The Number of Edges in Maximal 2-Planar Graphs

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Abstract

A graph is 2-*planar* if it has local crossing number two, that is, it can be drawn in the plane such that every edge has at most two crossings. A graph is *maximal* 2-*planar* if no edge can be added such that the resulting graph remains 2-planar. A 2-planar graph on n vertices has at most $5n - 10$ edges, and some (maximal) 2-planar graphs – referred to as *optimal* 2-*planar* – achieve this bound. However, in strong contrast to maximal planar graphs, a maximal 2-planar graph may have fewer than the maximum possible number of edges. In this paper, we determine the minimum edge density of maximal 2-planar graphs by proving that every maximal 2-planar graph on $n \geq 5$ vertices has at least $2n$ edges. We also show that this bound is tight, up to an additive constant. The lower bound is based on an analysis of the degree distribution in specific classes of drawings of the graph. The upper bound construction is verified by carefully exploring the space of admissible drawings using computer support.

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Supplementary Material *Software (Source Code)*: `https://github.com/meghanamreddy/Sparse-maximal-2-planar-graphs` [13]
archived at `swh:1:dir:a23d5245480bf9944c35362d4abc755c6148219f`

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1 Introduction

Maximal planar graphs a.k.a. (combinatorial) triangulations are a rather important and well-studied class of graphs with a number of nice and useful properties. To begin with, the number of edges is uniquely determined by the number of vertices, as every maximal planar graph on $n \geq 3$ vertices has $3n - 6$ edges. It is natural to wonder if a similar statement can be made for the various families of near-planar graphs, which have received considerable attention over the past decade; see, e.g. [11, 15].

In this paper we focus on k-planar graphs, specifically for $k = 2$. These are graphs with local crossing number at most k, that is, they admit a drawing in $\mathbb{R}^2$ where every edge has at most k crossings. The class of 1-planar graphs was introduced by Ringel [21] in the context of vertex-face colorings of planar graphs. Later, Pach and Tóth [20] used upper bounds on the number of edges in k-planar graphs to derive an improved version of the Crossing Lemma, which gives a lower bound on the crossing number of a simple (no loops or multi-edges) graph in terms of its number of vertices and edges. The class of k-planar graphs is not closed under edge contractions and already for $k = 1$ there are infinitely many minimal non-1-planar graphs, as shown by Korzhik [17].

The maximum number of edges in a k-planar graph on n vertices increases with k, but the exact dependency is not known. A general upper bound of $O(\sqrt{k}n)$ is known due to Ackerman and Pach and Tóth [1, 20] for graphs that admit a *simple* k-plane drawing, that is, a drawing where every pair of edges has at most one common point. Only for small k we have tight bounds. A 1-planar graph on n vertices has at most $4n-8$ edges and there are infinitely many *optimal* 1-planar graphs that achieve this bound, as shown by Bodendiek, Schumacher, and Wagner [7]. A 2-planar graph on n vertices has at most $5n-10$ edges and there are infinitely many *optimal* 2-planar graphs that achieve this bound, as shown by Pach and Tóth [20]. In fact, there are complete characterizations, for optimal 1-planar graphs by Suzuki [23] and for optimal 2-planar graphs by Bekos, Kaufmann, and Raftopoulou [6].

Much less is known about *maximal* k-planar graphs, that is, graphs for which adding any edge results in a graph that is not k-planar anymore. In contrast to planar graphs, where maximal and optimal coincide, it is easy to find examples of maximal k-planar graphs that are not optimal; a trivial example is the complete graph K_5. In fact, the difference between maximal and optimal can be quite large for k-planar graphs, even – perhaps counterintuitively – maximal k-planar graphs for $k \geq 1$ may have fewer edges than maximal planar graphs on the same number of vertices. Hudák, Madaras, and Suzuki [16] describe an infinite family of maximal 1-planar graphs with only $8n/3 + O(1) \approx 2.667n$ edges. An improved construction with $45n/17 + O(1) \approx 2.647n$ edges was given by Brandenburg, Eppstein, Gleißner, Goodrich, Hanauer, and Reislhuber [8] who also established a lower bound by showing that every maximal 1-planar graph has at least $28n/13 - O(1) \approx 2.153n$ edges. Later, this lower bound was improved to $20n/9 \approx 2.22n$ by Barát and Tóth [4].

Maximal 2-planar graphs were studied by Auer, Brandenburg, Gleißner, and Hanauer [3] who constructed an infinite family of maximal 2-planar graphs with n vertices and $387n/147 + O(1) \approx 2.63n$ edges.[1] We are not aware of any nontrivial lower bounds on the number of edges in maximal k-planar graphs, for $k \geq 2$.

Results. In this paper, we give tight bounds on the minimum number of edges in maximal 2-planar graphs, up to an additive constant.

▶ **Theorem 1.** *Every maximal 2-planar graph on $n \geq 5$ vertices has at least $2n$ edges.*

▶ **Theorem 2.** *There exists a constant $c \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there exists a maximal 2-planar graph on n vertices with at most $2n + c$ edges.*

Related work. Maximality has also been studied for drawings of simple graphs. Let $\mathcal{D}$ be a class of drawings. A drawing $D \in \mathcal{D}$ is *saturated* if no edge can be added to D so that the resulting drawing is still in $\mathcal{D}$. For the class of simple drawings, Kynčl, Pach, Radoičić and Tóth [18] showed that every saturated drawing on n vertices has at least $1.5n$ edges and there exist saturated drawings with no more than $17.5n$ edges. The upper bound was improved to $7n$ by Hajnal, Igamberdiev, Rote and Schulz [12]. Chaplick, Klute, Parada, Rollin, and Ueckerdt [9] studied saturated k-plane drawings, for $k \geq 4$, and obtained tight bounds linear in n, where the constant depends on k, for various types of crossing restrictions. For the class of 1-plane drawings, Brandenburg, Eppstein, Gleißner, Goodrich, Hanauer, and Reislhuber [8]

[1] Maximality is proven via uniqueness of the 2-plane drawing of the graph. However, there is no explicit proof of the uniqueness in this short abstract.

showed that there exist saturated drawings with no more than $7n/3 + O(1) \approx 2.33n$ edges. On the lower bound side, the abovementioned bound of $20n/9 \approx 2.22n$ edges by Barát and Tóth [4] actually holds for saturated 1-plane drawings. For the class of 2-plane drawings, Auer, Brandenburg, Gleißner, and Hanauer [3] describe saturated drawings with no more than $4n/3 + O(1) \approx 1.33n$ edges, and Barát and Tóth [5] show that every saturated 2-plane drawing on n vertices has at least $n - 1$ edges.

