An Improved Trickle down Theorem for Partite Complexes

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Abstract

We prove a strengthening of the trickle down theorem for partite complexes. Given a $(d+1)$-partite d-dimensional simplicial complex, we show that if "on average" the links of faces of co-dimension 2 are $\frac{1-\delta}{d}$-(one-sided) spectral expanders, then the link of any face of co-dimension k is an $O(\frac{1-\delta}{k\delta})$-(one-sided) spectral expander, for all $3 \leq k \leq d+1$. For an application, using our theorem as a black-box, we show that links of faces of co-dimension k in recent constructions of bounded degree high dimensional expanders have spectral expansion at most $O(1/k)$ fraction of the spectral expansion of the links of the worst faces of co-dimension 2.

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1 Introduction

A simplicial complex X on a finite ground set $[n] = \{0, \ldots, n\}$ is a downwards closed collection of subsets of $[n]$, i.e. if $\tau \in X$ and $\sigma \subset \tau$, then $\sigma \in X$. The elements of X are called *faces*, and the maximal faces are called *facets*. We say that a face τ is of dimension k if $|\tau| = k+1$ and write $\dim(\tau) = k$. A simplicial complex X is a *pure* d-dimensional complex if every facet has dimension d. In this paper, all simplicial complexes are assumed to be pure. Given a d-dimensional complex X, for any $0 \leq i \leq d$, define $X(i) = \{\tau \in X : \dim(\tau) = i\}$. Moreover, the co-dimension of a face $\tau \in X$ is defined as $\mathrm{codim}(\tau) = d - \dim(\tau)$. For a face $\tau \in X$, define the *link* of τ as the simplicial complex $X_\tau = \{\sigma \setminus \tau : \sigma \in X, \sigma \supset \tau\}$. Note that X_τ is a $(\mathrm{codim}(\tau) - 1)$- dimensional complex.

A $(d+1)$-partite complex is a a d-dimensional complex such that $X(0)$ can be (uniquely) partitioned into sets $T_0 \cup \cdots \cup T_d$ such that for every facet $\tau \in X(d)$, we have $|\tau \cap T_i| = 1$ for all $i \in [d]$. The type of any face $\tau \in X$ is defined as $\mathrm{type}(\tau) = \{i \in [d] : |\tau \cap T_i| = 1\}$.

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We equip X with a probability distribution π supported on all *facets* of X and we denote this pair by (X, π). For a face $\tau \in X$, π induces a conditional distribution π_τ on facets of X_τ where for each facet $\sigma \in X_\tau$,

$$\pi_\tau(\sigma) = \frac{\pi(\sigma \cup \tau)}{\sum_{\text{facet } \sigma' \in X_\tau} \pi(\sigma' \cup \tau)}.$$

For each face τ of co-dimension at least 2 the 1-skeleton of (X_τ, π_τ) is a weighted graph with vertices $X_\tau(0)$, edges $X_\tau(1)$, and edge weights given by $\mathbb{P}_{\sigma\sim\pi_\tau}[\{x, y\} \subseteq \sigma]$ for each edge $\{x, y\}$. Note that when τ is of co-dimension 2, the complex (X_τ, π_τ) is itself a weighted graph. We say that a complex X is *totally connected* if the 1-skeleton of the link of any face τ of co-dimension at least 2 is connected.

▶ **Definition 1** (Local Spectral High Dimensional Expander). *We say that the link of a face τ of co-dimension at least 2 of a d-dimensional (weighted) complex (X, π) is a λ-(one sided) spectral expander if the second largest eigenvalue of the simple random walk on the 1-skeleton of (X_τ, π_τ) is at most λ. We say that (X, π) is a $(\gamma_2, \gamma_3, \dots, \gamma_{d+1})$-local spectral expander if the link of any face τ of co-dimension at least 2 is a $\gamma_{\mathrm{codim}(\tau)}$-spectral expander. When the complex (X, π) is clear in the context, for an integer $2 \le k \le d + 1$, we write γ_k to denote the largest 2nd eigenvalue of the simple random walk on the 1-skeleton of all links of faces of co-dimension k of the complex.*

Over the last few years, the study of local spectral high dimensional expanders (HDX) has revolutionized several areas of Math and theoretical computer science, namely in analysis of Markov chains [4, 3, 6, 1], coding theory [9], and elsewhere [2, 11, 10]. One can generally divide the family of HDXes studied in recent works into two groups: (i) Dense Complexes. Here, we have a HDX with exponentially large number of facets, i.e., $|X(0)|^d$. One typically encounters these objects in studying Markov Chain Monte Carlo technique where we use a Markov Chain to sample from an exponentially large probability distribution. Perhaps the simplest such family is the complex of all independent sets of a matroid. (ii) Sparse/Ramanujan Complexes. Here we have a HDX where every vertex (of $X(0)$) only appear in constant number of facets, independent of $|X(0)|$. See, [15, 13, 17] for explicit constructions. These objects have been useful in constructing double samplers [11], agreement testers [8, 7], or locally testable codes [9].

One of the main aspects of local spectral expanders is their "local to global phenomenon", often referred to as the Garland's method or the trickle down theorem [18].

▶ **Theorem 2** (Trickle Down Theorem). *Given a totally connected complex (X, π), if $\gamma_2 \le \frac{1-\delta}{d}$ for some $0 < \delta \le 1$, then $\gamma_k \le \frac{1-\delta}{d-(k-2)(1-\delta)} \le \frac{1-\delta}{d\delta}$ for all $2 \le k \le d$.*

The trickle down theorem has found numerous applications in proving bounds on local spectral expansion of simplicial complexes. To invoke the theorem one needs to inspect all faces of co-dimension 2 to find the worst 2nd eigenvalue. If we get lucky and this number is below $1/d$, then, the trickle down theorem kicks in and inductively bounds the spectral expansion of all links of the complex.

There are, however, two pitfalls for the theorem: i) The required bound on γ_2 is too small and often not satisfiable. In particular, for many dense complexes in counting and sampling applications that satisfy $\gamma_k = O(1/k)$ for $k \ge \Omega(d)$ (see e.g., [3, 6]), the links of faces of co-dimension 2 are only $\Theta(1)$-spectral expanders. ii) Even if $\gamma_2 \ll 1/d$, the trickle down theorem only implies $\gamma_k \simeq \gamma_2$, i.e., γ_k does not increase too much as k increases. This is in contrast with the fact that, for many dense complexes, one can observe a steep decrease in spectral expansion as the co-dimension increases, i.e., $\gamma_k \lesssim \gamma_2/k$.

Such a decrease has not been known for any sparse complex. This led some experts to conjecture that, perhaps, dense and sparse complexes exhibit a different pattern of local spectral expansion; in particular, unlike dense HDX, local spectral expansion does not decay for sparse complexes.

In this paper, we prove a generalization of the trickle down theorem for *partite complexes* that shows that even if $\gamma_2 = \Theta(1)$, but "on average" the links of faces of co-dimension 2 are $< 1/d$-spectral expanders, then we have $\gamma_k \leq O(1/k)$ for all $3 \leq k \leq d+1$. Surprising to us, our average condition is satisfied by some recent construction of (sparse) bounded degree high dimensional expanders [13, 17]. In particular, as we explain below, one can use our theorem to prove a significantly better local spectral expansion for the Kaufman-Opennheim construction in a black-box manner.

1.1 Main Contribution

We start by stating two special cases of our theorem. We need the following definition.

▶ **Definition 3.** *Given a $(d+1)$-partite complex (X, π) with parts $[d]$, for every $i \in [d]$, define*

$$\Delta_{(X,\pi)}(i) = |\{j \in [d] \setminus i : \exists \tau \textit{ of } \operatorname{type}(\tau) = [d] \setminus \{i,j\} \textit{ s.t. } \lambda_2(P_\tau) > 0\}|,$$

i.e. $\Delta_{(X,\pi)}(i)$ is the number of parts $j \neq i$ for which there exists a face of type $[d] \setminus \{i,j\}$ whose link is not a 0-spectral expander. Moreover, define $\Delta_{(X,\pi)} = \max_{i\in[d]} \Delta_{(X,\pi)}(i)$. We drop the subscripts (X, π) when the complex is clear in the context.

▶ **Theorem 4.** *Let (X, π) be a $(d+1)$-partite (weighted) totally connected complex. For some $0 < \delta < 1$, assume that*

$$\gamma_2 \leq \frac{\delta^2}{10(1+\ln\Delta)} \quad \textit{and} \quad \gamma_2 \leq \frac{1-\delta}{\Delta + \ln\Delta}.$$

Then, the link of any face τ of co-dimension k of X has spectral expansion

$$\begin{cases} \frac{c(1-\delta)}{k\delta} & \textit{if } k \geq \Delta, \\ \frac{c(1-\delta)\frac{k+\ln k}{\Delta+\ln\Delta}}{k\delta} & \textit{if } k < \Delta, \end{cases}$$

for some constant $c \leq 2$ that depends on δ. .

