Decidability of Fully Quantum Nonlocal Games with Noisy Maximally Entangled States

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Abstract

This paper considers the decidability of fully quantum nonlocal games with noisy maximally entangled states. Fully quantum nonlocal games are a generalization of nonlocal games, where both questions and answers are quantum and the referee performs a binary POVM measurement to decide whether they win the game after receiving the quantum answers from the players. The quantum value of a fully quantum nonlocal game is the supremum of the probability that they win the game, where the supremum is taken over all the possible entangled states shared between the players and all the valid quantum operations performed by the players. The seminal work $\mathrm{MIP}^* = \mathrm{RE}$ [16, 17] implies that it is undecidable to approximate the quantum value of a fully nonlocal game. This still holds even if the players are only allowed to share (arbitrarily many copies of) maximally entangled states. This paper investigates the case that the shared maximally entangled states are noisy. We prove that there is a computable upper bound on the copies of noisy maximally entangled states for the players to win a fully quantum nonlocal game with a probability arbitrarily close to the quantum value. This implies that it is decidable to approximate the quantum values of these games. Hence, the hardness of approximating the quantum value of a fully quantum nonlocal game is not robust against the noise in the shared states.

This paper is built on the framework for the decidability of non-interactive simulations of joint distributions [12, 7, 11] and generalizes the analogous result for nonlocal games in [26]. We extend the theory of Fourier analysis to the space of super-operators and prove several key results including an invariance principle and a dimension reduction for super-operators. These results are interesting in their own right and are believed to have further applications.

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1 Introduction

Nonlocal games are a core model in the theory of quantum computing, which has found wide applications in quantum complexity theory, quantum cryptography, and the foundation of quantum mechanics. A nonlocal game is executed by three parties, a referee and two non-communicating players, which are usually named Alice and Bob. Before the game starts, the players may share an arbitrary bipartite quantum state. The referee samples a pair of questions and sends each of them to the players, separately. Each player is supposed to reply with a classical answer to the referee. They win the game if the questions and the answers satisfy a given predicate. The distribution of the questions and the predicate is known to the players. The *quantum value* is the supremum of the probability that the players win the game. It is a central topic in quantum computing to understand the computational complexity of computing the quantum value of a nonlocal game. After decades of efforts [6, 21, 20, 14, 15, 24, 9], it has been finally settled by the seminal work $\mathrm{MIP}^* = \mathrm{RE}$ [16, 17], where Ji, Natarajan, Vidick, Wright and Yuen proved that it is undecidable to approximately compute the quantum value of a nonlocal game with constant precision. This result implies that there is no computable upper bound on the preshared entanglement for the players to win the game with a probability close to the quantum value. Otherwise, the probability of success can be obtained by ε-netting all possible quantum strategies and brute-force searching for the optimal value. Ji et al. essentially proved that it is still uncomputable even if the players are only allowed to share (arbitrarily many) EPR states.

In [26], the authors investigated the robustness of the hardness of the nonlocal games under noise. More specifically, they considered a variant of nonlocal games, where the preshared quantum states are corrupted. It is shown that the quantum value of a nonlocal game is computable if the players are allowed to share arbitrarily many copies of *noisy maximally entangled states* (MES). Hence, the hardness of the nonlocal games collapses in the presence of noise from the preshared entangled states.

In this paper, we consider *fully quantum nonlocal games*, which are a broader class of games where both questions and answers are quantum and the predicates are replaced by quantum measurements with binary outcomes: win and loss. More specifically, a fully quantum nonlocal game

$$\mathfrak{G} = \left(\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{A}, \mathcal{B}, \phi_{\mathsf{in}}^{\mathcal{PQR}}, \left\{P_{\mathsf{win}} = M^{\mathcal{ABR}}, P_{\mathsf{loss}} = \mathbb{1} - M^{\mathcal{ABR}}\right\}\right)$$

consists of a referee and two non-communicating players: Alice and Bob, where $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{A}, \mathcal{B}$ are quantum systems, $\phi_{\mathsf{in}}^{\mathcal{PQR}}$ is a tripartite quantum state in $\mathcal{P} \otimes \mathcal{Q} \otimes \mathcal{R}$ and $\{P_{\mathsf{win}}, P_{\mathsf{loss}}\}$ is a measurement acting on $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{R}$. Alice, Bob, and the referee share the input state $\phi_{\mathsf{in}}^{\mathcal{PQR}}$, where Alice, Bob, and the referee hold $\mathcal{P}, \mathcal{Q}, \mathcal{R}$, respectively, at the beginning of the game. Alice and Bob are supposed to perform quantum operations mapping $\mathcal{P}$ to $\mathcal{A}$ and $\mathcal{Q}$ to $\mathcal{B}$, and then send the quantum states in $\mathcal{A}$ and $\mathcal{B}$ to the referee, respectively. After receiving the quantum messages from the players, the referee performs the POVM measurement $\{P_{\mathsf{win}}, P_{\mathsf{loss}}\}$. Again, the players are allowed to share arbitrary quantum states before the game starts. Both players know the description of ϕ_{in} and the POVM. The quantum value of the game G is the supremum of the probability that the players win the game. The supremum is over all possible preshared quantum states and the quantum operations that can be implemented by both parties. It is not hard to see if $\phi_{\mathsf{in}} = \sum_{x,y} \mu(x, y) |x\rangle\langle x|^{\mathcal{P}} \otimes |y\rangle\langle y|^{\mathcal{Q}} \otimes |xy\rangle\langle xy|^{\mathcal{R}}$ and both P_{win} and P_{loss} are projectors on computational basis, where μ is a bipartite distribution, then it boils down to a nonlocal game.

Fully quantum nonlocal games also capture the complexity class of two-prover one-round quantum multi-prover interactive proof systems $\mathsf{QMIP}(2, 1)$. The variants of nonlocal games, where either the questions or the answers are replaced by quantum messages have occurred in much literature [3, 22, 27, 5, 10, 4, 2, 18]. In [3], Buscemi introduced the so-called semi-quantum nonlocal games, which are nonlocal games with quantum questions and classical answers, and proved that semi-quantum nonlocal games can be used to characterize LOSR (local operations and shared randomness) paradigm. Such games are further used to study the entanglement verification in the subsequent work [4, 2]. In a different context, Regev and Vidick in [27] proposed quantum XOR games, where the questions are quantum and the answers are still classical. In [22], Leung, Toner, and Watrous introduced a communication task: coherent state exchange and its analogue in the setting of nonlocal games, where both questions and answers are quantum. In [10], Fitzsimons and Vidick demonstrated an efficient reduction that transforms a local Hamiltonian into a 5-players nonlocal game allowing classical questions and quantum answers. They showed that approximating the value of this game to a polynomial inverse accuracy is QMA-complete. In [5], Chung, Wu, and Yuen further proved a parallel repetition for nonlocal games where again questions are classical and answers are quantum.

As fully quantum nonlocal games are a generalization of nonlocal games, Ji et al.'s result [16, 17] implies that it is also undecidable to approximately compute the quantum value of a fully quantum nonlocal game, even if they are only allowed to share MESs.

In this paper, we continue the line of research in [26] to investigate whether the hardness of fully quantum nonlocal games can be maintained against the noise. More specifically, we consider the games where the players share arbitrarily many copies of noisy MES's $\psi^{\mathcal{S}\mathcal{T}}$. Each $\psi^{\mathcal{S}\mathcal{T}}$ is a bipartite state in quantum system $\mathcal{S}\otimes\mathcal{T}$, where Alice and Bob hold $\mathcal{S}$ and $\mathcal{T}$, respectively. The value of a game can be written as

$$\mathrm{val}_Q(\mathfrak{G},\psi)=\lim_{n\to\infty}\max_{\Phi_{\mathsf{Alice}},\Phi_{\mathsf{Bob}}}\mathrm{Tr}\left[P_{\mathsf{win}}\left(\left(\Phi_{\mathsf{Alice}}\otimes\Phi_{\mathsf{Bob}}\right)\left(\phi_{\mathsf{in}}^{\mathcal{P}\mathcal{Q}\mathcal{R}}\otimes\left(\psi^{\mathcal{S}\mathcal{T}}\right)^{\otimes n}\right)\right)\right].$$

where the maximum is taken over all quantum operations $\Phi_{\mathsf{Alice}}:\mathcal{P}\otimes\mathcal{S}^{\otimes n}\to\mathcal{A}$ and $\Phi_{\mathsf{Bob}}:\mathcal{Q}\otimes\mathcal{T}^{\otimes n}\to\mathcal{B}$. Noisy MESs were introduced in [26], which will be defined later. They include depolarized EPR states $(1-\varepsilon)\left|\Psi\right\rangle\!\left\langle\Psi\right|+\varepsilon\mathbb{1}/2\otimes\mathbb{1}/2$, where $\varepsilon>0$ and $\left|\Psi\right\rangle=\left(\left|00\right\rangle+\left|11\right\rangle\right)/\sqrt{2}$ is an EPR state. [16, 17] proved that it is undecidable to approximate $\mathrm{val}_Q(\mathfrak{G},\left|\Psi\right\rangle)$ within constant precision.

Main results

In this paper, we prove that it is computable to approximate $\mathrm{val}_Q(\mathfrak{G},\psi)$ within arbitrarily small precision if ψ is a noisy MES.

▶ **Theorem 1** (Main result, informal). *Given integer $m\geq 2$, $\delta\in(0,1)$ and a fully quantum nonlocal game $\mathfrak{G}$, where players are allowed to share arbitrarily many copies m-dimensional noisy MESs ψ, there exists an explicitly computable bound $D=D\left(\varepsilon,\delta,m,\mathfrak{G}\right)$ such that it suffices for the players to share D copies of ψ to achieve the winning probability at least $\mathrm{val}_Q(\mathfrak{G},\psi)-\delta$. Thus it is feasible to approximate the quantum value of the game $(\mathfrak{G},\psi)$ to arbitrarily precision.*

As mentioned above, the class of noisy MESs includes $(1-\varepsilon)\left|\Psi\right\rangle\!\left\langle\Psi\right|+\varepsilon\mathbb{1}/2\otimes\mathbb{1}/2$, where $\varepsilon>0$ and Ψ is an EPR state. It is as hard as Halting problem to approximate $\mathrm{val}_Q(\mathfrak{G},\left|\Psi\right\rangle)$ proved by [16, 17]. Therefore, our result implies that the hardness of fully quantum nonlocal games is also not robust against the noise in the preshared states.

This result generalizes [26] where the authors proved that it is feasible to approximate the values when both questions and answers are classical. Both works are built on the series of works for the decidability of *non-interactive simulations of joint distributions* [12, 11, 7]. In the setting of non-interactive simulations of joint distributions, two non-communicating players Alice and Bob are provided a sequence of independent samples $(x_1,y_1),(x_2,y_2),\ldots$ from a joint distribution μ, where Alice observes $x_1,x_2,\ldots$ and Bob observes $y_1,y_2,\ldots$. The question is to decide what joint distribution ν Alice and Bob can sample. The research on this problem has a long history and fruitful results (see, for example [19] and the references therein). The quantum analogue was first studied by Delgosha and Beigi [8], which is referred to as *local state transformation*. The decidability of local state transformation is still widely open. In this work, we prove that the local state transformation is decidable when the source states are noisy MESs.

