On Higher Dimensional Point Sets in General Position

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Abstract

A finite point set in $\mathbb{R}^d$ is in general position if no $d+1$ points lie on a common hyperplane. Let $\alpha_d(N)$ be the largest integer such that any set of N points in $\mathbb{R}^d$ with no $d+2$ members on a common hyperplane, contains a subset of size $\alpha_d(N)$ in general position. Using the method of hypergraph containers, Balogh and Solymosi showed that $\alpha_2(N) < N^{5/6+o(1)}$. In this paper, we also use the container method to obtain new upper bounds for $\alpha_d(N)$ when $d \geq 3$. More precisely, we show that if d is odd, then $\alpha_d(N) < N^{\frac{1}{2}+\frac{1}{2d}+o(1)}$, and if d is even, we have $\alpha_d(N) < N^{\frac{1}{2}+\frac{1}{d-1}+o(1)}$.

We also study the classical problem of determining the maximum number $a(d,k,n)$ of points selected from the grid $[n]^d$ such that no $k+2$ members lie on a k-flat. For fixed d and k, we show that

$$a(d,k,n) \leq O\left(n^{\frac{d}{2\lfloor (k+2)/4\rfloor}(1-\frac{1}{2\lfloor (k+2)/4\rfloor d+1})}\right),$$

which improves the previously best known bound of $O\left(n^{\frac{d}{\lfloor (k+2)/2\rfloor}}\right)$ due to Lefmann when $k+2$ is congruent to 0 or 1 mod 4.

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1 Introduction

A finite point set in $\mathbb{R}^d$ is said to be in *general position* if no $d+1$ members lie on a common hyperplane. Let $\alpha_d(N)$ be the largest integer such that any set of N points in $\mathbb{R}^d$ with no $d+2$ members on a hyperplane, contains $\alpha_d(N)$ points in general position.

In 1986, Erdős [8] proposed the problem of determining $\alpha_2(N)$ and observed that a simple greedy algorithm shows $\alpha_2(N) \geq \Omega(\sqrt{N})$. A few years later, Füredi [10] showed that

$$\Omega(\sqrt{N \log N}) < \alpha_2(N) < o(N),$$

where the lower bound uses a result of Phelps and Rödl [20] on partial Steiner systems, and the upper bound relies on the density Hales-Jewett theorem [11, 12]. In 2018, a breakthrough was made by Balogh and Solymosi [3], who showed that $\alpha_2(N) < N^{5/6+o(1)}$. Their proof was based on the method of hypergraph containers, a powerful technique introduced independently by Balogh, Morris, and Samotij [1] and by Saxton and Thomason [24], that reveals an underlying structure of the independent sets in a hypergraph. We refer interested readers to [2] for a survey of results based on this method.

In higher dimensions, the best lower bound for $\alpha_d(N)$ is due to Cardinal, Tóth, and Wood [5], who showed that $\alpha_d(N) \geq \Omega((N \log N)^{1/d})$, for every fixed $d \geq 2$. For upper bounds, Milićević [18] used the density Hales-Jewett theorem to show that $\alpha_d(N) = o(N)$ for every fixed $d \geq 2$. However, these upper bounds in [18], just like that in [10], are still almost linear in N. Our main result is the following.

▶ **Theorem 1.** *Let $d \geq 3$ be a fixed integer. If d is odd, then $\alpha_d(N) < N^{\frac{1}{2}+\frac{1}{2d}+o(1)}$. If d is even, then $\alpha_d(N) < N^{\frac{1}{2}+\frac{1}{d-1}+o(1)}$.*

Our proof of Theorem 1 is also based on the hypergraph container method. A key ingredient in the proof is a new supersaturation lemma for $(k+2)$-tuples of the grid $[n]^d$ that lie on a k-flat, which we shall discuss in the next section. Here, by a *k-flat* we mean a k-dimensional affine subspace of $\mathbb{R}^d$.

We also study the classical problem of determining the maximum number of points selected from the grid $[n]^d$ such that no $k+2$ members lie on a k-flat. The key ingredient of Theorem 1 mentioned above can be seen as a supersaturation version of this Turán-type problem. When $k = 1$, this is the famous *no-three-in-line problem* raised by Dudeney [7] in 1917: Is it true that one can select $2n$ points in $[n]^2$ such that no three are collinear? Clearly, $2n$ is an upper bound as any vertical line must contain at most 2 points. For small values of n, many authors have published solutions to this problem obtaining the bound of $2n$ (e.g. see [9]), but for large n, the best known general construction is due to Hall et al. [13] with slightly fewer than $3n/2$ points.

More generally, we let $a(d,k,r,n)$ denote the maximum number of points from $[n]^d$ such that no r points lie on a k-flat. Since $[n]^d$ can be covered by n^{d-k} many k-flats, we have the trivial upper bound $a(d,k,r,n) \leq (r-1)n^{d-k}$. For certain values d, k, and r fixed and n tends to infinity, this bound is known to be asymptotically best possible: Many authors [22, 4, 17] noticed that $a(d,d-1,d+1,n) = \Theta(n)$ by looking at the modular moment curve over a finite field $\mathbb{Z}_p$; In [21], Pór and Wood proved that $a(3,1,3,n) = \Theta(n^2)$; Very recently, Sudakov and Tomon [25] showed that $a(d,k,r,n) = \Theta(n^{d-k})$ when $r > d^k$.

We shall focus on the case when $r = k+2$ and write $a(d,k,n) := a(d,k,k+2,n)$. Surprisingly, Lefmann [17] (see also [16]) showed that $a(d,k,n)$ behaves much differently than $\Theta(n^{d-k})$. In particular, he showed that

$$a(d,k,n) \leq O\left(n^{\frac{d}{\lfloor (k+2)/2 \rfloor}}\right).$$

Our next result improves this upper bound when $k+2$ is congruent to 0 or 1 mod 4.