Note that, for $\Delta = d$, this theorem retrieves Theorem 2 up to a lower order term in the condition on γ_2 and a constant in the bounds on local spectral expansions.

When $\Delta \ll d$, this theorem is a significant improvement over Theorem 2. Roughly speaking, this theorem says that, if the complex has many faces of co-dimension 2 whose links are 0-expanders, one needs to satisfy a much weaker condition on γ_2 to get $O(1/k)$-spectral expansion for faces of co-dimension k. In other words, the faces of co-dimension 2 that have perfect spectral expansion can compensate for faces of co-dimension 2 that have bad spectral expansion.

Next, we state the second special case of our theorem. For every integers $1 \leq n$, let $H_n = \sum_{i=1}^{n} \frac{1}{i}$ be the n-th harmonic number. Moreover, for any $1 \leq i \leq n$ define $H_n(i) = \sum_{j=i}^{n} \frac{1}{j}$ and let $H_n(0) = H_n(1)$.

▶ **Theorem 5.** *Let (X, π) be a $(d+1)$-partite (weighted) totally connected complex. For any distinct $i, j \in [d]$, let $\epsilon_{\{i,j\}} = \max_{\tau:\text{type}(\tau)=[d]\setminus\{i,j\}} \lambda_2(P_\tau)$ be the 2nd largest eigenvalue of the simple random walk matrices on (X_τ, π_τ) for all τ of type $[d] \setminus \{i, j\}$. For some $0 < \delta < 1$, assume that for every $i \in [d]$,*

$$\begin{aligned} \epsilon_{\{i,j\}} \cdot H_d &\leq \frac{\delta^2}{10}, \forall j \neq i \quad \text{and} \\ \sum_{\ell=1}^{d} \epsilon_{\{i,j_\ell\}} \cdot \frac{H_d(\ell-1)}{d} &\leq \frac{1-\delta}{d}, \end{aligned}$$

where $j_0 \ldots, j_d$ is an ordering of $[d] \setminus i$ such that $\epsilon_{\{i,j_0\}} \leq \cdots \leq \epsilon_{\{i,j_d\}}$. Then, X is $(\frac{c(1-\delta)}{\delta}, \ldots, \frac{c(1-\delta)}{d\delta})$-local spectral expander for some constant $c \leq 2$ that depends on δ.

We remark that for every any $i \in [d]$, $1 \leq \frac{\sum_{\ell=1}^{d} H_d(\ell-1)}{d} \leq 1 + \frac{\ln d}{d}$. So, roughly speaking, the latter condition can be seen as $\mathbb{E}_j[\epsilon_{\{i,j\}}] \leq \frac{1-\delta}{d}$ for every $i \in [d]$, where the expectation is weighted according to $\frac{H_d(.)}{d}$. This is an improvement over the stronger condition in Theorem 2. Now, we state the main theorem.

▶ **Theorem 6** (Main). *Let (X, π) be a $(d+1)$-partite (weighted) totally connected complex. For any distinct $i, j \in [d]$, let $\epsilon_{\{i,j\}} = \max_{\tau:\text{type}(\tau)=[d]\setminus\{i,j\}} \lambda_2(P_\tau)$ be the 2nd largest eigenvalue of the simple random walk matrices on (X_τ, π_τ) for all τ of type $[d] \setminus \{i, j\}$. For some $0 < \delta < 1$, assume that for every $i \in [d]$,*

$$\epsilon_{\{i,j\}} \cdot H_{\Delta-1} \leq \frac{\delta^2}{10}, \forall j \neq i \quad \text{and} \tag{1}$$

$$\sum_{\ell=1}^{\Delta(i)} \epsilon_{\{i,j_\ell\}} \cdot H_{\Delta(i)-1}(\ell-1) \leq 1-\delta, \tag{2}$$

where $j_0 \ldots, j_d$ is an ordering of $[d] \setminus i$ such that $\epsilon_{\{i,j_0\}} \leq \cdots \leq \epsilon_{\{i,j_d\}}$. Then, (the link of the emptyset of) X is a $\frac{c(1-\delta)}{d\delta}$-expander for $c = \frac{2(1+\frac{\delta^2}{10})}{(1+\delta)}$.

▶ Remark 7. If, for some $\delta > 0$, the conditions of the above theorem hold for a complex (X, π), then the conditions also hold for the same δ for all links (X_τ, π_τ) (of faces of co-dimension at least 2). Therefore, this theorem implies that X is $(\frac{c(1-\delta)}{\delta}, \ldots, \frac{c(1-\delta)}{d\delta})$-local spectral expander for $c = \frac{2(1+\frac{\delta^2}{10})}{(1+\delta)}$. One can prove tighter bounds if they apply this theorem to any link (X_τ, π_τ) individually and possibly use better bounds on $\Delta_{(X_\tau,\pi_\tau)}(i)$.

Proof of Theorem 4. Fix a face τ of co-dimension k. For brevity we abuse notation and write Δ_τ denote $\Delta_{(X_\tau,\pi_\tau)}$. If $k \geq \Delta$ the statement follows from the above remark. In particular, for any $i, j \in [d]$

$$\begin{aligned} &\epsilon_{\{i,j\}} \cdot H_{\Delta_\tau-1} \leq \gamma_2 \cdot H_{\Delta-1} \leq \gamma_2 \cdot (1 + \ln \Delta) \leq \frac{\delta^2}{10}, \\ &\sum_{\ell=1}^{\Delta_\tau(i)} \epsilon_{\{i,j_\ell\}} \cdot H_{\Delta_\tau(i)-1}(\ell-1) \leq \gamma_2(\Delta + \ln \Delta) \leq 1-\delta. \end{aligned}$$

So, we can apply Theorem 6.

Otherwise, to bound the spectral expansion of (X_τ, π_τ), let $\delta_k = 1 - (1-\delta)\frac{k+\ln k}{\Delta+\ln\Delta} \geq \delta$. For $i, j \in [d]$

$$\epsilon_{\{i,j\}} \cdot H_{\Delta_\tau - 1} \leq \gamma_2 \cdot H_{k-1} \leq \frac{\delta^2 \cdot H_{k-1}}{10(1+\ln\Delta)} \underset{\delta \leq \delta_k}{\leq} \frac{\delta_k^2}{10},$$
$$\sum_{\ell=1}^{\Delta_\tau(i)} \epsilon_{\{i,j_\ell\}} \cdot H_{\Delta_\tau(i)-1} \underset{\epsilon_{i,j_\ell} \leq \gamma_2, \Delta_\tau(i) \leq k}{\leq} \gamma_2(k + \ln k) \leq \frac{(1-\delta)(k+\ln k)}{\Delta + \ln\Delta} = 1 - \delta_k.$$

Therefore, applying Theorem 6 to (X_τ, π_τ), we obtain that (X_τ, π_τ) is a $\frac{c(1-\delta_k)}{k\delta}$-expander. ◀

Applications to Graph Coloring

Consider a graph $G = ([n], E)$ with degree function $\Delta : [n] \to \mathbb{Z}_{\geq 0}$ and maximum degree Δ, paired with a collection of color lists $\{L(i)\}_{i \in [n]}$ satisfying $L(i) \geq \Delta(i) + (1+\eta)\Delta$ for all $i \in [n]$ and for some $0 < \eta \leq 0.9$ such that $\frac{1+\ln\Delta}{\Delta} \leq \frac{\eta^2}{40}$. We define the $(n+1)$-partite coloring complex $X(G, L)$ specified by the following facets: $\{i, \sigma(i)\}_{i \in [n]}$ is a facet if and only if σ is a proper L-coloring of G, i.e. $\sigma(i) \in L(i)$ for each $i \in [n]$ and $\sigma(i) \neq \sigma(j)$ if $\{i, j\} \in E$. It is not hard to see that if $\{i, j\} \notin E$, then $\epsilon_{\{i,j\}} = 0$. Moreover, if $\{i, j\} \in E$, then $\epsilon_{\{i,j\}} \leq \frac{1}{(1+\eta)\Delta} + \frac{1}{(1+\eta)^2\Delta^2}$ (see Theorem 4.4 in [1]). Once can verify that if we apply the above theorem to the coloring complex $X(G, L)$ with $\delta = \frac{\eta}{2}$, we get that $X(G, L)$ is a $\left(\frac{4}{\eta}, \frac{4}{2\eta}, \ldots, \frac{4}{(|V|-1)\cdot\eta}\right)$-local spectral expander, and thus the Glauber dynamics for sampling a random proper coloring mixes in polynomial time. This retrieves (up to constants) a theorem proved in [1].

