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# Complex Mathematics Education: An Integrated and Inquiry-Based Mathematics Teaching Method

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**Abstract** Little is available in mathematics education research about what the teacher can anticipate from the students when applying inquiry-based learning (IBL). Even less is known about how to recognize and exploit on the spot when a mathematical domain, other than the one in focus, is activated in the students' minds. Yet, in tests, in everyday life, and the labour market, it is common to face problems that require interrelated mathematical thinking. Although one of the unique advantages of complex mathematics education (CME) is the coherence between different domains and CME has been practiced for over half a century in Hungary, the Hungarian line of IBL has only recently joined the international methodological mainstream. In this paper, I summarize a segment of IBL correspondent to CME and integrated mathematics education, and I illustrate the possible divergence of solutions during implementation with an example that emerged about a probability game in a fifth-grade class.

**Résumé** La recherche portant sur l'enseignement des mathématiques ne révèle que peu de contenu en ce qui a trait à ce que l'enseignant peut s'attendre des élèves lorsqu'il utilise l'apprentissage basé sur l'enquête (ABE). On en sait encore moins sur la façon de reconnaître et d'exploiter de manière impromptue le moment où l'attention des élèves est avivée par un domaine mathématique autre que celui sur lequel l'effort est axé. Pourtant, dans les tests, dans la vie de tous les jours et sur le marché du travail, il est courant d'être confronté à des problèmes qui nécessitent un raisonnement mathématique corrélatif. Bien que l'un des avantages particuliers de l'enseignement complexe des mathématiques (ECM) soit la cohérence entre différents domaines, et alors que l'ECM est pratiqué depuis plus d'un demi-siècle en Hongrie, le courant hongrois de l'ABE n'a rallié que récemment la tendance méthodologique internationale. Dans cet article, je résume un aspect de l'ABE qui s'applique à l'ECM et à l'enseignement intégré des mathématiques, et j'illustre, avec un exemple qui s'est manifesté à l'égard d'un jeu de probabilité dans une classe de cinquième année les différentes solutions potentielles associées à la mise en œuvre.

**Keywords** Inquiry-based education · Complex mathematics education · Thinking processes · Probability

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## Introduction

Although there are many research papers and syllabi which promote and urge inquiry-based learning in schools, its effects on everyday classrooms have remained limited. One of the reasons for this phenomenon may be that teachers fear what would and could occur on their students' side in the classroom when they let go of control. I would like to shed more light on this issue in this paper. I first briefly summarize what mathematics education research shows about inquiry-based learning (IBL) and certain branches of IBL, with a special focus on its Hungarian branch, complex mathematics education (CME). One of the specialties of CME is that it puts a special emphasis on interrelationships of mathematical domains, and it views mathematics and mathematics teaching through integrating spectacles. This unique view is presented in the following primarily in light of IBL, with an example from teaching probability.

## Theoretical Background

CME is a type of IBL, in which students are guided in their inquiry. Mathematical concepts and problems are embedded into questions about the broader real world of the students, including games and tales. Students' own ideas are listened to and utilized; therefore, active learning takes place, while their thought-processes are channelized or guided through queries. This is called guided (or Level 2) inquiry in research and has been proved to be more fruitful in mathematics teaching and also leads to greater commitment towards mathematics than many other teaching methods (Freeman et al., 2014). Bruder and Prescott (2013) in their synthesis state that they believe it is unnecessary to carry out more studies comparing IBL to other teaching methods. It seems to be true not only for humankind but several animal species that we develop stronger attachment to what we had to cope with and work for (Czaczkes et al., 2018).

Even though IBL is an umbrella-term, researchers have managed to introduce a definition which is adequate and gives a common ground to practitioners first in the European PRIMAS project (http://www.primasproject.eu), and then, this definition was further refined in the succeeding projects (Artigue et al., 2020) The objective of the project was a widespread implementation of IBL in day-to-day teaching across all 10 partner countries. Therefore, the definition had to be broad enough to work well in different teaching traditions, narrow enough to allow selecting a common set of tasks, and complex enough to clarify what IBL means for those who do not yet have hidden understanding of it. The definition has five pillars: valued outcomes, classroom culture, what the teacher does, what students do, and the type of questions the class deals with (Fig. 1).

Many studies urge teachers to take on the role of a facilitator, instead of being an instructor, which includes letting go of control and content, and share authority (Kinser-Traut & Turner, 2020; Maaß & Artigue, 2013). Despite the facts that a more student-centred education has been discussed for decades, many theoretical constructs have been formed (realistic mathematics education and the problem solving tradition, for example), and there are an increasing number of studies connected to the field, and effects on everyday classrooms remained limited (Artigue & Blomhøj, 2013; Maaß & Artigue, 2013). Maaß and Artigue (2013) believe that one of the reasons behind this inefficiency is a sort of theory–practice gap. There is less attention paid on the dissemination (spreading the innovation widely) and implementation (setting the innovation in motion) process than would be appropriate.