Although the general spirit is similar, saturated drawings are quite different from maximal abstract graphs. To obtain a sparse saturated drawing, one can choose both the graph and the drawing, whereas for sparse maximal graphs one can choose the graph only and needs to get a handle on all possible drawings. Universal lower bounds for saturated drawings carry over to the maximal graph setting, and existential upper bounds for maximal graphs carry over to saturated drawings. But bounds obtained in this fashion are far from tight usually; compare, for instance, the range of between n and $4n/3$ edges for saturated 2-plane drawings to our bound of $2n$ edges for maximal 2-planar graphs.

2 Preliminaries

A *drawing* of a graph $G = (V, E)$ is a map $\gamma : G \to \mathbb{R}^2$ that maps each vertex $v \in V$ to a point $\gamma(v) \in \mathbb{R}^2$ and each each edge $uv \in E$ to a simple (injective) curve $\gamma(uv)$ with endpoints $\gamma(u)$ and $\gamma(v)$, subject to the following conditions: (1) γ is injective on V; (2) for all $uv \in E$ we have $\gamma(uv) \cap \gamma(V) = \{\gamma(u), \gamma(v)\}$; and (3) for each pair $e_0, e_1 \in E$ with $e_0 \neq e_1$ the curves $\gamma(e_0)$ and $\gamma(e_1)$ have at most finitely many intersections, and each such intersection is either a common endpoint or a proper, transversal *crossing* (that is, no touching points between these curves). The connected components of $\mathbb{R}^2 \setminus \gamma(G)$ are the *faces* of γ. The *boundary* of a face f is denoted by ∂f.

To avoid notational clutter we will often identify vertices and edges with their geometric representations in a given drawing. A drawing is *simple* if every pair of edges has at most one common point. A drawing is *k-plane*, for $k \in \mathbb{N}$, if every edge has at most k crossings. A graph is *k-planar* if it admits a k-plane drawing. A graph is *maximal k-planar* if no edge can be added to it so that the resulting graph is still k-planar.

To analyze a k-planar graph one often analyzes one of its k-plane drawings. It is, therefore, useful to impose additional restrictions on this drawing if possible. One such restriction is to consider a *crossing-minimal* k-plane drawing, that is, a drawing that minimizes the total number of edge crossings among all k-plane drawings of the graph. For small k, such a drawing is always simple; for $k \geq 4$ this is not the case in general [22, Footnote 112].

▶ **Lemma 3** (Pach, Radoičić, Tardos, and Tóth [19, Lemma 1.1]). *For $k \leq 3$, every crossing-minimal k-plane drawing is simple.*

In figures, we use the following convention to depict edges: Uncrossed edges are shown green, singly crossed edges are shown purple, doubly crossed edges are shown blue, and edges for which the number of crossings is undetermined are shown black.

Connectivity. Next let us collect some basic properties of maximal k-planar graphs and their drawings. Some of these may be folklore, but for completeness we include the (simple) proofs in the full version [14].

▶ **Lemma 4.** *Let D be a crossing-minimal k-plane drawing of a maximal k-planar graph G, and let u and v be two vertices that lie on (the boundary of) a common face in D. Then uv is an edge of G and it is uncrossed in D.*

▶ **Lemma 5.** *Let D be a crossing-minimal k-plane drawing of a maximal k-planar graph on n vertices, for $k \leq 2 \leq n$. Then every vertex is incident to an uncrossed edge in D.*

▶ **Lemma 6.** *For $k \leq 2$, every maximal k-planar graph on $n \geq 3$ vertices is 2-connected.*

3 The Lower Bound

In this section we develop our lower bound on the edge density of maximal 2-planar graphs by analyzing the distribution of vertex degrees. As we aim for a lower bound of $2n$ edges, we want to show that the average vertex degree is at least four. Then, the density bound follows by the handshaking lemma. However, maximal 2-planar graphs may contain vertices of degree less than four. By Lemma 6 we know that the degree of every vertex is at least two. But degree-two vertices, so-called *hermits*, may exist, as well as vertices of degree three.

In order to lower bound the average degree by four, we employ a charging scheme where we argue that every *low-degree* vertex, that is, every vertex of degree two and three claims a certain number of halfedges at an adjacent *high-degree* vertex, that is, a vertex of degree at least five. Claims are exclusive, that is, every halfedge at a high-degree vertex can be claimed at most once. We use the term *halfedge* because the claim is not on the whole edge but rather on its incidence to one of its high-degree endpoints. The incidence at the other endpoint may or may not be claimed independently (by another vertex). For an edge uv we denote by $\overrightarrow{uv}$ the corresponding halfedge at v and by $\overrightarrow{vu}$ the corresponding halfedge at u. A halfedge $\overrightarrow{uv}$ inherits the properties of its *underlying* edge uv, such as being crossed or uncrossed in a particular drawing. Vertices of degree four have a special role, as they are neither low– nor high-degree. However, a vertex of degree four that is adjacent to a hermit is treated like a low-degree vertex. More precisely, our charging scheme works as follows:

(C1) Every hermit claims two halfedges at each high-degree neighbor.
(C2) Every degree-three vertex claims three halfedges at some high-degree neighbor.
(C3) Every degree four vertex that is adjacent to a hermit h claims two halfedges at some neighbor v of degree ≥ 6. Further, the vertices h and v are adjacent, so h also claims two halfedges at v by (C1). If $\deg(v) = 6$, then v is adjacent to exactly one hermit.
(C4) At most one vertex claims (one or more) halfedges at a degree five vertex.

