On the complexity of packing rainbow spanning trees

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A B S T R A C T

One of the most important questions in matroid optimization is to find disjoint common bases of two matroids. The significance of the problem is well-illustrated by the long list of conjectures that can be formulated as special cases. Bérczi and Schwarcz showed that the problem is hard in general, therefore identifying the borderline between tractable and intractable instances is of interest.

In the present paper, we study the special case when one of the matroids is a partition matroid while the other one is a graphic matroid. This setting is equivalent to the problem of packing rainbow spanning trees, an extension of the problem of packing arborescences in directed graphs which was answered by Edmonds' seminal result on disjoint arborescences. We complement his result by showing that it is NP-complete to decide whether an edge-colored graph contains two disjoint rainbow spanning trees. Our complexity result holds even for the very special case when the graph is the union of two spanning trees and each color class contains exactly two edges. As a corollary, we give a negative answer to a question on the decomposition of oriented k-partition-connected digraphs.

1. Introduction

Let $M = (E, r)$ be a matroid over ground set E with rank function r. The *covering number* of M, denoted by $\beta(M)$, is the minimum number of independent sets needed to cover its ground set. The matroid partition theorem of Edmonds and Fulkerson [12] implies that $\beta(M) = \max\{\lceil |X|/r(X) \rceil : \emptyset \neq X \subseteq E\}$. Analogously, given two matroids $M_1 = (E, r_1)$ and $M_2 = (E, r_2)$, the *covering number* $\beta(M_1, M_2)$ *of their intersection* is the minimum number of common independent sets needed to cover E. It is not difficult to see that $\beta(M_1, M_2) \leq \beta(M_1) \cdot \beta(M_2)$ holds, but this gives a very weak upper bound on the covering number of the intersection. Thus, a central problem of matroid theory is to find upper bounds on $\beta(M_1, M_2)$ in terms of $\beta(M_1)$ and $\beta(M_2)$.

By replacing one of the matroids by a general simplicial complex, Aharoni and Berger [1] provided a generalization of Edmonds' matroid intersection theorem. As an application, they showed that $\beta(M_1, M_2) \leq 2 \max\{\beta(M_1), \beta(M_2)\}$, that is, the covering number of the intersection is at most twice the maximum of the covering numbers of the matroids. Establishing a constant multiplicative gap between the lower and the upper bounds on the covering number of the intersection of matroids was a significant milestone. Nevertheless, no example is known for which the true value would be even close to the upper bound. In fact, Aharoni and Berger conjectured that if M_1 and M_2 are loopless matroids on the same ground set,

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then $\beta(M_1, M_2) \leq \max\{\beta(M_1), \beta(M_2)\} + 1$. The conjecture was shown to be true when $\max\{\beta(M_1), \beta(M_2)\} \leq 2$ [2], and no counterexample is known even for the stronger statement $\beta(M_1, M_2) \leq \max\{\beta(M_1), \beta(M_2) + 1\}$.

A packing counterpart of the problem of covering by independent sets is to find disjoint common bases of two matroids. The *packing number* of a matroid M, denoted by $\gamma(M)$, is the maximum number of its pairwise disjoint bases. The matroid partition theorem of Edmonds and Fulkerson [12] implies $\gamma(M) = \min\{\lfloor |E - X|/(r(E) - r(X)) \rfloor : X \subseteq E, r(X) < r(E)\}$. Given two matroids $M_1 = (E, r_1)$ and $M_2 = (E, r_2)$, the *packing number* $\gamma(M_1, M_2)$ *of their intersection* is the maximum number of pairwise disjoint common bases. An easy upper bound for the packing number of the intersection is $\gamma(M_1, M_2) \leq \min\{\gamma(M_1), \gamma(M_2)\}$. However, unlike in the case of the covering number, no good bounds are known from the opposite direction, and giving lower bounds for $\gamma(M_1, M_2)$ in terms of $\gamma(M_1)$ and $\gamma(M_2)$ is an intriguing open problem. To be more precise, it might easily happen that the two matroids have no common bases at all; an easy example is when each element is a loop in at least one of them. Therefore, it is common to concentrate on instances where the ground set partitions into bases in both matroids.

