Linear Size Universal Point Sets for Classes of Planar Graphs

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Abstract

A finite set P of points in the plane is n-universal with respect to a class $\mathcal{C}$ of planar graphs if every n-vertex graph in $\mathcal{C}$ admits a crossing-free straight-line drawing with vertices at points of P.

For the class of all planar graphs the best known upper bound on the size of a universal point set is quadratic and the best known lower bound is linear in n.

Some classes of planar graphs are known to admit universal point sets of near linear size, however, there are no truly linear bounds for interesting classes beyond outerplanar graphs.

In this paper, we show that there is a universal point set of size $2n - 2$ for the class of bipartite planar graphs with n vertices. The same point set is also universal for the class of n-vertex planar graphs of maximum degree 3. The point set used for the results is what we call an exploding double chain, and we prove that this point set allows planar straight-line embeddings of many more planar graphs, namely of all subgraphs of planar graphs admitting a one-sided Hamiltonian cycle.

The result for bipartite graphs also implies that every n-vertex plane graph has a 1-bend drawing all whose bends and vertices are contained in a specific point set of size $4n - 6$, this improves a bound of $6n - 10$ for the same problem by Löffler and Tóth.

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1 Introduction

Given a family $\mathcal{C}$ of planar graphs and a positive integer n, a point set $P \subseteq \mathbb{R}^2$ is called an *n-universal point set* for the class $\mathcal{C}$ or simply *n-universal* for $\mathcal{C}$ if for every graph $G \in \mathcal{C}$ on n vertices there exists a straight-line crossing-free drawing of G such that every vertex of G is placed at a point of P.

To determine the minimum size of universal sets for classes of planar graphs is a fundamental problem in geometric graph theory, see e.g. Problem [17] in the Open Problem Garden. More specifically, the quest is for good bounds on the minimum size $f_{\mathcal{C}}(n)$ of an n-universal point set for a class $\mathcal{C}$.

Schnyder [21] showed that for $n \geq 3$ the $[n-1] \times [n-1]$-grid forms an n-universal point set for planar graphs, even if the combinatorial embedding of the planar graph is prescribed. This shows that $f(n) := f_{\mathcal{P}}(n) \leq n^2 \in O(n^2)$, where $\mathcal{P}$ is the class of all planar graphs. Asymptotically, the quadratic upper bound on $f(n)$ remains the state of the art. Only the multiplicative constant in this bound has seen some improvement, the current upper bound is $f(n) \leq \frac{1}{4}n^2 + O(n)$ by Bannister et al. [5]. For several subclasses $\mathcal{C}$ of planar graphs, better upper bounds are known: A classical result by Gritzmann et al. [13] is that every outerplanar n-vertex graph embeds straight-line on *any* set of n points in general position, and hence $f_{\text{out-pl}}(n) = n$. Near-linear upper bounds of $f_{\mathcal{C}}(n) = O(n \operatorname{polylog}(n))$ are known for 2-outerplanar graphs, simply nested graphs, and for the classes of bounded pathwidth [4, 5]. Finally, for the class $\mathcal{C}$ of planar 3-trees (also known as Apollonian networks or stacked triangulations), $f_{\mathcal{C}}(n) = O(n^{3/2} \log n)$ has been proved by Fulek and Tóth [12].

As for lower bounds, the trivial bounds $n \leq f_{\mathcal{C}}(n) \leq f(n)$ hold for all $n \in \mathbb{N}$ and all planar graph classes $\mathcal{C}$. The current lower bound $f(n) \geq 1.293n - o(n)$ from [20] has been shown using planar 3-trees, we refer to [6, 8, 9, 15] for earlier work on lower bounds.

Choi, Chrobak and Costello [7] recently proved that point sets chosen uniformly at random from the unit square must have size $\Omega(n^2)$ to be universal for n-vertex planar graphs with high probability. This suggests that universal point sets of size $o(n^2)$ -if they exist- will not look nice, e.g., they will have a large ratio between shortest and largest distances.

In this paper we study a specific ordered point set H (the exploding double chain) and denote the initial piece of size $2n-2$ in H as H_n. Let $\mathcal{C}$ be the class of all planar graphs G which have a plane straight-line drawing on the point set H_n where $n = |V(G)|$. That is, H_n forms an n-universal point set for $\mathcal{C}$.

A graph is POSH (partial one-sided Hamiltonian) if it is a spanning subgraph of a graph admitting a plane embedding with a one-sided Hamiltonian cycle (for definitions see Section 2). Triangulations with a one-sided Hamiltonian cycle have been studied before by Alam et al. [2] in the context of cartograms. They conjectured that every plane 4-connected triangulation has a one-sided Hamiltonian cycle. Later Alam and Kobourov [3] found a plane 4-connected triangulation on 113 vertices which has no one-sided Hamiltonian cycle.

Our main result (Theorem 3) is that every POSH graph is in $\mathcal{C}$. We let

$$\mathcal{C}' := \{\mathcal{G} : \mathcal{G} \text{ is POSH}\}.$$

Theorem 3 motivates further study of $\mathcal{C}'$. On the positive side we show that every bipartite plane graph is POSH (proof in Section 4). We proceed to use the construction for bipartite graphs to show that subcubic planar graphs have a POSH embedding in Section 5. On the negative side, we also show that not all 2-trees are POSH. We conclude with some conjectures and open problems in Section 7.

An exploding double chain was previously used by Löffler and Tóth [16]. They show that every planar graph with n vertices has a 1-bend drawing on a subset S_n of H with $|S_n| = 6n - 10$. Our result about bipartite graphs implies a better bound:

▶ **Corollary 1.** *There is a point set $P = H_{2n-2}$ of size $4n-6$ such that every n-vertex planar graph admits a 1-bend drawing with bends and vertices on P.*

Proof. The dual of a plane triangulation is a bridgeless 3-regular graph of $2n-4$ vertices; it has a perfect matching by Petersen's Theorem [19]. Hence, subdividing at most $n-2$ edges can make any planar graph on n vertices bipartite. Thus H_{n+n-2} of size $2(n+n-2)-2 = 4n-6$ is sufficient to accomodate 1-bend drawings of all n-vertex planar graphs. ◀

Universality for 1-bend and 2-bend drawings with no restriction on the placement of bends has been studied by Kaufmann and Wiese [14], they show that every n-element point set is universal for 2-bend drawings of planar graphs.

2 The point set and the class of POSH graphs

In this section we define the exploding double chain H and the class $\mathcal{C}'$ of POSH graphs and show that for every $n \geq 2$ the initial part H_n of size $2n-2$ of H is n-universal for $\mathcal{C}'$.