The remainder of this section is organized as follows. First, we present the proof of Theorem 1 in Section 3.1. Then we prove the validity of our charging scheme along with some useful properties of low-degree vertices in Section 3.2–3.5. Specifically, we will use the following statements in the proof of Theorem 1 below.

▶ **Lemma 7.** *Let G be a maximal 2-planar graph on $n \geq 5$ vertices, let h be a hermit, and let x, y be the neighbors of h in G. Then we have $\deg(x) \geq 4$ and $\deg(y) \geq 4$.*

▶ **Lemma 8.** *Let G be a maximal 2-planar graph on $n \geq 5$ vertices. Then a vertex of degree i in G is adjacent to at most $\lfloor i/3 \rfloor$ hermits.*

3.1 Proof of Theorem 1

Let G be a maximal 2-planar graph on $n \geq 5$ vertices, and let m denote the number of edges in G. We denote by v_i the number of vertices of degree i in G. By Lemma 6 we know that G is 2-connected and, therefore, we have $v_0 = v_1 = 0$. Thus, we have

$$n = \sum_{i=2}^{n-1} v_i \text{ and by the handshaking lemma } 2m = \sum_{i=2}^{n-1} i \cdot v_i. \tag{1}$$

Vertices of degree four or higher can be adjacent to hermits. Let $v_i^{\mathrm{h}j}$ denote the number of vertices of degree i incident to j hermits in G. By Lemma 8 we have

$$v_i = \sum_{j=0}^{\lfloor i/3 \rfloor} v_i^{\mathrm{h}j} \qquad \text{for all } i \geq 3. \tag{2}$$

By Lemma 7 both neighbors of a hermit have degree at least four. Thus, double counting the edges between hermits and their neighbors we obtain

$$2v_2 \leq v_4^{\mathrm{h}1} + v_5^{\mathrm{h}1} + v_6^{\mathrm{h}1} + 2v_6^{\mathrm{h}2} + v_7^{\mathrm{h}1} + 2v_7^{\mathrm{h}2} + 2v_8 + v_9^{\mathrm{h}1} + 2v_9^{\mathrm{h}2} + 3v_9^{\mathrm{h}3} + \sum_{i=10}^{n-1} \lfloor i/3 \rfloor v_i. \tag{3}$$

If a vertex u claims halfedges at a vertex v, we say that v *serves* u. According to (C2), every vertex of degree three claims three halfedges at a high-degree neighbor. Every degree four vertex that is adjacent to a hermit together with this hermit claims four halfedges at a high-degree neighbor by (C3). We sum up the number of these claims and assess how many of them can be served by the different types of high-degree vertices.

In general, a high-degree vertex of degree $i \geq 5$ can serve at most $\lfloor i/3 \rfloor$ such claims. For $i \in \{5, 6, 7, 9\}$, we make a more detailed analysis, taking into account the number of adjacent hermits. Specifically, by (C3) and (C4) a degree five vertex serves at most one low-degree vertex, which is either a hermit or a degree-three vertex. A degree six vertex can serve two degree-three vertices but only if it is not adjacent to a hermit. If a degree six vertex serves a degree four vertex, it is adjacent to exactly one hermit by (C3). In particular, a degree six vertex that is adjacent to two hermits does not serve any degree three or degree four vertex. Altogether we obtain the following inequality:

$$v_3 + v_4^{\mathrm{h}1} \leq v_5^{\mathrm{h}0} + 2v_6^{\mathrm{h}0} + v_6^{\mathrm{h}1} + 2v_7^{\mathrm{h}0} + 2v_7^{\mathrm{h}1} + v_7^{\mathrm{h}2} + 2v_8 + 3v_9^{\mathrm{h}0} + 2v_9^{\mathrm{h}1} + 2v_9^{\mathrm{h}2} + v_9^{\mathrm{h}3} + \sum_{i=10}^{n-1} \lfloor i/3 \rfloor v_i. \tag{4}$$

The combination $((3) + (4))/2$ together with (2) yields

$$v_2 + \frac{1}{2}v_3 \leq \frac{1}{2}v_5 + v_6 + \frac{3}{2}v_7 + 2v_8 + 2v_9 + \sum_{i=10}^{n-1} \lfloor i/3 \rfloor v_i. \tag{5}$$

Now, using these equations and inequalities, we can prove that $m - 2n \geq 0$, to complete the proof of Theorem 1. Let us start from the left hand side, using (1).

$$\begin{aligned} m - 2n &= \frac{1}{2}\sum_{i=2}^{n-1} i v_i - 2\sum_{i=2}^{n-1} v_i = \sum_{i=2}^{n-1} \frac{i-4}{2} v_i \\ &= -v_2 - \frac{1}{2}v_3 + \frac{1}{2}v_5 + v_6 + \frac{3}{2}v_7 + 2v_8 + \frac{5}{2}v_9 + \sum_{i=10}^{n-1} \frac{i-4}{2} v_i \end{aligned}$$

By (5) the right hand side is nonnegative, quod erat demonstrandum.

3.2 Admissible Drawings

So far we have worked with the abstract graph G. In order to discuss our charging scheme, we also use a suitably chosen drawing of G. Specifically, we consider a maximal 2-planar graph G on $n \geq 5$ vertices and a crossing-minimal 2-plane drawing D of G that, among all such drawings, minimizes the number of doubly crossed edges. We refer to a drawing with these properties as an *admissible* drawing of G. By Lemma 3 we know that D is simple.

3.3 Hermits and degree four vertices

▶ **Lemma 9.** *Let h be a hermit and let x, y be its neighbors in G. Then x and y are adjacent in G and all three edges xy, hx, hy are uncrossed in D.*

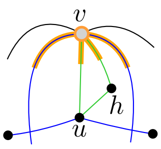
We refer to the edge xy as the *base* of the hermit h, which *hosts* h.

▶ **Lemma 10.** *Every edge of G hosts at most one hermit.*

By Lemma 7 both neighbors of a hermit have degree at least four. A vertex is of type *T4-H* if it has degree exactly four and it is adjacent to a hermit. The following lemma characterizes these vertices and ensures that every hermit has at least one high-degree neighbor.