It is worth mentioning that, by a result of Harvey, Király, and Lau [16], one of the matroids can be assumed to be a partition matroid, and Edmonds' matroid intersection theorem [10] implies that $\gamma(M_1, M_2) \geq 1$. Showing the existence of two common bases is already challenging, and it requires non-trivial ideas even in special cases such as proper edge-colorings of complete graphs [6] or Rota's basis conjecture [14]. An analogous finding for three common bases would be a landmark result towards answering Woodall's conjecture [33]. In [1], Aharoni and Berger considered a relaxation of the problem in which disjoint common spanning sets are required instead of bases, and showed that there always exist $\lfloor \min\{\gamma(M_1), \gamma(M_2)\}/2 \rfloor$ pairwise disjoint common spanning sets of M_1 and M_2.

Previous work Bérczi and Schwarcz [4] proved that there is no algorithm which decides if the ground set of two matroids can be partitioned into common bases by using a polynomial number of independence queries. Their result implies that determining the exact values of $\beta(M_1, M_2)$ and $\gamma(M_1, M_2)$ for two matroids is hard under the rank oracle model. Nevertheless, the hardness of the abstract problem has no implications on the complexity of its special cases.

A particularly well-investigated special case of packing common bases is the intersection of the graphic matroid of a complete graph K_n on n vertices and a partition matroid. By thinking of the partition classes as color classes, this problem can also be interpreted as finding disjoint rainbow spanning trees of an edge-colored complete graph. Here, a spanning tree is called rainbow colored if no two of its elements have the same color. Brualdi and Hollingsworth [6] conjectured that if $k \geq 3$ and each color class forms a perfect matching, then the edge set of the complete graph K_{2k} can be partitioned into rainbow spanning trees. A strengthening was proposed by Kaneko, Kano, and Suzuki [21], stating that for any proper edge-coloring of K_n with $n \geq 5$, there exists $\lfloor n/2 \rfloor$ disjoint rainbow spanning trees. Constantine [8] suggested that the spanning trees in the Brualdi-Hollingsworth conjecture can be chosen to be isomorphic to each other. Recently, Glock, Kühn, Montgomery, and Osthus [15] verified the conjecture as well as its strengthening by Constantine for large enough values of k.

The existence of disjoint rainbow spanning trees was also considered for not necessarily proper edge-colorings. Akbari and Alipour [3] showed that each K_n that is edge-colored such that no color appears more than $n/2$ times contains at least two disjoint rainbow spanning trees. Under the same assumption, Carraher, Hartke, and Horn [7] verified the existence of $\Omega(n/\log n)$ disjoint rainbow spanning trees.

In contrast to the extensive list of results on complete graphs, not much is known for general graphs. In the past years, arborescence packing problems have seen renewed interest due to the fact that branchings form the common independent sets of a graphic matroid and a partition matroid, hence all positive results characterize algorithmically tractable instances of the problem of packing common bases. A milestone result of this area is Edmonds' arborescences theorem [11], characterizing the existence of k pairwise disjoint arborescences in a directed graph. Bérczi and Schwarcz [5] studied rainbow circuit-free colorings of binary matroids in general, and showed that if an n-element rank r binary matroid M is colored with exactly r colors, then M either contains a rainbow colored circuit or a monochromatic cocircuit. Such a coloring can be identified with a so-called reduction to a partition matroid, which is closely related to the problem of packing rainbow spanning trees. For further details on recent developments, see e.g. [13,17,18,25,29].

Our results Despite impressive achievements, the complexity of finding disjoint rainbow spanning trees remained an intriguing open question that was raised by many, see e.g. [5,24].

> PACKING RAINBOW SPANNING TREES
> **Input:** Edge-colored graph $G = (V, E)$ and positive integer $k \in \mathbb{Z}_+$.
> **Goal:** Find k pairwise disjoint rainbow spanning trees.

Our first main result is a proof showing that PACKING RAINBOW SPANNING TREES is NP-complete even when $k = 2$, the graph is the union of two spanning trees, and each color class contains exactly two edges.[1] Therefore, determining $\beta(M_1, M_2)$ and $\gamma(M_1, M_2)$ is hard even when M_1 is a graphic matroid and M_2 is a partition matroid.

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[1] Just before the submission of the present paper, Hörsch, Kaiser, and Kriesell [19] published an independent work that considers analogous problems.

As a corollary, we give a negative answer to a problem that appeared in [26]. A graph $G = (V, E)$ is called *k-partition-connected* if for every partition $\mathcal{P}$ of V there are at least $k(|\mathcal{P}| - 1)$ edges in E that connect different classes of $\mathcal{P}$. Tutte [31] showed that this is equivalent to the property that G can be decomposed into k connected spanning subgraphs.