▶ **Theorem 2.** *For fixed d and k, as $n \to \infty$, we have*

$$a(d,k,n) \leq O\left(n^{\frac{d}{2\lfloor (k+2)/4 \rfloor}\left(1-\frac{1}{2\lfloor (k+2)/4 \rfloor d+1}\right)}\right).$$

For example, we have $a(4,2,n) \leq O(n^{\frac{16}{9}})$ while Lefmann's bound in [17] gives us $a(4,2,n) \leq O(n^2)$, which coincides with the trivial upper bound. In particular, Theorem 2 tells us that, if 4 divides $k+2$, then $a(d,k,n)$ only behaves like $\Theta(n^{d-k})$ when $d = k+1$. This is quite interesting compared to the fact that $a(3,1,n) = \Theta(n^2)$ proved in [21]. Lastly, let us note that the current best lower bound for $a(d,k,n)$ is also due to Lefmann [17], who showed that $a(d,k,n) \geq \Omega\left(n^{\frac{d}{k+1}-k-\frac{k}{k+1}}\right)$.

For integer $n > 0$, we let $[n] = \{1,\ldots,n\}$, and $\mathbb{Z}_n = \{0,1,\ldots,n-1\}$. We systemically omit floors and ceilings whenever they are not crucial for the sake of clarity in our presentation. All logarithms are in base two.

2 $(k+2)$-tuples of $[n]^d$ on a k-flat

In this section, we establish two lemmas that will be used in the proof of Theorem 1.

Given a set T of $k+2$ points in $\mathbb{R}^d$ that lie on a k-flat, we say that T is *degenerate* if there is a subset $S \subset T$ of size j, where $3 \le j \le k+1$, such that S lies on a $(j-2)$-flat. Otherwise, we say that T is *non-degenerate*. We establish a supersaturation lemma for non-degenerate $(k+2)$-tuples of $[n]^d$.

▶ **Lemma 3.** *For real number $\gamma > 0$ and fixed positive integers d, k, such that k is even and $d - 2\gamma > (k-1)(k+2)$, any subset $V \subset [n]^d$ of size $n^{d-\gamma}$ spans at least $\Omega(n^{(k+1)d-(k+2)\gamma})$ non-degenerate $(k+2)$-tuples that lie on a k-flat.*

Proof. Let $V \subset [n]^d$ such that $|V| = n^{d-\gamma}$. Set $r = \frac{k}{2} + 1$ and $E_r = \binom{V}{r}$ to be the collection of r-tuples of V. Notice that the sum of a r-tuple from V belongs to $[rn]^d$. For each $v \in [rn]^d$, we define

$$E_r(v) = \{\{v_1, \ldots, v_r\} \in E_r : v_1 + \cdots + v_r = v\}.$$

Then for $T_1, T_2 \in E_r(v)$, where $T_1 = \{v_1, \ldots, v_r\}$ and $T_2 = \{u_1, \ldots, u_r\}$, we have

$$v_1 + \cdots + v_r = v = u_1 + \cdots + u_r,$$

which implies that $T_1 \cup T_2$ lies on a common k-flat. Let

$$E_{2r} = \bigcup_{v \in [rn]^d} \bigcup_{T_1, T_2 \in E_r(v)} \{T_1, T_2\}.$$

Hence, for each $\{T_1, T_2\} \in E_{2r}$, $T_1 \cup T_2$ lies on a k-flat. Moreover, by Jensen's inequality, we have

$$|E_{2r}| = \sum_{v \in [rn]^d} \binom{|E_r(v)|}{2} \ge (rn)^d \binom{\frac{\sum_v |E_r(v)|}{(rn)^d}}{2} = (rn)^d \binom{|E_r|/(rn)^d}{2} \ge \frac{|E_r|^2}{4(rn)^d}.$$

Since k and d are fixed and $r = \frac{k}{2} + 1$ and $|V| = n^{d-\gamma}$,

$$|E_r|^2 = \binom{|V|}{r}^2 = \binom{|V|}{(k/2)+1}^2 \ge \Omega(n^{(k+2)(d-\gamma)}).$$

Combining the two inequalities above gives

$$|E_{2r}| \ge \Omega(n^{(k+1)d-(k+2)\gamma}).$$

We say that $\{T_1, T_2\} \in E_{2r}$ is *good* if $T_1 \cap T_2 = \emptyset$, and the $(k+2)$-tuple $(T_1 \cup T_2)$ is non-degenerate. Otherwise, we say that $\{T_1, T_2\}$ is *bad*. In what follows, we will show that at least half of the pairs (i.e. elements) in E_{2r} are good. To this end, we will need the following claim.

▷ Claim 4. If $\{T_1, T_2\} \in E_{2r}$ is bad, then $T_1 \cup T_2$ lies on a $(k-1)$-flat.

Proof. Write $T_1 = \{v_1, \ldots, v_r\}$ and $T_2 = \{u_1, \ldots, u_r\}$. Let us consider the following cases.

Case 1. Suppose $T_1 \cap T_2 \neq \emptyset$. Then, without loss of generality, there is an integer $j < r$ such that

$$v_1 + \cdots + v_j = u_1 + \cdots + u_j,$$

where $v_1, \ldots, v_j, u_1, \ldots, u_j$ are all distinct elements, and $v_t = u_t$ for $t > j$. Thus $|T_1 \cup T_2| = 2j + (r - j)$. The $2j$ elements above lie on a $(2j-2)$-flat. Adding the remaining $r - j$ points implies that $T_1 \cup T_2$ lies on a $(j - 2 + r)$-flat. Since $r = \frac{k}{2} + 1$ and $j \leq \frac{k}{2}$, $T_1 \cup T_2$ lies on a $(k-1)$-flat.

Case 2. Suppose $T_1 \cap T_2 = \emptyset$. Then $T_1 \cup T_2$ must be degenerate, which means there is a subset $S \subset T_1 \cup T_2$ of j elements such that S lies on a $(j-2)$-flat, for some $3 \leq j \leq k+1$. Without loss of generality, we can assume that $v_1 \notin S$. Hence, $(T_1 \cup T_2) \setminus \{v_1\}$ lies on a $(k-1)$-flat. On the other hand, we have

$$v_1 = u_1 + \cdots + u_r - v_2 - \cdots - v_r.$$

Hence, v_1 is in the affine hull of $(T_1 \cup T_2) \setminus \{v_1\}$ which implies that $T_1 \cup T_2$ lies on a $(k-1)$-flat. ◀

We are now ready to prove the following claim.

▷ Claim 5. At least half of the pairs in E_{2r} are good.