1.1 Contributions

The main contribution in this paper is developing a Fourier-analytic framework for the study of the space of super-operators. Here we list some conceptual or technical contributions, which are believed to be interesting in their own right and have further applications in quantum information science.

1. Analysis in the space of super-operators.
   The space of super-operators is difficult to understand in general. In this paper, we make a crucial observation that the quantum value of a fully quantum nonlocal game can be reformulated in terms of the *Choi representations* of the adjoint maps of the quantum operations. Instead of the space of super-operators, we apply Fourier analysis to the space spanned by those Choi representations. Then we prove an invariance principle for super-operators as well as a dimension reduction for quantum operations, which generalize the analogous results in [26].
   Our understanding of Fourier analysis in the space of super-operators is still very limited, although Boolean analysis has been studied extensively in both mathematics and theoretical computer science for decades. The approach taken in this paper may pave the way for the theory of Fourier analysis in the space of super-operators.
2. Invariance principle for super-operators.
   The classical invariance principle is a central limit theorem for polynomials [23], which asserts that the distribution of a low-degree and flat polynomial with random inputs uniformly drawn from $\{\pm 1\}^n$ is close to the distribution which is obtained by replacing the inputs with i.i.d. standard normal distributions. Here a polynomial is flat means that no variable has high influence on the value of the polynomial. In [26], the authors established an invariance principle for matrix spaces. This paper further proves an invariance principle for super-operators. This is essential to reduce the number of shared noisy MESs.
3. Dimension reduction for quantum operations.
   An important step in our proof is a dimension reduction for quantum operations, which enables us to reduce the dimensions of both players' quantum operations. It, in turn, reduces the number of noisy MESs shared between the players. Dimension reductions for quantum operations are usually difficult and sometimes even impossible [13, 28]. In this paper, we prove a dimension reduction via an invariance principle for super-operators and the dimension reduction for polynomials in Gaussian spaces [11]. we adopt the techniques in [11] with a delicate analysis. It leads to an exponential upper bound in the main theorem. which also improves the doubly exponential upper bound in [26].

1.2 Comparison with [26]

In [26], the authors applied Fourier analysis to the Hilbert space where both players' measurements stay, and proved hypercontractive inequalities, quantum invariance principles and dimension reductions for matrices and random matrices. In a fully quantum nonlocal game, both players perform quantum operations. Hence, a natural approach is to further extend the framework in [12, 26] to the space of super-operators.

The first difficulty occurs as the answers are quantum. In [26], the authors applied the framework to each pair of POVM elements (one from Alice and one from Bob). Further taking a union bound, the result concludes. Hence, it suffices to work on the space where the POVM elements stay, which is a tensor product of identical Hilbert spaces. This approach fails when considering fully quantum nonlocal games as the answers are quantum. Hence, we need to have a convenient representation of super-operators to work on. It is known that there are several equivalent representations of super-operators [29]. In this paper, we choose the Choi representations of super-operators, which view a super-operator as an operator in the tensor product of the input space and the output space. Hence, the underlying Hilbert space is a tensor product of a number of identical Hilbert spaces and the output Hilbert space. Thus, the analysis in [26] cannot be generalized here directly.

The second difficulty occurs as the questions are quantum. In [26], the authors essentially proved an upper bound on the number of noisy MESs for each pair of inputs. If the precision of the approximation is good enough, then we can obtain an upper bound for all inputs again by a union bound because the questions are finite in a nonlocal game. This argument cannot be directly generalized to fully nonlocal games as the questions are the marginal state of the input state with Alice and Bob. Fortunately, this difficulty can be avoided as the input state is in a bounded-dimensional space and thus it suffices to prove the theorem for each basis element from a properly chosen basis in the space, and then take a union bound.

The last difficulty is that the rounding argument in [26] does not apply to fully quantum nonlocal games. In the end of the construction, the new super-operators are no longer valid quantum operations. In [26], the construction gives a number of Hermitian operators in the end. The rounding argument proves that it is possible to round these Hermitian operators to valid POVMs with small deviation. For fully quantum nonlocal games we need a new rounding argument which is able to round super-operators to valid quantum operations with small deviation in the end of the construction.

1.3 Proof overview

The proof is built on the framework in [12, 11, 7] for the decidability of non-interactive simulation of joint distributions. To explain the high-level idea of our proof, we start with the decidability of a particular task of local state transformation. Then we explain how to generalize it to nonlocal games.

Local state transformation

We are interested in the decidability of the following local state transformation problem.

Given $\delta > 0$, a bipartite state σ and a noisy MES ψ, suppose Alice and Bob share arbitrarily many copies of ψ.

- **Yes**. Alice and Bob are able to jointly generate a bipartite state σ' using only local operations such that σ' is δ-close to σ, i.e., $\|\sigma - \sigma'\|_1 \leq \delta$.
- **No**. Any quantum state σ' that Alice and Bob can jointly generate using only local operations is 2δ-far from σ, i.e., $\|\sigma - \sigma'\|_1 \geq 2\delta$.

As there is no upper bound on the number of copies of ψ, the decidability of this question is unclear. If it were proved that any quantum operation could be simulated by a quantum operation in a bounded dimension, then the problem would be decidable as we could search all possible quantum operations in a bounded-dimensional space via an ε-net and brute force. More specifically, suppose Alice and Bob share n copies of noisy MESs ψ and they perform quantum operations Φ_{Alice} and Φ_{Bob}. For any precision parameter $\delta \in (0,1)$, we need to construct quantum operations $\widetilde{\Phi_{\mathsf{Alice}}}$ and $\widetilde{\Phi_{\mathsf{Bob}}}$ acting on D copies of ψ, where D is independent of n, such that

$$\left(\Phi_{\mathsf{Alice}} \otimes \Phi_{\mathsf{Bob}}\right)\left(\psi^{\otimes n}\right) \approx \left(\widetilde{\Phi_{\mathsf{Alice}}} \otimes \widetilde{\Phi_{\mathsf{Bob}}}\right)\left(\psi^{\otimes D}\right). \tag{1}$$

To explain the high-level ideas, we assume that ψ is a 2-qubit quantum state for simplicity. Let $\{\mathcal{X}_a\}_{a\in\{0,1,2,3\}}$ be an orthonormal basis in the space of 2×2 matrices. We observe that the left hand side of Equation (1) is determined by the following 4^{2n} values:

$$\left\{\mathrm{Tr}\left[\left(\mathcal{X}_a \otimes \mathcal{X}_b\right)\left(\left(\Phi_{\mathsf{Alice}} \otimes \Phi_{\mathsf{Bob}}\right)\left(\psi^{\otimes n}\right)\right)\right]\right\}_{a,b\in\{0,1,2,3\}^n},$$

where $\mathcal{X}_a = \mathcal{X}_{a_1} \otimes \cdots \otimes \mathcal{X}_{a_n}$. Notice that

$$\mathrm{Tr}\left[\left(\mathcal{X}_a \otimes \mathcal{X}_b\right)\left(\left(\Phi_{\mathsf{Alice}} \otimes \Phi_{\mathsf{Bob}}\right)\left(\psi^{\otimes n}\right)\right)\right] = \mathrm{Tr}\left[\left(\left(\Phi_{\mathsf{Alice}}\right)^*\left(\mathcal{X}_a\right) \otimes \left(\Phi_{\mathsf{Bob}}\right)^*\left(\mathcal{X}_b\right)\right)\left(\psi^{\otimes n}\right)\right],$$

where $\left(\Phi_{\mathsf{Alice}}\right)^*$ and $\left(\Phi_{\mathsf{Bob}}\right)^*$ are the adjoints of Φ_{Alice} and Φ_{Bob}, respectively. Hence, Equation (1) is equivalent to

$$\mathrm{Tr}\left[\left(\left(\Phi_{\mathsf{Alice}}\right)^*\left(\mathcal{X}_a\right) \otimes \left(\Phi_{\mathsf{Bob}}\right)^*\left(\mathcal{X}_b\right)\right)\psi^{\otimes n}\right] \approx \mathrm{Tr}\left[\left(\left(\widetilde{\Phi_{\mathsf{Alice}}}\right)^*\left(\mathcal{X}_a\right) \otimes \left(\widetilde{\Phi_{\mathsf{Bob}}}\right)^*\left(\mathcal{X}_b\right)\right)\psi^{\otimes D}\right]. \tag{2}$$

Equation (2) resembles the setting considered in [26]. It is proved in [26] that for any POVM $\{M_i \otimes N_j\}_{i,j}$ acting on $\psi^{\otimes n}$, there exists POVM $\left\{M_i' \otimes N_j'\right\}_{i,j}$ acting on $\psi^{\otimes D}$ such that

$$\mathrm{Tr}\left[\left(M_i \otimes N_j\right)\psi^{\otimes n}\right] \approx \mathrm{Tr}\left[\left(M_i' \otimes N_j'\right)\psi^{\otimes D}\right],$$

for all i,j. However, $\left(\Phi_{\mathsf{Alice}}\right)^*\left(\mathcal{X}_a\right)$ and $\left(\Phi_{\mathsf{Bob}}\right)^*\left(\mathcal{X}_b\right)$ are not positive. It is even not clear how to characterize $\left(\Phi_{\mathsf{Alice}}\right)^*\left(\mathcal{X}_a\right)$ and $\left(\Phi_{\mathsf{Bob}}\right)^*\left(\mathcal{X}_b\right)$ for valid quantum operations Φ_{Alice} and Φ_{Bob}. Thus we cannot directly apply the results in [26]. Instead of working on each of $\left(\Phi_{\mathsf{Alice}}\right)^*\left(\mathcal{X}_a\right)$ and $\left(\Phi_{\mathsf{Bob}}\right)^*\left(\mathcal{X}_b\right)$, we work on the Choi representations $J\left(\left(\Phi_{\mathsf{Alice}}\right)^*\right)$ and $J\left(\left(\Phi_{\mathsf{Bob}}\right)^*\right)$, which include the information of $\left(\Phi_{\mathsf{Alice}}\right)^*\left(\mathcal{X}_a\right)$ and $\left(\Phi_{\mathsf{Bob}}\right)^*\left(\mathcal{X}_b\right)$ for all a,b. One more advantage of Choi representations is that we have a neat characterization of the Choi representations of quantum operations. Thus it is more convenient to bound the deviations of the intermediate super-operators from valid quantum operations throughout the construction. We consider the Fourier expansions of $J\left(\left(\Phi_{\mathsf{Alice}}\right)^*\right)$ and $J\left(\left(\Phi_{\mathsf{Bob}}\right)^*\right)$, and reduce the dimensions of the super-operators via the framework for the decidability of non-interactive simulations of joint distributions in [12, 11, 7, 26]. To this end, we prove an invariance principle for super-operators, and combine it with the dimension reduction for polynomials in Gaussian spaces [11]. There are several prerequisites for the invariance principle. Firstly, the Choi representation should have low degree. Secondly, all but a constant number of systems are of low influence, that is, all but a constant number of subsystems do not influence the super-operators much. The construction takes several steps to adjust the Fourier coefficients of $J\left(\left(\Phi_{\mathsf{Alice}}\right)^*\right)$ and $J\left(\left(\Phi_{\mathsf{Bob}}\right)^*\right)$ to meet those prerequisites. Meanwhile, the new super-operators still need to be close to valid quantum operations so that the value of the game does not change much. Once these prerequisites are satisfied, the basis elements in those subsystems with low influence are replaced by properly chosen Gaussian variables, which only causes a small deviation by the invariance principle.