DECOMPOSITION OF k-PARTITION-CONNECTED DIGRAPHS

Input: Digraph $D = (V, A)$ whose underlying graph is k-partition-connected in which all but one vertex have in-degree at least k, the remaining vertex $r \in V$ has in-degree 0.
Goal: Find k weakly connected spanning subgraphs of D so that every vertex $v \in V - r$ has positive in-degree in each of them.

The problem can be rephrased using matroid terminology as follows. Let M_1 be the graphic matroid of the underlying undirected graph of D, and let M_2 be the partition matroid where each class consists of the set of edges of D entering a given vertex $v \in V - r$. Is it true that if both M_1 and M_2 contain k disjoint spanning sets, then there are k disjoint common spanning sets? When the degree of each vertex except r is exactly k, then D is in fact rooted k-edge-connected, and the existence of k weakly connected spanning subgraphs follows from Edmonds' disjoint arborescences theorem [11]. Unfortunately, this seems to be the only tractable case, as we show that deciding the existence of k subgraphs satisfying the conditions of the problem is NP-complete.

Given a matroid $M = (E, \mathcal{I})$ whose ground set is partitioned into two-element subsets called *pairs*, a set $X \subseteq E$ is called a *parity set* if it is the union of pairs. The *matroid parity problem*, introduced by Lawler [22], asks for a parity independent set of maximum size. Though matroid parity cannot be solved efficiently in general matroids [20,23], Lovász [23] developed a polynomial time algorithm for linear matroids that is applicable if a linear representation is available. The graphic matroid of a graph is linear whose representation is easy to construct, therefore the following arises naturally.

PACKING PARITY SPANNING TREES

Input: Graph $G = (V, E)$ whose edges are partitioned into pairs and positive integer $k \in \mathbb{Z}_+$.
Goal: Find k pairwise disjoint parity spanning trees.

We prove hardness of PACKING PARITY SPANNING TREES by reduction from PACKING RAINBOW SPANNING TREES. Interestingly, the direction of the reduction between the two problems is just the opposite of the one appearing in [4] where the hardness of packing disjoint common bases was proved by reduction from packing parity bases.

Paper organization The rest of the paper is organized as follows. Basic definitions and notation are given in Section 2. The complexity of PACKING RAINBOW SPANNING TREES is discussed in Section 3. Finally, we prove hardness of DECOMPOSITION OF k-PARTITION-CONNECTED DIGRAPHS in Section 4.1 and of PACKING PARITY SPANNING TREES in Section 4.2.

2. Preliminaries

General notation We denote the set of positive integers by $\mathbb{Z}_+$. For a positive integer k, we use $[k]$ to denote the set $\{1, \ldots, k\}$. Given a ground set E together with subsets $X, Y \subseteq E$, the *difference* of X and Y is denoted by $X - Y$. If Y consists of a single element y, then $X - \{y\}$ and $X \cup \{y\}$ are abbreviated as $X - y$ and $X + y$, respectively. A directed graph $G = (V, E)$ is *weakly connected* if its underlying undirected graph is connected.

Matroids Although the results are presented using graph terminology, we give a brief introduction into matroid theory; for further details, the interested reader is referred to [28]. Matroids were introduced independently by Whitney [32] and by Nakasawa [27]. A *matroid* $M = (E, \mathcal{I})$ is defined by its *ground set* E and its *family of independent sets* $\mathcal{I} \subseteq 2^E$ that satisfies the *independence axioms*: (I1) $\emptyset \in \mathcal{I}$, (I2) $X \subseteq Y$, $Y \in \mathcal{I}$ implies $X \in \mathcal{I}$, and (I3) $X, Y \in \mathcal{I}$, $|X| < |Y|$ implies that there exists $e \in Y - X$ such that $X + e \in \mathcal{I}$. The maximal independent subsets of E are called *bases*. A set $Z \subseteq E$ is a *spanning set* of M if it contains a basis of the matroid.