Applications to Sparse High Dimensional Expanders

Kaufman and Oppenheim [13] obtained a simple construction of sparse $(d+1)$-partite complexes with $|X(0)| \geq p^s$ for any integer $s > d$ and prime power p such that every $x \in X(0)$ is in at most $p^{O(d^3)}$ many facets (hence the degree is independent of s). They argued that for any non-consecutive pair of parts $i, j \in [d]$, i.e., $j \neq i+1$ and $i \neq j+1$ (mod $d+1$), we have $\epsilon_{\{i,j\}} = 0$ but $\epsilon_{\{i,i+1\}} \leq \frac{1}{\sqrt{p}}$ for any $i \in [d]$ ($i+1$ is taken modulo $d+1$). Consequently, $\Delta(i) = 2$ for any $i \in [d]$. Then, using Theorem 2, they show that the complex is a $(\frac{1}{\sqrt{p}-(d-2)}, \ldots, \frac{1}{\sqrt{p}-d-2})$-local spectral expander for $p > (d-2)^2$. Simply plugging in these values into the above theorem, for $\delta = 1 - \frac{2}{\sqrt{p}}$ and $p \geq 193$ (independent of d) the assumptions of the theorem are satisfied. The resulting complex is $(\frac{2c}{\sqrt{p}\delta}, \ldots, \frac{2c}{d\sqrt{p}\delta})$-local spectral expander for $c \approx 1.15$. In other words, not only does the Kaufman-Opennheim construction give a HDX for constant values of p independent of d, but also its local spectral expansion improves inverse linearly with the co-dimension.

O'Donnell and Pratt [17] constructed $(d+1)$-partite (sparse) high-dimensional expanders, with unbounded dimension d, via root systems of simple Lie Algebras, namely families A_d for $d \geq 1$, B_d for $d \geq 2$, C_d for $d \geq 3$ and D_d for $d \geq 4$. For explicit descriptions of these root systems, see e.g. [5, Sec. 3.6]. O'Donnell and Pratt [17] showed that, similar to the Kaufman-Oppenheim construction, the resulting d-dimensional complex X satisfies $|X(0)| \geq p^{\Theta(m)}$ whereas every vertex is only in $p^{\Theta(d^2)}$ many facets and for any $i, j \in [d]$, $\epsilon_{i,j} \leq \sqrt{2/p}$. Then, using Theorem 2 they concluded that the complex is a $(\frac{1}{\sqrt{p/2}-d+1}, \ldots, \frac{1}{\sqrt{p/2}-d+1})$-local spectral expander. Upon further inspection of the explicit set of roots, one can verify that $\Delta \leq 2$ for complexes based on A_d, B_d, C_d root systems and $\Delta \leq 3$ for the D_d root system. Plugging in these values in the above theorem and setting $\delta = 1 - 2\sqrt{2/p}$ for A_d, B_d, C_d

complexes and $\delta = 1 - 3.5\sqrt{2/p}$ for the D_d complex, if $p \geq 376$ for A_d, B_d, C_d complexes and $p \geq 729$ for the D_d complex, we get that these complexes are $(\frac{c'}{\sqrt{p}\delta}, \ldots, \frac{c'}{d\sqrt{p}\delta})$-local spectral expander for some constant $c' > 1$.

The well known Ramanujan complexes, also known as LSV complexes, are generalizations of Ramanujan graphs that were introduced by Lubotsky, Samuels, and Vishne in [14] and explicitly constructed in [16]. Any d-dimenssional LSV complex X that is q-thick for some fixed prime power q and $d \geq 2$ has a bounded degree (the number of facets that contain each $x \in X(0)$ only deponents on q and d, and is constant in the size of the ground set n which can be arbitrarily large) (e.g. see [12]). Moreover, the link of every proper face of type S is a spherical building complex in which $\Delta(i) = |\{j \neq i : \epsilon_{\{i,j\}} > 0\}|$ is at most 2 for every $i \in [d] \setminus S$. Furthermore, the worst expansion among links co-dimension 2 is $\frac{c}{\sqrt{q}}$, for some constant c independent of q, d, n. So, there is a constant q_0 such that if $q \geq q_0$, Theorem 6 implies that the link of any (proper) face of X of co-dimension k is a $\frac{c'}{(k-1)\sqrt{q}}$-spectral expander for some constant $c' > 0$ independent of q, d, n. This improves over the bound $\frac{C(d)}{\sqrt{q}}$ proved in [12], where $C(d) \geq 2^d(d+1)!$.

1.2 Proof Overview

At a high-level, our proof builds on the matrix trickle down framework introduced in the work of the authors with Liu [1]. The Oppenheim's trickle down theorem follows from an inductive argument that derives a bound on the second eigenvalue of the simple walk on 1-skeleton of each link (X_τ, π_τ) using the largest second eigenvalue of the simple walk on the 1-skeleton of links $(X_{\tau'}, \pi_{\tau'})$ for all faces $\tau' \supset \tau$ of size $|\tau| + 1$. The reason that one has to take the largest 2nd eigenvalue as opposed to the average in each inductive step is that the eigenspaces of these simple walks are very different. The matrix trickle down framework overcomes this issue by substituting the scalar bounds on the second eigenvalues with matrices that upper bound the transition probability matrices of the simple walks on the 1-skeletons of links. However, as opposed to Oppenhiem's trickle down theorem, the matrix trickle down framework cannot be applied in a black-box manner to bound the spectral expansion of the 1-skeletons of all links only by bounding the spectral expansion of the 1-skeletons of links of faces of co-dimension 2. The main result of this paper can be seen as applying the matrix trickle down framework with a carefully chosen set of upper-bound matrices to prove an improved trickle down theorem for partite complexes that can be applied in the same black-box fashion, just known an "average" second eigenvalue.

Our technical contribution in this paper are twofold: First, we observe that for any two disjoint sets of parts $S, T \subseteq [d]$, if the links of all faces of co-dimension 2 whose types intersect with both S, T are 0-spectral expanders, then for any $\sigma \in X$ of type S and τ of type T we get

$$\mathbb{P}_{\eta\sim\pi}[\sigma \subset \eta | \tau \subset \eta] = \mathbb{P}_{\eta\sim\pi}[\sigma \subset \eta] \quad \text{and} \quad \mathbb{P}_{\eta\sim\pi}[\tau \subset \eta | \sigma \subset \eta] = \mathbb{P}_{\eta\sim\pi}[\tau \subset \eta],$$

namely, the conditional distributions on these types are independent (see Lemma 18 for details). This observation significantly simplifies invoking the Matrix trickle down framework. Armed with this tool, we invoke the matrix trickle down theorem using a carefully chosen family of (diagonal) matrices as the matrix bounds. These matrices are recursively defined based on an "average" of the spectral expansions of the links of all faces of co-dimension 2, See the proof of Theorem 6 for the construction of these matrices.

2 Preliminaries

For any integer $n \geq 0$, let $[n] = \{0, \dots, n\}$. When it is clear from context, we write x to denote a singleton $\{x\}$. Given a set S, we write $v \in \mathbb{R}^S$ and $M \in \mathbb{R}^{S\times S}$ to respectively denote a vector and a matrix indexed by S. Given a probability distributions μ over a set S, we may view μ as a vector in $\mathbb{R}^S_{\geq 0}$. For a $n \times n$ matrix M with eigenvalues $\lambda_1, \dots, \lambda_n$, define $\rho(A) = \max_{1\leq i\leq n} |\lambda_i|$.

Graphs

Given a graph $G = (V, E)$, for any $v \in V$, let $\Delta_G(v)$ be the degree of v in G, and let Δ_G be the maximum degree of G. Moreover, given a subset $S \subseteq V$, $G[S]$ denotes the induced subgraph of G on the set of vertices S. For any $S \subseteq V$, define $G_S = G[V \setminus S]$. For simplicity of notation, when G is clear from context, we denote $\Delta_G(v)$ by $\Delta(v)$ for any $v \in V$, and for any $S \subseteq V$, we denote $\Delta_{G_S}(v)$ by $\Delta_S(v)$ for any $v \in V \setminus S$. Similarly, we denote the maximum degree of G and G_S by Δ and Δ_S respectively. Moreover, when G is clear from context, we write $u \sim v$ if u, v are adjacent vertices in G and $u \sim_S v$ if $u, v \in V \setminus S$ and $u \sim v$.