Each step is sketched as follows.

1. **Smoothing**
   This step is aimed to obtain bounded-degree approximations of $J\left((\Phi_{\mathsf{Alice}})^*\right)$ and $J\left((\Phi_{\mathsf{Bob}})^*\right)$. We apply a noise operator Δ_γ for some $\gamma \in (0,1)$ defined in Definition 10 to both $J\left((\Phi_{\mathsf{Alice}})^*\right)$ and $J\left((\Phi_{\mathsf{Bob}})^*\right)$ on the input spaces. Note that both Choi representations are positive operators. After smoothing the operation and truncating the high-degree parts, we get bounded-degree approximations $M^{(1)}$ and $N^{(1)}$, of $J\left((\Phi_{\mathsf{Alice}})^*\right)$ and $J\left((\Phi_{\mathsf{Bob}})^*\right)$, respectively. Though the bounded-degree approximations may no longer be positive, the deviation can be proved to be small.
2. **Regularity**
   This step is aimed to prove that the number of subsystems having high influence is bounded. The influence of a subsystem of a multipartite Hermitian operator is defined in Definition 3. Informally speaking, the influence measures how much the subsystem can affect the operator. For a bounded operator, the total influence, which is the summation of the influences of all subsystems, is upper bounded by the degree of the operator. This is a generalization of a standard result in Boolean analysis. Note that we have bounded-degree approximations after the first step. The desired result follows by a Markov inequality.
3. **Invariance principle**
   In this step, we use correlated Gaussian variables to substitute the basis elements in all the subsystems with low influence in $M^{(1)}$ and $N^{(1)}$, after which we get random operators $\mathbf{M}^{(2)}$ and $\mathbf{N}^{(2)}$, whose Fourier coefficients are low-degree multilinear polynomials in Gaussian variables. We also need to prove that, $\mathbf{M}^{(2)}$ and $\mathbf{N}^{(2)}$ are close to positive operators in expectation.
4. **Dimension reduction**
   This step is aimed to reduce the number of Gaussian variables. After applying a dimension reduction to $\mathbf{M}^{(2)}$ and $\mathbf{N}^{(2)}$, we get random operators $\mathbf{M}^{(3)}$ and $\mathbf{N}^{(3)}$ containing a bounded number of Gaussian random variables. Unlike [26], we get an upper bound independent of the number of quantum subsystems via a more delicate analysis. However, the Fourier coefficients of $\mathbf{M}^{(3)}$ and $\mathbf{N}^{(3)}$ are no longer low-degree polynomials after the dimension reduction.
5. **Smooth random operators**
   The remaining steps are mainly concerned with removing the Gaussian variables. This step is aimed to get low-degree approximations of the Fourier coefficients of $\mathbf{M}^{(3)}$ and $\mathbf{N}^{(3)}$. We apply the Ornstein-Uhlenbeck operator (aka noise operators in Gaussian space) to the Gaussian variables in $\mathbf{M}^{(3)}$ and $\mathbf{N}^{(3)}$ and truncate the high-degree parts to get $\mathbf{M}^{(4)}$ and $\mathbf{N}^{(4)}$. We should note that the Fourier coefficients of $\mathbf{M}^{(4)}$ and $\mathbf{N}^{(4)}$ are polynomials, but not multilinear.
6. **Multilinearization**
   This step is aimed to get multilinear approximations of the Fourier coefficients of $\mathbf{M}^{(4)}$ and $\mathbf{N}^{(4)}$. To this end, We apply the multilinearization lemma in [11] to get random operators $\mathbf{M}^{(5)}$ and $\mathbf{N}^{(5)}$. Now we are ready to use the invariance principle again to convert random operators back to operators.
7. **Invariance to operators**
   In this step we substitute the Gaussian variables with properly chosen basis elements, to get operators $M^{(6)}$ and $N^{(6)}$, which have a bounded number of quantum subsystems. Again, we need to apply a quantum invariance principle to ensure that $M^{(6)}$ and $N^{(6)}$ are close to positive operators.

8. **Rounding**
   We now have operators $M^{(6)}$ and $N^{(6)}$ that are close to positive operators. The last thing to do is to round them to the Choi representations of the adjoints of some quantum operations. After the rounding, the whole construction is done.

2 Preliminary

Given $n \in \mathbb{Z}_{>0}$, let $[n]$ and $[n]_{\geq 0}$ represent the sets $\{1, \ldots, n\}$ and $\{0, \ldots, n-1\}$, respectively. For all $a \in \mathbb{Z}_{\geq 0}^n$, we define $|a| = |\{i : a_i > 0\}|$. In this paper, the lower-cased letters in bold $\mathbf{g}, \mathbf{h}, \ldots$ are reserved for random variables, and the capital letters in bold $\mathbf{M}, \mathbf{N}$ are reserved for random operators.

2.1 Quantum mechanics

We denote the set of Hermitian operators in a quantum system $\mathcal{S}$ by $\mathcal{H}_{\mathcal{S}}$. The identity operator is denoted by $\mathbb{1}_{\mathcal{S}}$. We use the shorthand $\mathcal{S}\mathcal{A}$ to represent $\mathcal{S} \otimes \mathcal{A}$. The Hermitian space of the composition of n Hermitian space $\mathcal{H}_{\mathcal{S}}$ is denoted by $\mathcal{H}_{\mathcal{S}}^{\otimes n}$, or $\mathcal{H}_{\mathcal{S}}^n$ for short.

Given quantum systems $\mathcal{S}, \mathcal{A}$, let $\mathcal{L}(\mathcal{S}, \mathcal{A})$ denote the set of all linear maps from $\mathcal{M}_{\mathcal{S}}$ to $\mathcal{M}_{\mathcal{A}}$ A quantum operation from the input system $\mathcal{S}$ to the output system $\mathcal{A}$ is represented by a CPTP (completely positive and trace preserving) map $\Phi \in \mathcal{L}(\mathcal{S}, \mathcal{A})$. We define $\psi^{\mathcal{S}} = \mathrm{Tr}_{\mathcal{A}}\, \psi^{\mathcal{S}\mathcal{A}}$ to represent the state obtained by tracing out system $\mathcal{A}$ from $\psi^{\mathcal{S}\mathcal{A}}$.

For a given map $\Phi \in \mathcal{L}(\mathcal{S}, \mathcal{A})$, the adjoint of Φ is defined to be the unique map $\Phi^* \in \mathcal{L}(\mathcal{A}, \mathcal{S})$ that satisfies

$$\mathrm{Tr}\ \Phi^*(Q)^\dagger P = \mathrm{Tr}\ Q^\dagger \Phi(P) \quad \text{for all } P \in \mathcal{L}(\mathcal{S}) \text{ and } Q \in \mathcal{L}(\mathcal{A}). \tag{3}$$

Given $\Psi \in \mathcal{L}(\mathcal{A}, \mathcal{S})$, the Choi representation of Ψ is a linear map $J : \mathcal{L}(\mathcal{A}, \mathcal{S}) \to \mathcal{H}(\mathcal{S}\mathcal{A})$ defined as follows:

$$J(\Psi) = \sum_a \Psi\left(\widetilde{\mathcal{A}_a}\right) \otimes \widetilde{\mathcal{A}_a}, \tag{4}$$

where $\widetilde{\mathcal{A}_a} = \mathcal{A}_a / \sqrt{|\mathcal{A}|}$[1], and $\left\{\mathcal{A}_a : a \in \left[|\mathcal{A}|^2\right]_{\geq 0}\right\}$ is an orthonormal basis in $\mathcal{A}$. J is a linear bijection. Ψ can be recovered from its Choi representation $J(\Psi)$ as follows.

$$\Psi(P) = \mathrm{Tr}_{\mathcal{A}}\left(J(\Psi)\left(\mathbb{1}_{\mathcal{S}} \otimes P^\dagger\right)\right). \tag{5}$$

▶ **Fact 2.** *$\Phi \in \mathcal{L}(\mathcal{S}, \mathcal{A})$ is a quantum operation if and only if $J(\Phi^*) \geq 0$ and $\mathrm{Tr}_{\mathcal{A}} J(\Phi^*) = \mathbb{1}_{\mathcal{S}}$.*

2.2 Fourier analysis in Gaussian space

Given $n \in \mathbb{Z}_{>0}$, let γ_n represent a standard n-dimensional normal distribution. A function $f : \mathbb{R}^n \to \mathbb{R}$ is in $L^2(\mathbb{R}, \gamma_n)$ if $\int_{\mathbb{R}^n} f(x)^2 \gamma_n(\mathrm{d}x) < \infty$.

All the functions involved in this paper are in $L^2(\mathbb{R}, \gamma_n)$. We equip $L^2(\mathbb{R}, \gamma_n)$ with an inner product $\langle f, g \rangle_{\gamma_n} = \mathbb{E}_{x \sim \gamma_n}[f(x)g(x)]$.

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[1] The denominator is because of the demoninator in the definition of the inner product $\frac{1}{s}\mathrm{Tr}\ P^\dagger Q$.

The set of *Hermite polynomials* forms an orthonormal basis in $L^2(\mathbb{R},\gamma_1)$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\gamma_1}$. The Hermite polynomials $H_r:\mathbb{R}\to\mathbb{R}$ for $r\in\mathbb{Z}_{\geq 0}$ are defined as

$$H_0(x)=1; H_1(x)=x; H_r(x)=\frac{(-1)^r}{\sqrt{r!}}\mathrm{e}^{x^2/2}\frac{\mathrm{d}^r}{\mathrm{d}x^r}\mathrm{e}^{-x^2/2}.$$

For any $\sigma\in(\sigma_1,\ldots,\sigma_n)\in\mathbb{Z}^n_{\geq 0}$, define $H_\sigma:\mathbb{R}^n\to\mathbb{R}$ as $H_\sigma(x)=\prod_{i=1}^n H_{\sigma_i}(x_i)$.

The set $\left\{H_\sigma:\sigma\in\mathbb{Z}^n_{\geq 0}\right\}$ forms an orthonormal basis in $L^2(\mathbb{R},\gamma_n)$. Every function $f\in L^2(\mathbb{R},\gamma_n)$ has an *Hermite expansion* as

$$f(x)=\sum_{\sigma\in\mathbb{Z}^n_{\geq 0}}\widehat{f}(\sigma)\cdot H_\sigma(x),$$

where $\widehat{f}(\sigma)$'s are the *Hermite coefficients* of f, which can be obtained by $\widehat{f}(\sigma)=\langle H_\sigma,f\rangle_{\gamma_n}$.