A *partition matroid* is a matroid $N = (E, \mathcal{J})$ such that $\mathcal{J} = \{X \subseteq E : |X \cap E_i| \leq 1 \text{ for } i \in [q]\}$ for some partition $E = E_1 \cup \cdots \cup E_q$.[2] For a graph $G = (V, E)$, the *graphic matroid* $M = (E, \mathcal{I})$ of G is defined as $\mathcal{I} = \{F \subseteq E : F \text{ does not contain a cycle}\}$, that is, a subset $F \subseteq E$ is independent in M precisely if it is a forest in G.

Rainbow spanning trees Let $G = (V, E)$ be an edge-colored graph. A subset $F \subseteq E$ of edges is *rainbow colored* if no two edges share the same color. For short, we call a rainbow colored spanning tree of G a *rainbow spanning tree*. Let M_1 denote the graphic matroid of G and let M_2 be the partition matroid defined by the color classes of the coloring. Then the rainbow

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[2] In general, the upper bounds might be different for the different partition classes. As all the partition matroids used in the paper have all-ones upper bounds, we make this restriction without explicitly mentioning it.
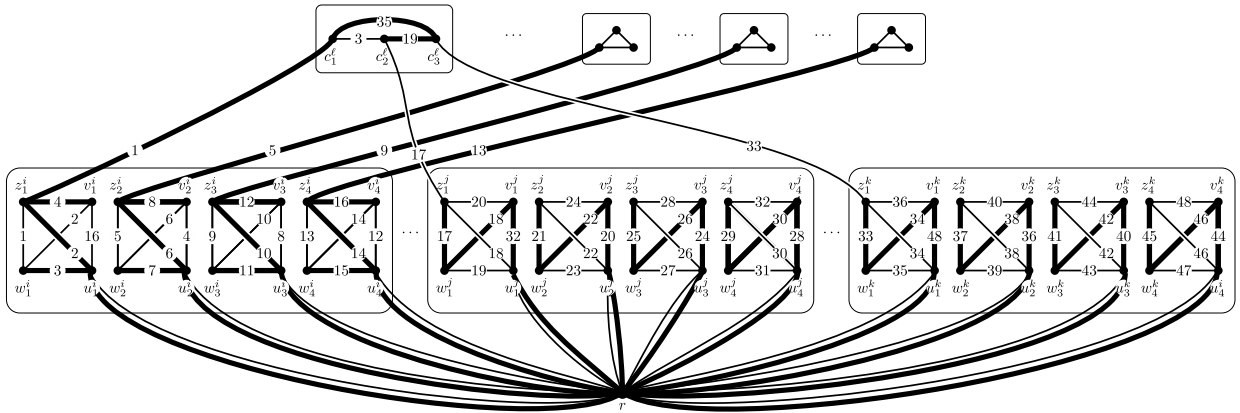
Fig. 1. An illustration of the proof of Theorem 1. Clause C_ℓ contains variables x_i, x_j and x_k. Relevant color classes are represented by numbers. Thick and thin edges correspond to rainbow spanning trees T_1 and T_2, respectively, when the value of x_i is set to true and the values of x_j and x_k are set to false.

forests correspond to the common independent sets of M_1 and M_2. Therefore, the existence of a single rainbow spanning tree is characterized by Edmonds' matroid intersection theorem [10], which gives the following: there exists a rainbow spanning tree in G if and only if for any partition $\mathcal{P}$ of V, the edges going between the classes of $\mathcal{P}$ use at least $|\mathcal{P}| - 1$ colors.

3. Packing rainbow spanning trees

This section is devoted to the proof of the main result of the paper, the hardness of PACKING RAINBOW SPANNING TREES. The proof is by reduction from Monotone Not-All-Equal 3-Sat in which each variable appears exactly four times. In such a problem, we are given a CNF formula in which no negated variable appears, all clauses contain exactly three distinct variables, and each variable appears exactly four times, and the goal is to decide whether there is a truth assignment such that for each clause at least one literal evaluates to true and at least one to false, respectively. This problem is known to be NP-complete, see e.g. [9,30].

Theorem 1. PACKING RAINBOW SPANNING TREES *is NP-complete even when $k = 2$, the graph is the union of two spanning trees, and each color class contains exactly two edges.*

Proof. Let $\Phi = (U, \mathcal{C})$ be an instance of Monotone Not-All-Equal 3-Sat where $U = \{x_1, \ldots, x_n\}$ is the set of variables and $\mathcal{C} = \{C_1, \ldots, C_m\}$ is the set of clauses, and each variable x_i is contained in exactly four members of $\mathcal{C}$. We construct an instance of PACKING RAINBOW SPANNING TREES as follows.