Proof. For the sake of contradiction, suppose at least half of the pairs in E_{2r} are bad. Let H be the collection of all the j-flats spanned by subsets of V for all $j \leq k-1$. Notice that if $S \subset V$ spans a j-flat h, then h is also spanned by only $j+1$ elements from S. So we have

$$|H| \leq \sum_{j=0}^{k-1} |V|^{j+1} \leq kn^{k(d-\gamma)}.$$

For each bad pair $\{T_1, T_2\} \in E_{2r}$, $T_1 \cup T_2$ lies on a j-flat from H by Claim 4. By the pigeonhole principle, there is a j-flat h with $j \leq k-1$ such that at least

$$\frac{|E_{2r}|/2}{|H|} \geq \frac{\Omega(n^{(k+1)d-(k+2)\gamma})}{2kn^{k(d-\gamma)}} = \Omega(n^{d-2\gamma})$$

bad pairs from E_{2r} have the property that their union lies in h. On the other hand, since h contains at most n^{k-1} points from $[n]^d$, h can correspond to at most $O(n^{(k-1)(k+2)})$ bad pairs from E_{2r}. Since we assumed $d - 2\gamma > (k-1)(k+2)$, we have a contradiction for n sufficiently large. ◀

Each good pair $\{T_1, T_2\} \in E_{2r}$ gives rise to a non-degenerate $(k+2)$-tuple $T_1 \cup T_2$ that lies on a k-flat. On the other hand, any such $(k+2)$-tuple in V will correspond to at most $\binom{k+2}{r}$ good pairs in E_{2r}. Hence, by Claim 5, there are at least

$$\frac{|E_{2r}|}{2} \Big/ \binom{k+2}{r} = \Omega(n^{(k+1)d-(k+2)\gamma})$$

non-degenerate $(k+2)$-tuples that lie on a k-flat, concluding the proof. ◀

In the other direction, we will use the following upper bound.

▶ **Lemma 6.** *For real number $\gamma > 0$ and fixed positive integers d, k, ℓ, such that $\ell < k+2$, suppose $U, V \subset [n]^d$ satisfy $|U| = \ell$ and $|V| = n^{d-\gamma}$, then V contains at most $n^{(k+1-\ell)(d-\gamma)+k}$ non-degenerate $(k+2)$-tuples that lie on a k-flat and contain U.*

Proof. If U spans a j-flat for some $j < \ell - 1$, then by definition no non-degenerate $(k+2)$-tuple contains U. Hence we can assume U spans a $(\ell-1)$-flat. Observe that a non-degenerate $(k+2)$-tuple T, which lies on a k-flat and contains U, must contain a $(k+1)$-tuple $T' \subset T$ such that T' spans a k-flat and $U \subset T'$. Then there are at most $n^{(k+1-\ell)(d-\gamma)}$ ways to add $k+1-\ell$ points to U from V to obtain such T'. After T' is determined, there are at most n^k ways to add a final point from the affine hull of T' to obtain T. So we conclude the proof by multiplication. ◀

3 The container method: Proof of Theorem 1

In this section, we use the hypergraph container method to prove Theorem 1. We follow the method outlined in [3]. Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ denote a $(k+2)$-uniform hypergraph. For any $U \subset V(\mathcal{H})$, its degree $\delta(U)$ is the number of edges containing U. For each $\ell \in [k+2]$, we use $\Delta_\ell(\mathcal{H})$ to denote the maximum $\delta(U)$ among all U of size ℓ. For parameter $\tau > 0$, we define the following quantity

$$\Delta(\mathcal{H}, \tau) = \frac{2^{\binom{k+2}{2}-1}|V(\mathcal{H})|}{(k+2)|E(\mathcal{H})|} \sum_{\ell=2}^{k+2} \frac{\Delta_\ell(\mathcal{H})}{\tau^{\ell-1} 2^{\binom{\ell-1}{2}}}.$$

Then we have the following hypergraph container lemma from [3], which is a restatement of Corollary 3.6 in [24].

▶ **Lemma 7.** *Let $\mathcal{H}$ be a $(k+2)$-uniform hypergraph and $0 < \epsilon, \tau < 1/2$. Suppose that $\tau < 1/(200 \cdot (k+2) \cdot (k+2)!)$ and $\Delta(\mathcal{H}, \tau) \le \epsilon/(12 \cdot (k+2)!)$. Then there exists a collection $\mathcal{C}$ of subsets (containers) of $V(\mathcal{H})$ such that*

1. *Every independent set in $\mathcal{H}$ is a subset of some $C \in \mathcal{C}$;*
2. $\log|\mathcal{C}| \le 1000 \cdot (k+2) \cdot ((k+2)!)^3 \cdot |V(\mathcal{H})| \cdot \tau \cdot \log(1/\epsilon) \cdot \log(1/\tau)$*;*
3. *For every $C \in \mathcal{C}$, the induced subgraph $\mathcal{H}[C]$ has at most $\epsilon|E(\mathcal{H})|$ many edges.*

The main result in this section is the following theorem.

▶ **Theorem 8.** *Let k, r be fixed integers such that $r \ge k \ge 2$ and k is even. Then for any $0 < \alpha < 1$, there are constants $c = c(\alpha, k, r)$ and $d = d(\alpha, k, r)$ such that the following holds. For infinitely many values of N, there is a set V of N points in $\mathbb{R}^d$ such that no $r+3$ members of V lie on an r-flat, and every subset of V of size $cN^{\frac{r+2}{2(k+1)}+\alpha}$ contains $k+2$ members on a k-flat.*

Before we prove Theorem 8, let us show that it implies Theorem 1. In dimensions $d_0 \ge 3$ where d_0 is odd, we apply Theorem 8 with $k = r = d_0 - 1$ to obtain a point set V in $\mathbb{R}^d$ with the property that no d_0+2 members lie on a (d_0-1)-flat, and every subset of size $cN^{\frac{1}{2}+\frac{1}{2d_0}+\alpha}$ contains d_0+1 members on a (d_0-1)-flat. By projecting V to a generic d_0-dimensional subspace of $\mathbb{R}^d$, we obtain N points in $\mathbb{R}^{d_0}$ with no d_0+2 members on a common hyperplane, and no $cN^{\frac{1}{2}+\frac{1}{2d_0}+\alpha}$ members in general position.