A sequence $(y_i)_{i\in\mathbb{N}}$ of real numbers satisfying $y_1 = 0$, $y_2 = 0$ is *exploding* and the corresponding point set $H = \{p_i, q_i | i \in \mathbb{N}\}$, where $p_i = (i, y_i), q_i = (i, -y_i)$, is an *exploding double chain*, if for all $n \in \mathbb{N}$, y_{n+1} is large enough that all intersections of lines going through two points of $H_n = \{p_i, q_i | i \in [n]\}$ with the line $x = n+1$ lie strictly between y_{n+1} and $-y_{n+1}$. It is $p_1 = q_1$ and $p_2 = q_2$, thus $|H_n| = 2n-2$. Figure 1 shows H_6. This fully describes the order type of the exploding double chain. Note that the coordinates given here can be made integers, but the largest coordinate of H_n is exponential in n, which is unavoidable for the order type. However, the ratio of largest to smallest distance does not have to be: We can alter the construction setting $y_i = i$, but letting the x-coordinates grow slowly enough as to achieve the same order type, but with a linear ratio.

An explicit construction of a point set H in this order type is given in the full version.

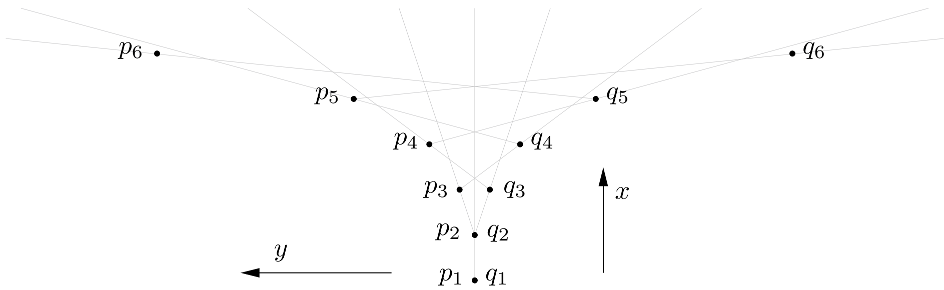

Figure 1 An example of a point set H_6 in a rotated coordinate system.

A plane graph G has a *one-sided Hamiltonian cycle with special edge vu* if it has a Hamiltonian cycle $(v = v_1, v_2, \ldots, v_n = u)$ such that vu is incident to the outer face and for every $j = 2, \ldots, n$, the two edges incident to v_j in the Hamiltonian cycle, i.e., edges $v_{j-1}v_j$ and $v_{j+1}v_j$, are consecutive in the rotation of v_j in the subgraph induced by $v_1, \ldots, v_j, v_{j+1}$ in G. In particular, the one-sided condition depends on the Hamiltonian cycle, its direction and its special edge. A more visual reformulation of the second condition is obtained using the closed bounded region D whose boundary is the Hamiltonian cycle. It is that in the embedding of G for every j either all the back-edges v_iv_j with $i < j$ are drawn inside D or in the open exterior of D. We let V_I be the set of vertices v_j which have a back-edge v_iv_j with $i < j-1$ drawn inside D and $V_O = V \setminus V_I$. The set V_I is the set of vertices having back-edges only inside D while vertices in V_O have back-edges only outside D.

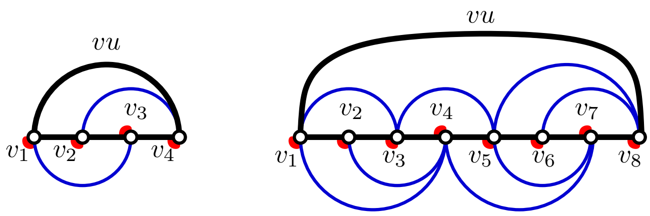

Figure 2 K_4 and a slightly larger graph both with a one-sided Hamiltonian cycle. Red angles indicate a side with no back-edge.

Recall that $\mathcal{C}'$ is the class of planar graphs which are *spanning* subgraphs of plane graphs admitting a one-sided Hamiltonian cycle. It is worth noting all subgraphs are POSH.

▶ **Proposition 2.** *Any subgraph of a POSH graph is POSH.*

Proof. As edge deletions preserve the POSH property by definition, it suffices to show that deleting a vertex preserves it as well. Let G be a POSH graph and let G' be its supergraph with a one-sided Hamiltonian cycle. Now after deleting v from G', adding an edge between its neighbours on the Hamiltonian cycle (if it does not exist) can be done along the two edges of v along the cycle. This is a supergraph of $G \setminus v$ with a one-sided Hamiltonian cycle. ◀

3 The embedding strategy

Our interest in POSH graphs is motivated by the following theorem.

▶ **Theorem 3.** *Let G' be POSH and let $v_1, \ldots, v_n$ be a one-sided Hamiltonian cycle of a plane supergraph G of G' on the same vertex set. Then there is a crossing-free embedding of G' on H_n with the property that v_i is placed on either p_i or q_i.*

Proof. It is sufficient to describe the embedding of the supergraph G on H_n. For the proof we assume that in the plane drawing of G the sequence $v_1, \ldots, v_n$ traverses the boundary of D in counter-clockwise direction. For each i vertex v_i is embedded at $\bar{v}_i = p_i$ if $v_i \in V_I$ and at $\bar{v}_i = q_i$ if $v_i \in V_O$.

Let $G_i = G[v_1, \ldots, v_i]$ be the subgraph of G induced by $\{v_1, \ldots, v_i\}$. The path $\Lambda_i = v_1, \ldots, v_i$ separates G_i. The *left part* GL_i consists of the intersection of G_i with D, the *right part* GR_i is G_i minus all edges which are interior to D. The intersection of GL_i and GR_i is Λ_i and their union is G_i. The counter-clockwise boundary walk of G_i consists of a path ∂R_i from v_1 to v_i which is contained in GR_i and a path from v_i to v_1 which is contained in GL_i, let ∂L_i be the reverse of this path.

Let $\bar{G}_i$ be the straight-line drawing of the plane graph G_i obtained by placing each vertex v_j at the corresponding $\bar{v}_j$. A vertex $\bar{v}$ of $\bar{G}_i$ is said to *see a point p* if there is no crossing between the segment $\bar{v}p$ and an edge of $\bar{G}_i$. By induction on i we show:

1. The drawing $\bar{G}_i$ is plane, i.e., non-crossing.
2. $\bar{G}_i$ and G_i have the same outer boundary walks.
3. Every vertex of ∂L_i in $\bar{G}_i$ sees all the points p_j with $j > i$ and every vertex of ∂R_i in $\bar{G}_i$ sees all the points q_j with $j > i$.