One of the symptoms of the latter can be that although there has been remarkable research about important issues such as differentiating instructions in middle schools (Hackenberg et al., 2021), the five phases of inquiry (Bybee, 2014; Gillies, 2020), and teacher-student dialogues (Erath et al., 2021), there is little description and analysis of what happens on the students' side of the classroom and what teachers can anticipate on the spot when they apply the proposed methods including letting go of control.

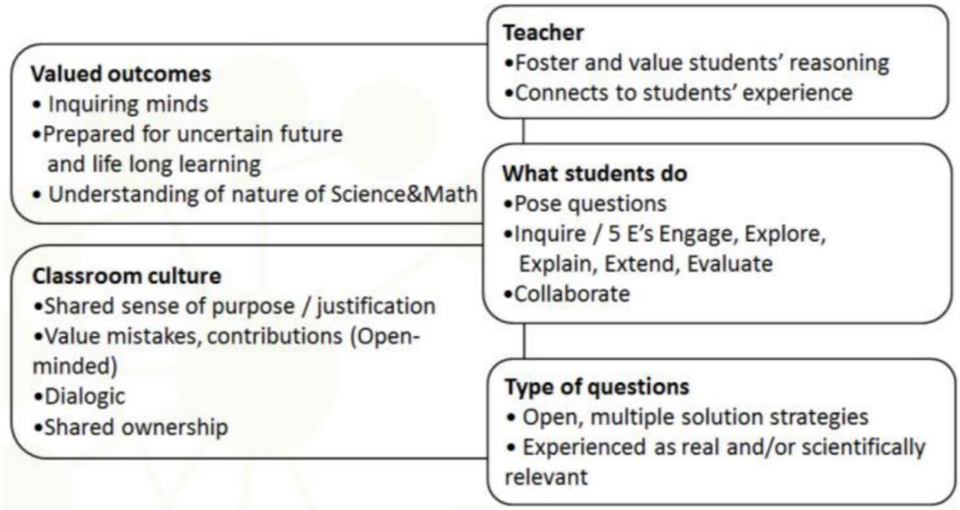


**Fig. 1** The definition of IBL in Artigue et al. (2020)

Based on Madden and Bickel (2010), I believe to convince teachers it is not enough to deal with their fears whether their students would perform well enough or whether they would like mathematics on the long term. These are in the focus of many former studies. But it would also be necessary to show what *the teachers can anticipate on the short term*, and how they can repossess control in those few moments when it is due. This could help encourage teachers to exchange the role of an instructor to that of a facilitator, which is on the one hand highly challenging for the teachers (Erath et al., 2021; Varga, 1988), and on the other hand one of the premises of a wider dissemination of IBL (Maaß & Artigue, 2013).

It is a frequent requirement of modern mathematics teaching to educate flexible generations that are able to think and combine. IBL methods are in line with this requirement by focusing on the improvement of students' thinking skills instead of teaching fixed methods to solve mathematical tasks (Artigue & Blomhøj, 2013). In tests and in the labour market, it is common to pose problems that incorporate segments of multiple mathematical domains. According to Maaß and Artigue (2013) for IBL to be implemented successfully, an extensive number of resources across subjects and age groups are needed. It is a widely researched current topic, especially connected to STEM (STEAM) education, how IBL can be implemented to interweave different school subject areas. Yet, in research literature there is little information about integrated teaching of mathematics itself and that IBL can be used to interweave different domains within mathematics. Some research articles consider it a distinctive perception to see mathematics as a system and to require from mathematics teaching to focus on the net of connections within this system. "Chinese teachers hold unique beliefs that mathematics is a system and that mathematics teaching should target the web of connections" (Cai & Ding, 2017, p. 9).

The Hungarian branch of IBL, which developed from the 1960s but has only become internationally known and recognized nowadays, clearly fits Cai and Ding's (2017) depiction of the Chinese approach concerning the quoted characteristic. Even in CME's name, "Complex" refers to this "unique" feature (Gosztonyi, 2020). In CME, the teacher sets mathematically meaningful problems, which lead to the general mathematical idea; the coherence and interrelatedness of different domains are particularly in focus. The children understand not only the topic in focus but also its links with other topics (Gosztonyi, 2020). It would be interesting to study whether this trait is responsible for the Hungarian students performing better in questions connected to everyday situations even if it requires modelling at the Hungarian final exams (Csapodi & Koncz, 2016). Lessons in CME may contain problems from different mathematical domains which can be found in different thematic chapters of the book. There are many internal and implicit references within the book; therefore, teachers have to know the entire book well to be able to use it successfully (Gosztonyi, 2020). I would like to illustrate this complexity and interrelatedness with students' various solutions and thought-processes in an example game later in this paper.