▶ **Lemma 11.** *Let u be a T4-H vertex with neighbors h, v, w, x in G such that h is a hermit and v is the second neighbor of h. Then both uw and ux are doubly crossed in D, and the two faces of $D \setminus h$ incident to uv are triangles that are bounded by (parts of) edges incident to u and doubly crossed edges incident to v. Furthermore, we have $\deg(v) \geq 6$, and if $\deg(v) = 6$, then h is the only hermit adjacent to v in G.*

In our charging scheme, each hermit h claims two halfedges at each high-degree neighbor v: the halfedge $\overrightarrow{hv}$ and the halfedge $\overrightarrow{uv}$, where uv denotes the edge that hosts h. Each T4-H vertex u claims the two doubly crossed halfedges at v that bound the triangular faces incident to uv in D.



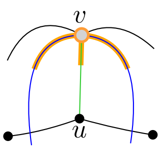
3.4 Degree-three vertices

We distinguish four different types of degree-three vertices in G, depending on their neighborhood and on the crossings on their incident edges in D. Consider a degree-three vertex u in G. By Lemma 5 every vertex is incident to at least one uncrossed edge in D.

T3-1: exactly one uncrossed edge. The two other edges incident to u are crossed.

▶ **Lemma 12.** *Let u be a T3-1 vertex with neighbors v, w, x in G such that the edge uv is uncrossed in D. Then the two faces of D incident to uv are triangles that are bounded by (parts of) edges incident to u and doubly crossed edges incident to v. Furthermore, we have $\deg(v) \geq 5$.*

In our charging scheme, each T3-1 vertex u claims three halfedges at its adjacent high-degree vertex v: the uncrossed halfedge $\overrightarrow{uv}$ along with the two neighboring halfedges at v, which are doubly crossed by Lemma 12.



T3-2: exactly two uncrossed edges. The third edge incident to u is crossed.

▶ **Lemma 13.** *Let u be a T3-2 vertex in D with neighbors v, w, x such that uv is crossed at a point α. Then α is the only crossing of uv in D. Further, the edge that crosses uv is doubly crossed, it is incident to w or x, and its part between w or x and α is uncrossed.*

By the following lemma, we are free to select which of the two edges uv and ux incident to a T3-2 vertex are singly crossed; see Figure 1 (left and middle).

▶ **Lemma 14.** *Let u be a T3-2 vertex in D, and let the neighbors of u be v, w, x such that the edge uv is (singly) crossed by a (doubly crossed) edge wb. Then there exists an admissible drawing D' of G such that (1) D' is identical to D except for the edge wb and (2) the edge wb crosses the edge ux in D'.*

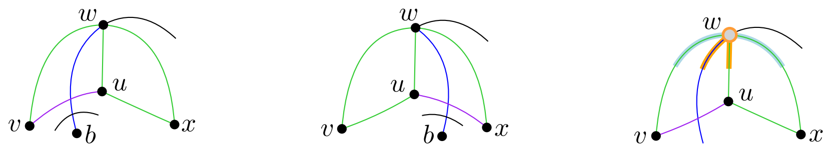


Figure 1 Illustration of Lemma 14 (left and middle); halfedges claimed (marked orange) and assessed (marked lightblue) by a T3-2 vertex u (right).

▶ **Lemma 15.** *Let u be a T3-2 vertex with neighbors v, w, x s.t. the edge uv is singly crossed by a doubly crossed edge wb in D. Then $\deg(w) \geq 5$ and $\min\{\deg(v), \deg(x)\} \geq 4$.*

A halfedge $\overrightarrow{wx}$ is *peripheral* for a vertex u of G if (1) u is a common neighbor of w and x; (2) $\deg(w) \geq 5$; and (3) $\deg(x) \geq 4$. In our charging scheme, every T3-2 vertex u claims three halfedges at the adjacent high-degree vertex w: the halfedge $\overrightarrow{uw}$, the doubly crossed halfedge $\overrightarrow{bw}$, and one of the uncrossed peripheral halfedges $\overrightarrow{vw}$ or $\overrightarrow{xw}$; see Figure 1 (right). While the former two are closely tied to u, the situation is more complicated for the latter two halfedges. Eventually, we need to argue that u can exclusively claim (at least) one of the two peripheral halfedges. But for the time being we say that it *assesses* both of them.

T3-3: all three incident edges uncrossed. We say that such a vertex is of type T3-3. As an immediate consequence of Lemma 4 each T3-3 vertex u together with its neighbors $\mathrm{N}(u)$ induces a plane K_4 in D. We further distiguish two subtypes of T3-3 vertices.

The first subtype accounts for the fact that there may be two adjacent T3-3 vertices in D. We refer to such a pair as an *inefficient hermit*. Observe that two T3-3 vertices z, z' that form an inefficient hermit have the same neighbors in $G \setminus \{z, z'\}$ by Lemma 4. A T3-3 vertex that is part of an inefficient hermit is called a *T3-3 hermit*.

▶ **Lemma 16.** *Let z be a T3-3 vertex in D, and let z' be a neighbor of z in G with $\deg(z') \leq 3$. Then z' is also a T3-3 vertex, that is, the pair z, z' forms an inefficient hermit in D.*

▶ **Lemma 17.** *Let z, z' be an inefficient hermit in D, and let x, y be their (common) neighbors in G. Then xy is an uncrossed edge in D, and the degree of x and y is at least five each.*

In particular, Lemma 17 implies that every T3-3 hermit is part of exactly one inefficient hermit. In our charging scheme, each T3-3 hermit claims three halfedges at one of its (two) adjacent high-degree vertices. More precisely, let z, z' be an inefficient hermit and let x, y be its neighbors in G. Then the vertices x, y, z, z' induce a plane K_4 subdrawing Q of D. The vertex z claims the three halfedges of Q at x, and z' claims the three halfedges of Q at y.