For each variable x_i, we add a complete graph on vertices $\{u_p^i, v_p^i, w_p^i, z_p^i\}$ to G for $p \in [4]$, that is, for each variable four complete graphs on four vertices are added. For each clause C_ℓ, we add a triangle on vertices $\{c_1^\ell, c_2^\ell, c_3^\ell\}$. Assume that C_ℓ contains variables x_{i_1}, x_{i_2} and x_{i_3}. If x_{i_q} is the jth occurrence of the variable x_{i_q} with respect to the ordering of the clauses, then we add an edge between $z_j^{i_q}$ and c_q^ℓ for $q \in [3]$. Finally, we add an extra vertex r and connect it to u_j^i with two parallel edges for $i \in [n]$, $j \in [4]$; see Fig. 1 for an example. Let $G = (V, E)$ denote the graph thus arising. Note that the number of vertices is $|V| = 16 \cdot n + 3 \cdot m + 1$, while the number of edges is $|E| = 32 \cdot n + 6 \cdot m$. It is not difficult to check that the edge set of G can be decomposed into two spanning trees.

Now we define an edge-coloring of G in which each color class consists of exactly two edges. The pairs of parallel edges leaving r form distinct color classes. Consider a variable x_i, and let C_{ℓ_1}, C_{ℓ_2}, C_{ℓ_3} and C_{ℓ_4} be the clauses containing x_i. Furthermore, let $c_{q_p}^{\ell_p}$ be the neighbor of z_p^i in the triangle corresponding to clause C_{ℓ_p} for $p \in [4]$. Then the coloring contains the pairs $\{w_p^i z_p^i, z_p^i c_{q_p}^{\ell_p}\}$, $\{v_p^i z_p^i, u_{p+1}^i v_{p+1}^i\}$, $\{u_p^i z_p^i, v_p^i w_p^i\}$ and $\{u_p^i w_p^i, c_{q_p}^{\ell_p} c_{q_p+1}^{\ell_p}\}$ as color classes, where indices are meant in a cyclic order. It is worth mentioning that among these, only the first two pairs play a role in the reduction, and the remaining two are added only to ensure that each color class has size exactly two.

We claim that Φ has a not-all-equal truth assignment if and only if the edge-set of G can be partitioned into two rainbow spanning trees. To see this, we prove the two directions separately.

Claim 2. *If Φ has a not-all-equal truth assignment, then the edge-set of G can be partitioned into two rainbow spanning trees.*

Proof. Take an assignment of truth values to the variables such that each clause contains at least one true and one false variable. We construct two spanning trees T_1 and T_2 of G as follows.

Firstly, for each pair of parallel edges leaving r, we add one of them to T_1. Let x_i be a variable, C_{ℓ_1}, C_{ℓ_2}, C_{ℓ_3} and C_{ℓ_4} be the clauses containing x_i, and $c_{q_p}^{\ell_p}$ be the neighbor of z_p^i in the triangle corresponding to clause C_{ℓ_p} for $p \in [4]$. If x_i is a true variable, then we add the edges $u_p^i w_p^i$, $u_p^i z_p^i$, $v_p^i z_p^i$ and $z_p^i c_{q_p}^{\ell_p}$ to T_1 for $p \in [4]$. If x_i is a false variable, then we add the edges $u_p^i v_p^i$, $v_p^i w_p^i$, $w_p^i z_p^i$, and $c_{q_p}^{\ell_p} c_{q_p+1}^{\ell_p}$ to T_1 for $p \in [4]$. Finally, define $T_2 := E - T_1$.

Observe that both T_1 and T_2 are rainbow colored by definition, hence it remains to show that both T_1 and T_2 are spanning trees. As each clause of Φ contains three variables and each variable is contained in exactly four clauses, we have $4 \cdot n = 3 \cdot m$. Since $|T_i| = 20 \cdot n = 16 \cdot n + 3 \cdot m = |V| - 1$, it suffices to show that T_i forms a connected graph on V for $i \in \{1, 2\}$, or equivalently, for any vertex $v \in V$, there exists a $v - r$ path in T_i. We show this for T_1; the proof for T_2 goes similarly.