In the past decade, there has been an upsurge in the research on how a teacher listens at a lesson. Hintz and Tyson (2015) in their engaging paper discuss complex listening. There they describe three distinctive types of listening a teacher may use in a lesson: evaluative, interpretive, and hermeneutic. In traditional teaching methods, evaluative listening tends to predominate, which is listening *for* something, often following a question for which the teacher has a particular answer in mind. Interpretive listening, which is seeking to understand another's thinking, and hermeneutic listening, which is listening openly and critically to one's own and others' speech through which also new thinking emerges, are rather features of IBL-related methods. Hintz and Tyson (2015) introduce complex listening for multiple types of listening within an instance and over time. They state that complex listening supports students' mathematical sensemaking and teachers can help students acquire the skills for complex listening if they learn and practice it themselves. I believe that it is not a coincidence that in Hintz and Tyson's (2015) sample hermeneutic listening and evaluative listening complement each other. I would like to reflect on some points in the example game where although evaluative listening would be possible, the teacher intentionally refrains from it to promote CME.

## Methods

In this section, I would like to present the methods of collecting data, while pointing out some reasons why the experiment took place in the given school, with the given classes and connected to the given topic.

### Why This School?

The school chosen for the experiment can no way be considered an average Hungarian school. However, there are good reasons why this school has been chosen to carry out the experiment and why the researcher decided to take on the role of both the teacher and the observer in one person. Table 1 shows the criteria behind the choice.

Let me give detailed reasons for my choice in some cases and show how the chosen school, teacher, and class meet the above requirements.

**Table 1** Criteria for the school, the teacher, and the class in the experiment

| | |
|---|---|
| The school | - Its spirit is in harmony with CME (IBL) methodology, not only in mathematics classes.<br>- It supports education for democracy.<br>- Is under the pressure of external measurements and requirements. |
| The teacher | - Regularly uses CME (IBL) methodology in her practice |
| The class | - Has CME experiences (Varga, 1988)<br>- Has been working with the teacher for at least a semester<br>- Has not been selected by ability or financial status<br>- Has an average class size (24–28 in Hungarian middle schools) |

*The Spirit of the School as a Whole Is in Line with IBL Methods and Promotes Education for Democracy*

Regarding the relationship between the teacher and the children, as well as their activity, the founder of the school points out that they engage in any activity together, yet they have the opportunity to weave in their own special ideas. The teacher gives and takes ideas; she is a playmate able to perform multiple voices and roles. The teacher does not make children play, but plays with children; this is one of her most important roles. Children's solutions are suggested by their fantasy, coloured by their activity, and the teacher and children build the work together to be complete. In this natural realization, in this open-minded work there are plenty of experience, challenge, and the joy of movement and fulfillment. The school is open up to today, working in this spirit.

Therefore, the school matches IBL requirements in its core ideas, not only in mathematics classrooms. The description of the spirit of the school taken from the school's website includes almost every item of the refined definition of IBL (Artigue et al., 2020), shown in Table 2 (the only exceptions being the items connected to motivation induced by the world of work which is obviously very distant for this age group). At the same time, it requires from the teacher to participate in the learning process in the role of a facilitator (Maaß & Artigue, 2013). The school also matches the IBL branch CME composed by Tamás Varga, an essential feature of which is democratic milieu in education (Gosztonyi, 2020).

*The School Is Under the Pressure of External Measurements and Requirements*

This point is based on the practical consideration that most schools are not free from external measurements and requirements worldwide. It would be quite easy to let go of control and content (Kinser-Traut & Turner, 2020), given that there is no output requirement. However, for the results to be widely applicable, it was necessary to examine CME in a situation that is not out of reach for the multitudes of other schools. Children in the chosen school all take the state entrance exams to secondary school at the end of grade 6, the results of which decide which schools they can go to, what they can specify in, and as a result, what kind of and how much mathematics they are going to learn.