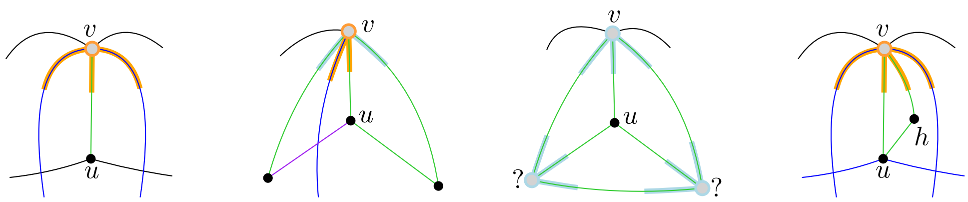
The second subtype is formed by those T3-3 vertices that are not T3-3 hermits; we call them *T3-3 minglers*. By Lemma 16 all neighbors of a T3-3 mingler have degree at least four.

▶ **Lemma 18.** *Let u be a T3-3 mingler in D, and let v, w, x be its neighbors. Then each of v, w, x has degree at least four. Further, at least one vertex among v, w, x has degree at least six, or at least two vertices among v, w, x have degree at least five.*

Let Q denote the plane K_4 induced by u, v, w, x in D. In our charging scheme, the T3-3 mingler u claims the three halfedges of Q at one of its high-degree neighbors. That is, the vertex u assesses all of its (up to six) peripheral halfedges at high-degree neighbors.

3.5 The charging scheme

In this section we argue that our charging scheme works out, that is, all claims made by low-degree vertices and T4-H vertices can be served by adjacent high-degree vertices. Figure 2 presents a summary of the different types of vertices and their claims.



(a) T3-1. **(b)** T3-2. **(c)** T3-3. **(d)** T4-H.

Figure 2 A vertex u with $\deg(u) \in \{3, 4\}$ and an adjacent high-degree vertex v at which u claims halfedges. Claimed halfedges are marked orange. Assessed halfedges are marked lightblue: A T3-2 vertex claims one of the two lightblue peripheral edges, and a T3-3 vertex claims a triple of halfedges at one of its high-degree neighbors.

For some halfedges it is easy to see that they are claimed at most once; these halfedges are shown orange in Figure 2. In particular, it is clear that a halfedge that is incident to the vertex that claims it is claimed at most once. We also need to consider the claims by hermits, which are not shown in the figure (except for the hermit adjacent to a T4-H vertex).

▶ **Lemma 19.** *Every halfedge claimed by a hermit is claimed by this hermit only.*

The next lemma settles the validity of our charging scheme for T3-1 and T4-H vertices.

▶ **Lemma 20.** *Every doubly crossed halfedge is claimed at most once.*

It remains to argue about the claims to peripheral halfedges by T3-2 and T3-3 vertices. Every T3-2 vertex assesses two peripheral halfedges of which it needs to claim one, and every T3-3 vertex assesses three pairs of halfedges of which it needs to claim one. In order to find a suitable assignment of claims for these vertices it is crucial that not too many vertices compete for the same sets of halfedges. Fortunately, we can show that this is not the case. We say that an edge of G is *assessed* by a low-degree vertex u if (at least) one of its corresponding halfedges is assessed by u.

▶ **Lemma 21.** *Every edge is assessed by at most two vertices.*

Proof. For a contradiction consider three vertices u_0, u_1, u_2 of type T3-2 or T3-3 that assess one of the halfedges of an edge uv. Then the edge uv is uncrossed in D, and all of u_0, u_1, u_2 are common neighbors of u and v in G. Moreover, we may suppose that all edges between u_0, u_1, u_2 and u, v are uncrossed in D: For T3-3 vertices all incident edges are uncrossed, anyway, and for T3-2 vertices this follows by Lemma 14. In other words, we have a plane $K_{2,3}$ subdrawing B in D between u_0, u_1, u_2 and u, v. Let ϕ_0, ϕ_1, ϕ_2 denote the three faces of B such that $\partial\phi_i = uu_ivu_{i\oplus 1}$, where $\oplus$ denotes addition modulo 3.

Consider some $i \in \{0, 1, 2\}$. As $\deg(u_i) = 3$, there is exactly one vertex $x_i \notin \{u, v\}$ that is adjacent to u_i in G. The edge from u_i to x_i in D enters the interior of exactly one of ϕ_i or $\phi_{i\oplus 2}$. In other words, for exactly one of ϕ_i or $\phi_{i\oplus 2}$, no edge incident to u_i enters its interior. It follows that $G^- := G \setminus \{u, v\}$ is disconnected, in particular, the vertices u_0, u_1, u_2 are split into at least two components. Suppose without loss of generality that u_0 is separated from both u_1 and u_2 in G^-, and let C_0 denote the component of G^- that contains u_0. Let D_0 denote the subdrawing of D induced by C_0 along with all edges between C_0 and u, v. Observe that uu_0v is an uncrossed path along the outer face of D_0.

We remove D_0 from D and put it back right next to the uncrossed path uu_1v, in the face (ϕ_0 or ϕ_1) incident to u_1 that is not entered by any edge incident to u_1; see Figure 3 for illustration. Furthermore, we flip D_0 with respect to u, v if necessary so as to ensure that the two uncrossed paths uu_1v and uu_0v appear consecutively in the circular order of edges incident to u and v, respectively, in the resulting drawing D', effectively creating a quadrilateral face uu_1vu_0. The drawing D' is an admissible drawing of G, to which we can add an uncrossed edge u_0u_1 in the face uu_1vu_0, a contradiction to the maximality of G. Therefore, no such triple u_0, u_1, u_2 of vertices exists in G. ◀

Figure 3 Redrawing in case that three vertices u_0, u_1, u_2 claim a halfedge of the edge uv.

Note that Lemma 21 settles the claims by T3-3 hermits, as they come in pairs that assess the same halfedges. By Lemma 21 no other vertex assesses these halfedges, so our scheme of assigning the halfedges at one endpoint to each works out. It remains to consider T3-2 vertices and T3-3 minglers. Let us start with the T3-2 vertices. Consider an edge or halfedge e that is assessed by a low-degree vertex u. We say that e is *contested* if there exists another low-degree vertex $u' \neq u$ that also assesses e. An edge or halfedge that is not contested is *uncontested*.