The statement clearly holds for vertices of the K_4 subgraphs corresponding to the variables. Let C_ℓ be a clause and, for $q \in [3]$, let x_{i_q} be the variable in C_ℓ such that c_q^ℓ has a neighbor in one of the complete graphs on four vertices corresponding to variable x_{i_q}, denoted by $z_{p_q}^{i_q}$. If x_{i_q} is a true variable, then the edge $z_{p_q}^{i_q} c_q^\ell$ is in T_1, hence c_q^ℓ can reach r in T_1. If x_{i_q} is a false variable, then the edge $c_q^\ell c_{q+1}^\ell$ is in T_1. Now if $x_{i_{q+1}}$ is a true variable, then the edge $z_{p_{q+1}}^{i_{q+1}} c_{q+1}^\ell$ is in T_1, hence c_q^ℓ can reach r in T_1 through c_{q+1}^ℓ. Otherwise, the edge $c_{q+1}^\ell c_{q+2}^\ell$ is in T_1. As each clause contains at least one true variable, $x_{i_{q+2}}$ necessarily has true value, and so the edge $z_{p_{q+2}}^{i_{q+2}} c_{q+2}^\ell$ is in T_1, hence c_q^ℓ can reach r in T_1 through c_{q+1}^ℓ and then c_{q+2}^ℓ. This concludes the proof of the claim. □

Claim 3. *If the edge-set of G can be partitioned into two rainbow spanning trees, then Φ has a not-all-equal truth assignment.*

Proof. Take a partitioning of the edge-set of G into two rainbow spanning trees T_1 and T_2. Let x_i be a variable, C_{ℓ_1}, C_{ℓ_2}, C_{ℓ_3} and C_{ℓ_4} be the clauses containing x_i, and $c_{q_p}^{\ell_p}$ be the neighbor of z_p^i in the triangle corresponding to clause C_{ℓ_p} for $p \in [4]$.

We claim that the edges $z_p^i c_{q_p}^{\ell_p}$ are all contained in the same spanning tree T_1 or T_2; the truth assignment will be based on this distinction. Without loss of generality, let us assume that $z_1^i c_{q_1}^{\ell_1}$ is in T_1. By the definition of the color classes, this implies that $w_1^i z_1^i$ is in T_2. Since $u_1^i z_1^1$ and $v_1^i w_1^i$ have the same color, they are contained in different spanning trees. Any partitioning of a complete graph on four vertices into two spanning trees consists of two paths of length three, hence necessarily $u_1^i v_1^i$ is in T_2 and $v_1^i z_1^i$ is in T_1. The latter implies that $u_2^i v_2^i$ is in T_2 which, using similar arguments, shows that $z_2^i c_{q_2}^{\ell_2}$ and $v_2^i z_2^i$ are in T_1. Continuing this, we get that $u_3^i v_3^i$ is in T_2, $z_3^i c_{q_3}^{\ell_3}$ and $v_3^i z_3^i$ are in T_1, $u_4^i v_4^i$ is in T_2, and $z_4^i c_{q_4}^{\ell_4}$ and $v_4^i z_4^i$ are in T_1. This proves that each of the edges $z_p^i c_{q_p}^{\ell_p}$ are all contained in the same spanning tree, namely T_1 in this case.

We define a truth assignment as follows. If the edges $z_p^i c_{q_p}^{\ell_p}$ are contained in T_1, then we set the value of x_i to true, otherwise we set it to false. Let C_ℓ be a clause and for $q \in [3]$, let x_{i_q} be the variable in C_ℓ such that c_q^ℓ has a neighbor in one of the complete graphs on four vertices corresponding to variable x_{i_q}, denoted by $z_{p_q}^{i_q}$. As both T_1 and T_2 are spanning trees, we have $T_i \cap \{c_1^\ell z_{q_1}^{i_1}, c_2^\ell z_{q_2}^{i_2}, c_3^\ell z_{q_3}^{i_3}\} \neq \emptyset$ for $i \in \{1, 2\}$, meaning that each clause contains at least one true and at least one false variable as required. □

The theorem follows by Claims 2 and 3. □

4. Further results

As an application of Theorem 1, in this section we prove hardness of Decomposition of k-partition-connected Digraphs and Packing Parity Spanning Trees.