We say that a graph $G = (V, E)$ paired with a weight function $w : E \to \mathbb{R}_{\geq 0}$ is ϵ-expander if $\lambda_2(P) \leq \epsilon$, where $P \in \mathbb{R}^{V\times V}$ is the transition probability matrix of the simple random walk on (G, w) defined as $P(x, y) = \frac{w(\{x,y\})}{\sum_z w(\{x,z\})}$ for any $x, y \in V$. For such a graph we write $d_w(x) = \sum_{y\sim x} w(\{x, y\})$ to denote the weighted degree of a vertex x and $\mathrm{vol}(S) = \sum_{x\in S} d_w(x)$ to denote the volume of a set $S \subseteq V$.

2.1 Linear Algebra

▶ **Lemma 8** (Cheeger's Inequality). *For any graph $G = (V, E)$ with weights $w : E \to \mathbb{R}_{\geq 0}$ and any $S \subseteq V$,*

$$\frac{w(E(S, \overline{S}))}{\min\{\mathrm{vol}(S), \mathrm{vol}(\overline{S})\}} \leq \sqrt{2(1 - \lambda_2)}$$

where λ_2 is the second largest eigenvalue of the simple random walk on G

▶ **Lemma 9** (Expander Mixing Lemma). *Given a (weighted) graph $G = (V, E, w)$, for any set $S \subseteq V$,*

$$\left| w(E(S)) - \frac{\mathrm{vol}(S)^2}{\mathrm{vol}(V)} \right| \leq \lambda_2 \mathrm{vol}(S),$$

where λ_2 is the second largest eigenvalue of the simple random walk on G.

2.2 Simplicial Complexes

We say that a simplicial complex X is *gallery connected* if for any face τ of co-dimension at least 2 and any pair of facets σ, σ' of X_τ there is a sequence of facets of X_τ, $\sigma = \sigma_0, \sigma_1, \dots, \sigma_\ell = \sigma'$, such that for all $0 \leq i < \ell$, $|\sigma_i \Delta \sigma_{i+1}| = 2$. It is shown in [18, Prop 3.6] that if X is totally connected, then it is gallery connected.

▶ **Lemma 10.** *Consider a totally connected $(d + 1)$-partite complex X with parts indexed by $[d]$. For any $S \subseteq [d]$, The induced subgraph of the 1-skeleton of X on vertices of type S is connected.*

Proof. Take x, y of type $i, j \in S$ and facets η, η' such that $x \in \eta$, $y \in \eta'$. Total connectivity implies that there is a sequence $\eta = \eta_1, \dots, \eta_t = \eta'$ such that $\eta_i \cap \eta_{i+1} \neq \emptyset$ for all $1 \leq i \leq t-1$. Let $\sigma_1 \subseteq \eta_1, \dots, \sigma_t \subseteq \eta_t$ be faces of type $\{i, j\}$. Then $\sigma_1, \dots, \sigma_t$ gives a path between x, y. ◀

Given a (weighted) complex (X, π), for integer $-1 \leq i \leq \dim(X) - 1$, π induces a distribution π_i on $X(i)$,

$$\pi_i(\sigma) = \frac{1}{\binom{\dim(X)+1}{i+1}} \Pr_{\tau \sim \pi}[\sigma \subset \tau] \quad \forall \sigma \in X(i).$$

Let $P_{(X,\pi),\tau} \in \mathbb{R}^{X(0) \times X(0)}$ denote the transition probability matrix of the simple random walk on the 1-skeleton of (X_τ, π_τ) padded with zeros outside the $X_\tau(0) \times X_\tau(0)$ block, i.e. $P_{(X,\pi),\tau}(x, y) = \frac{\mathbb{P}_{\sigma \sim \pi_\tau}[\{x,y\} \subset \sigma]}{\sum_{z \in x_\tau(0)} \mathbb{P}_{\sigma \sim \pi_\tau}[\{x,z\} \subset \sigma]}$ for $x, y \in X_\tau(0)$, and $P_\tau(x, y) = 0$ otherwise. When the weighted complex (X, π) is clear from context, we write P_τ to denote $P_{(X,\pi),\tau}$. For any τ of co-dimension at least 2, we define the diagonal matrix $\Pi_{(X,\pi),\tau} \in \mathbb{R}^{X(0) \times X(0)}$ as follows: $\Pi_{(X,\pi),\tau}(x, x) = \pi_{\tau,0}(x)$ for $x \in X_\tau(0)$, and $\Pi_{(X,\pi),\tau}(x, x) = 0$ otherwise. When (X, π) is clear from context, we write Π_τ to denote $\Pi_{(X,\pi),\tau}$. Note that $\Pi_\tau P_\tau$ is a symmetric matrix.

Given a $(d + 1)$-partite complex,

we say that an $x \in X(0)$ is of type i and write $\text{type}(x) = i$ if $x \in T_i$. Similarly, the type of a face $\tau \in X$ is defined as $\text{type}(\tau) = \{i \in [d] : |\tau \cap T_i| = 1\}$. The following facts hold for weighted partite complexes.

▶ **Observation 11.** *Consider a weighted $(d + 1)$ partite complex (X, π) and a face τ of co-dimension $k \geq 1$. We have $k\pi_{\tau,0}(x) = \Pr_{\sigma \sim \pi_\tau}[x \in \sigma]$ for all $x \in X_\tau(0)$.*

▶ **Observation 12.** *Consider a weighted $(d + 1)$ partite complex (X, π) with parts indexed by $[d]$ and a face τ of co-dimension $k \geq 1$. For any $i \in [d]$, $\sum_{x:\text{type}(x)=i} \Pr_{\sigma \sim \pi_\tau}[x \in \sigma] = 1$.*

The following definition is useful for proving the main theorem.

▶ **Definition 13.** *For any $(d + 1)$-partite complex (X, π) with parts indexed by $[d]$, define a graph $G_{(X,\pi)}$ on the set of vertices $[d]$, where any distinct $i, j \in [d]$ are adjacent in $G_{(X,\pi)}$ if there exists τ of type $[d] \setminus \{i, j\}$ such that the second eigenvalue of (X_τ, π_τ) is positive.*

▶ Remark 14. For any $(d + 1)$-partite complex (X, π) with parts indexed by $[d]$, for every $i \in [d]$, $\Delta(i)$ (see Definition 3) is the degree of i in graph $G_{(X,\pi)}$ and Δ is the maximum degree of $G_{(X,\pi)}$.

Note that if $\text{codim}(\tau) = k$, the link X_τ is a k-partite complex with parts indexed by $[d] \setminus S$. One can verify that given a face τ of type S, the set of edges of $G_{(X_\tau,\pi_\tau)}$ is a subset of the edges of $(G_{(X,\pi)})_S$, i.e., the induced subgraph of $G_{(X,\pi)}$ on $[d] \setminus S$. When (X, π) is clear from context, we write G for $G_{(X,\pi)}$ and G_S for $(G_{(X,\pi)})_S$.

Product of Weighted Complexes

Given weighted complexes $(Y_1, \mu_1), \dots, (Y_\ell, \mu_\ell)$ defined on disjoint ground sets and of dimensions $d_1, \dots, d_\ell$ respectively, and a weighted complexes (X, π) of dimension d, we write $(X, \pi) = (Y_1, \mu_1) \times \cdots \times (Y_\ell, \mu_\ell)$ if $X(d) = \{\cup_{i \in [\ell]} \tau_i : \tau_1 \in Y_1(d_1), \dots, \tau_\ell \in Y_\ell(d_\ell)\}$ and $\pi(\cup_{i \in [\ell]} \tau_i) = \prod_{i \in [\ell]} \mu_i(\tau_i)$ for all $\tau_1 \in Y_1(d_1), \dots, \tau_\ell \in Y_\ell(d_\ell)$. We denote the generating polynomial of (X, π) by $g_{(X,\pi)}$, i.e. $g_{(X,\pi)} = \sum_{\tau \in X(d)} \pi(\tau) \prod_{x \in \tau} z_x$. One can verify that $(X, \pi) = (X_1, \mu_1) \times \cdots \times (X_\ell, \mu_\ell)$ if and only if $g_{(X,\pi)} = g_{(X_1,\mu_1)} \times \cdots \times g_{(X_\ell,\mu_\ell)}$. Note that this is true because we assume that for any weighted simplicial complex, the given distribution on facets is non-zero on all facets.

Matrix Trickle Down Theorem

We use the following theorem which is the main technical theorem in [1].