We say $f\in L^2(\mathbb{R},\gamma_n)$ is *multilinear* if $\widehat{f}(\sigma)=0$ for $\sigma\notin\{0,1\}^n$.

▶ **Definition 3.** *The* influence *of the i-th coordinate(variable) on f, denoted by $\mathrm{Inf}_i(f)$, is defined by*

$$\mathrm{Inf}_i(f)=\mathop{\mathbb{E}}_{\mathbf{x}\sim\gamma_n}\left[\mathrm{Var}_{\mathbf{x}'_i\sim\gamma_1}\left[f(\mathbf{x}_1,\ldots,\mathbf{x}_{i-1},\mathbf{x}'_i,\mathbf{x}_{i+1},\ldots\mathbf{x}_n)\right]\right].$$

The following fact summarizes some basic properties of variance and influence.

▶ **Fact 4** ([25, Proposition 8.16 and Proposition 8.23])**.** *Given $f\in L^2(\mathbb{R},\gamma_n)$, it holds that*

1. $\mathrm{Var}[f]=\sum_{\sigma\neq 0^n}\widehat{f}(\sigma)^2\leq\sum_\sigma\widehat{f}(\sigma)^2=\|f\|_2^2$.
2. $\mathrm{Inf}_i(f)=\sum_{\sigma_i\neq 0}\widehat{f}(\sigma)^2\leq\mathrm{Var}[f]$.

2.3 Fourier analysis in matrix space

Given $1\leq m,p\leq\infty$, and $H\in\mathcal{H}_m$, the p-norm of H is defined to be

$$\|H\|_p=\left(\mathrm{Tr}\ \left(H^2\right)^{p/2}\right)^{1/p}.$$

The *normalized p-norm* of H is defined as

$$|||H|||_p=\left(\frac{1}{m}\mathrm{Tr}\ \left(H^2\right)^{p/2}\right)^{1/p}.$$

Given $P,Q\in\mathcal{H}_m$, we define an inner product over $\mathbb{R}$:

$$\langle P,Q\rangle=\frac{1}{m}\mathrm{Tr}\ PQ.$$

We need the following particular classes of bases in $\mathcal{H}_m$ on which our Fourier analysis is based.

▶ **Definition 5.** *Let $\{\mathcal{B}_i\}_{i\in[m^2]_{\geq 0}}$ be an orthonormal basis in $\mathcal{H}_m$ over $\mathbb{R}$. We say $\{\mathcal{B}_i\}_{i\in[m^2]_{\geq 0}}$ is a standard orthonormal basis if $\mathcal{B}_0=\mathbb{1}_m$.*

▶ **Fact 6.** *Let $\{\mathcal{B}_i\}_{i=0}^{m^2-1}$ be a standard orthonormal basis in $\mathcal{H}_m$. Then the set $\left\{\mathcal{B}_\sigma=\otimes_{i=1}^n\mathcal{B}_{\sigma_i}:\sigma\in[m^2]^n_{\geq 0}\right\}$ is a standard orthonormal basis in $\mathcal{H}_m^{\otimes n}$.*

Given a standard orthonormal basis $\{\mathcal{B}_i\}_{i=0}^{m^2-1}$ in $\mathcal{H}_m$, every $H \in \mathcal{H}_m^{\otimes n}$ has a Fourier expansion:

$$H = \sum_{\sigma \in [m^2]_{\geq 0}^n} \widehat{H}(\sigma)\,\mathcal{B}_\sigma,$$

where $\widehat{H}(\sigma) \in \mathbb{R}$ are the Fourier coefficients. The basic properties of $\widehat{H}(\sigma)$'s are summarized in the following fact, which can be easily derived from the orthonormality of $\{\mathcal{B}_\sigma\}_{\sigma\in[m^2]_{\geq 0}^n}$.

▶ **Fact 7** ([26, Fact 2.11]). *Given a standard orthonormal basis $\{\mathcal{B}_i\}_{i\in[m^2]_{\geq 0}}$ in $\mathcal{H}_m$ and $M, N \in \mathcal{H}_m$, it holds that*

1. $\langle M, N\rangle = \sum_\sigma \widehat{M}(\sigma)\,\widehat{N}(\sigma)$.
2. $\|M\|_2^2 = \langle M, M\rangle = \sum_\sigma \widehat{M}(\sigma)^2$.
3. $\langle \mathbb{1}_m, M\rangle = \widehat{M}(0)$.

▶ **Definition 8.** *Let $\mathcal{B} = \{\mathcal{B}_i\}_{i\in[m^2]_{\geq 0}}$ be a standard orthonormal basis in $\mathcal{H}_m$, $P, Q \in \mathcal{H}_m^{\otimes n}$*

1. *The degree of P is defined to be* $\deg P = \max\left\{|\sigma| : \widehat{P}(\sigma) \neq 0\right\}$.
2. *For any $i \in [n]$, the influence of i-th coordinate is defined to be*

   $$\mathrm{Inf}_i(P) = \|P - \mathbb{1}_m \otimes \mathrm{Tr}_i P\|_2^2,$$

   where $\mathbb{1}_m$ is in the i'th quantum system, and Tr_i means tracing out the i'th system.
3. *The total influence of P is defined to be* $\mathrm{Inf}(P) = \sum_i \mathrm{Inf}_i(P)$.

▶ **Fact 9** ([26, Lemma 2.16]). *Given $P \in \mathcal{H}_m^{\otimes n}$, a standard orthonormal basis $\mathcal{B} = \{\mathcal{B}_i\}_{i\in[m^2]_{\geq 0}}$ in $\mathcal{H}_m$, it holds that*

1. $\mathrm{Inf}_i(P) = \sum_{\sigma:\sigma_i\neq 0} \left|\widehat{P}(\sigma)\right|^2$.
2. $\mathrm{Inf}(P) = \sum_\sigma |\sigma| \left|\widehat{P}(\sigma)\right|^2 \leq \deg P \cdot \|P\|_2^2$.

▶ **Definition 10.** *Given a quantum system $\mathcal{S}$ with dimension $|\mathcal{S}| = \mathsf{s}$, $\gamma \in [0,1]$, the depolarizing operation $\Delta_\gamma : \mathcal{H}_\mathcal{S} \to \mathcal{H}_\mathcal{S}$ is defined as follows. For any $P \in \mathcal{H}_\mathcal{S}$,*

$$\Delta_\gamma(P) = \gamma P + \frac{1-\gamma}{\mathsf{s}}(\mathrm{Tr}\ P)\cdot \mathbb{1}_\mathcal{S}.$$

▶ **Fact 11** ([26, Lemma 3.6 and Lemma 6.1]). *Given $n, m \in \mathbb{Z}_{>0}$, $\gamma \in [0,1]$, a standard orthonormal basis of $\mathcal{H}_m$: $\mathcal{B} = \{\mathcal{B}_i\}_{i=0}^{m^2-1}$, the following holds:*

1. *For any $P \in \mathcal{H}_m^{\otimes n}$ with a Fourier expansion $P = \sum_{\sigma\in[m^2]_{\geq 0}^n} \widehat{P}(\sigma)\,\mathcal{B}_\sigma$, it holds that*

   $$\Delta_\gamma(P) = \sum_{\sigma\in[m^2]_{\geq 0}^n} \gamma^{|\sigma|}\widehat{P}(\sigma)\,\mathcal{B}_\sigma.$$

2. *For any $P \in \mathcal{H}_m^{\otimes n}$, $\|\Delta_\gamma(P)\|_2 \leq \|P\|_2$.*
3. *Δ_γ is a quantum operation.*
4. *For any $d \in \mathbb{Z}_{>0}, P \in \mathcal{H}_m^{\otimes n}$, it holds that $\left\|(\Delta_\gamma(P))^{>d}\right\|_2 \leq \gamma^d\|P\|_2$.*

▶ **Definition 12** (Maximal correlation). *[1] Given quantum systems $\mathcal{S}, \mathcal{T}$ with dimensions $\mathsf{s} = |\mathcal{S}|$ and $\mathsf{t} = |\mathcal{T}|$, $\psi^{\mathcal{S}\mathcal{T}} \in \mathcal{H}_{\mathcal{S}\mathcal{T}}$ with $\psi^\mathcal{S} = \mathbb{1}_\mathcal{S}/\mathsf{s}, \psi^\mathcal{T} = \mathbb{1}_\mathcal{T}/\mathsf{t}$, the maximal correlation of $\psi^{\mathcal{S}\mathcal{T}}$ is defined to be*

$$\rho\left(\psi^{\mathcal{S}\mathcal{T}}\right) = \sup\left\{ {\left|\mathrm{Tr}\left((P\otimes Q)\psi^{\mathcal{S}\mathcal{T}}\right)\right| : P\in\mathcal{H}_\mathcal{S}, Q\in\mathcal{H}_\mathcal{T}, \atop \mathrm{Tr}\ P = \mathrm{Tr}\ Q = 0, \|P\|_2 = \|Q\|_2 = 1} \right\}.$$

2.4 Random operators

In this subsection, we introduce random operators defined in [26], which unifies Gaussian variables and operators.

▶ **Definition 13** ([26]). *Given $p, h, n, m \in \mathbb{Z}_{>0}$, we say $\mathbf{P}$ is a random operator if it can be expressed as*

$$\mathbf{P} = \sum_{\sigma \in [m^2]^h_{\geq 0}} p_\sigma(\mathbf{g}) \mathcal{B}_\sigma,$$

where $\{\mathcal{B}_i\}_{i \in [m^2]_{\geq 0}}$ is a standard orthonormal basis in $\mathcal{H}_m$, $p_\sigma : \mathbb{R}^n \to \mathbb{R}$ for all $\sigma \in [m^2]^h_{\geq 0}$ and $\mathbf{g} \sim \gamma_n$. $\mathbf{P} \in L^p\left(\mathcal{H}_m^{\otimes h}, \gamma_n\right)$ if $p_\sigma \in L^p(\mathbb{R}, \gamma_n)$ for all $\sigma \in [m^2]^h_{\geq 0}$.

We say $\mathbf{P}$ is multilinear if $p_\sigma(\cdot)$ is multilinear for all $\sigma \in [m^2]^h_{\geq 0}$.

▶ **Fact 14** ([26, Lemma 2.23]). *Given $n, h, m \in \mathbb{Z}_{>0}$, let $\mathbf{P} \in L^2\left(\mathcal{H}_m^{\otimes h}, \gamma_n\right)$ with an associated vector-valued function p under a standard orthonormal basis. It holds that $\mathbb{E}\left[\|\|\mathbf{P}\|\|_2^2\right] = \|p\|_2^2$.*

2.5 Rounding maps

Define a function $\zeta : \mathbb{R} \to \mathbb{R}$ as follows.

$$\zeta(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ 0 & \text{otherwise} \end{cases}. \tag{6}$$

The function ζ measures the distance between an Hermitian operator and the set of positive semi-definite operators in 2-norm.

