$.

Our extremal problems remain open for $d \ge 4$, $k \ge 3$.

Note added in proof: in a recent manuscript we verified that $\epsilon_4 = 0$, see Q. Chen, S. Jajodia, T. Jordán, K. Perkins, Redundantly globally rigid braced triangulations, Egerváry Research Group, Budapest, TR-2021-12.

Acknowledgements

We thank Csaba Király for several useful comments.

This work was supported by the Hungarian Scientific Research Fund, Hungary grants no. K 109240, K135421, and the project Application Domain Specific Highly Reliable IT Solutions which has been implemented with the support provided from the National Research, Development and Innovation Fund of Hungary, financed under the Thematic Excellence Programme, Hungary TKP2020-NKA-06 (National Challenges Subprogramme) funding scheme.

Table 1
The extremal values in $\mathbb{R}^3$ for the four versions and for all $k \geq 1$. The values obtained in this paper are in boldface. In four special cases the bounds are not tight but get close to the right value. We show that $\epsilon_1 \leq 15$, $\epsilon_3 \leq 18$, and $\epsilon_2, \epsilon_4 \leq 1$.

Redundancy	1	2	3	4	5	...	k
Vertex rigidity	$3n-6$	$3n-3$	$3n$	$\mathbf{3n+5+\epsilon_1}$	$\lceil \mathbf{3.5n} \rceil$	...	$\lceil \frac{(k+2)n}{2} \rceil$
Edge rigidity	$3n-6$	$3n-5$	$\mathbf{3n-4}$	$\mathbf{3n}$	$\lceil \mathbf{3.5n} \rceil$	...	$\lceil \frac{(k+2)n}{2} \rceil$
Vertex global rigidity	$3n-5$	$\mathbf{3n-2+\epsilon_2}$	$\mathbf{3n+2+\epsilon_3}$	$\lceil \mathbf{3.5n} \rceil$	$\mathbf{4n}$	...	$\lceil \frac{(k+3)n}{2} \rceil$
Edge global rigidity	$3n-5$	$\mathbf{3n-4+\epsilon_4}$	$\mathbf{3n}$	$\lceil \mathbf{3.5n} \rceil$	$\mathbf{4n}$	...	$\lceil \frac{(k+3)n}{2} \rceil$

References

[1] G. Brinkmann, B.D. McKay, Construction of planar triangulations with minimum degree 5, Discrete Math. 301 (2005) 147–163.
[2] R. Connelly, Generic global rigidity, Discrete Comput. Geom. 33 (2005) 549–563.
[3] R. Connelly, W. Whiteley, Global rigidity: the effect of coning, Discrete Comput. Geom. 43 (2010) 717–735.
[4] J. Cruickshank, D. Kitson, S.C. Power, The generic rigidity of triangulated spheres with blocks and holes, J. Comb. Theory Ser. B. 122 (2017) 550–577.
[5] W. Finbow-Singh, W. Whiteley, Isostatic block and hole frameworks, SIAM J. Discrete Math. 27 (2013) 991–1020.
[6] B. Hendrickson, Conditions for unique graph realizations, SIAM J. Comput. 21 (1992) 65–84.
[7] B. Hendrickson, The molecule problem: exploiting structure in global optimization, SIAM J. Optim. 5 (4) (1995) 835–857.
[8] B. Jackson, T. Jordán, Graph theoretic techniques in the analysis of uniquely localizable sensor networks, in: G. Mao, B. Fidan (Eds.), Localization Algorithms and Strategies for Wireless Sensor Networks, IGI Global, 2009, pp. 146–173.
[9] T. Jordán, Combinatorial rigidity: graphs and matroids in the theory of rigid frameworks, in: Discrete Geometric Analysis, in: MSJ Memoirs, vol. 34, 2016, pp. 33–112.
[10] T. Jordán, Minimum size highly redundantly rigid graphs in the plane, Graphs Combin. 37 (2021) 1415–1431.
[11] T. Jordán, C. Poston, R. Roach, Extremal Families of Redundantly Rigid Graphs in Three Dimensions, Egerváry Research Group, Budapest, TR-2020-22.
[12] T. Jordán, S. Tanigawa, Global rigidity of triangulations with braces, J. Comb. Theory Ser. B. 136 (2019) 249–288.
[13] T. Jordán, W. Whiteley, Global rigidity, in: J.E. Goodman, J. O'Rourke, C. Tóth (Eds.), Handbook of Discrete and Computational Geometry, third ed., CRC Press, 2018.
[14] V. Kaszanitzky, Cs. Király, On minimally highly vertex-redundantly rigid graphs, Graphs Combin. 32 (2016) 225–240.
[15] R. Kohta, M. Yamakawa, N. Katoh, Y. Araki, M. Ohsaki, A design method for optimal truss structures with certain redundancy based on combinatorial rigidity theory, in: 10th World Congress on Structural and Multidisciplinary Optimization, Orlando, Florida, USA, 2013.
[16] S.A. Motevallian, C. Yu, B.D.O. Anderson, On the robustness to multiple agent losses in 2D and 3D formations, Internat. J. Robust Nonlinear Control (2014).
[17] B. Schulze, W. Whiteley, Rigidity and scene analysis, in: J.E. Goodman, J. O'Rourke, C.D. Tóth (Eds.), Handbook of Discrete and Computational Geometry, third ed., CRC Press, 2018.
[18] T.H. Summers, C. Yu, B.D.O. Anderson, Addressing agent loss in vehicle formations and sensor networks, Internat. J. Robust Nonlinear Control 19 (15) (2009) 1673–1696.
[19] S. Tanigawa, Sufficient conditions for the global rigidity of graphs, J. Combin. Theory Ser. B 113 (2015) 123–140.
[20] W. Whiteley, Infinitesimally rigid polyhedra II: Modified spherical frameworks, Trans. Amer. Math. Soc. 306 (1988) 115–139.
[21] W. Whiteley, Vertex splitting in isostatic frameworks, Struct. Topol. 16 (1991) 23–30.
[22] W. Whiteley, Ome matroids from discrete applied geometry, in: Matroid Theory (Seattle, WA, 1995), in: Contemp. Math., 197, Amer. Math. Soc., Providence, RI, 1996, pp. 171–311.
[23] W. Whiteley Cones, Infinity, and one-story buildings, Struct. Topol. 8 (1983) 53–70.
[24] C. Yu, B.D.O. Anderson, Development of redundant rigidity theory for formation control, Internat. J. Robust Nonlinear Control 19 (13) (2009) 1427–1446.