4.1. Decomposition of k-partition-connected digraphs

In matroid terms, Decomposition of k-partition-connected Digraphs aims at finding pairwise disjoint common spanning sets of two matroids, one of them being a graphic matroid of some graph $G = (V, E)$ while the other is a partition matroid. When the in-degree of each vertex $v \in V - r$ is exactly k, then the existence of k pairwise disjoint arborescences rooted at r follows by Edmonds' disjoint arborescences theorem [11], and these arborescences are actually common spanning sets of the underlying two matroids. However, we now show that deciding the existence of k disjoint common spanning sets is hard in general.

Theorem 4. Decomposition of k-partition-connected Digraphs *is NP-complete even when $k = 2$.*
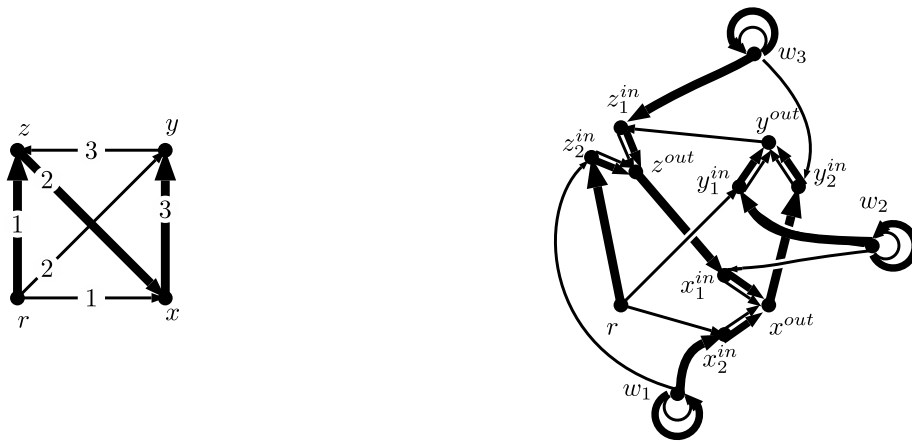
(a) An instance of Packing Rainbow Spanning Trees. Color classes are represented by numbers. Thick and thin edges form rainbow spanning trees T_1 and T_2, respectively. The edges are oriented in such a way that the in-degree of each vertex in $V - r$ is two.

(b) The corresponding Decomposition of k-partition-connected Digraphs instance $D = (U, F)$. Thick and thin edges form connected subgraphs F_1 and F_2 with each vertex in $U - r$ having positive in-degree.

Fig. 2. An illustration of Theorem 4.

Proof. Consider an instance of Packing Rainbow Spanning Trees where the graph $G = (V, E)$ is the union of two spanning trees, and each color class contains exactly two edges. For ease of discussion, we denote the set of colors by $\mathcal{C}$. We construct an instance $D = (U, F)$ of Decomposition of 2-partition-connected Digraphs as follows.

Let $r \in V$ be an arbitrary vertex of G. As E can be decomposed into two spanning trees, G admits an orientation in which each vertex has in-degree exactly two, except r having in-degree zero. Let $\overrightarrow{G} = (V, \overrightarrow{E})$ denote the digraph obtained by taking such an orientation. We add a copy of r to U that, by abuse of notation, we denote also by r. For each vertex $v \in V - r$, we add three vertices $v_1^{in}, v_2^{in}, v^{out}$ to U and, for $i \in \{1, 2\}$, add two parallel arcs from v_i^{in} to v^{out}. Furthermore, if $u_1 v$ and $u_2 v$ are the two arcs entering v in $\overrightarrow{G}$, then we add the arcs $u_1^{out} v_1^{in}$ and $u_2^{out} v_2^{in}$ to F if $u_1, u_2 \neq r$, while if one of them (or even both), say u_1, is r then we add the arc $r v_1^{in}$ instead. This way, each edge of G has a corresponding 'image' in D. Finally, if the edges $uv, xy \in E$ formed a color class $c \in \mathcal{C}$ in G and the corresponding arcs in D are $u^{out} v_p^{in}$ and $x^{out} y_q^{in}$ for some $p, q \in [2]$, then we add a vertex w_c to U together with arcs $w_c v_p^{in}, w_c y_q^{in}$ and two loops on w_c; see Fig. 2 for an example.

We claim that the edge-set of G can be decomposed into two rainbow spanning trees if and only if D can be partitioned into two weakly connected spanning subgraphs such that every vertex $u \in U - r$ has positive in-degree in both of them. To see this, we prove the two directions separately.