▶ **Fact 15** ([26, Lemma 9.1]). *Given $m \in \mathbb{Z}_{>0}$, $H \in \mathcal{H}_m$, it holds that*

$$\operatorname{Tr} \zeta(H) = \min\left\{\|H - X\|_2^2 : X \geq 0\right\}.$$

3 Main results

▶ **Theorem 16.** *Given $\epsilon \in (0,1)$, $n, s \in \mathbb{Z}_{>0}$, and quantum systems $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{A}, \mathcal{B}$ with dimensions $\mathsf{p} = |\mathcal{P}|, \mathsf{q} = |\mathcal{Q}|, \mathsf{r} = |\mathcal{R}|, \mathsf{s} = |\mathcal{S}|, \mathsf{t} = |\mathcal{T}|, \mathsf{a} = |\mathcal{A}|, \mathsf{b} = |\mathcal{B}|$. Let $\{\mathcal{A}_a\}_{a \in [\mathsf{a}^2]_{\geq 0}}$, $\{\mathcal{B}_b\}_{b \in [\mathsf{b}^2]_{\geq 0}}$, $\{\mathcal{R}_r\}_{r \in [\mathsf{r}^2]_{\geq 0}}$ be orthonormal bases in $\mathcal{H}_\mathcal{A}$, $\mathcal{H}_\mathcal{B}$ and $\mathcal{H}_\mathcal{R}$, respectively. Let $\psi^{\mathcal{ST}} \in \mathcal{H}_{\mathcal{ST}}$ be a noisy MES with the maximal correlation $\rho = \rho\left(\psi^{\mathcal{ST}}\right) < 1$, which is defined in Definition 12. Let $\phi_{in}^{\mathcal{PQR}} \in \mathcal{H}_{\mathcal{PQR}}$ be an arbitrary tripartite quantum state. Then there exists an explicitly computable $D = D(\rho, \epsilon, s, \mathsf{p}, \mathsf{q}, \mathsf{r}, \mathsf{s}, \mathsf{t}, \mathsf{a}, \mathsf{b})$, such that for all quantum operations $\Phi_{Alice} \in \mathcal{L}(\mathcal{S}^n\mathcal{P}, \mathcal{A})$, $\Phi_{Bob} \in \mathcal{L}(\mathcal{T}^n\mathcal{Q}, \mathcal{B})$, there exist quantum operations $\widetilde{\Phi_{Alice}} \in \mathcal{L}\left(\mathcal{S}^D\mathcal{P}, \mathcal{A}\right)$, $\widetilde{\Phi_{Bob}} \in \mathcal{L}\left(\mathcal{T}^D\mathcal{Q}, \mathcal{B}\right)$ such that for all $a \in \left[\mathsf{a}^2\right]_{\geq 0}$, $b \in \left[\mathsf{b}^2\right]_{\geq 0}$, $r \in \left[\mathsf{r}^2\right]_{\geq 0}$,* [2]

$$\Bigg|\operatorname{Tr}\left[\left(\Phi_{Alice}^*\left(\widetilde{\mathcal{A}_a}\right) \otimes \Phi_{Bob}^*\left(\widetilde{\mathcal{B}_b}\right) \otimes \widetilde{\mathcal{R}_r}\right)\left(\phi_{in}^{\mathcal{PQR}} \otimes \left(\psi^{\mathcal{ST}}\right)^{\otimes n}\right)\right] - \operatorname{Tr}\left[\left(\widetilde{\Phi_{Alice}^*}\left(\widetilde{\mathcal{A}_a}\right) \otimes \widetilde{\Phi_{Bob}^*}\left(\widetilde{\mathcal{B}_b}\right) \otimes \widetilde{\mathcal{R}_r}\right)\left(\phi_{in}^{\mathcal{PQR}} \otimes \left(\psi^{\mathcal{ST}}\right)^{\otimes D}\right)\right]\Bigg| \leq \epsilon.$$

In particular, one may choose

$$D = \exp\left(\operatorname{poly}\left(\mathsf{a}, \mathsf{b}, \mathsf{p}, \mathsf{q}, \mathsf{r}, \log \mathsf{s}, \log \mathsf{t}, \frac{1}{1-\rho}, \frac{1}{\epsilon}\right)\right).$$

[2] Remind that $\widetilde{\mathcal{A}_a} = \mathcal{A}_a/\sqrt{\mathsf{a}}$, $\widetilde{\mathcal{B}_b} = \mathcal{B}_b/\sqrt{\mathsf{b}}$ and $\widetilde{\mathcal{R}_r} = \mathcal{R}_r/\sqrt{\mathsf{r}}$.

▶ **Theorem 17.** *Given parameters $0 < \epsilon, \rho < 1$, and a fully quantum nonlocal game*

$$\mathfrak{G} = \left(\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{A}, \mathcal{B}, \phi_{in}, \left\{M^{\mathcal{ABR}}, \mathbb{1} - M^{\mathcal{ABR}}\right\}\right),$$

with dimensions $\mathsf{p} = |\mathcal{P}|, \mathsf{q} = |\mathcal{Q}|, \mathsf{r} = |\mathcal{R}|, \mathsf{s} = |\mathcal{S}|, \mathsf{t} = |\mathcal{T}|, \mathsf{a} = |\mathcal{A}|, \mathsf{b} = |\mathcal{B}|$*, suppose the two players are restricted to share an arbitrarily finite number of noisy MES states $\psi^{\mathcal{ST}}$, i.e., $\psi^{\mathcal{S}} = \mathbb{1}_{\mathcal{S}}/\mathsf{s}$, $\psi^{\mathcal{T}} = \mathbb{1}_{\mathcal{T}}/\mathsf{t}$ with the maximal correlation $\rho < 1$ as defined in Definition 12. Let $\mathrm{val}_Q(\mathfrak{G}, \psi^{\mathcal{ST}})$ be the supremum of the winning probability that the players can achieve. Then there exists an explicitly computable bound $D = D(\rho, \epsilon, \mathsf{p}, \mathsf{q}, \mathsf{r}, \mathsf{s}, \mathsf{t}, \mathsf{a}, \mathsf{b})$ such that it suffices for the players to share D copies of $\psi^{\mathcal{ST}}$ to achieve the winning probability at least $\mathrm{val}_Q(\mathfrak{G}, \psi^{\mathcal{ST}}) - \epsilon$. In particular, one may choose*

$$D = \exp\left(\mathrm{poly}\left(\mathsf{a}, \mathsf{b}, \mathsf{p}, \mathsf{q}, \mathsf{r}, \log\mathsf{s}, \log\mathsf{t}, \frac{1}{1-\rho}, \frac{1}{\epsilon}\right)\right).$$

Proof. To keep the notations short, the superscripts will be omitted whenever it is clear from the context. Suppose the players share n copies of $\psi^{\mathcal{ST}}$ and employ the strategy $(\Phi_{\mathsf{Alice}}, \Phi_{\mathsf{Bob}})$ with the winning probability $\mathrm{val}_Q(\mathfrak{G}, \psi^{\mathcal{ST}})$. We apply Theorem 16 to $(\Phi_{\mathsf{Alice}}, \Phi_{\mathsf{Bob}})$ with $\epsilon \leftarrow \epsilon/(\mathsf{abr})^{3/2}$ to obtain $\left(\widetilde{\Phi_{\mathsf{Alice}}}, \widetilde{\Phi_{\mathsf{Bob}}}\right)$. We claim that the strategy $\left(\widetilde{\Phi_{\mathsf{Alice}}}, \widetilde{\Phi_{\mathsf{Bob}}}\right)$ wins the game with probability at least $\mathrm{val}_Q(\mathfrak{G}, \psi^{\mathcal{ST}}) - \epsilon$.

Let $\{\mathcal{A}_a\}_{a\in[\mathsf{a}^2]_{\geq 0}}$, $\{\mathcal{B}_b\}_{b\in[\mathsf{b}^2]_{\geq 0}}$, $\{\mathcal{R}_r\}_{r\in[\mathsf{r}^2]_{\geq 0}}$ be orthonormal bases in $\mathcal{H}_{\mathcal{A}}$, $\mathcal{H}_{\mathcal{B}}$ and $\mathcal{H}_{\mathcal{R}}$, respectively. From Theorem 16, for all $a \in \left[\mathsf{a}^2\right]_{\geq 0}$, $b \in \left[\mathsf{b}^2\right]_{\geq 0}$, $r \in \left[\mathsf{r}^2\right]_{\geq 0}$, we have

$$\Bigg|\mathrm{Tr}\left[\left(\Phi^*_{\mathsf{Alice}}\left(\widetilde{\mathcal{A}_a}\right) \otimes \Phi^*_{\mathsf{Bob}}\left(\widetilde{\mathcal{B}_b}\right) \otimes \widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes n}\right)\right] \\ - \mathrm{Tr}\left[\left(\widetilde{\Phi^*_{\mathsf{Alice}}}\left(\widetilde{\mathcal{A}_a}\right) \otimes \widetilde{\Phi^*_{\mathsf{Bob}}}\left(\widetilde{\mathcal{B}_b}\right) \otimes \widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes D}\right)\right]\Bigg| \leq \epsilon/(\mathsf{abr})^{3/2}.$$

By Equation (3), it is equivalent to

$$\Bigg|\mathrm{Tr}\left[\left(\left(\Phi_{\mathsf{Alice}} \otimes \Phi_{\mathsf{Bob}}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes n}\right)\right)\left(\widetilde{\mathcal{A}_a} \otimes \widetilde{\mathcal{B}_b} \otimes \widetilde{\mathcal{R}_r}\right)\right] \\ - \mathrm{Tr}\left[\left(\left(\widetilde{\Phi_{\mathsf{Alice}}} \otimes \widetilde{\Phi_{\mathsf{Bob}}}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes D}\right)\right)\left(\widetilde{\mathcal{A}_a} \otimes \widetilde{\mathcal{B}_b} \otimes \widetilde{\mathcal{R}_r}\right)\right]\Bigg| \leq \epsilon/(\mathsf{abr})^{3/2}.$$

We finally get the desired result:

$$\begin{aligned}
&\left|\mathrm{Tr}\left[M^{\mathcal{ABR}}\left(\left(\Phi_{\mathsf{Alice}} \otimes \Phi_{\mathsf{Bob}}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes n}\right) - \left(\widetilde{\Phi_{\mathsf{Alice}}} \otimes \widetilde{\Phi_{\mathsf{Bob}}}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes D}\right)\right)\right]\right| \\
\overset{(\star)}{\leq}\ & \left\|M^{\mathcal{ABR}}\right\| \cdot \left\|\left(\Phi_{\mathsf{Alice}} \otimes \Phi_{\mathsf{Bob}}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes n}\right) - \left(\widetilde{\Phi_{\mathsf{Alice}}} \otimes \widetilde{\Phi_{\mathsf{Bob}}}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes D}\right)\right\|_1 \\
\leq\ & (\mathsf{abr})^{1/2}\left\|\left(\Phi_{\mathsf{Alice}} \otimes \Phi_{\mathsf{Bob}}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes n}\right) - \left(\widetilde{\Phi_{\mathsf{Alice}}} \otimes \widetilde{\Phi_{\mathsf{Bob}}}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes D}\right)\right\|_2 \\
=\ & \Bigg(\mathsf{abr}\sum_{a,b,r}\Bigg(\mathrm{Tr}\left[\left(\left(\Phi_{\mathsf{Alice}} \otimes \Phi_{\mathsf{Bob}}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes n}\right)\right)\left(\widetilde{\mathcal{A}_a} \otimes \widetilde{\mathcal{B}_b} \otimes \widetilde{\mathcal{R}_r}\right)\right] \\
&-\mathrm{Tr}\left[\left(\left(\widetilde{\Phi_{\mathsf{Alice}}} \otimes \widetilde{\Phi_{\mathsf{Bob}}}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes D}\right)\right)\left(\widetilde{\mathcal{A}_a} \otimes \widetilde{\mathcal{B}_b} \otimes \widetilde{\mathcal{R}_r}\right)\right]\Bigg)^2\Bigg)^{1/2} \leq \epsilon,
\end{aligned}$$

where $(\star)$ is by Hölder's inequality. ◀

3.1 Notations and setup

The proof of Theorem 16 involves a number of notations. To keep the proof succinct, we introduce the setup and the notations that are used frequently in the rest of the paper.