In dimensions $d_0 \ge 4$ where d_0 is even, we apply Theorem 8 with $k = d_0 - 2$ and $r = d_0 - 1$ to obtain a point set V in $\mathbb{R}^d$ with the property that no d_0+2 members on a (d_0-1)-flat, and every subset of size $cN^{\frac{1}{2}+\frac{1}{d_0-1}+\alpha}$ contains d_0 members on a (d_0-2)-flat. By adding another point from this subset, we obtain d_0+1 members on a (d_0-1)-flat. Hence, by projecting to V a generic d_0-dimensional subspace of $\mathbb{R}^d$, we obtain N points in $\mathbb{R}^{d_0}$ with no d_0+2 members on a common hyperplane, and no $cN^{\frac{1}{2}+\frac{1}{d_0-1}+\alpha}$ members in general position. This completes the proof of Theorem 1.

Proof of Theorem 8. We set $d = d(\alpha, k, r)$ to be a sufficiently large integer depending on α, k, and r. Let $\mathcal{H}$ be the hypergraph with $V(\mathcal{H}) = [n]^d$ and $E(\mathcal{H})$ consists of non-degenerate $(k+2)$-tuples T such that T lies on a k-flat. Let $C^0 = [n]^d$, $\mathcal{C}^0 = \{C^0\}$, and $\mathcal{H}^0 = \mathcal{H}$. In what follows, we will apply the hypergraph container lemma to $\mathcal{H}^0$ to obtain a family of containers $\mathcal{C}^1$. For each $C^1_j \in \mathcal{C}^1$, we consider the induced hypergraph $\mathcal{H}^1_j = \mathcal{H}[C^1_j]$, and we apply the hypergraph container lemma to it. The collection of containers obtained from all $\mathcal{H}^1_j$ will form another collection of containers $\mathcal{C}^2$. We iterate this process until each container in $\mathcal{C}^i$ is sufficiently small, and moreover, we will only produce a small number of containers. As a final step, we apply the probabilistic method to show the existence of the desired point set. We now flesh out the details of this process.

We start by setting $C^0 = [n]^d, \mathcal{C}^0 = \{C^0\}$, and set $\mathcal{H}^0 = \mathcal{H}[C^0] = \mathcal{H}$. Having obtained a collection of containers $\mathcal{C}^i$, for each container $C^i_j \in \mathcal{C}^i$ with $|C^i_j| \geq n^{\frac{k}{k+1}d+k}$, we set $\mathcal{H}^i_j = \mathcal{H}[C^i_j]$. Let $\gamma = \gamma(i,j)$ be defined by $|V(\mathcal{H}^i_j)| = n^{d-\gamma}$. So, $\gamma \leq \frac{d}{k+1} - k$. We set $\tau = \tau(i,j) = n^{-\frac{k}{k+1}d+\gamma+\alpha}$ and $\epsilon = \epsilon(i,j) = c_1 n^{-\alpha}$, where $c_1 = c_1(d,k)$ is a sufficiently large constant depending on d and k. Then we can verify the following condition.

▷ Claim 9. $\Delta(\mathcal{H}^i_j, \tau) \leq \epsilon/(12 \cdot (k+2)!)$.

Proof. Since $|V(\mathcal{H}^i_j)| = n^{d-\gamma}$, $\gamma \leq \frac{d}{k+1} - k$, and d is sufficiently large, Lemma 3 implies that $|E(\mathcal{H}^i_j)| \geq c_2 n^{(k+1)d-(k+2)\gamma}$ for some constant $c_2 = c_2(d,k)$. Hence, we have

$$\frac{|V(\mathcal{H}^i_j)|}{|E(\mathcal{H}^i_j)|} \leq \frac{n^{d-\gamma}}{c_2 n^{(k+1)d-(k+2)\gamma}} = \frac{1}{c_2 n^{kd-(k+1)\gamma}}.$$

On the other hand, by Lemma 6, we have

$$\Delta_\ell(\mathcal{H}^i_j) \leq n^{(d-\gamma)(k+1-\ell)+k} \quad \text{for } \ell < k+2,$$

and obviously $\Delta_{k+2}(\mathcal{H}^i_j) \leq 1$.

Applying these inequalities together with the definition of Δ, we obtain

$$\begin{aligned}
\Delta(\mathcal{H}^i_j, \tau) &= \frac{2^{\binom{k+2}{2}-1}|V(\mathcal{H}^i_j)|}{(k+2)|E(\mathcal{H}^i_j)|} \sum_{\ell=2}^{k+2} \frac{\Delta_\ell(\mathcal{H}^i_j)}{\tau^{\ell-1} 2^{\binom{\ell-1}{2}}} \\
&\leq \frac{c_3}{n^{kd-(k+1)\gamma}} \left(\sum_{\ell=2}^{k+1} \frac{n^{(k+1-\ell)(d-\gamma)+k}}{\tau^{\ell-1}} + \frac{1}{\tau^{k+1}} \right) \\
&= \sum_{\ell=2}^{k+1} \frac{c_3}{\tau^{\ell-1} n^{(\ell-1)d-k-\ell\gamma}} + \frac{c_3}{\tau^{k+1} n^{kd-(k+1)\gamma}},
\end{aligned}$$

for some constant $c_3 = c_3(d,k)$. Let us remark that the summation above is where we determined our τ and γ. In order to make the last term small, we choose $\tau = n^{-\frac{k}{k+1}d+\gamma+\alpha}$. Having determined τ, in order for the first term in the summation to be small, we choose $\gamma \leq \frac{d}{k+1} - k$.