For $i = 2$ the graph G_i is just an edge and the three claims are immediate, for Property 3 just recall that the line spanned by p_1 and p_2 separates the p-side and the q-side of H_n.

Now assume that $i \in \{3, \ldots, n\}$, the properties are true for $\bar{G}_{i-1}$ and suppose that $v_i \in V_I$ (the argument in the case $v_i \in V_O$ works symmetrically). This implies that all the back-edges of v_i are in the interior of D whence all the neighbors of v_i belong to ∂L_{i-1}. Since $v_i \in V_I$ we have $\bar{v}_i = p_i$ and Property 3 of $\bar{G}_{i-1}$ implies that the edges connecting to $\bar{v}_i$ can be added to $\bar{G}_{i-1}$ without introducing a crossing. This is Property 1 of $\bar{G}_i$.

Since G_{i-1} and $\bar{G}_{i-1}$ have the same boundary walks and v_i (respectively $\bar{v}_i$) belong to the outer faces of G_i (respectively $\bar{G}_i$) and since v_i has the same incident edges in G_i as $\bar{v}_i$ in $\bar{G}_i$, the outer walks of G_i and $\bar{G}_i$ again equal each other, i.e., Property 2.

Let j be minimal such that $v_j v_i$ is an edge and note that ∂L_i is obtained by taking the prefix of ∂L_{i-1} whose last vertex is v_j and append v_i. The line spanned by $\bar{v}_j$ and $\bar{v}_i = p_i$ separates all the edges incident to $\bar{v}_i$ in $\bar{G}_i$ from all the segments $\bar{v}_\ell p_k$ with $\ell < j$ and $\bar{v}_\ell \in \partial L_i$ and $k > i$. This shows that every vertex of ∂L_i in $\bar{G}_i$ sees all the points p_k with $k > i$. For the proof of the second part of Property 3 assume some edge $\bar{v}_i\bar{v}_j$ crosses the line of sight from $\bar{v}_l$ to $q_k, k > i$, we refer to Figure 3. First note that this is only possible if $l \leq j$, since otherwise $\bar{v}_j\bar{v}_l$ separates $\bar{v}_i = p_i$ and q_k, because p_i is on the left as can be seen at $x = i$ and q_k is on the right as can be seen at $x = k$ by definition. Since $j = l$ is impossible by construction, we are left with the case $l < j$. Then one of $\bar{v}_i$ and $\bar{v}_l$, say $\bar{v}$, lies to the right of the oriented line $\bar{v}_j q_k$. However that implies that $\bar{v}_j\bar{v}$ has q_k on its left, which is a contradiction to the definition of q_k at $x = k$. This completes the proof of Property 3 and thus the inductive step.

Finally, Property 1 for $\bar{G}_n$ implies the theorem. ◀

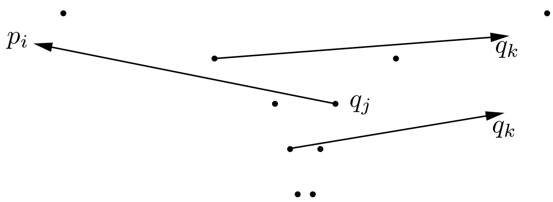

Figure 3 Vertices from ∂R_i see q_k

4 Plane bipartite graphs

In this section we consider bipartite plane graphs and show that they are POSH.

▶ **Theorem 4.** *Every bipartite plane graph $G = (V, E)$ is a subgraph of a plane graph G' on the same vertex set V which has a one-sided Hamiltonian cycle, i.e., G is POSH.*

Proof. Quadrangulations are the plane graphs with all faces of degree four. Equivalently they are the maximal plane bipartite graphs, i.e., any bipartite plane graph except stars is a subgraph of a quadrangulation. Thus since POSH graphs are closed under taking subgraphs, it suffices to prove the theorem for quadrangulations.

Let Q be a quadrangulation and let V_B and V_W be the *black* and *white* vertices of a 2-coloring. Label the two black vertices of the outer face as s and t. Henceforth, when talking about a quadrangulation we think of an embedded quadrangulation endowed with s and t. A *separating decomposition* is a pair $D = (Q, Y)$ where Q is a quadrangulation and Y is an orientation and coloring of the edges of Q with colors red and blue such that:

1. The edges incident to s and t are incoming in color red and blue, respectively.
2. Every vertex $v \notin \{s, t\}$ is incident to a non-empty interval of red edges and a non-empty interval of blue edges. If v is white, then, in clockwise order, the first edge in the interval of a color is outgoing and all the other edges of the interval are incoming. If v is black, the outgoing edge is the clockwise last in its color (see Figure 4).

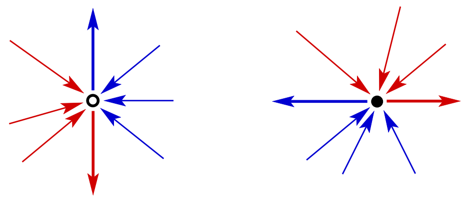

Figure 4 Edge orientations and colors at white and black vertices.[10]

Separating decompositions of a quadrangulation Q have been defined by de Fraysseix and Ossona de Mendez [18]. They show a bijection between separating decompositions and 2-orientations (orientations of the edges of Q such that every vertex $v \notin \{s, t\}$ has out-degree 2) and show the existence of a 2-orientation of Q with an argument related to flows and matchings. An inductive proof for the existence of separating decompositions was given by Felsner et al. [11], this proof is based on identifying pairs of opposite vertices on faces.

In a separating decomposition the red edges form a tree directed towards s, and the blue edges form a tree directed towards t. Each of the trees connects all the vertices $v \notin \{s, t\}$ to the respective root. Felsner et al. ([10, 11]) show that the edges of the two trees can be separated by a curve which starts in s, ends in t, and traverses every vertex and every inner face of Q. This curve is called the *equatorial line*.

If Q is redrawn such that the equatorial line is mapped to the x-axis with s being the left end and t being the right end of the line, then the red tree and the blue tree become *alternating trees* ([11], defined below) drawn in the upper respectively lower half-plane defined by the x-axis. Note that such a drawing of Q is a 2-page book embedding, we call it an *alternating 2-page book embedding* to emphasize that the graphs drawn on the two pages of the book are alternating trees.