▶ **Lemma 22.** *The claims of all T3-2 vertices can be resolved in a greedy manner.*

Proof. Let u be a T3-2 vertex in D, and let $\overrightarrow{wv}$ and $\overrightarrow{xv}$ denote the halfedges that u assesses. We start a sequence of greedy selections for the claims of vertices $u_1, u_2, \ldots, u_k$ by letting $u_1 := u$ claim one of $\overrightarrow{wv}$ and $\overrightarrow{xv}$ arbitrarily, say, let u claim $\overrightarrow{wv}$ (and withdraw its assessment of $\overrightarrow{xv}$). More generally, at the i-th step of our selection procedure we have a vertex u_i that has just claimed one of its assessed halfedges $\overrightarrow{w_iv_i}$. By Lemma 21 there is at most one other vertex u_{i+1} that also assesses $\overrightarrow{w_iv_i}$. If no such vertex u_{i+1} exists, then we are done and the selection procedure ends here with $i = k$. Otherwise, we consider two cases.

Case 1: *u_{i+1} is a T3-2 vertex.* Then there is only one other (than $\overrightarrow{w_i v_i}$) halfedge that u_{i+1} assesses, denote it by $\overrightarrow{x_{i+1} v_i}$. We let u_{i+1} claim $\overrightarrow{x_{i+1} v_i}$ and proceed with the next step. ◁

Case 2: *u_{i+1} is a T3-3 mingler.* Then u_{i+1} also assesses $\overrightarrow{v_i w_i}$, which is uncontested now, and it also assesses a second halfedge $\overrightarrow{x_{i+1} w_i}$ at w_i. We let u_{i+1} claim both $\overrightarrow{v_i w_i}$ and $\overrightarrow{x_{i+1} w_i}$ and then proceed with the next step. ◁

For the correctness of the selection procedure it suffices to note that at every step exactly one halfedge is claimed that is (still) contested, and the claims of the (unique) vertex that assesses this halfedge are resolved in the next step. In particular, at the end of the procedure, all (still) assessed edges are unclaimed. As long as there exists another T3-2 vertex in D that has not claimed one of the two halfedges it requires, we start another selection procedure from there. Thus, eventually the claims of all T3-2 vertices are resolved. ◀

At this point it only remains to handle the claims of the remaining T3-3 minglers. They are more tricky to deal with compared to the T3-2 vertices because they require *two* halfedges at a single high-degree vertex. We may restrict our attention to a subclass of T3-3 minglers which we call tricky, as they assess a directed 3-cycle of contested halfedges. Consider a T3-3 mingler u, and let v, w, x be its neighbors in G. We say that u is *tricky* if (1) it assesses all six halfedges among its neighbors and (2) all of the halfedges $\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xv}$ or all of the halfedges $\overrightarrow{vx}, \overrightarrow{xw}, \overrightarrow{wv}$ (or both sets) are contested. A T3-3 mingler that is not tricky is *easy.*

▶ **Lemma 23.** *The claims of all easy T3-3 minglers can be resolved in a postprocessing step.*

Proof. Let M denote the set of easy T3-3 minglers in D. We remove M along with all the corresponding assessments from consideration, and let all other (that is, tricky) T3-3 minglers make their claims. We make no assumption about preceding claims, other than that every vertex (1) claims edges incident to one vertex only and (2) claims only edges it assesses. After all tricky T3-3 minglers have made their claims, we process the vertices from M, one by one, in an arbitrary order. In the following, the terms *assessed* and *(un)contested* refer to the initial situation, before any claims were made. The current state of a halfedge is described as either *claimed* or *unclaimed.*

Consider a vertex $u \in M$. If not all six halfedges are assessed by u, then not all of its neighbors are high-degree, in which case, at most one peripheral edge of u is contested. Thus, there always exists one pair of halfedges that is unclaimed and can be claimed by u. In the other case, let H denote the set of six halfedges that are assessed by u. By (2) every uncontested halfedge in H is unclaimed. By Lemma 21 every edge is assessed by at most two vertices. Thus, for each of the edges vx, xw, wv at most one vertex other than u assesses this edge. This other vertex may have claimed a corresponding halfedge, but by (1) for every edge vx, xw, wv at least one of its two corresponding halfedges is unclaimed.

As u is easy, at least one of $\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xv}$ and at least one of $\overrightarrow{vx}, \overrightarrow{xw}, \overrightarrow{wv}$ is uncontested. Suppose without loss of generality that $\overrightarrow{vw}$ is uncontested. We conclude with three cases.

Case 1: *$\overrightarrow{vx}$ is uncontested.* At least one of $\overrightarrow{xw}$ or $\overrightarrow{wx}$ is unclaimed. Thus, we can let u claim one of the pairs $\overrightarrow{xw}, \overrightarrow{vw}$ or $\overrightarrow{wx}, \overrightarrow{vx}$. ◁

Case 2: *$\overrightarrow{xw}$ is uncontested.* Then we let u claim $\overrightarrow{vw}, \overrightarrow{xw}$. ◁

Case 3: *$\overrightarrow{wv}$ is uncontested.* If one of $\overrightarrow{xw}$ or $\overrightarrow{xv}$ is unclaimed, then we let u claim it together with the matching halfedge of the edge vw, which is uncontested by assumption. Otherwise, both $\overrightarrow{xw}$ and $\overrightarrow{xv}$ are claimed. Then both $\overrightarrow{wx}$ and $\overrightarrow{vx}$ are unclaimed, and so u can safely claim these two halfedges. ◀

It remains to resolve the claims of tricky T3-3 minglers. Note that the classification tricky vs. easy depends on the other T3-3 minglers. For instance, a T3-3 mingler that is tricky initially may become easy after removing another easy T3-3 mingler. Here we have to deal with those T3-3 minglers only that remain tricky after all easy T3-3 minglers have been iteratively removed from consideration.