By setting $\epsilon = c_1 n^{-\alpha}$ with $c_1 = c_1(d,k)$ sufficiently large, we have

$$\begin{aligned}
\Delta(\mathcal{H}^i_j, \tau) &\leq c_3 \left(\sum_{\ell=2}^{k+1} n^{-\frac{\ell-1}{k+1}d+\gamma+k-(\ell-1)\alpha} + n^{-(k+1)\alpha} \right) \\
&\leq c_3 k n^{-\alpha} + c_3 n^{-(k+1)\alpha} \\
&< \frac{\epsilon}{12(k+2)!}.
\end{aligned}$$

This verifies the claimed condition. ◀

Given the condition above, we can apply Lemma 7 to $\mathcal{H}_j^i$ with chosen parameters τ and ϵ. Hence we obtain a family of containers $\mathcal{C}_j^{i+1}$ such that

$$\begin{aligned}|\mathcal{C}_j^{i+1}| &\leq 2^{10^3(k+2)((k+2)!)^3|V(\mathcal{H}_i^j)|\tau\log(1/\epsilon)\log(1/\tau)}\\ &\leq 2^{c_4 n^{\frac{d}{k+1}+\alpha}\log^2 n},\end{aligned}$$

for some constant $c_4 = c_4(d,k)$. In the other case where $|C_j^i| < n^{\frac{k}{k+1}d+k}$, we just define $\mathcal{C}_j^{i+1} = \{C_j^i\}$. Then, for each container $C \in \mathcal{C}_j^{i+1}$, we have either $|C| < n^{\frac{k}{k+1}d+k}$ or $|E(\mathcal{H}[C])| \leq \epsilon|E(\mathcal{H}_j^i)| \leq \epsilon^i|E(\mathcal{H})|$. After applying this procedure for each container in $\mathcal{C}^i$, we obtain a new family of containers $\mathcal{C}^{i+1} = \bigcup \mathcal{C}_j^i$ such that

$$|\mathcal{C}^{i+1}| \leq |\mathcal{C}^i| 2^{c_4 n^{\frac{d}{k+1}+\alpha}\log^2 n} \leq 2^{(i+1)c_4 n^{\frac{d}{k+1}+\alpha}\log^2 n}.$$

Notice that the number of edges in $\mathcal{H}_j^i$ shrinks by a factor of $c_1 n^{-\alpha}$ whenever i increases by one, while on the other hand, Lemma 3 tells us that every large subset $C \subset [n]^d$ induces many edges in $\mathcal{H}$. Hence, after at most $t \leq c_5/\alpha$ iterations, for some constant $c_5 = c_5(d,k)$, we obtain a collection of containers $\mathcal{C} = \mathcal{C}^t$ such that: each container $C \in \mathcal{C}$ satisfies $|C| < n^{\frac{k}{k+1}d+k}$; every independent set of $\mathcal{H}$ is a subset of some $C \in \mathcal{C}$; and

$$|\mathcal{C}| \leq 2^{(c_5/\alpha)c_4 n^{\frac{d}{k+1}+\alpha}\log^2 n}.$$

Before we construct the desired point set, we make the following crude estimate.

▷ Claim 10. The grid $[n]^d$ contains at most $O(n^{(r+1)d+2r})$ many $(r+3)$-tuples that lie on a r-flat.

Proof. Let T be an arbitrary $(r+3)$-tuple that spans a j-flat. There are at most $n^{(j+1)d}$ ways to choose a subset $T' \subset T$ of size $j+1$ that spans the affine hull of T. After this T' is determined, there are at most $n^{(r+2-j)j}$ ways to add the remaining $r+2-j$ points from the j-flat spanned by T'. Then the total number of $(r+3)$-tuples that lie on a r-flat is at most

$$\sum_{j=1}^{r} n^{(j+1)d+(r+2-j)j} \leq \sum_{j=1}^{r} n^{(j+1)d+(r+2-j)r} \leq rn^{(r+1)d+2r},$$

since we can assume $d > r$. ◀

Now, we randomly select a subset of $[n]^d$ by keeping each point independently with probability p. Let S be the set of selected elements. Then for each $(r+3)$-tuple T in S that lies on an r-flat, we delete one point from T. We denote the resulting set of points by S'. By the claim above, the number of $(r+3)$-tuples in $[n]^d$ that lie on a r-flat is at most $c_6 n^{(r+1)d+2r}$ for some constant $c_6 = c_6(r)$. Therefore,

$$\mathbb{E}[|S'|] \geq pn^d - c_6 p^{r+3} n^{(r+1)d+2r}.$$

By setting $p = (2c_6)^{-\frac{1}{r+2}} n^{-\frac{r}{r+2}(d+2)}$, we have

$$\mathbb{E}[|S'|] \geq \frac{pn^d}{2} = \Omega(n^{\frac{2(d-r)}{r+2}}).$$

Finally, we set $m = (c_7/\alpha)n^{\frac{d}{k+1}+2\alpha}$ for some sufficiently large constant $c_7 = c_7(d,k,r)$. Let X denote the number of independent sets of size m in S'. Using the family of containers

$\mathcal{C}$, we have

$$\begin{aligned}
\mathbb{E}[X] &\leq |\mathcal{C}| \cdot \binom{n^{\frac{k}{k+1}d+k}}{m} p^m \\
&\leq \left(2^{(c_5/\alpha)c_4 n^{\frac{d}{k+1}+\alpha}\log^2 n}\right)\left(\frac{en^{\frac{k}{k+1}d+k}p}{m}\right)^m \\
&\leq \left(2^{(c_5/\alpha)c_4 n^{\frac{d}{k+1}+\alpha}\log^2 n}\right)\left(c_8\alpha\frac{n^{\frac{k}{k+1}d+k}\cdot n^{-\frac{r}{r+2}(d+2)}}{n^{\frac{d}{k+1}+2\alpha}}\right)^m \\
&\leq \left(2^{(c_5/\alpha)c_4 n^{\frac{d}{k+1}+\alpha}\log^2 n}\right)\left(c_8\alpha n^{\frac{2(k-r-1)d}{(k+1)(r+2)}+k-\frac{2r}{r+2}-2\alpha}\right)^{(c_7/\alpha)n^{\frac{d}{k+1}+2\alpha}},
\end{aligned}$$

for some constant $c_8 = c_8(d,k,r)$. Since $r \geq k$, $0 < \alpha < 1$, and d is large, for n sufficiently large, we have

$$c_8\alpha n^{\frac{2(k-r-1)d}{(k+1)(r+2)}+k-\frac{2r}{r+2}-2\alpha} < 1/2.$$

Hence, we have $\mathbb{E}[X] \leq o(1)$ as n tends to infinity. Notice that $|S'|$ is exponentially concentrated around its mean by Chernoff's inequality. Therefore, some realization of S' satisfies: $|S'| = N = \Omega(n^{2(d-r)/(r+2)})$; S' contains no $(r+3)$-tuples on a r-flat; and $\mathcal{H}[S']$ does not contain an independent set of size