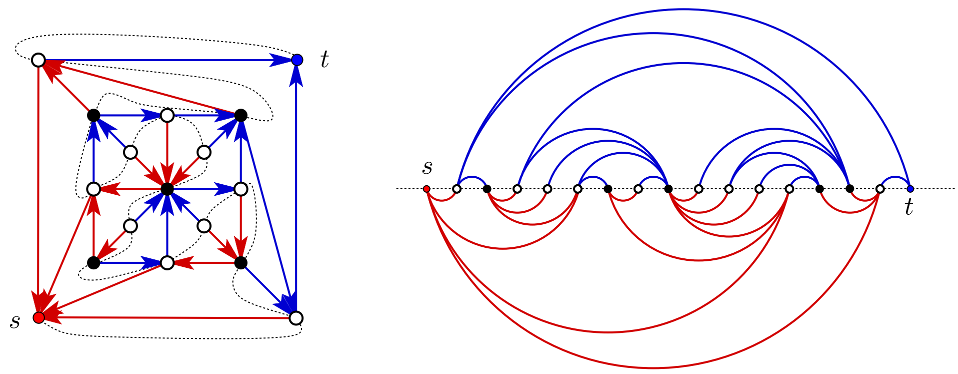

Figure 5 A quadrangulation Q with a separating decomposition S, and the alternating 2-page book embedding induced by the equatorial line of S[10].

An *alternating tree* is a plane tree T with a plane drawing such that the vertices of T are placed at different points of the x-axis and all edges are embedded in the half-plane above the x-axis (or all below). Moreover, for every vertex v it holds that all its neighbors are on one side, either they are all left of v or all right of v. In these cases we call the vertex v respectively a *right* or a *left vertex* of the alternating layout. Note that every vertex is a left vertex in one of the two trees and a right vertex in the other.

Let Q be a plane quadrangulation on n vertices and let S be a separating decomposition of Q. Let $s = v_1, v_2, \ldots, v_n = t$ be the spine of the alternating 2-page book embedding of Q based on S. Let Q^+ be obtained from Q by adding $v_n v_1$ and all the edges $v_i v_{i+1}$ which do not yet belong to the edge set of Q. By construction $v_1, v_2, \ldots, v_n$ is a Hamiltonian cycle of Q^+ and since the trees are alternating, black vertices have only blue edges to the left and white vertices have only red edges to the left. Thus this Hamiltonian cycle is one-sided with reverse edge $v_n v_1 = ts$. Hence Q is POSH. ◀

It is worth noting that the Hamiltonian cycle read in the reverse direction, i.e., as $v_n, v_{n-1}, \ldots, v_1$, is again one-sided, now the reverse edge is $v_1 v_n = st$.

5 Planar subcubic graphs

In this section we identify another large subclass of the $\mathcal{C}'$. Recall that 3-regular graphs are also known as cubic graphs and in subcubic graphs all vertices have degree at most 3.

▶ **Theorem 5.** *Every planar subcubic graph G is a spanning subgraph of a planar graph G' which has an embedding with a one-sided Hamiltonian cycle, i.e., G has a POSH embedding.*

▶ Remark 6. Note that we do *not* claim the theorem for all *plane* subcubic graphs. However, we are not aware of any connected subcubic plane graph, which is not POSH.

To prove this, we use Theorem 4 and the following lemmas:

▶ **Lemma 7.** *Let G be a subcubic graph. Then G admits a matching M such that contracting all the edges of M results in a bipartite multi-graph.*

Proof. Let (X, Y) be a partition the vertex-set of G such that the size of the cut, i.e., the number of edges in G with one endpoint in X and one endpoint in Y, is maximized. We claim that the induced subgraphs $G[X]$ and $G[Y]$ of G are matchings. Suppose that a vertex $v \in X$ has at least two neighbors in $G[X]$. Then v has at most one neighbor in Y, and hence moving v from X to Y increases the size of the cut by at least one, a contradiction. The same argument works for $G[Y]$.

Let M be the matching in G consisting of all the edges in $G[X]$ and $G[Y]$. Contracting the edges in M transforms $G[X]$ and $G[Y]$ into independent sets, and hence results in a bipartite multi-graph G/M. ◀

A *separating k-cycle* of a plane graph D is a simple cycle of length k, i.e., k edges, such that there are vertices of D inside the cycle.

▶ **Lemma 8.** *Let G be a subcubic planar graph. Then G admits a plane embedding D_G and a matching M such that contracting all the edges of M in D_G results in a bipartite multi-graph without separating 2-cycles.*

Proof. Let G be a subcubic planar graph. Without loss of generality G is connected, otherwise we just deal with the components first, then embed G in a way that all components are incident to the outer face.

Note that a 2-cycle can only arise by contracting one matching edge of a triangle or two matching edges of a quadrilateral. Consider an embedding D of G which minimizes the number of separating 3-cycles and among those minimizes the number of separating 4-cycles.

▷ Claim 9. D has no separating 3-cycle.

Proof. For illustration, see Figure 6. We will first show D has no *separating diamond*, that is, two triangles sharing an edge $e = uv$, at least one of which is a separating 3-cycle. Otherwise place u very closely to v. Now e is short and we reroute the other two edges of u such that

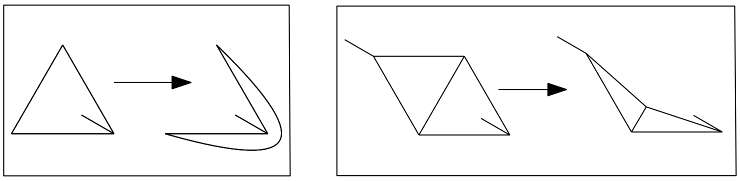

Figure 6 Procedure to eliminate triangles with an inner vertex. The procedure on the left eliminates isolated separating triangles, while the one on the right deals with separating diamonds.

they stay close to the corresponding edge of v. Since one of the triangles containing e was assumed to be separating the new drawing has fewer separating 3-cycles, a contradiction.

We are ready to show D has no separating 3-cycle. If T is a separating 3-cycle some edge has to go from a vertex v of T into its interior. Since v has degree at most 3 it has no edge to the outside of T. We can then redraw the edge e of T not incident to v outside of T closely to its two other edges. Again the new drawing has fewer separating 3-cycles: indeed, if the redrawn edge would be part of another 3-cycle, T is part of a separating diamond. $\lhd$

Now choose an edge set M of minimum cardinality, such that contracting it yields a bipartite multi-graph. The proof of Lemma 7 implies that M is a matching. Among those matchings, we choose M such that the number of separating 4-cycles which have 2 edges in M is minimized. Such separating 4-cycles are said to be *covered* by M.