▶ **Theorem 15** ([1, Thm III.5]). *Let (X, π) be a totally connected weighted complex. Suppose $\{M_\tau \in \mathbb{R}^{X(0)\times X(0)}\}_{\tau\in X(\le d-2)}$ is a family of symmetric matrices satisfying the following:*

1. **Base Case:** *For every τ of co-dimension 2, we have the spectral inequality*

$$\Pi_\tau P_\tau - 2\pi_{\tau,0}\pi_{\tau,0}^\top \preceq M_\tau \preceq \frac{1}{5}\Pi_\tau.$$

2. **Recursive Condition:** *For every τ of co-dimension at least $k \ge 3$, at least one of the following holds: M_τ satisfies*

$$M_\tau \preceq \frac{k-1}{3k-1}\Pi_\tau \quad \text{and} \quad \mathbb{E}_{x\sim\pi_\tau} M_{\tau\cup\{x\}} \preceq M_\tau - \frac{k-1}{k-2} M_\tau \Pi_\tau^{-1} M_\tau. \tag{3}$$

Or, $(X_\tau, \pi_{\tau,k-1})$ is a product of weighted simplicial complexes $(Y_1, \mu_1), \dots, (Y_t, \mu_t)$ and for every $\eta \in X_\tau(k-1)$,

$$M_\tau = \bigoplus_{1\le i\le t: d_{Y_i}\ge 1} \frac{d_{Y_i}(d_{Y_i}+1)}{k(k-1)} M_{\tau\cup\eta_{-i}},$$

where $\eta_{-i} = \eta \setminus Y_i(0)$.

Then for every $\tau \in X(\le d-2)$, we have the bound $\lambda_2(\Pi_\tau P_\tau) \le \rho(\Pi_\tau^{-1} M_\tau)$.

3 Simplifying Matrix Trickle Down's Conditions to Scalar Inequalities

In this section, given a $(d+1)$-partite complex (X, π), we apply the matrix trickle down theorem to derive a set of conditions on a family of vectors $\{f_S \in \mathbb{R}^{[d]}\}_{S\subset[d],|S|<d}$ that will guarantee that $\lambda_2(P_\tau) \le \frac{\max_{i\in[d]} f_S(i)}{k-1}$ for all $k \ge 2$ and τ of co-dimension k and type S. We prove the following theorem.

▶ **Theorem 16.** *Consider a totally connected $(d+1)$-partite complex (X, π) with parts indexed by $[d]$ and graph $G = G_{(X,\pi)}$. Suppose we are given a family of vectors $\{f_S \in \mathbb{R}^{[d]}\}_{S\subset[d],|S|<d}$ such that for all $S \subset [d]$ of size $(d+1)-k$, the support of f_S is a subset of $[d] \setminus S$, and the following holds:*

- *If G_S is disconnected, then $f_S = \sum_{1\le i\le \ell: |I_i|\ge 2} f_{[d]\setminus I_i}$, where $I_1 \cup \dots \cup I_\ell$ are the vertices of the connected components of G_S. Note that if all connected components are of size 1, then $f_S = 0$.*
- *Otherwise if G_S is connected, we have $\max_{i\in[d]} f_S(i) \le \frac{(k-1)^2}{3k-1}$ and*
  - **(i)** *Base Case: If $k = 2$, then for every face τ of type S, $\lambda_2(P_\tau) \le \max_{i\in[d]\setminus S} f_S(i)$.*
  - **(ii)** *Recursive Condition: If $k \ge 3$, then*

$$\sum_{j\in[d]\setminus(S\cup i)} f_{S\cup j}(i) \le (k-2) f_S(i) - f_S^2(i),$$

for all $i \in [d] \setminus S$.

Then, for all $k \ge 2$ and τ of co-dimension k and type S, $\lambda_2(P_\tau) \le \frac{\max_{i\in[d]} f_S(i)}{k-1}$.

The main sets of conditions in the above theorem are the inequalities in Item i and Item ii. To get some intuition about these conditions, it is helpful to compare the above with the standard trickle down theorem (Theorem 2). There, one shows that if $\lambda_2(P_{\tau\cup\{x\}}) \le \lambda$ for all $x \in X_\tau(0)$, then $\lambda_2(P_\tau) \le \alpha$, where satisfies

$$\lambda \le \alpha - \alpha^2(1-\lambda). \tag{4}$$

Then, Theorem 2 follows by recursively applying this inequalities.

In the above theorem, instead of a single upper bound on $\lambda_2(P_\tau)$ for faces τ of co-dimension 2, one bounds the expansion of the links of all faces of co-dimension 2 of each type separately, allowing higher degrees of freedom. For any face τ of type S and co-dimension $k = |S|$, the function $\frac{f_S(.)}{k-1}$ will serve as the digonal entries of a matrix upper-bound P_τ.

Then, the inequality $\frac{\sum_{j\in[d]\setminus(S\cup i)} f_{S\cup j}(i)}{k-2} \leq f_S(i) - \frac{f_S^2(i)}{k-2}$ is the natural analogue of (4) which requires f_S to be at least "the average" of $f_{S\cup j}$ for all $j \in [d] \setminus S$ plus an square error term.

Before proving the above theorem, we show that if G_S is disconnected with parts $G[I_1], \ldots, G[I_\ell]$ for some $S \subset [d]$ of size at most $d-1$, then for any τ of type S, $(X_\tau, \pi_{\tau,k-1})$ can be written as a product of family of its links of types $[d] \setminus I_i$ for all $1 \leq i \leq \ell$. This allows us to prove a better upper-bound on $\lambda_2(P_\tau)$ for such faces τ by simply "concatenating" upper-bounds on each connected component of G_S.

▶ **Lemma 17.** *Consider a 2-partite complex (X, π) with parts S, T. If (X, π) is 0-expander, then $(X, \pi) = (X_z, \pi_z) \times (X_y, \pi_y)$ for any $y \in S$ and $z \in T$.*

Proof. Note that (X, π) is a weighted bipartite graph with parts S, T. Let $A \in \mathbb{R}^{X(0)\times X(0)}$ be the adjacency matrix of (X, π). Let $A_{S,T}(y, z) = A(y, z)$ for $y \in S$, $z \in T$ and 0 on other entries. Moreover, let $A_{T,S} = A - A_{S,T}$. Then, for any vector $v \in \mathbb{R}^{X(0)}$, we get $A = A_{S,T}v_T + A_{T,S}v_S$, where v_S, v_T are respectively supported on S, T and $v = v_S + v_T$. Thus, if $Av = \lambda v$, then $Av' = -\lambda v'$, for $v' = (-v_S + v_T)$. So if μ is an eigenvalue of A, then $-\mu$ is also an eigenvalue of A. Thus, if (X, π) is 0-expander, the rank of A is 2. This implies that there are vectors $w_S \in \mathbb{R}^S$ and $w_T \in \mathbb{R}^T$ such that $\pi(\{y, z\}) = A(y, z) = A(z, y) = w_S(y)w_T(z)$ for $y \in S$, $z \in T$. Without loss of generality, assume $\|w_S\|_1 = \|w_T\|_1 = 1$. Then, for any $y \in S$ and $z \in T$, we have $\pi_z(y) = \frac{\pi(\{y,z\})}{\sum_{x\in S}\pi(\{x,z\})} = w_S(y)$. Similarly $\pi_y(z) = w_T(z)$. Thus $\pi(\{y, z\}) = \pi_y(z)\pi_z(y)$. This finishes the proof. ◀

▶ **Lemma 18.** *Consider a totally connected $(d+1)$-partite complex (X, π) with parts indexed by $[d]$ and its associated graph $G = G_{(X,\pi)}$. Let $I_1 \cup \cdots \cup I_\ell$ be a partition of $[d]$ such that for any $1 \leq i \leq \ell$ the induced graph $G[I_i]$ is a connected component or the union of several connected components of G. Then $(X, \pi) = (X_{\sigma_{-1}}, \pi_{\sigma_{-1}}) \times \cdots \times (X_{\sigma_{-\ell}}, \pi_{\sigma_{-\ell}})$, where σ_{-i} is an arbitrary face of type $[d] \setminus I_i$ for any $1 \leq i \leq \ell$.*