*The Teacher Regularly Uses CME (IBL) Methodology in Her Practice*

For the teacher to be able to apply IBL methods properly, a certain amount of practice is required (Andrews-Larson et al., 2019). What sort of right and wrong considerations and solutions children give thought to can only really be revealed to the outside observer if a classroom culture is already established where children can freely share ideas, no matter if they are right or wrong (cf. the definition in Artigue et al., 2020, and Gosztonyi, 2020). Therefore, an experienced teacher at CME was needed to find out what instinctive connections between mathematical domains appear in children's thinking. According

**Table 2** Part of the refined definition of IBL in Artigue et al. (2020)

| Classroom culture | Teachers |
|---|---|
| - Shared sense of purpose/justification<br>- Value mistakes, contributions (open-minded)<br>- Dialogic<br>- Shared ownership and responsibilities<br>- Working together as a group<br>- Meaningful outcome which is useful for all students | - Foster and value upon students' reasoning<br>- Connects to students' experience<br>- From telling to supporting and scaffolding<br>- Motivating by connecting to the world of work |

to Hackenberg et al. (2021), at the heart of differentiated mathematical instructions—apart from using research-based knowledge–there is inquiry into children's thinking. As a result, these findings can later serve as a ground for research to examine teachers' instructions.

Likewise, the basis for my expectation that the teacher should have worked with the chosen class for at least a semester by the beginning of the experiment is that the classroom culture described in Artigue et al. (2020) should be well grounded before the experiment, and that the children should have some former CME (IBL) experience.

#### *The Class Is Not Selected by Ability or Financial Status, and Is of an Average Size*

The underlying thought of this requirement is the effort to inform teachers and researchers of a wide range. In a class selected by ability, a teacher may experience different challenges than in a mixed ability class. In a (private) class where finances play an important role in selection, students may have very different opportunities for learning which cannot be considered general.

A speciality of the chosen school is that the third of each class of 24–28 students have official paper that they need special help to be integrated into the school system. These students had been chosen with the help of a trained psychologist in a purposeful proportion from students coping with hyperactivity, attention deficit, autism, hearing or visual impairment, or psychological disorders—to name a few. This specialty of the school I believe does not alter the essence of the findings of this study, as many of these problems do occur in any state school.

### Why This Topic?

In everyday life, we often encounter situations in which we need to interpret the likelihood of something, or the probability of something contributes to our decision, yet there are records of multiple preconceptions, misconceptions, and biases concerning this mathematical domain (Chernoff & Sriraman, 2014; Lecoutre et al., 1990; Pratt & Kazak, 2018; Tversky & Kahnemann, 1974). Probabilistic decisions and the terms associated with them (probably, likely, is expected to, etc.) are familiar to children from a very early age. Playing games is an integral part of children's daily life, and it is an ever-recurring topic whether something is "just," "right," or "fair" in a game, what is regular in a game, and whether the game itself is fair at all. Concerning when probability should be introduced in school education, there is no consensus (Hourigan & Leavy, 2020). In IBL methods, however, including CME, discovery is related to students' experience; it grows out of it (Artigue et al., 2020); therefore, probability is a suitable area to teach from the lower grades onwards.

Probability teaching differs from teaching other mathematical domains by much, as probability is rather different from the deterministic situations that occur in other mathematical areas (Stohl, 2005). This domain is not well integrated into mathematics teaching either, except from its close relations to statistics (Borovcnik, 2011). Even Batanero and Borovcnik in their comprehensive probability book pay little attention to the idea of interweaving probability with other mathematical domains, even though they do analyze the opportunities to connect concepts within the domain and they also address the benefits of teaching related to real-world problems (Batanero & Borovcnik, 2016). As this present paper is connected to integration through many layers, including integrating IBL into educational practice, integrating CME into IBL, and integrating different ability children into the school system, to name a few, an example of the effort to integrate probability into the web of mathematics fits well the pattern of this paper.

#### How Was the Data Collected?

The study focuses on the first game of the two lessons set for probability in the fifth grade (10–11-year-old students) in Hungary and data was collected in 5 consecutive years in five different 5th-grade classes. Based on field notes and some copies of students' work, the solutions and partial solutions shared by children in the class-discussion period are going to be presented and analyzed, primarily on terms of what mathematical domains other than the one in focus are instinctively mobilized in children's thinking.

### A Brief Introduction to the Game in Light of Research

The analysis is based on a game (shown in Table 3) that is widely known and used, or at least could easily be adapted, at different levels of mathematics teaching. It is often set in the form of a problem. Some obvious advantages of the problem are that it can take the form of a game, can be played in groups, needs little need for equipment, and offers many opportunities for differentiation.