$$m = (c_7/\alpha)n^{\frac{d}{k+1}+2\alpha} \leq cN^{\frac{r+2}{2(k+1)}+\frac{(r+2)r}{2(k+1)(d-r)}+\frac{r+2}{d}2\alpha} \leq cN^{\frac{r+2}{2(k+1)}+\alpha},$$

for some constant $c = c(\alpha,d,k,r)$. Here we assume d is sufficiently large so that

$$\frac{(r+2)r}{2(k+1)(d-r)} + \frac{r+2}{d}2\alpha \leq \alpha.$$

This completes the proof. ◀

4 Avoiding non-trivial solutions: Proof of Theorem 2

In this section, we will give a proof of Theorem 2. Let $V \subset [n]^d$ such that there are no $k+2$ points that lie on a k-flat. In [17], Lefmann showed that $|V| \leq O\left(n^{\frac{d}{\lfloor (k+2)/2 \rfloor}}\right)$. To see this, assume that k is even and consider all elements of the form $v_1 + \cdots + v_{\frac{k}{2}+1}$, where $v_i \neq v_j$ and $v_i \in V$. All of these elements are distinct, since otherwise we would have $k+2$ points on a k-flat. In other words, the equation

$$\left(\mathbf{x}_1 + \cdots + \mathbf{x}_{\frac{k}{2}+1}\right) - \left(\mathbf{x}_{\frac{k}{2}+2} + \cdots + \mathbf{x}_{k+2}\right) = \mathbf{0},$$

does not have a solution with $\{\mathbf{x}_1, \ldots, \mathbf{x}_{\frac{k}{2}+1}\}$ and $\{\mathbf{x}_{\frac{k}{2}+2}, \ldots, \mathbf{x}_{k+2}\}$ being two different $(\frac{k}{2}+1)$-tuples of V. Therefore, we have $\binom{|V|}{\frac{k}{2}+1} \leq (kn)^d$, and this implies Lefmann's bound.

More generally, let us consider the equation

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_r\mathbf{x}_r = \mathbf{0}, \tag{1}$$

with constant coefficients $c_i \in \mathbb{Z}$ and $\sum_i c_i = 0$. Here, the variables $\mathbf{x}_i$ takes value in $\mathbb{Z}^j$. A solution $(\mathbf{x}_1, \ldots, \mathbf{x}_r)$ to equation (1) is called *trivial* if there is a partition $\mathcal{P} : [r] = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_t$, such that $\mathbf{x}_j = \mathbf{x}_\ell$ if and only if $j, \ell \in \mathcal{I}_i$, and $\sum_{j\in\mathcal{I}_i} c_j = 0$ for all $i \in [t]$. In other words, being trivial means that, after combining like terms, the coefficient of each $\mathbf{x}_i$ becomes zero. Otherwise, we say that the solution $(\mathbf{x}_1, \ldots, \mathbf{x}_r)$ is *non-trivial.* A natural extremal problem is to determine the maximum size of a set $A \subset [n]^d$ with only trivial solutions to (1). When $d = 1$, this is a classical problem in additive number theory, and we refer the interested reader to [23, 19, 15, 6].

By combining the arguments of Cilleruelo and Timmons [6] and Jia [14], we establish the following theorem.

▶ **Theorem 11.** *Let d, r be fixed positive integers. Suppose $V \subset [n]^d$ has only trivial solutions to each equation of the form*

$$c_1\left((\boldsymbol{x}_1 + \cdots + \boldsymbol{x}_r) - (\boldsymbol{x}_{r+1} + \cdots + \boldsymbol{x}_{2r})\right) = c_2\left((\boldsymbol{x}_{2r+1} + \cdots + \boldsymbol{x}_{3r}) - (\boldsymbol{x}_{3r+1} + \cdots + \boldsymbol{x}_{4r})\right), \tag{2}$$

for integers c_1, c_2 such that $1 \leq c_1, c_2 \leq n^{\frac{d}{2rd+1}}$. Then we have

$$|V| \leq O\left(n^{\frac{d}{2r}\left(1-\frac{1}{2rd+1}\right)}\right).$$

Notice that Theorem 2 follows from Theorem 11. Indeed, when $k+2$ is divisible by 4, we set $r = (k+2)/4$. If $V \subset [n]^d$ contains $k+2$ points $\{v_1, \ldots, v_{k+2}\}$ that is a non-trivial solution to (2) with $\mathbf{x}_i = v_i$, then $\{v_1, \ldots, v_{k+2}\}$ must lie on a k-flat. Hence, when $k+2$ is divisible by 4, we have

$$a(d,k,n) \leq O\left(n^{\frac{d}{(k+2)/2}\left(1-\frac{1}{(k+2)d/2+1}\right)}\right).$$

Since we have $a(d,k,n) < a(d,k-1,n)$, this implies that for all $k \geq 2$, we have

$$a(d,k,n) \leq O\left(n^{\frac{d}{2\lfloor (k+2)/4\rfloor}\left(1-\frac{1}{2\lfloor (k+2)/4\rfloor d+1}\right)}\right).$$

In the proof of Theorem 11, we need the following well-known lemma (see e.g. [6]Lemma 2.1 and [23]Theorem 4.1). For $U, T \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, we define

$$\Phi_{U-T}(x) = \{(u,t) : u - t = x, u \in U, t \in T\}.$$

▶ **Lemma 12.** *For finite sets $U, T \subset \mathbb{Z}^d$, we have*

$$\frac{(|U||T|)^2}{|U+T|} \leq \sum_{x \in \mathbb{Z}^d} |\Phi_{U-U}(x)| \cdot |\Phi_{T-T}(x)|.$$

Proof of Theorem 11. Let d, r, and V be as given in the hypothesis. Let $m \geq 1$ be an integer that will be determined later. We define

$$S_r = \{v_1 + \cdots + v_r : v_i \in V, v_i \neq v_j\},$$

and a function

$$\sigma : \binom{V}{r} \to S_r,\ \{v_1, \ldots, v_r\} \mapsto v_1 + \cdots + v_r.$$

Notice that σ is a bijection. Indeed, suppose on the contrary that

$$v_1 + \cdots + v_r = v_1' + \cdots + v_r'$$

for two different r-tuples in V. Then by setting $(\mathbf{x}_1, \ldots, \mathbf{x}_r) = (v_1, \ldots, v_r)$, $(\mathbf{x}_{r+1}, \ldots, \mathbf{x}_{2r}) = (v'_1, \ldots, v'_r)$, $(\mathbf{x}_{2r+1}, \ldots, \mathbf{x}_{3r}) = (\mathbf{x}_{3r+1}, \ldots, \mathbf{x}_{4r})$ arbitrarily, and $c_1 = c_2 = 1$, we obtain a non-trivial solution to (2), which is a contradiction. In particular, we have $|S_r| = \binom{|V|}{r}$.