Proof. We prove the statement by induction on d. For $d = 1$, the statement simply follows from Lemma 17. Now, assume that $d > 1$. If $|I_i| = 1$ for all $1 \leq i \leq \ell$, then $\ell \geq 3$. In this case, let $S = I_1 \cup I_2$. Otherwise, WLOG assume that $|I_1| \geq 2$ and let $S = I_1$. First, we show that $g_{(X,\pi)}$ can be written as $g_{(X,\pi)} = h \cdot h'$, where h is a polynomial in $\{z_y : \text{type}(y) \in I \setminus S\}$ and h' is a polynomial in terms of variables in $\{z_y : \text{type}(y) \in S\}$. By induction hypothesis, for any $i \in S$, $x \in T_i$, and any face $\sigma \in X$ of type S such that $x \in \sigma$

$$\partial_{z_x} g_{(X,\pi)} = f^x \cdot g^x \tag{5}$$

where f^x is a polynomial in terms of variables in $\{z_y : \text{type}(y) \in S \setminus i\}$ and g^x is a polynomial in terms of variables in $\{z_y : \text{type}(y) \in I \setminus S\}$. Now, take arbitrary $i, j \in S$ such that $i \neq j$. Then, (5) implies that for any face $\{x, y\}$ of type $\{i, j\}$

$$\partial_{z_x}\partial_{z_y} g_{(X,\pi)} = (\partial_{z_y} f^x)g^x = (\partial_{z_x} f^y)g^y$$

It thus follows that g^x is a multiple of g^y. One can see this simply by substituting 1 for all variables in $\{z_y : \text{type}(y) \in S \setminus \{i, j\}\}$. Moreover, since g^x and g^y are generating polynomials of distributions, i.e. the coefficients sum up to 1, we get $g^x = g^y$. Therefore, we get that for any distinct x, y such that $\mathrm{type}(x), \mathrm{type}(y) \in S$ and $\{x, y\}$ is a face, $g^x = g^y$. Applying Lemma 10, we get $g^x = g^y$ for all $x, y \in \cup_{i\in S} T_i$. Thus, there exist a polynomial h in variables $\{z_y : \mathrm{type}(y) \in I \setminus S\}$ such that we can rewrite (5) for any x with $\mathrm{type}(x) \in S$ as

$$\partial_{z_x} g_{(X,\pi)} = f^x \cdot h,$$

where f^x is a polynomial in terms of variables in $\{z_y : \mathrm{type}(y) \in S \setminus i\}$. Finally, since X is a partite complex,

$$|S| g_{(X,\pi)} = \sum_{i\in S} \sum_{x\in T_i} z_x \partial_{z_x} g_{(X,\pi)} = h \cdot \sum_{i\in S} \sum_{x\in T_i} z_x f^x = h \cdot h', \tag{6}$$

where $h' = \sum_{i\in S} \sum_{x\in T_i} z_x f^x$ is a polynomial in $\{z_y : \mathrm{type}(y) \in S\}$. It remains to show that for any face σ of type S, we have $h = g_{(X_\sigma,\pi_\sigma)}$, and for any τ of type $[d] \setminus S$, we have $h' = g_{(X_\tau,\pi_\tau)}$. Fix arbitrary faces σ of type S and τ of type $[d] \setminus S$. Noting that $g_{(X,\pi)}$ is a multiple of $h \cdot h'$, and that h' is in variables associated to elements whose types are in S and h is in variables associated to elements whose types are in $[d] \setminus S$, we conclude that h' has a monomial that is a multiple of $\prod_{x\in\sigma} z_x$ and h has a monomial that is a multiple of $\prod_{x\in\tau} z_x$. First, take $(\prod_{x\in\sigma} \partial_{z_x})$ from both sides of (6). We get that $g_{(X_\sigma,\pi_\sigma)}$ is a positive multiple of h. Similarly, taking $(\prod_{x\in\tau} \partial_{z_x})$ from both sides of (6), we get that $g_{(X_\tau,\pi_\tau)}$ is a positive multiple of h'. Thus, noting that the coefficients of generating polynomials sum up to 1, we get $h = g_{(X_\sigma,\pi_\sigma)}$ and $h' = g_{(X_\tau,\pi_\tau)}$ as desired. Repeating the same argument inductively on the complex (X_σ, π_σ) proves the claim. ◀

Now we are ready to prove Theorem 16.

Proof of Theorem 16. We apply Theorem 15. For every $S \subset [d]$ such that $|S| < d$, define a diagonal matrix $D_S \in \mathbb{R}^{X(0)\times X(0)}$ as $D_S(x, x) = f_S(\mathrm{type}(x))$ for all $x \in X(0)$. We prove that the conditions of Theorem 15 hold for $M_\tau = \frac{\Pi_\tau D_S}{k-1}$ for an arbitrary face $\tau \in X$ of co-dimension at least $k \geq 2$ and type S. If G_S is connected, $\max_{i\in[d]} f_S(i) \leq \frac{(k-1)^2}{3k-1}$ holds by assumption. If G_S is disconnected, $\max_{i\in[d]} f_S(i) \leq \frac{(k-1)^2}{3k-1}$ follows from the assumptions that $f_S = \sum_{1\leq i\leq \ell : |I_i|\geq 2} f_{[d]\setminus I_i}$, where $I_1 \cup \cdots \cup I_\ell$ are the vertices of connected components of G_S. That is because the supports of vectors $f_{[d]\setminus I_i}$ are disjoint by assumption and $\frac{(k-1)^2}{3k-1}$ is an increasing function for $k \geq 2$. So, we get $D_\tau \preceq \frac{(k-1)^2}{3k-1} I$, and thus, $M_\tau \preceq \frac{k-1}{3k-1}\Pi_\tau$. To prove the rest of the conditions hold, first assume that $k = 2$. If G_S is two disconnected vertices, we get $f_S = 0$, and therefore, $D_S = 0$. Thus, we get $\Pi_\tau P_\tau - \pi_{\tau,0}\pi_{\tau,0}^\top \preceq 0 = \Pi_\tau D_S = M_\tau$, as desired. If G_S is connected, the base case assumption (Item i) implies that $\lambda_2(P_\tau) \leq D_S(x, x)$ for all $x \in X_\tau(0)$. Therefore, $\Pi_\tau P_\tau - \pi_{\tau,0}\pi_{\tau,0}^\top \preceq \Pi_\tau D_S = M_\tau$. Now, assume $k \geq 3$. First assume that G_S is disconnected and $G[I_1], \ldots, G[I_\ell]$ are its connected components for some partition $I_1 \cup \cdots \cup I_\ell$ of $[d] \setminus S$. Fix any $\sigma \in X_\tau(k-1)$. By Lemma 18, $(X_\tau, \pi_\tau) = (X_{\tau\cup\sigma_{-1}}, \pi_{\tau\cup\sigma_{-1}}) \times \cdots \times (X_{\tau\cup\sigma_{-\ell}}, \pi_{\tau\cup\sigma_{-\ell}})$ where for every $1 \leq j \leq \ell$, σ_{-j} is a subset of σ that has type $[d] \setminus (S \cup I_j)$. Therefore, we get $\Pr_{\eta\sim\pi_{\tau\cup\sigma_{-j}}}[x \in \eta] = \Pr_{\eta\sim\pi_\tau}[x \in \eta]$ for all $1 \leq j \leq \ell$ and $x \in X_{\tau\cup\sigma_{-j}}(0)$. Combining this with Observation 11, we get $k_j \cdot \pi_{\tau\cup\sigma_{-j},0}(x) = k \cdot \pi_{\tau,0}(x)$, where $k_j = |I_j|$ for all $1 \leq j \leq \ell$. Thus we can write

$$\begin{aligned}\sum_{1\leq j\leq \ell : |I_j|\geq 2} \frac{(k_j-1)k_j}{(k-1)k} M_{\tau\cup\sigma_{-j}} &\underset{\text{def of } M_{\tau\cup\sigma_{-j}}}{=} \sum_{1\leq j\leq \ell : |I_j|\geq 2} \frac{(k_j-1)k_j}{(k-1)k} \frac{\Pi_{\tau\cup\sigma_{-j}} D_{[d]\setminus I_j}}{k_j - 1} \\ &= \sum_{1\leq j\leq \ell : |I_j|\geq 2} \frac{k_j}{k(k-1)} \frac{k}{k_j} \Pi_\tau D_{[d]\setminus I_j} \\ &= \frac{\Pi_\tau}{k-1} \sum_{1\leq j\leq \ell : |I_j|\geq 2} D_{[d]\setminus I_j} = \frac{\Pi_\tau D_S}{k-1} = M_\tau,\end{aligned}$$

where in the second to last equality, we used the fact that $\sum_{1\le j\le \ell:|I_j|\ge 2} f_{[d]\setminus I_j} = f_S$, and thus $\sum_{1\le j\le \ell:|I_j|\ge 2} D_{[d]\setminus I_j} = D_S$ by definition of D_S. Now, assume that G_S is connected. It is enough to show that $\mathbb{E}_{x\sim\pi_{\tau,0}} M_{\tau\cup x} \preceq M_\tau - M_\tau \Pi_\tau^{-1} M_\tau$. This is equivalent to showing that for any $x \in X_\tau(0)$

$$\mathbb{E}_{y\sim\pi_{\tau,0}}\left[\frac{(\Pi_\tau^{-1}\Pi_{\tau\cup y} D_{S\cup \mathrm{type}(y)})(x,x)}{k-2}\right] \le \frac{D_S(x,x)}{k-1} - \frac{D_S^2(x,x)}{(k-2)(k-1)} \tag{7}$$