$\rhd$ Claim 10. M covers no separating 4-cycle.

Proof. Suppose $Q = v_1v_2v_3v_4$ is a separating 4-cycle such that v_1v_2 and $v_3v_4 \in M$ and v_1 has an edge e_I to the inside, thus no edge to the outside.

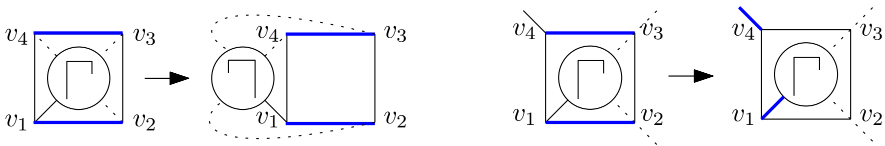

Figure 7 Procedure to eliminate quadrilaterals with an inner vertex. The redrawing (left) cannot be applied in the right case, where we are changing the blue matching to avoid a separating 2-cycle.

If v_4 has no edge to the outside either, we change D to a drawing D' by redrawing the part Γ of D inside Q outside of it reflected across v_1v_4, see Figure 7. In D' the original separating 4-cycle is no longer separating. We claim that no new separating 3-cycle or 4-cycle that is covered by M was created. The claim contradicts the choice of D or M.

To prove the claim note that $S = \{v_2, v_3\}$ is a 2-separator, unless Q is the outer face of D, so let's assume first that it is not. Thus a separating 3- or 4-cycle has to live on one side of S, since the shortest path between them in $Q \cup \Gamma$ except their edge is of length 3 except if both v_2 and v_3 are adjacent to the same vertex of Γ, in which case Q is the outer face, a contradiction. Let X be the component of $G \setminus S$ containing Γ. Then the number of vertices inside 3- or 4-cycles that are not part of X is unchanged in D', since the face X is located in is still the same. The only 3- or 4-cycles in $X \cup S$ that were not reflected in their entirety are the ones containing the edge v_2v_3. Since Q is assumed not to be the outer face, at least one of v_2 and v_3 is not connected to Γ. Thus such a cycle C is a 4-cycle consisting of v_2, v_3, one of v_1 or v_4 as well as a common neighbour of v_2 and v_4 or v_1 and v_3 in Γ. However v_1v_2 or v_3v_4 respectively would be the only edge in $M \cap C$. This is a contradiction to the fact that contracting M yields a bipartite graph.

Now if Q is the outer face of D, it is still true that the only cycles not reflected in their entirety contain v_2v_3. However v_2 and v_3 could both be adjacent to a vertex in Γ, either a common neighbour for a 3-cycle or two adjacent neighbours for a 4-cycle. Since v_2 and v_3 are already covered by M, this 3-cycle would contain no edge in M, whereas the 4-cycle would contain at most one. Therefore both of these contradict the definition of M.

Therefore, we know that v_4 has an edge e_O to the outside. This edge does not go to any vertex of the quadrilateral, because the only candidate left would be v_2, but this would yield that one of the triangles $v_2v_3v_4$ and $v_1v_2v_4$ is separating.

Change the matching M to an edge set M' by removing v_1v_2 and v_3v_4 from it and adding e_O and e_I. Contracting M' still results in a bipartite graph, because the same four facial cycles that contained our previous edges contain exactly one new edge each as well, so their size after contraction does not change. Thus M' is a matching, because it has the same cardinality as M and is therefore minimal as well. We conclude M' does not cover v_2 or v_3, because M did not contain any other edge than v_1v_2 and v_3v_4 at them either. Since M' does not contain two edges from quadrilateral $v_1, \ldots, v_4$ but M is minimal, there has to be a separating quadrilateral, of which M' contains two edges, but M doesn't. If such a separating quadrilateral Q contains e_I, then it has to contain another edge incident to v_1. It cannot contain v_1v_2, because we know v_2 is not covered by M'. Therefore it contains v_1v_4 and consequently e_O. The same argumentation works to show that if it contains e_O, then it also contains e_I. This is a contradiction to the existence of M' because the endpoints of e_O and e_I are on the outside and the inside of the quadrilateral respectively and therefore non-adjacent. ◁

So we proved that our choice of M makes sure that no separating 2-cycles will be present in the contracted plane bipartite multi-graph. ◀

▶ Remark 11. The embedding D and the matching M can be constructed starting from an arbitrary embedding and matching by iterative application of the operations used in the proof.

Proof of Theorem 5. Now let B be the plane bipartite multi-graph obtained from G by contracting the edges in M without changing the embedding any further. Let B' be the underlying simple graph of B and let Q be a quadrangulation or a star which has B' as a spanning subgraph. The proof of Theorem 4 shows that there is a left to right placement $v_1, \ldots, v_s$ of the vertices of Q on the x-axis such that for each $i \in [s]$ all the edges v_jv_i with $j < i - 1$ are in one half-plane and all edges v_iv_j with $j > i + 1$ are in the other half-plane. Delete all the edges from Q which do not belong to B', and duplicate the multi-edges of B in the drawing. This yields a 2-page book embedding Γ of B.

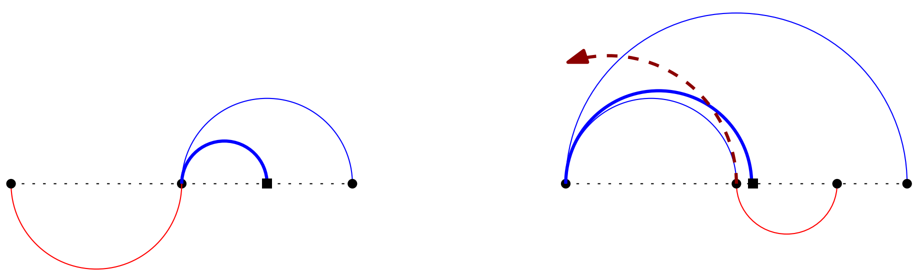

Figure 8 How to add leaves: The leaf is plotted as a square, its new adjacent edge fat.