Out of the three outcomes, C is twice as probable as the other two, but all three outcomes (A, B, C) are conceived to be equiprobable because of equiprobable bias theorized by Lecoutre et al. (1990). In their experimental research containing questions similar to this game, nearly 60% of the adult participants gave a false equiprobability answer. They discuss how this bias is very resistant, and even when correct representations are available, it is hard to activate them. They find concealing the probabilistic nature of the question a possible solution. At the beginning of the game above, students usually choose one of the roles (A, B, C) thinking that their choice does not matter in the game. However, at some point of the game they realize that it does matter, as "somehow" C is gaining more points, yet they have to stay in their original role until the end of the game, which makes those who chose A or B experience cognitive dissonance (Festinger, 1957). It would be an interesting question to examine whether and how students in the different roles of the game shown in Table 3 relate to the equiprobable bias after the lesson, in light of whether and when they begin to experience cognitive dissonance during the above game.

For the teacher, the game can be helpful in building the concept of elementary probability by children recognizing the atomic events of the game (which contain only a single outcome) that will indeed be equally probable in combinatorial probability. The statistics provided by the class approximate the theoretical distribution of the victories of the three outcomes (A, B, C) according to the sample size, therefore ensuring that C is very likely to win. There is some additional complexity, beyond the level of a fifth grader, considering that the winner must accumulate 8 points. Thereafter, I am going to focus mostly on students' reasoning concerning a single dice roll.

This is the first game out of the three games that construct the frame of the lesson. At the beginning, when children have shared the roles A, B, and C among themselves, the probabilistic aim of the game for children is to realize that although the game seems to be a fair one at first glance, "somehow" the winner is almost always C. This is a meaningful problem situation that serves as the engagement phase out of the 5 Es of inquiry: engaging learners, exploring phenomena, explaining phenomena, elaborating scientific concepts and abilities, and evaluating learners (Gillies, 2020). Furthermore, just as in the RME branch of IBL, this problem situation motivates children to mathematize the situation and to find some sort of an explanation to it (Artigue et al., 2020).

**Table 3** Description of the game behind the research

| | |
|---|---|
| Three players, two regular dice | A gets a point if both are even,<br>B gets a point if both are odd,<br>C gets a point if one is even and one is odd.<br>Winner is the one who first collects 8 points. |

The “exploration” phase takes place in parallel with playing the game, as well as for some time afterwards. In this phase, children are exploring the phenomenon they experience. This inquiry is rooted in their own motivation and is based on their own ideas and approach. They may invite their playing partners to inquire the question together (Gillies, 2020). This phase leads to the “explanation” phase which requires more involvement on the part of the teacher.

In the explanation phase, the most frequent reason the students gave on their own was that there are four different outcomes: even-even, odd-odd, odd–even, and even–odd, out of which A and B got one, and C got two. Although this single explanation not necessarily convinces every member of the class, this can serve here to show a very common answer to the problem.

As in CME, it is frequent that the exploration and explanation phases take place hand in hand, the class moving back and forth within them; I am going to refer to these two phases in the following as the discussion phase.

## The Discussion Phase of CME

It is a frequent process in the discussion phase of CME that the teacher triggers the discussion, draws other students into the conversation, and invites yet other students to express their ideas either by acting as if she did not understand properly or by asking other questions, while leading students to give reasons. Example questions can be seen in Table 4. Students’ ideas are not evaluated right or wrong in the process by the teacher.

It rarely happens that the class as a whole is about to agree on something wrong. In these rare cases, it is the teacher’s responsibility to intervene. However, in CME in these cases the teacher still does not leave the role of a co-inquirer stepping into another role of a judge, for example, but either sets a task for students or asks a question, answering which can show them that their thinking direction was wrong. (For example, if the class as a whole is about to agree that a fair game is one in which it is neck and neck as to who would win, the teacher can set an obviously unfair game, check if everyone sees that it is unfair, and ask students to try to make results neck and neck. A board game of say 20 fields, in which player A rolls a regular die to step, while player B can step as many steps as she wishes, can be such a game.)

After listening to students’ suggestions and reasons, the teacher may recall the original question (“I am still a little concerned about what I should call a fair game”) and invite the whole class to get involved in the question. In primary and lower middle school, in this phase the class is often not expected to verbalize their opinion about some abstract question (fair or not fair), which still can be out of reach for some of the students. Instead, the class is asked to rely on their former practical experience (for example, games based on rolling a regular die). This attitude has the potential to involve less interested or talented students as well. The concept of fairness is formed and refined based on examples of fair and not fair games (Skemp, 2012).