One can check that for any $x \in X_\tau(0)$ of type i

$$\begin{aligned}\mathbb{E}_{y\sim\pi_{\tau,0}}\left[\frac{\Pi_\tau^{-1}\Pi_{\tau\cup y} D_{S\cup \mathrm{type}(y)}(x,x)}{k-2}\right] &= \frac{\sum_{y\in X_{\tau\cup x}(0)} \Pr_{\sigma\sim\pi_{\tau\cup x}}[y\in\sigma] D_{S\cup\mathrm{type}(y)}(x,x)}{(k-1)(k-2)} \\ &= \sum_{j\in[d]\setminus S} \frac{f_{\tau\cup j}(i)}{(k-1)(k-2)} \sum_{\substack{y\in X_{\tau\cup x}(0):\\ \mathrm{type}(y)=j}} \Pr_{\sigma\sim\pi_{\tau\cup x}}[y\in\sigma] \\ &= \frac{\sum_{j\in[d]\setminus S} f_{\tau\cup j}(i)}{(k-1)(k-2)},\end{aligned}$$

where in the last equality, we used Observation 12. Thus, substituting $D_S(x,x) = f_S(\mathrm{type}(x))$ in the RHS of (7), it is enough to show that for any $i \in [d]\setminus S$

$$\frac{\sum_{j\in[d]\setminus S} f_{\tau\cup j}(i)}{(k-1)(k-2)} \le \frac{f_S(i)}{k-1} - \frac{f_S^2(i)}{(k-1)(k-2)},$$

which holds by assumption Item ii. ◀

4 Proof of Main Theorem

We are ready to prove Theorem 6.

Proof of Theorem 6. We find a family of vectors $\{f_S \in \mathbb{R}^{[d]}\}_{S\subset[d]:|S|<d}$ that satisfy the conditions of theorem Theorem 16. Let $G = G_{(X,\pi)}$. Based on the conditions of Theorem 16, vectors $\{f_S \in \mathbb{R}^{[d]}\}_{S\subset[d]:|S|<d}$ can be defined as functions of $\{\epsilon_{\{i,j\}}\}_{i,j\in[d],i\ne j}$. Recall that edges of G capture pairs $\{i,j\}$ for which $\epsilon_{\{i,j\}} > 0$. Assign every edge $\{i,j\}$ of G with weight $\epsilon_{\{i,j\}}$. We restrict our attention to functions that are very local with respect to G, i.e. for every S and $i \in [d]\setminus S$, we assume $f_S(i)$ only depends on $\Delta_S(i)$ and the weights of edges adjacent to i in G_S if $\Delta(i) > 1$. It turns out that if $\Delta(i) = 1$, we would need to also take into account the degree of the unique neighbor of i. More formally, consider the following family of vectors $\{f_S \in \mathbb{R}^{[d]}\}_{S\subset[d]:|S|<d}$: for any $S \subset [d]$ such that $|S| < d$, let f_S be of the following form: for any $i \in S$, let $f_S(i) = 0$, and for any $i \in [d]\setminus S$ define

$$f_S(i) = \begin{cases} 0 & \text{if } \Delta_S(i) = 0, \\ \epsilon_{\{i,j\}} \cdot g_{i,j}(\Delta_S(j)) & \text{if } \Delta_S(i) = 1 and i \sim_S j, \\ \sum_{j\sim_S i} \epsilon_{\{i,j\}} \cdot h_i\left(\Delta_S(i)\right) & \text{if } \Delta_S(i) \ge 2, \end{cases}$$

where for every $i \in [d]$ and $j \sim i$, functions $g_{i,j}, h_i : \{1,\dots,\Delta\} \to \mathbb{R}_{\ge 0}$ are defined later in a way that guarantees that $\{f_S\}_{S\subset[d]:|S|<d}$ satisfies the assumptions of Theorem 16 (see (10), (12)).

First, consider the case that G_S is disconnected. Note that for any $S, S' \subset [d]$ such that $|S|, |S'| < d$, if $\{j \in [d] : j \sim_S i\} = \{j \in [d] : j \sim_{S'} i\}$ for some $i \notin S, S'$, then $f_S(i) = f_{S'}(i)$. Let $I_1, \ldots, I_\ell$ be the vertices of connected components of G_S. Since the neighborhood of each vertex in any connected component of G_S is the same as its neighborhood in G_S, we get $f_S = \sum_{1 \le i \le \ell : |I_i| \ge 2} f_{[d] \setminus I_i}$.

Now, assume G_S is connected. Take an arbitrary $k \ge 2$ and $S \subset [d]$ of size $(d+1) - k$. First we verify the set of conditions given in Item i and Item ii. First, assume that $k = 2$. Let $[d] \setminus S = \{i, j\}$. By definition of $\epsilon_{\{i,j\}}$, for any τ of type S, $\lambda_2(P_\tau) \le \epsilon_{\{i,j\}}$. Thus, if we define $g_{\ell,t}(1) = 1$ for all distinct $\ell, t \in [d]$, then we get $\lambda_2(P_\tau) \le \epsilon_{\{i,j\}} = \epsilon_{\{i,j\}} g_{i,j}(1) = f_S(i) = f_S(j)$, as desired. Now, assume that $k \ge 3$. Fix an arbitrary $i \in [d] \setminus S$. Our goal is to define $g_{i,j}, h_i : \{1, \ldots, \Delta\} \to \mathbb{R}_{\ge 0}$ for all $j \sim i$ such that $g_{i,j}(1) = 1$ for all $j \sim i$ and the following inequality is satisfied:

$$\sum_{j \in [d] \setminus (S \cup i)} f_{S \cup j}(i) \le (k-2) f_S(i) - f_S^2(i). \tag{8}$$

To keep the notation concise, relabel the elements such that i is relabeled to 0 and $\epsilon_{\{0,1\}} \ge \cdots \ge \epsilon_{\{0,d\}}$. Moreover, define $\epsilon_j = \epsilon_{\{0,j\}}$ for any $j \in [d] \setminus 0$.

Case 1: $\Delta_S(0) = 1$, and $j \sim_S 0$. Since G_S is connected and $(d+1) - |S| \ge 3$, we have $\Delta_S(j) \ge 2$. Define $t = \Delta_S(j)$. We have

$$\begin{aligned}
\sum_{\ell \in [d] \setminus (S \cup 0)} f_{S \cup \ell}(0) &= f_{S \cup j}(0) + \sum_{\ell \in [d] \setminus (S \cup 0) : \ell \sim_S j} f_{S \cup \ell}(0) + \sum_{\ell \in [d] \setminus (S \cup 0) : \ell \not\sim_S j, \ell \ne j} f_{S \cup \ell}(0) \\
&= 0 + (t-1) \cdot \epsilon_j \cdot g_{0,j}(t-1) + (k-t-1) \cdot \epsilon_j \cdot g_{0,j}(t).
\end{aligned}$$

On the other hand, $(k-2) f_S(0) - f_S(0)^2 = (k-2) \cdot \epsilon_j \cdot g_{0,j}(t) - \epsilon_j^2 \cdot g_{0,j}^2(t)$. So it is enough to satisfy

$$(t-1) \cdot \epsilon_j \cdot (g_{0,j}(t) - g_{0,j}(t-1)) \ge \epsilon_j^2 \cdot g_{0,j}^2(t). \tag{9}$$

Now, define $g_{0,j} : \{1, \ldots, \Delta\} \to \mathbb{R}_{\ge 0}$ as follows: recall that we defined $g_{0,j}(1) = 1$. For any $2 \le \ell \le \Delta$, let

$$g_{0,j}(\ell) = 1 + 1.3 \cdot \epsilon_j \cdot H_{\ell-1}. \tag{10}$$

Using assumption (1), $\epsilon_j H_{\Delta-1} \le \frac{\delta^2}{10} \le \frac{1}{10}$. Thus

$$\epsilon_j^2 \cdot g_{0,j}^2(t) \le \epsilon_j^2 \left(1 + 1.3\epsilon_j \left(1 + H_{\Delta-1}\right)\right)^2 < 1.3\epsilon_j^2.$$

Substituting $g_{0,j}(t)$ according to (10) and using the above bound, one can verify that (9) holds.