Let v be a contracted vertex of B. Vertex v was obtained by contracting an edge $uw \in M$. If u and/or w did not have degree 3, we add edges at the appropriate places into the embedding that end in leaves, see Figure 8. To add an edge to u for instance, choose a face f incident to u that is not contracted into a 2-cycle. Let e and e' be the two edges incident to both v and f. If the angle between e and e' contains part of the spine (the x-axis), we put the leaf on the spine close to v connected to v with a short edge below or above the spine, in a way to accomodate the local vertex condition of v. If it doesn't, assume without loss of generality it is in the upper half-plane and that edge e is the edge closer to the spine. This edge is unique because both edges at v delimiting f go upwards and therefore both to the same side, say right of v. Route the new edge closely along e then put the leaf just next to the other endpoint x of e. Edges that would cross this new edge cannot cross e, thus the only possibility are edges incident to x that emanate into the upper halfspace. However those edges have to go to the left of x by its local vertex condition. These edges do not exist, as any such edge would have to cross e', see the dashed line in Figure Figure 8. Thus the new edge is uncrossed. This procedure will be done to every vertex first. Note that the resulting graph stays bipartite and the local vertex conditions are still fulfilled, but now every contracted vertex has degree 4. This makes the case distinction of splitting the vertices easier.

We now show how to undo the contractions, i.e., *split* vertices, in the drawing Γ in such a way that at the end we arrive at a one-sided 2-page book drawing $\Gamma^\star$ of G, that is, a 2-book embedding of G with vertex-sequence $v_1, \ldots, v_n$ such that for every $j \in \{1, \ldots, n\}$ the incident back-edges v_iv_j with $1 \leq i < j$ are all drawn either on the spine or on the same page of the book embedding (all above or all below the spine). Once we have obtained such a book embedding, we can delete the artificial added leaves, then add the spine edges (including the back edge from the rightmost to the leftmost vertex) to G to obtain a supergraph G^+ of G which has a one-sided Hamiltonian cycle, showing that G is POSH.

Before we advance to show how we split a single vertex v of degree four into an edge $uw \in M$, we first want to give an overview of the order in which the different splits, the *far splits* and *local splits* are applied. We will then describe what these different splits actually mean. To split all the degree four vertices we proceed as follows:

First we split all vertices which are subject to a far split, from the outside inwards. More precisely, define a partially ordered set on the edges incident[1] to vertices subject to a far split in the following way: Every edge e defines a region R_e which is enclosed by e and the spine. Now order the edges by the containment order of regions R_e. From this poset, choose a maximum edge and then a vertex that needs a far split incident to that edge. When no further far split is possible we do all the local splits. These splits are purely local, so they cannot conflict with each other. Therefore their order can be chosen arbitrarily.

We label the edges of v in clockwise order as e_1, e_2, e_3, e_4 such that in G the edges e_1, e_2 are incident to u and e_3, e_4 are incident to w. If the two angles $\angle e_2e_3$ and $\angle e_4e_1$ together take part of both half-planes defined by the spine, then it is possible to select two points left and right of the point representing v in Γ and to slightly detour the edges e_i such that no crossings are introduced and one of the two points is incident to e_1, e_2 and the other to e_3, e_4. The addition of an edge connecting the two points completes the split of v into the edge $uw \in M$. Figure 9 shows a few instances of this *local* split.

The above condition about the two angles is not fulfilled if and only if all four edges of v emanate into the same halfspace, say the upper one, and the clockwise numbering starting at the x-axis is either e_4, e_1, e_2, e_3 or e_2, e_3, e_4, e_1. The two cases are the same up to exchanging

[1] There will be a clarification later as to what this means exactly.

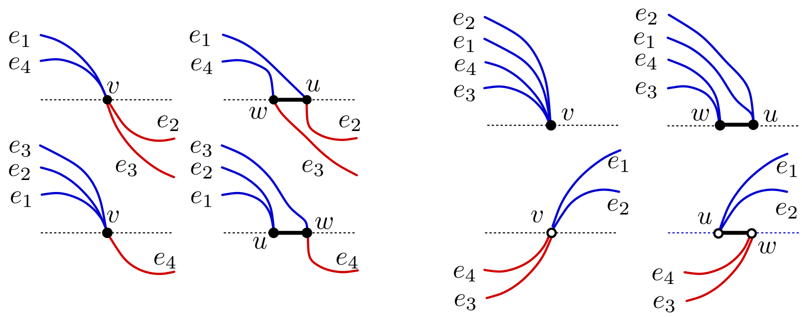

Figure 9 Four cases for the local split of a vertex v.

the names of u and w, therefore we can assume the first one. A more important distinction is whether most e_i end to the left or right of v. Note that in the ordering given by Γ, all e_i go to the same side, since they are all in the same halfplane. However, if v is not the first vertex we are splitting, it may happen, that a single edge on the spine to the other side exists, see Figure 10. For all $i \in [4]$ let v_i be the other endpoint of e_i than v. While it can happen that some of the v_i coincide due to multi-edges, we will first discuss the case that they don't. In the left case we put u slightly left of v_1 while in the right case u is put slightly right of v_2, connecting u to this close vertex by a spine edge. In both cases we leave w at the former position of v. Figure 10 shows the right case and Figure 11 the left.

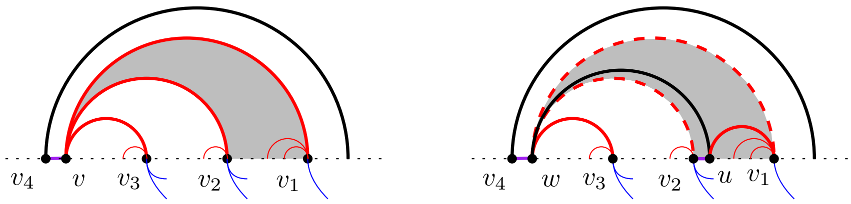

Figure 10 Far split with v_i to the right except for the spine edge neighbor.

To see that in the left case edges uv_2 and uw are completely free of crossings, observe that we can route them close to the path v_2vv_1 and the edge v_1v respectively in the original drawing (dashed in Figure 11). It is important to note here, that due to the order in which we chose to do the splits, v_1 and v_2 are still original vertices of B, that is, they have not been split in the upper half-plane and thus still don't have two edges emanating to the upper half-plane to both sides. Therefore, similarly to the argumentation for adding leaves, no edge incident to v_1 crosses uw or uv_2. The right case is analogous, just exchange the roles of v_1 and v_2.

This kind of split is a *far* split. For the purposes of incidence in the poset structure mentioned above, vertices are not only considered incident to any edge they are an endpoint of, but the spine neighbour of u (v_1 or v_2) is also considered to be incident to the edge uw. For illustration, consider the outermost black edge in Figure 10 (left), it is considered incident to v.

In the following we describe how the different kinds of splits are affected by the presence of multi-edges. The first thing to note is that local splits can be done in the same way, since we did not mention the end vertices at all.