▶ **Setup 18.** *Given quantum systems $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{A}, \mathcal{B}$ with dimensions*

$$\mathsf{p} = |\mathcal{P}|, \mathsf{q} = |\mathcal{Q}|, \mathsf{r} = |\mathcal{R}|, \mathsf{s} = |\mathcal{S}|, \mathsf{t} = |\mathcal{T}|, \mathsf{a} = |\mathcal{A}|, \mathsf{b} = |\mathcal{B}|,$$

let $\phi_{\mathsf{in}}^{\mathcal{PQR}}$ be the input state in $\mathcal{P} \otimes \mathcal{Q} \otimes \mathcal{R}$ shared among Alice, Bob and the referee, where Alice, Bob and the referee hold $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$, respectively. Let $\psi^{\mathcal{ST}} \in \mathcal{H}_{\mathcal{ST}}$ be the noisy MES shared between Alice and Bob, where Alice has $\mathcal{S}$ and Bob has $\mathcal{T}$. Let $\rho < 1$ be the maximal correlation of $\psi^{\mathcal{ST}}$. Let $\mathcal{A}$ and $\mathcal{B}$ be the answer registers of Alice and Bob, respectively.

Let $\{\mathcal{S}_s\}_{s\in[\mathsf{s}^2]_{\geq 0}}, \{\mathcal{T}_t\}_{t\in[\mathsf{t}^2]_{\geq 0}}$ be standard orthonormal bases in $\mathcal{H}_{\mathcal{S}}, \mathcal{H}_{\mathcal{T}}$, respectively. Let $\{\mathcal{A}_a\}_{a\in[\mathsf{a}^2]_{\geq 0}}, \{\mathcal{B}_b\}_{b\in[\mathsf{b}^2]_{\geq 0}}, \{\mathcal{P}_p\}_{p\in[\mathsf{p}^2]_{\geq 0}}, \{\mathcal{Q}_q\}_{q\in[\mathsf{q}^2]_{\geq 0}}, \{\mathcal{R}_r\}_{r\in[\mathsf{r}^2]_{\geq 0}}$ be orthonormal bases (not necessary to be standard orthonormal) in $\mathcal{H}_{\mathcal{A}}, \mathcal{H}_{\mathcal{B}}, \mathcal{H}_{\mathcal{P}}, \mathcal{H}_{\mathcal{Q}}, \mathcal{H}_{\mathcal{R}}$, respectively. For convenience, we denote $\widetilde{\mathcal{A}_a}$ to be $\mathcal{A}_a/\sqrt{\mathsf{a}}$. The same for $\widetilde{\mathcal{B}_b}$, $\widetilde{\mathcal{P}_p}$, $\widetilde{\mathcal{Q}_q}$, $\widetilde{\mathcal{R}_r}$.

When we use universal quantifiers, we omit the ranges of the variables whenever they are clear in the context. For example, we say "for all a, b" to mean "for all $a \in [\mathsf{a}^2]_{\geq 0}$, $b \in [\mathsf{b}^2]_{\geq 0}$".

Given $M \in \mathcal{H}_{\mathcal{S}^n\mathcal{P}\mathcal{A}}$, for all p, a, we define M_a to be $\mathrm{Tr}_{\mathcal{A}}\left[\left(\mathbb{1}_{\mathcal{S}^n\mathcal{P}} \otimes \widetilde{\mathcal{A}_a}\right) M\right]$, and $M_{p,a}$ to be $\mathrm{Tr}_{\mathcal{P}}\left[\left(\mathbb{1}_{\mathcal{S}^n} \otimes \widetilde{\mathcal{P}_p}\right) M_a\right]$. Similar for $N, N_b, N_{q,b}$. In other words,

$$M = \sum_{a\in[\mathsf{a}^2]_{\geq 0}} M_a \otimes \widetilde{\mathcal{A}_a}, \quad N = \sum_{b\in[\mathsf{b}^2]_{\geq 0}} N_b \otimes \widetilde{\mathcal{B}_b}. \tag{7}$$

and

$$M_a = \sum_{p\in[\mathsf{p}^2]_{\geq 0}} M_{p,a} \otimes \widetilde{\mathcal{P}_p}, \quad N_b = \sum_{q\in[\mathsf{q}^2]_{\geq 0}} N_{q,b} \otimes \widetilde{\mathcal{Q}_q}. \tag{8}$$

3.2 Proof of Theorem 16

Proof of Theorem 16. Let δ, θ be parameters which are chosen later. The proof is composed of several steps.

- **Smoothing**
  We apply a noise operator defined in Definition 10 to $J\left(\Phi^*_{\mathsf{Alice}}\right)$ and $J\left(\Phi^*_{\mathsf{Bob}}\right)$, and truncate the high-degree parts to get $M^{(1)}$ and $N^{(1)}$, respectively. [3] They satisfy the following.

  1. For all a, b, $\left\|\left|M_a^{(1)}\right|\right\|_2 \leq 1$ and $\left\|\left|N_b^{(1)}\right|\right\|_2 \leq 1$, where $M_a^{(1)}$ and $N_b^{(1)}$ are defined in Equation (7).
  2. For all a, b, r:

$$\Big|\mathrm{Tr}\left[\left(\Phi^*_{\mathsf{Alice}}\left(\widetilde{\mathcal{A}_a}\right) \otimes \Phi^*_{\mathsf{Bob}}\left(\widetilde{\mathcal{B}_b}\right) \otimes \widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes n}\right)\right] - \mathrm{Tr}\left[\left(M_a^{(1)} \otimes N_b^{(1)} \otimes \widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes n}\right)\right]\Big| \leq \delta.$$

[3] Readers may refer to the full version for details.

3. For all a, b, p, q, $M^{(1)}_{p,a}$ and $N^{(1)}_{q,b}$ have degree at most d_1, where $M^{(1)}_{p,a}$ and $N^{(1)}_{q,b}$ are defined in Equation (8).
4. $$\frac{1}{\mathsf{s}^n}\mathrm{Tr}\ \zeta\left(M^{(1)}\right) \leq \delta \quad \text{and} \quad \frac{1}{\mathsf{t}^n}\mathrm{Tr}\ \zeta\left(N^{(1)}\right) \leq \delta,$$
   where ζ is defined in Equation (6).
5. $M^{(1)}_0 = \mathbb{1}_{\mathcal{S}^n\mathcal{P}}/\sqrt{\mathsf{a}}$ and $N^{(1)}_0 = \mathbb{1}_{\mathcal{T}^n\mathcal{Q}}/\sqrt{\mathsf{b}}$.

Here $d_1 = O\left(\frac{\mathsf{a}^2\mathsf{b}^2\mathsf{pq}}{\delta(1-\rho)}\right)$.

- **Regularization**
  We denote $H \subseteq [n]$ to be the set of registers with high influence, satisfying $h = |H| \leq d_1\left(\mathsf{a}+\mathsf{b}\right)/\theta$ such that for all $i \notin H$:
  $$\mathrm{Inf}_i\left(M\right) \leq \theta, \quad \mathrm{Inf}_i\left(N\right) \leq \theta.$$
- **Invariance to random operators**
  Substituting the basis elements in the subsystems with low influence in $M^{(1)}$ and $N^{(1)}$, we obtain joint random operators $\mathbf{M}^{(2)}$ and $\mathbf{N}^{(2)}$ satisfying the following.

1. For all a, b, p, q:
   $$\mathbb{E}\left[\left\|\mathbf{M}^{(2)}_{p,a}\right\|_2^2\right]^{1/2} = \left\|M^{(1)}_{p,a}\right\|_2 \quad \text{and} \quad \mathbb{E}\left[\left\|\mathbf{N}^{(2)}_{q,b}\right\|_2^2\right]^{1/2} = \left\|N^{(1)}_{q,b}\right\|_2.$$
2. For all a, b, r:
   $$\mathbb{E}\left[\mathrm{Tr}\left[\left(\mathbf{M}^{(2)}_a \otimes \mathbf{N}^{(2)}_b \otimes \widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes h}\right)\right]\right] = \mathrm{Tr}\left[\left(M^{(1)}_a \otimes N^{(1)}_b \otimes \widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes n}\right)\right].$$
3. $$\left|\frac{1}{\mathsf{s}^h}\mathbb{E}\left[\mathrm{Tr}\ \zeta\left(\mathbf{M}^{(2)}\right)\right] - \frac{1}{\mathsf{s}^n}\mathrm{Tr}\ \zeta\left(M^{(1)}\right)\right| \leq O\left(\mathsf{p}^{10/3}\mathsf{a}^4\left(3^{d_1}\mathsf{s}^{d_1/2}\sqrt{\theta}d_1\right)^{2/3}\right)$$
   and
   $$\left|\frac{1}{\mathsf{t}^h}\mathbb{E}\left[\mathrm{Tr}\ \zeta\left(\mathbf{N}^{(2)}\right)\right] - \frac{1}{\mathsf{t}^n}\mathrm{Tr}\ \zeta\left(N^{(1)}\right)\right| \leq O\left(\mathsf{q}^{10/3}\mathsf{b}^4\left(3^{d_1}\mathsf{t}^{d_1/2}\sqrt{\theta}d_1\right)^{2/3}\right).$$
4. $\mathbf{M}^{(2)}_0 = \mathbb{1}_{\mathcal{S}^h\mathcal{P}}/\sqrt{\mathsf{a}}$ and $\mathbf{N}^{(2)}_0 = \mathbb{1}_{\mathcal{T}^h\mathcal{Q}}/\sqrt{\mathsf{b}}$.