Case 2: $\Delta_S(0) \ge 2$. For simplicity of notation, define $t = \Delta_S(0)$ and $\alpha = \sum_{j : j \sim_S 0} \epsilon_j$. Define $h_0(1) = \max_{j : j \sim 0} g_{0,j}(\Delta)$.

$$\begin{aligned}
\sum_{j \in [d] \setminus (S \cup 0)} f_{S \cup j}(0) &= \sum_{j \in [d] \setminus (S \cup 0) : j \sim_S 0} f_{S \cup j}(0) + \sum_{j \in [d] \setminus (S \cup 0) : j \not\sim_S 0} f_{S \cup j}(0) \\
&\le \left(\sum_{j \in [d] \setminus (S \cup 0) : j \sim_S 0} (\alpha - \epsilon_{\{0,j\}}) \right) \cdot h_0(t-1) + (k-t-1) \cdot \alpha \cdot h_0(t) \\
&= (t-1) \cdot \alpha \cdot h_0(t-1) + (k-t-1) \cdot \alpha \cdot h_0(t).
\end{aligned}$$

Note that if $t \geq 3$, the first inequality is an equality by definition. If $t = 2$, the first inequality follows from the definition of $h_0(1)$. Thus, it is enough to satisfy

$$\sum_{j \in [d] \setminus (S \cup 0)} f_{S \cup j}(0) = (t-1) \cdot \alpha \cdot h_0(t-1) + (k-t-1) \cdot \alpha \cdot h_0(t)$$
$$\leq (k-2) \cdot \alpha \cdot h_0(t) - \alpha^2 \cdot h_0^2(t) = (k-2) f_S(0) - f_S^2(0).$$

Equivalently, it suffices to satisfy

$$(t-1)(h_0(t) - h_0(t-1)) \geq \alpha \cdot h_0^2(t). \tag{11}$$

Now, define $h_0 : \{1, \dots, \Delta\} \to \mathbb{R}_{\geq 0}$ as follows: recall that we defined $h_0(1) = \max_{j: j \sim 0} g_{0,j}(\Delta)$. For any $2 \leq \ell \leq \Delta$, define

$$h_0(\ell) = \frac{h_0(1)}{1 - c\left(\sum_{j=1}^{\ell} \epsilon_j H_{\ell-1}(j-1)\right)}. \tag{12}$$

We need to prove (11) for a carefully chosen c. Let β be such that $h_0(t) = \frac{h_0(1)}{\beta}$. We get $h_0(t-1) = \frac{h_0(1)}{\beta + c(\sum_{j=1}^{t} \frac{\epsilon_j}{t-1})}$, and thus,

$$(t-1)(h_0(t) - h_0(t-1)) = \frac{h_0(1) \cdot c \sum_{j=1}^{t} \epsilon_j}{\beta \cdot (\beta + \frac{c \sum_{j=1}^{t} \epsilon_j}{t-1})}.$$

Note that $\alpha \cdot h_0^2(t) = \frac{\alpha \cdot h_0^2(1)}{\beta^2}$. Thus, to satisfy (11), it is enough to show that

$$\beta \cdot c \cdot \left(\sum_{j=1}^{t} \epsilon_j\right) \geq \alpha \cdot h_0(1) \cdot \left(\beta + \frac{c \sum_{j=1}^{t} \epsilon_j}{t-1}\right).$$

Note that

$$h_0(1) \leq \max_{j \sim i} g_{0,j}(\Delta) = 1 + 1.3\epsilon_1 H_{\Delta-1} \underset{\text{by (1)}}{\leq} 1 + 1.3\frac{\delta^2}{10}. \tag{13}$$

Moreover, $\sum_{j=1}^{t} \epsilon_j \geq \sum_{j: j \sim_S 0} \epsilon_j = \alpha$. Thus, letting $c = 1 + c'\delta$ for some $c' > 0$ that we choose later, it is enough to show that

$$\beta \cdot (c' - 0.13\delta)\delta \geq (1 + 0.13\delta) \cdot \frac{(1 + c'\delta) \sum_{j=1}^{t} \epsilon_j}{t-1}.$$

Using $\frac{\sum_{j=1}^{t} \epsilon_j}{t-1} \leq 2\epsilon_1 \underset{(1)}{\leq} \frac{\delta^2}{5}$, it is enough to show that

$$\beta \cdot (c' - 0.13\delta) \geq (1 + 0.13\delta)(1 + c'\delta)\frac{\delta}{5}. \tag{14}$$

On the other hand,

$$\beta \geq 1 - (1 + c'\delta)\left(\sum_{j=1}^{\Delta(0)} \epsilon_j H_{\Delta(0)-1}(j-1)\right) \underset{(2)}{\geq} 1 - (1 + c'\delta)(1 - \delta) = \delta(1 - c' + c'\delta), \tag{15}$$

Thus, to satisfy (14), it is enough to show that $(1-c'+c'\delta)(c'-0.13\delta) \geq (1+1.13\delta)(1+c'\delta)\frac{1}{5}$. Letting $c' = \frac{1}{2}$, this inequality holds for every $0 < \delta < 1$. This establishes *Equation* (8). So we verified conditions Item i and Item ii are satisfied.

To show that all conditions of Theorem 16 are satisfied, it remains to show that $\max_{i \in [d]} f_S(i) \leq \frac{(k-1)^2}{3k-1}$. Note that $\sum_{j:j\sim i} \epsilon_{\{i,j\}} \leq \Delta_S \cdot \epsilon_1 \underset{(1)}{\leq} \Delta_S \cdot \frac{\delta^2}{10}$ for all $i \in [d] \setminus S$. Thus, we get $\max_{i\in[d]} f_S(i) \leq \Delta_S \cdot \frac{\delta^2}{10} \max_{i \in [d]\setminus S} \cdot h_i(\Delta_S(i))$. Moreover, using (13) and (15) with $c' = \frac{1}{2}$ (we can write this inequality for every i), we get

$$h_i(\Delta_S(i)) \leq h_i(\Delta(i)) \leq \frac{1+\frac{\delta^2}{10}}{\delta(\frac{1}{2}+\frac{\delta}{2})}, \tag{16}$$

Thus, we can write

$$\max_{i\in[d]} f_S(i) \leq \Delta_S \cdot \frac{\delta^2}{10} \frac{1+\frac{\delta^2}{10}}{\delta(\frac{1}{2}+\frac{\delta}{2})} \leq \frac{\Delta_S}{5} \leq \frac{k-1}{5} \leq \frac{(k-1)^2}{3k-1},$$

as desired. So we proved that $\{f_S\}_{S\subset[d]:|S|<d}$ satisfies the conditions of Theorem 16. Now, we are ready to bound $\lambda_2(P_\tau)$ for any face τ of co-dimension $k \geq 2$ and type S. First, we show that for every $i \in [d] \setminus S$, $\sum_{j:j\sim_S i} \epsilon_{\{i,j\}} \leq 1-\delta$. Note that

$$\sum_{\ell=1}^{\Delta(i)} H_{\Delta(i)-1}(\ell-1) = \sum_{\ell=2}^{\Delta(i)} \frac{\ell}{\ell-1} = 2 + \sum_{\ell=3}^{\Delta(i)} \frac{\ell}{\ell-1} \geq \Delta(i).$$

Thus, we can write

$$\sum_{j:j\sim_S i} \epsilon_{\{i,j\}} \leq \left(\sum_{\ell=1}^{\Delta(i)} \frac{H_{\Delta(i)-1}(\ell-1)}{\Delta(i)}\right)\left(\sum_{j\sim i} \epsilon_{\{i,j\}}\right) \leq \sum_{\ell=1}^{\Delta(i)} H_{\Delta(i)-1}(\ell-1)\cdot\epsilon_{\{i,j_\ell\}} \leq 1-\delta. \tag{17}$$

where we assumed that $i_1, \ldots, j_d$ is an ordering of $[d] \setminus S$ such that $\epsilon_{j_1} \leq \cdots \leq \epsilon_{j_d}$. Using this inequality and (16), we get

$$\begin{aligned}\lambda_2(P_\tau) &\leq \frac{\max_{i\in[d]\setminus S} f_S(i)}{k-1} \leq \frac{\max_{i\in[d]}(\sum_{j\sim_S i}\epsilon_{\{i,j\}})\cdot h_i(\Delta_S(i))}{k-1} \\ &\leq \frac{(1-\delta)\cdot\max_{i\in[d]} h_i(\Delta(i))}{k-1} \leq \frac{(1-\delta)\cdot\frac{2(1+\frac{\delta^2}{10}\delta)}{\delta(\delta+1)}}{k-1},\end{aligned}$$

as desired. ◀

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