Claim 5. *If the edge-set of G can be partitioned into two rainbow spanning trees, then D can be partitioned into two weakly connected spanning subgraphs such that every vertex $u \in U - r$ has positive in-degree in both of them.*

Proof. Take a partitioning of the edge-set of G into two rainbow spanning trees T_1 and T_2. For $i \in \{1, 2\}$, define F_i as follows. Take the arcs of D corresponding to the edges of T_i. Add one arc from each parallel pair going between v_i^{in} and v^{out} for $i \in \{1, 2\}$, $v \in V - r$. For each color class $c \in \mathcal{C}$, add one of the loops on w_c to F_i. Finally, since exactly one edge of c appeared in T_i, exactly one of the two neighbors of w_c has in-degree zero in F_i so far; add the arc leaving w_c that goes to this vertex. It is easy to check that F_i is a weakly connected spanning subgraph and each vertex $U - r$ has positive in-degree in it, concluding the proof of the claim. □

Claim 6. *If D can be partitioned into two weakly connected spanning subgraphs such that every vertex $u \in U - r$ has positive in-degree in both of them, then the edge-set of G can be partitioned into two rainbow spanning trees.*

Proof. Take a partitioning of D into two weakly connected spanning subgraphs F_1 and F_2 such that every vertex $u \in U - r$ has positive in-degree in both of them. For each color class $c \in \mathcal{C}$, the vertex w_c is connected to the rest of D through two outgoing arcs, hence one of them is contained in F_1 while the other is contained in F_2. This implies two things. Firstly, for $i \in \{1, 2\}$, the set of edges $T_i \subseteq E$ of G whose images are contained in F_i form a connected subgraph of G. Secondly, for the two edges in color class c, the image of one is in F_1 while the image of the other is in F_2. Thus, one member of c is in T_1

(a) An instance of Packing Rainbow Spanning Trees. Color classes are represented by numbers. Thick and thin edges form rainbow spanning trees T_1 and T_2, respectively.

(b) The corresponding Packing Parity Spanning Trees instance $G' = (V', E')$. Edges in the same pair share the same number. Thick and thin edges form parity spanning trees T'_1 and T'_2.

Fig. 3. An illustration of Theorem 7.

while the other is in T_2. As $|E| = 2 \cdot |V| - 2$, T_1 and T_2 necessarily partitions the edge-set of G into rainbow spanning trees as required. □

The theorem follows by Theorem 1 and Claims 5 and 6. □

4.2. Packing parity spanning trees

In the parity base packing problem we are given a matroid M whose ground-set is partitioned into two-element subsets, and the goal is to find pairwise disjoint parity bases of M. The problem was shown to be hard in [4], which in turn implied the hardness of packing disjoint common bases of two matroids. However, the proof there did not settle the complexity of packing parity bases in graphic matroids, which remained an interesting open problem.

Now we close this gap by showing that the problem remains hard in graphic matroids. The reduction is from Packing Rainbow Spanning Trees; note that the two problems are orthogonal in the sense that the members of a pair should be contained in the same spanning tree in Packing Parity Spanning Trees, while in different spanning trees in Packing Rainbow Spanning Trees.

Theorem 7. Packing Parity Spanning Trees *is NP-complete even when $k = 2$ and the graph is the union of two spanning trees.*

Proof. Consider an instance of Packing Rainbow Spanning Trees where the graph $G = (V, E)$ is the union of two spanning trees, and each color class contains exactly two edges. For ease of discussion, we denote the set of colors by $\mathcal{C}$. We construct an instance $G' = (V', E')$ of Packing Parity Spanning Trees as follows.

We extend G by adding a new vertex w_c for each color class $c \in \mathcal{C}$. Furthermore, if the color class c contains the edges e and f, then we add an edge e_c between w_c and one of the end vertices of e, and an edge f_c between w_c and one of the end vertices of f. Finally, we define the pairs of e and f to be e_c and f_c, respectively; see Fig. 3 for an example. Let $G' = (V', E')$ denote the graph thus obtained. Note that $V \subseteq V'$ and $E \subseteq E'$.

Note that since G is the union of two spanning trees, the same holds for G'. Furthermore, it is routine to check that T'_1 and T'_2 give a partitioning of the edge-set of G' into two parity spanning trees if and only if $T_1 := T'_1 \cap E$ and $T_2 := T'_2 \cap E$ give a partitioning of the edge-set of E into two rainbow spanning trees, hence the theorem follows. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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