**Table 4** Example questions concerning the game

| Didactical function | Teacher’s utterance |
|---|---|
| Triggering discussion | “Was this a fair game?” |
| Drawing more students into the conversation | “Do you agree?” |
| Acting as if not understanding properly | “I am not sure that it is clear for me.” |
| Asking for other approaches | “Has someone been thinking differently?” |
| Leading students to give reasons | “How could we decide if it was fair or not?” |

Typically, there is a point in the inquiry when some students are ready to confirm their ideas to a certain extent, and some are not, but it is the teacher's responsibility to keep the conversation flowing and interesting for the entire class, while slowing down the pace of inquiry so that more students can keep track. For example, prompting the class to find examples on their own or in pairs that they consider to be a fair game is a way of differentiation, an opportunity for slower students to keep up, as well as an opportunity for the teacher to gain useful information about the students' former knowledge and experience. In classes in which some students find it too easy to come up with a fair game, it is a possible way of differentiation to ask them to find examples that they think are deceptive in the same way as the game presented here (that they seem fair at first but are in fact not fair). However, it is a far more difficult task, and students probably have far less experience concerning this; therefore, it is recommended to set it for groups of three, after checking whether students already boiled down in what way the presented game was deceptive and giving a set frame for the game (for example, stating that they have to design this using two coins).

In CME, summarizing the findings of the class is delayed until it is a common knowledge of the class. At this point, the discussion period can be finished, and the teacher leaves the role of a co-inquirer, may reinforce findings as stating that they are correct, and summarizes the findings of the class. Students write down the gist of the conversation. Even in this period, the class is invited into the process, for example, by the teacher beginning a sentence: "We call a game fair if … if what?" And students continue with their own words, for example, "if the players are equally likely to win." In CME, it is not a concern that at this point "equally likely" may rather be an intuition than a computed value, provided that the intuition is in line with higher mathematics, and the concept image it conveys is correct. Neither is it a concern that students formulate the gist of ideas using their own vocabulary. In CME, although technical vocabulary is introduced, in some cases its introduction is delayed, and its use is not forced.

After summarizing the findings of a question, the class may return to the exploration phase again and inquire about another layer of the game, as it is indicated in Table 5. For example, in this game, after students decided what they call a fair game and agreed on the fact that it was not a fair game, the class may try to find out about what made it unfair, or why it was unfair. In CME, discussion about intuitions always precedes (formal) calculations, even if the latter is in reach of the class.

**Table 5** Coordination of the 5E model, CME, and the teacher's roles in the example lesson

| **5E** | **CME** | | | **Teacher's role in the lesson** |
|---|---|---|---|---|
| engagement | engagement | | | - sets the scene |
| exploration | discussion | exploration | new layer | - triggers the discussion<br>- plays the role of a co-inquirer |
| explanation | | explanation | | - draws more and more students into the conversation<br>- leads students to give reasons |
| elaboration | delayed summary and elaboration | | | - steps out of the role of a co-inquirer<br>- reinforces findings<br>- leads summary<br>- sets the scene for more questions some of which may remain open |
| evaluation | delayed evaluation, often not in the same lesson | | | - sets a scene in which students' choices highlight their advancement in conceptualizing the main mathematical contents of the lesson |

**Fig. 2** The empty 6×6 chart

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |
| 4 | | | | | | |
| 5 | | | | | | |
| 6 | | | | | | |

## Analysis of Students' Explanations and References Across Domains

Zahner and Corter (2010) discuss that probability problem solvers rely on both internal visualizations, and external sketches and diagrams. In this special case, it is very interesting to see how the reasonings changed and a divergence in solutions was observed after the teacher suggested to draw a 6×6 chart (Fig. 2).

This is the elaboration phase (Gillies, 2020). In accordance with Maaß and Artigue's (2013) description of IBL, I would like to point out here the significance of the teacher not stating whether the idea is "right" or "wrong," not instructing children how to fill in the chart, not forcing probabilistic thinking, and allowing children to come up with ideas and solutions that she may not have expected. In his 1988 paper, Varga points out the positive effects of intentionally placing students in a situation where there is not one correct answer (Varga, 1988).

A common explanation given by the students was to fill in the chart, and then count the number of As, Bs, and Cs, and discover that the ratio was 9:9:18. This discovery showed that the mathematical domain of *ratios* was mobilized, and could be used to help students realize that it was actually the same as 1:1:2.

On the basis of this explanation, another domain was mobilized. No wonder the ratio is 1:1:2, said many students, as if we split the chart into 2×2 squares, in every square there is one A, one B, and two Cs (Fig. 3). This explanation showed that students could break up the big structure into identical parts in their minds, which ability is connected to their *geometrical* thinking.