For $j \in [m]$ and $w \in \mathbb{Z}_j^d$, we let

$$U_{j,w} = \{u \in \mathbb{Z}^d : ju + w \in S_r\}.$$

Notice that for fixed $j \in [m]$, we have

$$\sum_{w \in \mathbb{Z}_j^d} |U_{j,w}| = \sum_{w \in \mathbb{Z}_j^d} |\{v \in S_r : v \equiv w \bmod j\}| = |S_r|.$$

Applying Jensen's inequality to above, we have

$$\sum_{w \in \mathbb{Z}_j^d} |U_{j,w}|^2 \geq |S_r|^2 / j^d. \tag{3}$$

For $i \geq 0$, we define

$$\Phi^i_{U_{j,w}-U_{j,w}}(x) = \{(u_1, u_2) \in \Phi_{U_{j,w}-U_{j,w}}(x) : |\sigma^{-1}(ju_1 + w) \cap \sigma^{-1}(ju_2 + w)| = i\}.$$

It's obvious that these sets form a partition of $\Phi_{U_{j,w}-U_{j,w}}(x)$. We also make the following claims.

▷ Claim 13. For a fixed $x \in \mathbb{Z}^d$, we have

$$\sum_{j \in [m]} \sum_{w \in \mathbb{Z}_j^d} |\Phi^0_{U_{j,w}-U_{j,w}}(x)| \leq 1,$$

Proof. For the sake of contradiction, suppose the summation above is at least two, then we have $(u_1, u_2) \in \Phi^0_{U_{j,w}-U_{j,w}}(x)$ and $(u_3, u_4) \in \Phi^0_{U_{j',w'}-U_{j',w'}}(x)$ such that either $(u_1, u_2) \neq (u_3, u_4)$ or $(j, w) \neq (j', w')$.

Let $s_1, s_2, s_3, s_4 \in S_r$ such that $s_1 = ju_1 + w$, $s_2 = ju_2 + w$, $s_3 = j'u_3 + w'$, $s_4 = j'u_4 + w'$ and write $\sigma^{-1}(s_i) = \{v_{i,1}, \ldots, v_{i,r}\}$. Notice that $u_1 - u_2 = x = u_3 - u_4$. Putting these equations together gives us

$$j'((v_{1,1} + \cdots + v_{1,r}) - (v_{2,1} + \cdots + v_{2,r})) = j((v_{3,1} + \cdots + v_{3,r}) - (v_{4,1} + \cdots + v_{4,r})). \tag{4}$$

It suffices to show that (4) can be seem as a non-trivial solution to (2). The proof now falls into the following cases.

Case 1. Suppose $j \neq j'$. Without loss of generality we can assume $j' > j$. Notice that $(u_1, u_2) \in \Phi^0_{U_{j,w}-U_{j,w}}(x)$ implies

$$\{v_{1,1}, \ldots, v_{1,r}\} \cap \{v_{2,1}, \ldots, v_{2,r}\} = \emptyset.$$

Then after combining like terms in (4), the coefficient of v_1^1 is at least $j' - j$, which means this is indeed a non-trivial solution to (2).

Case 2. Suppose $j = j'$, then we must have $s_1 \neq s_3$. Indeed, if $s_1 = s_3$, we must have $w = w'$ (as s_1 modulo j equals s_3 modulo j') and $s_2 = s_4$ (as $j'(s_1 - s_2) = j(s_3 - s_4)$). This is a contradiction to either $(u_1, u_2) \neq (u_3, u_4)$ or $(j, w) \neq (j', w')$.

Given $s_1 \neq s_3$, we can assume, without loss of generality, $v_{1,1} \notin \{v_{3,1}, \ldots, v_{3,r}\}$. Again, we have $\{v_{1,1}, \ldots, v_{1,r}\} \cap \{v_{2,1}, \ldots, v_{2,r}\} = \emptyset$. Hence, after combining like terms in (4), the coefficient of v_1^1 is positive and we have a non-trivial solution to (2). ◀

▷ Claim 14. For a finite set $T \subset \mathbb{Z}^d$, and fixed integers $i, j \geq 1$, we have

$$\sum_{w\in\mathbb{Z}_j^d}\sum_{x\in\mathbb{Z}^d} |\Phi^i_{U_{j,w}-U_{j,w}}(x)| \cdot |\Phi_{T-T}(x)| \leq |V|^{2r-i}|T|.$$

Proof. The summation on the left-hand side counts all (ordered) quadruples (u_1, u_2, t_1, t_2) such that $(u_1, u_2) \in \Phi^i_{U_{j,w}-U_{j,w}}(t_1 - t_2)$. For each such a quadruple, let $s_1, s_2 \in S_r$ such that

$$s_1 = ju_1 + w \quad \text{and} \quad s_2 = ju_2 + w.$$

There are at most $|V|^{2r-i}$ ways to choose a pair (s_1, s_2) satisfying $|\sigma^{-1}(s_1) \cap \sigma^{-1}(s_2)| = i$. Such a pair (s_1, s_2) determines (u_1, u_2) uniquely. Moreover, (s_1, s_2) also determines the quantity

$$t_1 - t_2 = u_1 - u_2 = \frac{s_1 - w}{j} - \frac{s_2 - w}{j} = \frac{1}{j}(s_1 - s_2).$$

After such a pair (s_1, s_2) is chosen, there are at most $|T|$ ways to choose t_1 and this will also determine t_2. So we conclude the claim by multiplication. ◀