- **Dimension Reduction**
  We then reducing the number of Gaussian variables in $\left(\mathbf{M}^{(2)}, \mathbf{N}^{(2)}\right)$ randomly. With probability at least $3/4 - \delta/2 > 0$, we get joint random operators $\left(\mathbf{M}^{(3)}, \mathbf{N}^{(3)}\right)$ such that the following holds:

1. For all a, b, p, q:
   $$\mathbb{E}\left[\left\|\mathbf{M}^{(3)}_{p,a}\right\|_2^2\right] \leq (1+\delta)\,\mathbb{E}\left[\left\|\mathbf{M}^{(2)}_{p,a}\right\|_2^2\right] \quad \text{and} \quad \mathbb{E}\left[\left\|\mathbf{N}^{(3)}_{q,b}\right\|_2^2\right] \leq (1+\delta)\,\mathbb{E}\left[\left\|\mathbf{N}^{(2)}_{q,b}\right\|_2^2\right].$$

2.

$$\mathop{\mathbb{E}}_{\mathbf{x}}\left[\mathrm{Tr}\ \zeta\left(\mathbf{M}^{(3)}\right)\right] \leq 8\mathop{\mathbb{E}}_{\mathbf{g}}\left[\mathrm{Tr}\ \zeta\left(\mathbf{M}^{(2)}\right)\right] \text{ and } \mathop{\mathbb{E}}_{\mathbf{y}}\left[\mathrm{Tr}\ \zeta\left(\mathbf{N}^{(3)}\right)\right] \leq 8\mathop{\mathbb{E}}_{\mathbf{h}}\left[\mathrm{Tr}\ \zeta\left(\mathbf{N}^{(2)}\right)\right].$$

3. For all a, b, r:

$$\left|\mathop{\mathbb{E}}_{\mathbf{x},\mathbf{y}}\left[\mathrm{Tr}\left[\left(\mathbf{M}_a^{(3)} \otimes \mathbf{N}_b^{(3)} \otimes \widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}} \otimes \left(\psi^{\mathcal{ST}}\right)^{\otimes h}\right)\right]\right] - \mathop{\mathbb{E}}_{\mathbf{g},\mathbf{h}}\left[\mathrm{Tr}\left[\left(\mathbf{M}_a^{(2)} \otimes \mathbf{N}_b^{(2)} \otimes \widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}} \otimes \left(\psi^{\mathcal{ST}}\right)^{\otimes h}\right)\right]\right]\right| \leq \delta.$$

4. $\mathbf{M}_0^{(3)} = \mathbb{1}_{\mathcal{S}^h\mathcal{P}}/\sqrt{\mathsf{a}}$ and $\mathbf{N}_0^{(3)} = \mathbb{1}_{\mathcal{T}^h\mathcal{Q}}/\sqrt{\mathsf{b}}$.

Here $n_0 = O\left(\frac{(\mathsf{abr})^{12}(\mathsf{pq})^{20}d_1^{O(d_1)}}{\delta^6}\right)$.

- **Smoothing random operators**
To get low-degree approximations of the Fourier coefficients of $\mathbf{M}^{(3)}$ and $\mathbf{N}^{(3)}$, we obtain joint random operators $\left(\mathbf{M}^{(4)}, \mathbf{N}^{(4)}\right)$ satisfying the following.

1. For all a, b, p, q:

$$\deg\left(\mathbf{M}_{p,a}^{(4)}\right) \leq d_2 \quad \text{and} \quad \deg\left(\mathbf{N}_{q,b}^{(4)}\right) \leq d_2.$$

2. For all a, b, p, q:

$$\mathbb{E}\left[\left\|\left|\mathbf{M}_{p,a}^{(4)}\right\|\right|_2^2\right]^{1/2} \leq \mathbb{E}\left[\left\|\left|\mathbf{M}_{p,a}^{(3)}\right\|\right|_2^2\right]^{1/2} \quad \text{and} \quad \mathbb{E}\left[\left\|\left|\mathbf{N}_{q,b}^{(4)}\right\|\right|_2^2\right]^{1/2} \leq \mathbb{E}\left[\left\|\left|\mathbf{N}_{q,b}^{(3)}\right\|\right|_2^2\right]^{1/2}.$$

3.

$$\mathbb{E}\left[\mathrm{Tr}\ \zeta\left(\mathbf{M}^{(4)}\right)\right] \leq \mathbb{E}\left[\mathrm{Tr}\ \zeta\left(\mathbf{M}^{(3)}\right)\right] + \delta \text{ and } \mathbb{E}\left[\mathrm{Tr}\ \zeta\left(\mathbf{N}^{(4)}\right)\right] \leq \mathbb{E}\left[\mathrm{Tr}\ \zeta\left(\mathbf{N}^{(3)}\right)\right] + \delta.$$

4. For all a, b, r:

$$\left|\mathbb{E}\left[\mathrm{Tr}\left[\left(\mathbf{M}_a^{(4)} \otimes \mathbf{N}_b^{(4)} \otimes \widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes h}\right)\right]\right] - \mathbb{E}\left[\mathrm{Tr}\left[\left(\mathbf{M}_a^{(3)} \otimes \mathbf{N}_b^{(3)} \otimes \widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes h}\right)\right]\right]\right| \leq \delta.$$

5. $\mathbf{M}_0^{(4)} = \mathbb{1}_{\mathcal{S}^h\mathcal{P}}/\sqrt{\mathsf{a}}$ and $\mathbf{N}_0^{(4)} = \mathbb{1}_{\mathcal{T}^h\mathcal{Q}}/\sqrt{\mathsf{b}}$.

Here $d_2 = O\left(\frac{\mathsf{a}^2\mathsf{b}^2\mathsf{pq}}{\delta(1-\rho)}\right)$.

- **Multilinearization**
Suppose that

$$\mathbf{M}_{p,a}^{(4)} = \sum_{s\in[\mathsf{s}^2]_{\geq 0}^h} m_{s,p,a}^{(4)}(\mathbf{x})\,\mathcal{S}_s \quad \text{and} \quad \mathbf{N}_{q,b}^{(4)} = \sum_{t\in[\mathsf{t}^2]_{\geq 0}^h} n_{t,q,b}^{(4)}(\mathbf{y})\,\mathcal{T}_t.$$

To get multilinear approximations of the Fourier coefficients of $\mathbf{M}^{(4)}$ and $\mathbf{N}^{(4)}$, we obtain multilinear random operators $\left(\mathbf{M}^{(5)}, \mathbf{N}^{(5)}\right)$ such that the following holds:

1. For all a,b,p,q, $\mathbf{M}^{(5)}_{p,a}$ and $\mathbf{N}^{(5)}_{q,b}$ are degree-d_2 multilinear random operators.
2. Suppose that
$$\mathbf{M}^{(5)}_{p,a}=\sum_{s\in[\mathsf{s}^2]^h_{\geq 0}} m^{(5)}_{s,p,a}\left(\mathbf{x}\right)\mathcal{S}_s \quad\text{and}\quad \mathbf{N}^{(5)}_{q,b}=\sum_{t\in[\mathsf{t}^2]^h_{\geq 0}} n^{(5)}_{t,q,b}\left(\mathbf{y}\right)\mathcal{T}_t,$$
where $(\mathbf{x},\mathbf{y})\sim\mathcal{G}_\rho^{\otimes n_0\cdot n_1}$. For all $(i,j)\in[n_0]\times[n_1], a,b,p,q,s,t,$
$$\mathrm{Inf}_{(i-1)n_1+j}\left(m^{(5)}_{s,p,a}\right)\leq\theta\cdot\mathrm{Inf}_i\left(m^{(4)}_{s,p,a}\right)\quad\text{and}\quad \mathrm{Inf}_{(i-1)n_1+j}\left(n^{(5)}_{t,q,b}\right)\leq\theta\cdot\mathrm{Inf}_i\left(n^{(4)}_{t,q,b}\right).$$
3. For all a,b:
$$\mathbb{E}\left[\left\|\left|\mathbf{M}^{(5)}_a\right\|\right|_2^2\right]\leq\mathbb{E}\left[\left\|\left|\mathbf{M}^{(4)}_a\right\|\right|_2^2\right]\quad\text{and}\quad \mathbb{E}\left[\left\|\left|\mathbf{N}^{(5)}_b\right\|\right|_2^2\right]\leq\mathbb{E}\left[\left\|\left|\mathbf{N}^{(4)}_b\right\|\right|_2^2\right].$$
4. $$\frac{1}{\mathsf{s}^h}\left|\mathbb{E}\left[\mathrm{Tr}\ \zeta\left(\mathbf{M}^{(5)}\right)\right]-\mathbb{E}\left[\mathrm{Tr}\ \zeta\left(\mathbf{M}^{(4)}\right)\right]\right|\leq\delta$$
and
$$\frac{1}{\mathsf{t}^h}\left|\mathbb{E}\left[\mathrm{Tr}\ \zeta\left(\mathbf{N}^{(5)}\right)\right]-\mathbb{E}\left[\mathrm{Tr}\ \zeta\left(\mathbf{N}^{(4)}\right)\right]\right|\leq\delta.$$
5. For all a,b,r:
$$\Big|\mathbb{E}\left[\mathrm{Tr}\left[\left(\mathbf{M}^{(5)}_a\otimes\mathbf{N}^{(5)}_b\otimes\widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}}\otimes\psi^{\otimes h}\right)\right]\right] - \mathbb{E}\left[\mathrm{Tr}\left[\left(\mathbf{M}^{(4)}_a\otimes\mathbf{N}^{(4)}_b\otimes\widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}}\otimes\psi^{\otimes h}\right)\right]\right]\Big|\leq\delta.$$
6. $\mathbf{M}^{(5)}_0=\mathbb{1}_{\mathcal{S}^h\mathcal{P}}/\sqrt{\mathsf{a}}$ and $\mathbf{N}^{(5)}_0=\mathbb{1}_{\mathcal{T}^h\mathcal{Q}}/\sqrt{\mathsf{b}}$.

Here $n_1=O\left(\frac{\mathsf{a}^4\mathsf{b}^4\mathsf{p}^2\mathsf{q}^2d_2^2}{\theta^2}\right)$.