Concerning the far splits, firstly we talk about the case that exactly two edges go from one vertex to another: As depicted in Figures 10 and 11 the case $v_2 = v_3$ and/or $v_4 = v_1$ is unproblematic, in this case we keep the dashed line(s) in the drawing. Double-edges are

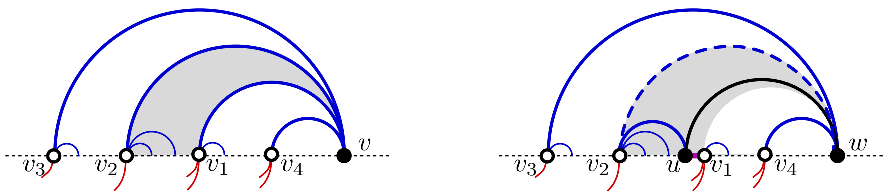

Figure 11 Far split within the gray region with v_i to the left in the upper half-plane.

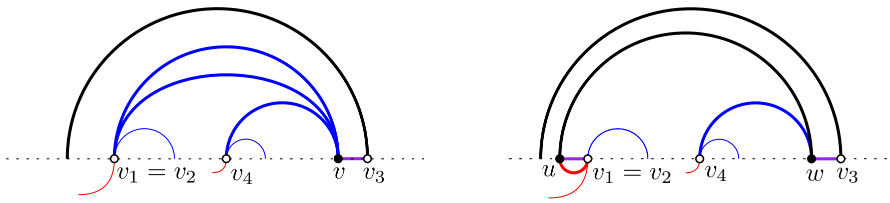

Figure 12 If $v_1 = v_2$, a double spine edge is created. Here $e_3 = vv_3$ is a spine edge.

consecutive because non-consecutive double-edges are separating 2-cycles, which we avoided in the construction. Thus the last case of a double-edge to consider is $v_1 = v_2$. In this case, we follow the same strategy of placement of u and w, but this results in a double-edge on the spine between u and $v_1 = v_2$, see Figure 12. As in later local splits, we might be interested what half-space the angle between the two spine edges is part of, we interpret one of these edges as a spine edge and the other as an edge which is above or below the spine depending on the right vertex of the two. This might be u or v_1, depending on whether we are in the left or right case. It is important for the one-sidedness condition to choose this direction so that all left neighbours of the right vertex of the two are reached by edges emanating into the same halfspace and/or spine edges.

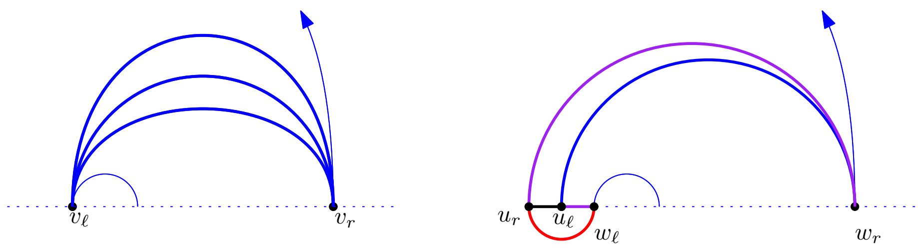

Figure 13 Doing a double split means splitting two vertices simultaneously.

Secondly, if there are three edges between a left vertex v_ℓ and a right vertex v_r, say in the upper half-plane, we will split both simultaneously, for illustration, see Figure 13. Since three edges go between these two vertices, there is just one more edge e left for v_ℓ. Therefore we can find a place on the spine just to the right or to the left of v_ℓ which is free, because the edge e is on the other side. Now we split v_ℓ into u_ℓ and w_ℓ and v_r into u_r and w_r simultaneously where w_ℓ and w_r are the vertices with the edge that goes somewhere else on both sides. From left to right we put u_r then u_ℓ just left of the position of v_l, which is the new position of w_ℓ. The three of them are connected by spine edges, just u_r and w_ℓ have an edge in the lower half-plane. These edges are not crossed, because the vertices are close enough together. Finally we put w_r at the position of v_r and add edges to w_r and w_ℓ in the upper half-plane. These edges are not crossed, because any edge crossing them would have crossed the triple edge in the original drawing.

This kind of split is a *double* split. These splits are purely local, so they can be performed together with the local splits in the end.

The last case is that all four edges of a given vertex go to the same vertex, this is a full connected component of the bipartite graph, because it has maximum degree 4. This component goes back to a K_4 component in the cubic graph that had two independent edges contracted. A one-sided Hamiltonian cycle of K_4 is illustrated in Figure 2. We apply another local double split which consists of replacing the 4 parallel edges by this drawing, embedded close to the place of one of the original vertices.

This completes the proof of Theorem 5. ◀

6 2-Trees

From the positive results in Sections 4 and 5 one might expect that "sufficiently sparse" planar graphs are POSH. This section shows that 2-trees are not.

A 2-*tree* is a graph which can be obtained, starting from a K_3, by repeatedly selecting an edge of the current graph and adding a new vertex which is made adjacent to the endpoints of that edge. We refer to this operation as *stacking* a vertex over an edge.

From the recursive construction it follows that a 2-tree on n vertices is a planar graph with $2n-3$ edges. We also mention that 2-trees are series-parallel planar graphs. Another well studied class which contains 2-trees as a subclass is the class of (planar) Laman graphs.

Fulek and Tóth have shown that planar 3-trees admit n-universal point sets of size $O(n^{3/2}\log n)$. Since every 2-tree is an induced subgraph of a planar 3-tree the bound carries over to this class.

▶ **Theorem 12.** *There is a 2-tree G on 499 vertices that is not POSH.*

Proof. Throughout the proof we assume that a 2-tree G is given together with a left to right placement $v_1,\dots,v_n$ of the vertices on the x-axis such that adding the spine edges and the reverse edge v_nv_1 to G yields a plane graph with a one-sided Hamiltonian cycle.

For an edge e of G we let $X(e)$ be the set of vertices which are stacked over e and $S(e)$ the set of edges which have been created by stacking over e, i.e., each edge in $S(e)$ has one vertex of e and one vertex in $X(e)$. We partition the set $X(e)$ of an edge $e=v_iv_j$ with $i<j$ into a left part $XL(e)=\{v_k\in X(e):k<i\}$, a middle part $XM(e)=\{v_k\in X(e):i<k<j\}$, and a right part: $XR(e)=\{v_k\in X(e):j<k\}$.

▷ Claim 13. For every edge $|XR(e)|\le 2$.