**Fig. 3** Splitting the chart into 2×2 squares

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | B | C | B | C | B | C |
| 2 | C | A | C | A | C | A |
| 3 | B | C | B | C | B | C |
| 4 | C | A | C | A | C | A |
| 5 | B | C | B | C | B | C |
| 6 | C | A | C | A | C | A |

**Fig. 4** The sequence BCB-CBC—CACACA in the chart

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | B | C | B | C | B | C |
| 2 | C | A | C | A | C | A |
| 3 | B | C | B | C | B | C |
| 4 | C | A | C | A | C | A |
| 5 | B | C | B | C | B | C |
| 6 | C | A | C | A | C | A |

Some of the explanations rooted in the way certain students started to fill in the chart. This is one of the reasons why CME recommends for the teacher to try to stay in the background as much as possible, which in this case means not starting to fill in the chart with the students. In the third case, some students started to fill in the chart starting with filling in the first row. One student explained that she first thought that the sequence B, C, B, C… would continue throughout the chart and maybe A would never occur. However, then she realized that A meant that both dice show an even number, like two and four, and it *should* occur then. She then realized that the *sequence* BCBCBC—CACACA was repeated three times in the chart (Fig. 4), and this way there were twice as many Cs as As or Bs. Here I would like to underline the importance of the teacher not using evaluative listening (Hintz & Tyson, 2015). This has a two-fold significance: on the one hand, reducing evaluative listening opens space for students to dare to tell the teacher and the class about their wrong thoughts and misconceptions ("she first thought…"), and on the other hand, it leaves the communication channel open for the students even after sharing a wrong thought, and this way allows the student to reflect on and correct misconceptions ("then she realized").

In the fourth case, some students started to fill in only As in the chart (Fig. 5).

Then, they realized that there are 9 As, "sort of a $3 \times 3$ chart, but there's always one gap left." Then, some of these students realized that Bs were placed in the same structure. Some of them realized that the structure could be "shifted" to show the place of Bs. Here, different forms of *geometrical* thinking can be tracked again. First students formed a *visual structure* in which As are placed in the chart, independent of the original $6 \times 6$ chart.

In their explanation of where to draw As, "like a three by three table but each A is one farther from the middle," some students showed traces of the antecedent of *central enlargement* (Fig. 6). In Hungarian public education, this type of homothety is taught in 8th grade and 10th grade, according

**Fig. 5** Mental shifting

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | | A | | A | | A |
| 3 | | | | | | |
| 4 | | A | | A | | A |
| 5 | | | | | | |
| 6 | | A | | A | | A |

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | B | | B | | B | |
| 2 | | A | | A | | A |
| 3 | B | | B | | B | |
| 4 | | A | | A | | A |
| 5 | B | | B | | B | |
| 6 | | A | | A | | A |

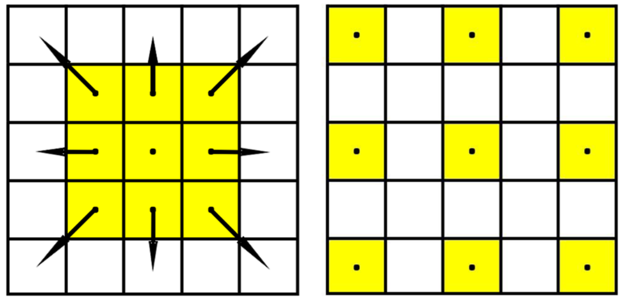

**Fig. 6** Mental antecedent of central enlargement

to the curriculum. As for CME, point transformations, including homothety, are taught based on concrete experience and students give commands to all the points of the plane in the process, such as "run to the centre the shortest way, and continue running twice/three times/half as much the same/opposite direction." Fifth grade students experience reflections as a point transformation in a similarly visual and motion geometrical way. In the lesson reported here, the convex hull of the As (a 5 times 5 chart) is represented as an independent structure in the quoted students' minds and they are instinctively using this motion geometrical approach in their explanation about the location of As. Taking cells of the table as points (vertices of the rectangle and midpoints of the sides), it is the central enlargement process of the inside rectangle into the outside rectangle about the middle cell. This approach of the students is not yet mathematically precise, as the cells are obviously not enlarged in the process. However, the mathematically precise homogeneous dilation can be observed on the midpoints of the coloured cells in Fig. 6.

Then, they either realized that the structures of As and Bs were *identical*, while there was noise in the chart (As in this case act as noise; see the second item in Fig. 5), or they intuitively used *shifting* in their explanation (Fig. 5).

Some students with this reasoning finished the same way, using the same "enlarged $3 \times 3$" structure of Cs, and then shifting it. It is interesting to mention that some other students realized that all the rest of the chart consisted of Cs, and there were 6 by 6 squares; therefore, there were $36 - (9 + 9) = 18$ Cs. In this thinking, students mobilized their knowledge about *sets:* they instinctively used the *union* of As and Bs in the chart and then found the *complementer* set (Cs).