- **Invariance to operators**

Applying item 2 above, Fact 4 and Fact 14, we have
$$\sum_{s,p,a}\mathrm{Inf}_i\left(m^{(5)}_{s,p,a}\right)\leq\theta\cdot\mathsf{p}\cdot\mathsf{a}\cdot\mathbb{E}\left[\left\|\left|\mathbf{M}^{(4)}\right\|\right|_2^2\right].$$
Similarly, we have
$$\sum_{t,q,b}\mathrm{Inf}_i\left(n^{(5)}_{t,q,b}\right)\leq\theta\cdot\mathsf{q}\cdot\mathsf{b}\cdot\mathbb{E}\left[\left\|\left|\mathbf{N}^{(4)}\right\|\right|_2^2\right].$$
Let
$$\theta_0=\max\left\{\theta\,\mathbb{E}\left[\left\|\left|\mathbf{M}^{(4)}\right\|\right|_2^2\right],\theta\,\mathbb{E}\left[\left\|\left|\mathbf{N}^{(4)}\right\|\right|_2^2\right]\right\}.$$
Substituting the Gaussian variables in $\left(\mathbf{M}^{(5)},\mathbf{N}^{(5)}\right)$ with matrix basis elements to get $\left(M^{(6)},N^{(6)}\right)$ satisfying that:

1. For all a, b, p, q:

$$\left\|M^{(6)}_{p,a}\right\|_2 = \mathbb{E}\left[\left\|\mathbf{M}^{(5)}_{p,a}\right\|_2^2\right]^{1/2} \quad \text{and} \quad \left\|N^{(6)}_{q,b}\right\|_2 = \mathbb{E}\left[\left\|\mathbf{N}^{(5)}_{q,b}\right\|_2^2\right]^{1/2}.$$

2. For all a, b, r:

$$\mathrm{Tr}\left[\left(M^{(6)}_a \otimes N^{(6)}_b \otimes \widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes n_0 n_1 + h}\right)\right] = \mathbb{E}\left[\mathrm{Tr}\left[\left(\mathbf{M}^{(5)}_a \otimes \mathbf{N}^{(5)}_b \otimes \widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}} \otimes \psi^{\otimes h}\right)\right]\right].$$

3.

$$\left|\frac{1}{\mathsf{s}^{n_0 n_1 + h}}\mathrm{Tr}\ \zeta\left(M^{(6)}\right) - \frac{1}{\mathsf{s}^h}\mathbb{E}\left[\mathrm{Tr}\ \zeta\left(\mathbf{M}^{(5)}\right)\right]\right| \le O\left(\mathsf{p}^{10/3}\mathsf{a}^4\left(3^{d_2}\mathsf{s}^{d_2/2}\sqrt{\theta_0}d_2\right)^{2/3}\right)$$

and

$$\left|\frac{1}{\mathsf{t}^{n_0 n_1 + h}}\mathrm{Tr}\ \zeta\left(N^{(6)}\right) - \frac{1}{\mathsf{t}^h}\mathbb{E}\left[\mathrm{Tr}\ \zeta\left(\mathbf{N}^{(5)}\right)\right]\right| \le O\left(\mathsf{q}^{10/3}\mathsf{b}^4\left(3^{d_2}\mathsf{t}^{d_2/2}\sqrt{\theta_0}d_2\right)^{2/3}\right).$$

4. $M^{(6)}_0 = \mathbb{1}_{\mathcal{S}^{n_0 n_1 + h}\mathcal{P}}/\sqrt{\mathsf{a}}$ and $N^{(6)}_0 = \mathbb{1}_{\mathcal{T}^{n_0 n_1 + h}\mathcal{Q}}/\sqrt{\mathsf{b}}$.

- **Rounding**
At last, we round $M^{(6)}$ and $N^{(6)}$ to the Choi representations of the adjoints of some quantum operations, $\widetilde{M}$ and $\widetilde{N}$, satisfying

$$\sum_a \left\|M^{(6)}_a - \widetilde{M_a}\right\|_2^2 = \mathsf{a}\cdot\left\|M^{(6)} - \widetilde{M}\right\|_2^2 \le O\left(\left(\frac{\mathsf{a}^7}{\mathsf{ps}^{\mathsf{D}}}\mathrm{Tr}\ \zeta\left(M^{(6)}\right)\right)^{1/2}\right), \tag{9}$$

$$\sum_b \left\|N^{(6)}_b - \widetilde{N_b}\right\|_2^2 = \mathsf{b}\cdot\left\|N^{(6)} - \widetilde{N}\right\|_2^2 \le O\left(\left(\frac{\mathsf{b}^7}{\mathsf{qt}^{\mathsf{D}}}\mathrm{Tr}\ \zeta\left(N^{(6)}\right)\right)^{1/2}\right). \tag{10}$$

Let $D = h + n_0 n_1$. Then

$$\begin{aligned}
&\left|\mathrm{Tr}\left[\left(M^{(6)}_a \otimes N^{(6)}_b \otimes \widetilde{\mathcal{R}_r} - \widetilde{M_a}\otimes\widetilde{N_b}\otimes\widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}}\otimes\psi^{\otimes D}\right)\right]\right| \\
\le &\left|\mathrm{Tr}\left[\left(M^{(6)}_a \otimes \left(N^{(6)}_b - \widetilde{N_b}\right)\otimes\widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}}\otimes\psi^{\otimes D}\right)\right]\right| \\
&+\left|\mathrm{Tr}\left[\left(\left(M^{(6)}_a - \widetilde{M_a}\right)\otimes\widetilde{N_b}\otimes\widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}}\otimes\psi^{\otimes D}\right)\right]\right| \\
\le &\ (\mathsf{pq})^{1/2}\left(\left\|M^{(6)}_a\right\|_2\left\|N^{(6)}_b - \widetilde{N_b}\right\|_2 + \left\|M^{(6)}_a - \widetilde{M_a}\right\|_2\left\|\widetilde{N_b}\right\|_2\right) \\
\le &\ (\mathsf{pq})^{1/2}\left(\left\|M^{(6)}_a\right\|_2\left(\sum_b\left\|N^{(6)}_b - \widetilde{N_b}\right\|_2^2\right)^{1/2} + \left(\sum_a\left\|M^{(6)}_a - \widetilde{M_a}\right\|_2^2\right)^{1/2}\left\|\widetilde{N_b}\right\|_2\right) \\
\overset{(\star)}{\le} &\left\|M^{(6)}_a\right\|_2 O\left(\left(\frac{\mathsf{b}^7\mathsf{p}^2\mathsf{q}}{\mathsf{t}^D}\mathrm{Tr}\ \zeta\left(N^{(6)}\right)\right)^{1/4}\right) \\
&+\left\|\widetilde{N_b}\right\|_2 O\left(\left(\frac{\mathsf{a}^7\mathsf{pq}^2}{\mathsf{s}^D}\mathrm{Tr}\ \zeta\left(M^{(6)}\right)\right)^{1/4}\right),
\end{aligned}$$

where $(\star)$ is by Equation (9) and Equation (10).

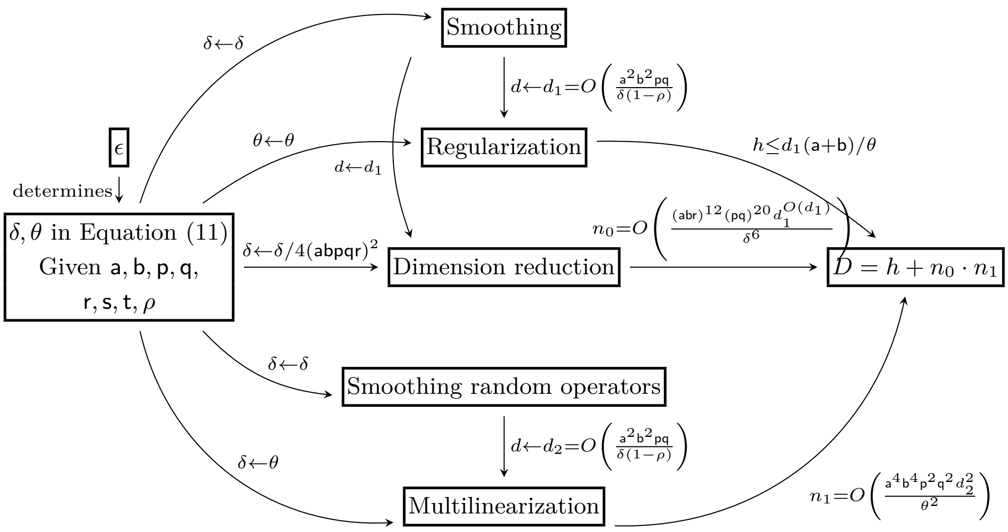

Figure 1 Dependency of parameters in the proof of Theorem 16.

Keeping track of the parameters in the construction, we are able to upper bound $\mathrm{Tr}\ \zeta\left(M^{(6)}\right)/\mathsf{s}^D$ and $\mathrm{Tr}\ \zeta\left(N^{(6)}\right)/\mathsf{t}^D$. Finally, by the triangle inequality we are able to upper bound

$$\left|\mathrm{Tr}\left[\left(\Phi^*_{\mathsf{Alice}}\left(\widetilde{\mathcal{A}_a}\right)\otimes\Phi^*_{\mathsf{Bob}}\left(\widetilde{\mathcal{B}_b}\right)\otimes\mathcal{R}_r\right)\left(\phi_{\mathsf{in}}\otimes\psi^{\otimes n}\right)\right]-\mathrm{Tr}\left[\left(\widetilde{M_a}\otimes\widetilde{N_b}\otimes\mathcal{R}_r\right)\left(\phi_{\mathsf{in}}\otimes\psi^{\otimes D}\right)\right]\right|$$

The dependency of the parameters is pictorially described in Figure 1.

We define $\Psi_{\mathsf{Alice}} \in \mathcal{L}\left(\mathcal{A}, \mathcal{S}^D\mathcal{P}\right)$, $\Psi_{\mathsf{Bob}} \in \mathcal{L}\left(\mathcal{B}, \mathcal{T}^D\mathcal{Q}\right)$ as follows:

$$\Psi_{\mathsf{Alice}}(X) = \mathrm{Tr}_{\mathcal{A}}\left(\widetilde{M}\left(\mathbb{1}_{\mathcal{S}^D\mathcal{P}}\otimes X^\dagger\right)\right), \quad \Psi_{\mathsf{Bob}}(Y) = \mathrm{Tr}_{\mathcal{B}}\left(\widetilde{N}\left(\mathbb{1}_{\mathcal{T}^D\mathcal{Q}}\otimes Y^\dagger\right)\right),$$

just as Equation (5). Let $\widetilde{\Phi_{\mathsf{Alice}}} = \Psi^*_{\mathsf{Alice}}$ and $\widetilde{\Phi_{\mathsf{Bob}}} = \Psi^*_{\mathsf{Bob}}$. Then by Fact 2, $\widetilde{\Phi_{\mathsf{Alice}}}$ and $\widetilde{\Phi_{\mathsf{Bob}}}$ are quantum operations. Furthermore,

$$\begin{aligned}\mathrm{Tr}&\left[\left(\left(\widetilde{\Phi_{\mathsf{Alice}}}\right)^*\left(\widetilde{\mathcal{A}_a}\right)\otimes\left(\widetilde{\Phi_{\mathsf{Bob}}}\right)^*\left(\widetilde{\mathcal{B}_b}\right)\otimes\widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}}\otimes\psi^{\otimes D}\right)\right]\\ &= \mathrm{Tr}\left[\left(\widetilde{M_a}\otimes\widetilde{N_b}\otimes\widetilde{\mathcal{R}_r}\right)\left(\phi_{\mathsf{in}}\otimes\psi^{\otimes D}\right)\right].\end{aligned}$$

Choosing

$$\delta = O(\epsilon), \quad \theta = \frac{\epsilon^{12}}{\exp\left(\frac{\mathsf{a}^2\mathsf{b}^2\mathsf{pq}\log\mathsf{s}\log\mathsf{t}}{\epsilon(1-\rho)}\right)}, \tag{11}$$

we finally conclude the result. ◀

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