Suppose that $|XR(e)|\ge 3$. Each vertex in this set has all its back-edges on the same side. Two of them use the same side for the back edges to the vertices of e. This implies a crossing pair of edges, a contradiction.

▷ Claim 14. If for all $e'\in S(e)$ we have $|X(e')|\ge 3$, then $|XM(e)|\le 3$.

Suppose that $e=v_iv_j$ with $i<j$ is in the upper half-plane and there are four vertices x_1,x_2,x_3,x_4 in $XM(e)$. One-sidedness implies that the four edges x_kv_j are in the upper half-plane. Thus if x_1,x_2,x_3,x_4 is the left to right order, then the edges v_ix_2, v_ix_3, and v_ix_4 have to be in the lower half-plane. Now let $e'=v_ix_3$ and consider the three vertices in $X(e')$. Two of them, say y_1,y_2, are on the same side of x_3. First suppose $y_1,y_2\in X(e')$ are left of x_3. The edges of v_ix_2 and x_2v_j enforce that y_1,y_2 are between x_2 and x_3. Due to edge x_2v_j the edges v_iy_1,v_iy_2 are in the lower half-plane. One-sidedness at x_3 requires that y_1x_3 and y_2x_3 are also in the lower half-plane. This makes a crossing unavoidable.

Now suppose that $y_1, y_2 \in X(e')$ are right of x_3. The edges v_ix_4 and x_4v_j enforce that y_1, y_2 are between x_3 and x_4. Due to the edge x_3v_j the edges v_iy_1 and v_iy_2 are in the lower half-plane. Now let y_1 be left of y_2. One-sidedness at y_2 requires that x_3y_2 is also in the lower half-plane, whence, there is a crossing between v_iy_1 and x_3y_2. This completes the proof of the claim.

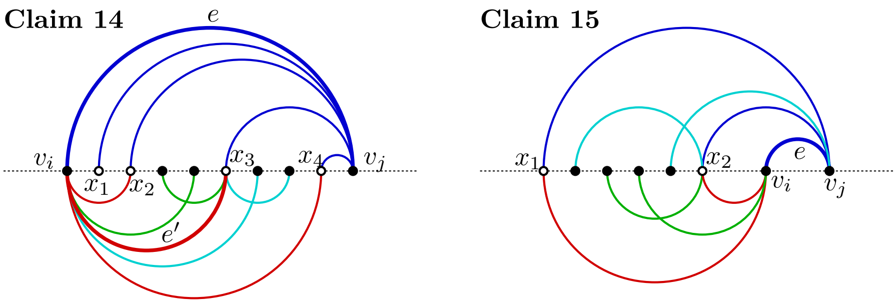

Figure 14 Illustrating the proofs of the claims.

▷ Claim 15. If $XL(e) \geq 2$ and x is the rightmost element of $XL(e)$, then $XL(e') \leq 1$ for some $e' \in S(e)$ incident with x and $XR(e') = \emptyset$ for both.

Suppose that $e = v_iv_j$ with $i < j$ is in the upper half-plane and there are two vertices x_1, x_2 in $XL(e)$. We assume that x_2 is the rightmost element of $XL(e)$. From one-sidedness at v_j we know that x_1v_j and x_2v_j are in the upper half-plane. Now x_1v_i and hence also x_2v_i are in the lower half-plane. All the vertices of $X(x_2v_i)$ and $X(x_2v_j)$ are in the region bounded by x_1v_j, v_jv_i, v_ix_1, in particular $XR(e') = \emptyset$ for both. Suppose for contradiction that we have $y_1, y_2 \in XL(x_2v_i)$ and $z_1, z_2 \in XL(x_2v_j)$. By one-sidedness the edges from x_2 to the four vertices y_1, y_2, z_1, z_2 are in the same half-plane. If they are in the lower half-plane and y_1 is left of y_2 there is a crossing between y_1x_2 and y_2v_i. If they are in the upper half-plane and z_1 is left of z_2 there is a crossing between z_1x_2 and z_2v_j. The contradiction shows that $XL(x_2v_i) \leq 1$ or $XL(x_2v_j) \leq 1$, since $x = x_2$ this completes the proof of the claim.

We are ready to define the graph G and then use the claims to prove that G is not POSH. The graph G contains a *base edge* e and seven vertices stacked on e, i.e., $|X(e)| = 7$. For each edge $e' \in S(e)$ there are five vertices stacked on e'. Finally, for each edge e'' introduced like that three vertices are stacked on e''. Note that there are $7 \cdot 2 = 14$ edges e', $14 \cdot 5 \cdot 2 = 140$ edges e'' and $140 \cdot 3 \cdot 2 = 840$ edges introduced by stacking on an edge e''. In total the number of edges is $995 = 2n - 3$, hence, the graph has 499 vertices.

Now suppose that G is POSH and let $v_1, \ldots, v_n$ be the order of vertices on the spine of a certifying 2-page book embedding. Let $e = v_iv_j$ with $i < j$ be the base edge. Assume by symmetry that e is in the upper half-plane. From Claim 13 we get $|XR(e)| \leq 2$ and from Claim 14 we get $|XM(e)| \leq 3$, it follows that $|XL(e)| \geq 2$. Let x_1 and x_2 be elements of $XL(e)$ such that x_2 is the rightmost element of $XL(e)$. Let $e' = x_2v_i$ and $e'' = x_2v_j$ then $XR(e') = \emptyset = XR(e'')$ by Claim 15. From Claim 14 applied to e' and e'' we deduce that $|XM(e')| \leq 3$ and $|XM(e'')| \leq 3$. Hence $|XL(e')| \geq 2$ and $|XL(e'')| \geq 2$. This is in contradiction with Claim 13. Thus there is no spine ordering for G which leads to a one-sided crossing-free 2-page book embedding. ◀

7 Concluding remarks

We have examined the exploding double chain as a special point set (order type) and shown that the initial part H_n of size $2n-2$ is n-universal for graphs on n vertices that are POSH. We believe that the class of POSH graphs is quite rich. On the sparse side, the result on bipartite graphs might be generalized, while for triangulations, the sheer number of Hamiltonian cycles in 5-connected graphs [1] makes it likely one of them is one-sided.

▶ **Conjecture 16.** *Every triangle-free planar graph is POSH.*

▶ **Conjecture 17.** *Every 5-connected planar triangulation is POSH.*

We have shown that 2-trees and their superclasses series-parallel and planar Laman graphs are not contained in the class $\mathcal{C}'$ of POSH graphs. The question whether these classes admit universal point sets of linear size remains intriguing.

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