It is admittedly a challenge for teachers, in accordance with research results discussed in the theoretical framework, to withhold themselves from evaluative listening and allow students to explain their ideas freely. Accepting this challenge certainly places the teacher in situations where she has to think on her feet. The following solution of a student is an example to this unexpected way of thinking. After filling in the chart, the student first checked the four corners, because "those were the *extreme* cases." Here, he could see BCCA. (Hereafter, letters are enumerated in the order in which we would read them from the chart.) Then, he "*moved one in*" and "made another *rectangle*," which meant moving to an adjacent cell with each "vertex" in a way to form another rectangle; it was CBAC (Fig. 7).

**Fig. 7** Motion rectangles

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | B | C | B | C | B | C |
| 2 | C | A | C | A | C | A |
| 3 | B | C | B | C | B | C |
| 4 | C | A | C | A | C | A |
| 5 | B | C | B | C | B | C |
| 6 | C | A | C | A | C | A |

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | B | C | B | C | B | C |
| 2 | C | A | C | A | C | A |
| 3 | B | C | B | C | B | C |
| 4 | C | A | C | A | C | A |
| 5 | B | C | B | C | B | C |
| 6 | C | A | C | A | C | A |

If we moved in "the other *direction*," the rectangle went as CABC (Fig. 7). You could "consume" all the chart by making these rectangles, and each of them would be some variation of ABCC. Traces of *extremal thinking*, *motion geometrical thinking*, and *structured experimenting* can be seen here. In fifth grade, it can be an interesting optional question for interested students to think about to see if making different "moves" each time would alter the end result or not.

This example is perfect for underlying that as opposed to traditional teaching methods; IBL (including CME) is far from a linear path towards finding the simplest solution. The main focus is rather different: IBL is not about teaching techniques to find the simplest solutions to set tasks, but to improve children's thinking by keeping them motivated and giving the freedom to think unexpectedly, to make mistakes and learn from them, and to listen to others' ideas curiously (Artigue & Blomhøj, 2013).

Finally, it may be interesting to note that the teacher asked at the end whether it mattered if the number on the first die was written in the columns and the second in the rows, or vice versa. The students reasoned that it does not matter as "you could switch letters over diagonally and still have the same result" and "fold the chart to meet opposite corners," which both refer to early concepts of *reflections.*

I would like to reinforce here that the discussions recorded in this paper did not all take place in one lesson of one single class, but they are a selection from the accumulation of data collected in the 5 consecutive years.

Bybee (2014) suggests not to eliminate any phase out of the 5 Es. In light of this suggestion, it is desirable to mention that although the inquiry presented here is denser than what Bybee (2014) proposes, it is in fact part of a cyclically ascending inquiry, where it is for the sake of dialogues and mathematical interconnections that the fifth E (evaluation) is carried out later through another game (Table 6).

**Table 6** The evaluation period

| |
|---|
| Game: Each round the teacher (or a student) tosses two coins. Students have to write down each round in advance A – two heads, B – a head and a tail, or C – two tails. Whoever first collects 8 points wins. |

## Contributions

The aim of the present study was to widen the concept of IBL by embedding the Hungarian branch of IBL (CME) into research literature in regard of its unique feature that it considers mathematics to be an interconnected system and mathematics teaching an effort to facilitate students to get acquainted with this web of connections. The study can contribute to the spread of application of IBL as well as to its dissemination strategies.

For research, the study can induce further research questions such as the following: Which domains are naturally more interconnected? How frequently do connections between two given domains occur? What ways can occurring connections be strengthened? What domains are activated when teaching other (not probabilistic) topics? Would it be possible to create a "recognition model" or a "recognition guideline" for teachers to help teachers recognize and utilize the moments when connections between different domains may be strengthened?

For teachers and educators, the study can highlight what broad opportunities there are in IBL teaching, and the sub-branch CME in particular. The study may contribute to the education of teachers by preparing them to what they can anticipate when they let go of control and inquire into their students' thinking (Kinser-Traut & Turner, 2020; Maaß & Artigue, 2013). Showing example situations and possible questions in the discussion phase, it is shown how the teacher can prompt the students based on the ideas that emerge in the class. The method presented also introduces a rarely discussed level of integration—integration of mathematical domains.

The study reinforces the view of mathematicians and well-known thinkers cited by Gosztonyi (2020) by presenting an example from school practice in which “the source of mathematics is intuition and experience; mathematical activity is basically dialogical and teaching mathematics is a joint activity of the students and of the teacher, where the teacher acts as an aid in students’ rediscovery of mathematics.” (p.17).

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## Declarations

**Conflict of Interest** The author declares no competing interests.



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