▶ **Lemma 24.** *The claims of all tricky T3-3 minglers can be resolved in a greedy manner.*

Proof. Let u be a tricky T3-3 mingler, and let v, w, x be its neighbors in G. As for each tricky T3-3 mingler all three peripheral edges are contested by other tricky T3-3 minglers, there exists a circular sequence $u = u_1, \ldots, u_k$, with $k \geq 2$, of tricky T3-3 minglers that are neighbors of v in G and whose connecting edges appear in this order around v in D. We distinguish two cases, depending on the parity of k.

Case 1: *k is even.* Then we let each u_i, for i odd, claim the two halfedges at v that it assesses. This resolves the claims for all u_i, with i odd, and we claim that now all u_i, with i even, are easy. To see this, consider a vertex u_i, with i even. Both of its assessed halfedges at v are now claimed; denote these halfedges by $\overrightarrow{w_i v}$ and $\overrightarrow{x_i v}$. It follows that both $\overrightarrow{v w_i}$ and $\overrightarrow{v x_i}$ are unclaimed and u_i is the only vertex that still assesses them. As there is no directed 3-cycle of contested halfedges among the halfedges assessed by u_i anymore, the vertex u_i is easy. ◁

Case 2: *k is odd.* Then we let each u_i, for $i < k$ odd, claim the two halfedges at v that it assesses. This resolves the claims for these u_i and makes all u_i, with $i < k - 1$ even, easy, as in Case 1 above. It remains to argue about u_{k-1} and u_k. Let $x_i v$ denote the edge assessed by both u_i and u_{i+1}, for $1 \leq i < k$, and let $x_k v$ denote the edge assessed by both u_k and u_1. As $\overrightarrow{x_k v}$ is claimed by u_1, we can let u_k claim $\overrightarrow{v x_k}$ and $\overrightarrow{x_{k-1} x_k}$ along with it. This makes u_{k-1} easy, as both $\overrightarrow{v x_{k-2}}$ and $\overrightarrow{v x_{k-1}}$ are uncontested now. However, we still need to sort out the bold claim on $\overrightarrow{x_{k-1} x_k}$ by u_k. To this end, we apply the same greedy selection procedure as in the proof of Lemma 22, except that here we start with the selection of $\overrightarrow{x_{k-1} x_k}$, as the only contested halfedge that is claimed, and here we can only encounter (tricky) T3-3 minglers over the course of the procedure. ◁

We get rid of at least two tricky T3-3 minglers, either by resolving their claims or by making them easy. Thus, after a finite number of steps, no tricky T3-3 mingler remains. ◀

Our analysis of the charging scheme is almost complete now. However, we still need to justify our claim about degree five vertices in Property (C4). In principle it would be possible that two halfedges at a degree five vertex are claimed by a hermit and the remaining three by a degree-three vertex. But we can show that this is impossible.

▶ **Lemma 25.** *At most one low-degree vertex claims halfedges at a degree five vertex.*

4 The Upper Bound: Proof outline of Theorem 2

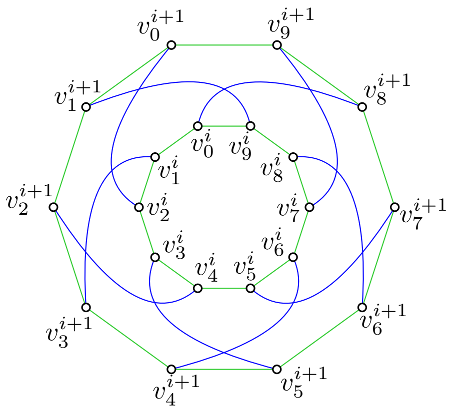
In this section we describe a construction for a family of maximal 2-planar graphs with few edges. We give a complete description of this family. But due to space constraints we give a very rough sketch only for the challenging part of the proof: to show that these graphs are *maximal* 2-planar. The full version [14] provides a complete version of this section, with all proofs.

The graphs can roughly be described as braided cylindrical grids. More precisely, for a given $k \in \mathbb{N}$ we construct our graph G_k on $10k + 140$ vertices as follows.

- Take k copies of C_{10}, the cycle on 10 vertices, and denote them by $D_1, \ldots, D_k$. Denote the vertices of D_i, for $i \in \{1, \ldots, 10\}$, by $v_0^i, \ldots, v_9^i$ so that the edges of D_i are $\{v_j^i v_{j\oplus 1}^i : 0 \leq j \leq 9\}$, where $\oplus$ denotes addition modulo 10.
- For every $i \in \{1, \ldots, k-1\}$, connect the vertices of D_i and D_{i+1} by a braided matching, as follows. For j even, add the edge $v_j^i v_{j\oplus 8}^{i+1}$ to G_k and for j odd, add the edge $v_j^i v_{j\oplus 2}^{i+1}$ to G_k. See Figure 4 (left) for illustration.
- To each edge of D_1 and D_k we attach a gadget $X \simeq K_9 \setminus (K_2 + K_2 + P_3)$ so as to forbid crossings along these edges. Denote the vertices of X by $x_0, \ldots, x_8$ such that $\deg_X(x_0) = \deg_X(x_1) = 8$, $\deg_X(x_8) = 6$ and all other vertices have degree seven. Let x_6, x_7 be the non-neighbors of x_8. To an edge e of D_1 and D_k we attach a copy of X so that e takes the role of the edge $x_6 x_7$ in this copy of X. As altogether there are 20 edges in D_1 and D_k and each copy of X adds seven more vertices, a total of $20 \cdot 7 = 140$ vertices are added to G_k with these gadgets.
- Finally, we add the edges $v_j^i v_{j\oplus 2}^i$, for all $0 \leq j \leq 9$ and $i \in \{1, k\}$.

This completes the description of the graph G_k. Note that G_k has $10k + 140$ vertices and $10k + 10(k-1) + 20 \cdot 31 + 2 \cdot 10 = 20k + 630$ edges. So to prove Theorem 2 asymptotically it suffices to choose $c \geq 630 - 2 \cdot 140 = 350$ and show that G_k is maximal 2-planar. Using some small local modifications we can then obtain the statement for all values of n.