Now, we set $T = \mathbb{Z}_\ell^d$ for some integer ℓ to be determined later. Notice that $U_{j,w} + T \subset \{0, 1, \ldots, \lfloor rn/j \rfloor + \ell - 1\}^d$, which implies

$$|U_{j,w} + T| \leq (rn/j + \ell)^d. \tag{5}$$

By Lemma 12, we have

$$\frac{|U_{j,w}|^2|T|^2}{|U_{j,w} + T|} \leq \sum_{x\in\mathbb{Z}^d} |\Phi_{U_{j,w}-U_{j,w}}(x)| \cdot |\Phi_{T-T}(x)|.$$

Summing over all $j \in [m]$ and $w \in \mathbb{Z}_j^d$, and using Claims 13 and 14, we can compute

$$\begin{aligned}
\sum_{j\in[m]}\sum_{w\in\mathbb{Z}_j^d} \frac{|U_{j,w}|^2|T|^2}{|U_{j,w} + T|} &\leq \sum_{j\in[m]}\sum_{w\in\mathbb{Z}_j^d}\sum_{x\in\mathbb{Z}^d} |\Phi_{U_{j,w}-U_{j,w}}(x)| \cdot |\Phi_{T-T}(x)| \\
&= \sum_{x\in\mathbb{Z}^d}\sum_{j\in[m]}\sum_{w\in\mathbb{Z}_j^d} \left(|\Phi^0_{U_{j,w}-U_{j,w}}(x)| + \sum_{i=1}^{r} |\Phi^i_{U_{j,w}-U_{j,w}}(x)|\right) |\Phi_{T-T}(x)| \\
&\leq \sum_{x\in\mathbb{Z}^d} |\Phi_{T-T}(x)| \sum_{j\in[m]}\sum_{w\in\mathbb{Z}_j^d} |\Phi^0_{U_{j,w}-U_{j,w}}(x)| + \sum_{j\in[m]}\sum_{i=1}^{r} |V|^{2r-i}\ell^d \\
&\leq \sum_{x\in\mathbb{Z}^d} \Phi_{T-T}(x) + \sum_{j\in[m]}\sum_{i=1}^{r-1} |V|^{2r-i}\ell^d \\
&\leq \ell^{2d} + rm|V|^{2r-1}\ell^d,
\end{aligned}$$

On the other hand, using (3) and (5), we can compute

$$\begin{aligned}\sum_{j\in[m]}\sum_{w\in\mathbb{Z}_j^d}\frac{|U_{j,w}|^2|T|^2}{|U_{j,w}+T|} &\geq \sum_{j\in[m]}\sum_{w\in\mathbb{Z}_j^d}\frac{|U_{j,w}|^2\ell^{2d}}{(rn/j+\ell)^d}\\ &\geq \sum_{j\in[m]}\frac{|S_r|^2\ell^{2d}}{j^d(rn/j+\ell)^d}\\ &= \sum_{j\in[m]}\frac{|S_r|^2\ell^{2d}}{(rn+j\ell)^d}\\ &\geq \frac{m|S_r|^2\ell^{2d}}{(rn+m\ell)^d},\end{aligned}$$

Combining the two inequalities above gives us

$$\begin{aligned}&\frac{m|S_r|^2\ell^{2d}}{(rn+m\ell)^d} \leq \ell^{2d}+rm|V|^{2r-1}\ell^d\\ \Longrightarrow\ &|S_r|^2 \leq \frac{(rn+m\ell)^d}{m}+r|V|^{2r-1}\frac{(rn+m\ell)^d}{\ell^d}.\end{aligned}$$

By setting $m = n^{\frac{d}{2rd+1}}$ and $\ell = n^{1-\frac{d}{2rd+1}}$, we get

$$\binom{|V|}{r}^2 = |S_r|^2 \leq cn^{d-\frac{d}{2rd+1}} + c|V|^{2r-1}n^{\frac{d^2}{2rd+1}},$$

for some constant c depending only on d and r. We can solve from this inequality that

$$|V| = O\left(n^{\frac{d}{2r}\left(1-\frac{1}{2rd+1}\right)}\right),$$

completing the proof. ◀

5 Concluding remarks

1. One can consider a generalization of the quantity $\alpha_d(N)$. We let $\alpha_{d,s}(N)$ be the largest integer such that any set of N points in $\mathbb{R}^d$ with no $d+s$ members on a hyperplane, contains $\alpha_{d,s}(N)$ points in general position. Hence, $\alpha_d(N) = \alpha_{d,2}(N)$. Following the arguments in our proof of Theorem 1 with a slight modification, we show the following.

▶ **Theorem 15.** *Let $d, s \geq 3$ be fixed integers. If d is odd and $\frac{2d+s-2}{2d+2s-2} < \frac{d-1}{d}$, then $\alpha_{d,s}(N) \leq N^{\frac{1}{2}+o(1)}$. If d is even and $\frac{2d+s-2}{2d+2s-2} < \frac{d-2}{d-1}$, then $\alpha_{d,s}(N) \leq N^{\frac{1}{2}+o(1)}$.*

For example, when we fix $d = 3$ and $s \geq 5$, we have $\alpha_{d,s}(N) \leq N^{\frac{1}{2}+o(1)}$. In the other direction, it is easy to show that $\alpha_{d,s}(N) \geq \Omega(N^{1/d})$ for any fixed $d, s \geq 2$ (see [8]).

▶ **Problem 16.** *Are there fixed integers $d, s \geq 3$ such that $\alpha_{d,s}(N) \leq o(N^{\frac{1}{2}})$?*

2. We call a subset $V \subset [n]^d$ an *m-fold B_g-set* if V only contains trivial solutions to the equations

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_g\mathbf{x}_g = c_1\mathbf{x}_1' + c_2\mathbf{x}_2' + \cdots + c_g\mathbf{x}_g',$$

with constant coefficients $c_i \in [m]$. We call 1-fold B_g-sets simply *B_g-sets*. By counting distinct sums, we have an upper bound $|V| \leq O(n^{\frac{d}{g}})$ for any B_g-set $V \subset [n]^d$.

Our Theorem 11 can be interpreted as the following phenomenon: by letting m grow as some proper polynomial in n, we have an upper bound for m-fold B_g-sets, where g is even, which gives a polynomial-saving improvement from the trivial $O(n^{\frac{d}{g}})$ bound. We believe this phenomenon should also hold without the parity condition on g.

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