To show that G_k is 2-planar it suffices to give a 2-plane drawing of it. Such a drawing can be deduced from Figure 4: (1) We nest the cycles $D_1, \ldots, D_k$ with their connecting edges using the drawing depicted in Figure 4 (left), (2) draw all copies of X attached to the edges of D_1 and D_k using the drawing depicted in Figure 4 (right), and (3) draw the remaining edges among the vertices of D_1 and D_k inside and outside D_1 and D_k, respectively.



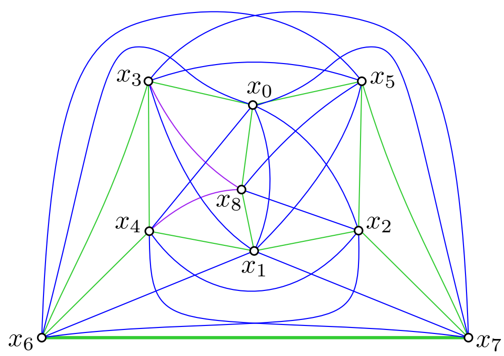


Figure 4 The braided matching between two consecutive ten-cycles in G_k, shown in blue (left); the gadget graph X that we attach to the edges of the first and the last ten-cycle of G_k (right).

It is much more challenging, though, to argue that G_k is *maximal* 2-planar. In fact, we do not know of a direct argument to establish this claim. Instead, we prove that G_k admits essentially only one 2-plane drawing, which is the one described above. Then maximality follows by just inspecting this drawing and observing that no edge can be added there because every pair of non-adjacent vertices is separated by a cycle of doubly-crossed edges.

This leaves us having to prove that G_k has a unique 2-plane drawing. We solve the problem in a somewhat brute-force way: by enumerating all 2-plane drawings of G_k, using computer support. Still, it is not immediately clear how to do this, given that (1) the space of (even 2-plane) drawings of a graph can be vast; (2) (G_k) is an infinite family; and (3) already for small k, even a single graph G_k is quite large.

First of all, the gadget graph X is of small constant size. So it can be analyzed separately, and there is no need to explicitly include these gadgets into the analysis of G_k. Instead, we account for the effect of the gadgets by considering the edges of D_1 and D_k as uncrossable. In this way, we also avoid counting all the variations of placing the attached copy of X on either side of the corresponding cycle as different drawings. In fact, the gadget X itself has a few formally different 2-plane drawings due to its automorphisms. But for our purposes of arguing about the maximality of G_k, these differences do not matter. However, these variations are the reason that the 2-plane drawing of G_k is *essentially* unique only.

We also disregard the length two edges along D_1 and D_k. Denote the resulting subgraph of G_i by G_i^-. We iteratively enumerate the 2-plane drawings of G_i^-, for all $i \in \mathbb{N}$, where only the edges of the first cycle D_1 are labeled uncrossable (but not the edges of the last cycle D_i). All drawings are represented as a doubly-connected edge list (DCEL) [10, Chapter 2]. As a base case, we use the unique (up to orientation, which we select to be counterclockwise, without loss of generality) plane drawing of $G_1^- = D_1$. For each drawing Γ computed, with a specific ten-cycle of vertices labeled as D, we consider all possible ways to extend Γ by adding another ten-cycle of new, labeled vertices and connect it to D using a braided matching, as in the construction of G_k and depicted in Figure 4 (left).

So in each iteration we have a partial drawing Γ and a collection H of vertices and edges still to be drawn. We then exhaustively explore the space of simple 2-plane drawings of $\Gamma \cup H$. Our approach is similar to the one used by Angelini, Bekos, Kaufmann, and Schneck [2] for complete and complete bipartite graphs. We consider the edges to be drawn in some order such that whenever an edge is considered, at least one of its endpoints is in the drawing already. When drawing an edge, we go over (1) all possible positions in the rotation at the source vertex and for each such position all options to (2) draw the edge with zero, one or two crossings. Each option to consider amounts to a traversal of some face incident to the source vertex, and up to two more faces in the neighborhood. At every step we ensure that the drawing constructed remains 2-plane and simple, and backtrack whenever an edge cannot be added or the drawing is complete (that is, it is a 2-plane drawing of G_i^-, for some $i \in \mathbb{N}$).

Every drawing for $\Gamma \cup H$ obtained in this fashion is then tested, as described below. If the tests are successful, then the drawing is added to the list of drawings to be processed, as a child of Γ, and such that the ten-cycle in H takes the role of D for future processing.

As for the testing a drawing Γ, we are only interested in a drawing that can – eventually, after possibly many iterations – be extended in the same way with an uncrossable ten-cycle D_k. In particular, all vertices and edges of D_k must lie in the same face of Γ. Hence, we test whether there exists a suitable *potential final* face in Γ where D_k can be placed; if not, then we discard Γ from further consideration. We also go over the faces of Γ and remove *irrelevant* faces and vertices that are too far from any potential final face to ever be able to interact with vertices and edges to be added in future iterations. Finally, we check whether the resulting *reduced* drawing has already been discovered by comparing it to all the already discovered drawings (by testing for an isomorphism that preserves the cycle D). If not, then we add it to the list of valid drawings.

For each drawing for which we found at least one child drawing, we also test whether there exists a similar extension where the cycle in H is uncrossable. Whenever such an extension is possible, we found a 2-plane drawing of G_i, for some $i \in \mathbb{N}$. The algorithm for G_i^- runs for about 1.5 days and discovers 86 simple 2-plane drawings of G_i^-. In only one of these drawings the last ten-cycle is uncrossed: the drawing described above (see Figure 4). The algorithm for the gadget X runs for about 3min. and discovers 32 simple 2-plane drawings, as expected. The full source code is available in our repository [13].

5 Conclusions

We have obtained tight bounds on the number of edges in maximal 2-planar graphs, up to an additive constant. Naturally, one would expect that our approach can also be applied to other families of near-planar graphs, specifically, to maximal 1- and 3-planar graphs. Intuitively, for k-planar graphs the challenge with increasing k is that the structure of the drawings gets more involved, whereas with decreasing k we aim for a higher bound.

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